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to obtain freeze-out value of n_x , go to limit:

$$\frac{dn_x}{dt} \rightarrow 0 \text{ for } t \rightarrow \infty$$

We have:

$$\frac{1}{a^3} \frac{d(n_x a^3)}{dt} = \frac{dn_x}{dt} + 3 \frac{\dot{a}}{a} \cdot n_x$$

$$= \frac{dn_x}{dt} + 3H \cdot n_x$$

thus we demand:

$$\frac{dn_x}{dt} = 0 \text{ at late times}$$

furthermore, at late times:

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$$\frac{m}{T} \rightarrow \infty$$

$$\Rightarrow n_x^{(0)} \sim e^{-\frac{m}{T}} \rightarrow 0$$

exponentially fast at late times

\Rightarrow (74) becomes:

$$3H \cdot n_x \simeq \langle \sigma_x v \rangle \cdot n_x^2 \quad (75)$$

if we realize,

$$\langle \sigma_x v \rangle n_x = \langle \Gamma_x \rangle$$

\uparrow
thermally averaged
reaction rate

this is nothing else than: ||0

$$H \simeq \langle \Gamma_x \rangle$$

our old freeze-out criterion!

We get:

$$(75) \Leftrightarrow n_x \sim \frac{H}{\langle \sigma_x v \rangle}$$

thus we get for ρ_x :

$$\rho_x = m_x n_x \sim \frac{m_x H}{\langle \sigma_x v \rangle} \quad (76)$$

and at freeze-out:

$$H \sim \frac{T_{f.o.}^2}{M_P}, \quad T_{f.o.} \sim m_x$$

$$(76) \Rightarrow \rho_x \sim \frac{m_x T_{f.o.}^2}{M_P \langle \sigma_x v \rangle} \quad ||1$$
$$\sim \frac{T_{f.o.}^3}{M_P \langle \sigma_x v \rangle} \quad (77)$$

now, ρ_x is ρ_x at freeze-out,
dilutes until now \rightarrow non-relat.
matter:

$$\rho_{x,0} = \rho_x \left(\frac{a_{f.o.}}{a_0} \right)^3 \sim$$
$$\sim \frac{T_0^3}{M_P \langle \sigma_x v \rangle} \left(\frac{T_{f.o.}}{T_0} \cdot \frac{a_{f.o.}}{a_0} \right)^3 \quad (78)$$

now, because:

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$$S = \text{const.} = \rho \cdot a^3 \Rightarrow T \sim \frac{1}{a}$$

$$\Rightarrow \frac{T_{f.o.}}{T_0} \cdot \frac{a_{f.o.}}{a_0} \sim O(1)$$

$$\Rightarrow \rho_{x,0} \sim \frac{T_0^3}{M_P \langle \sigma_{x\nu} \rangle} \quad (79)$$

We can now calculate to-day's density parameter $\Omega_{x,0}$ of the X particles:

$$(80) \quad \Omega_{x,0} = \frac{\rho_{x,0}}{\rho_{cr,0}} \sim \frac{T_0^3}{M_P^3} \cdot \frac{1}{H_0^2 \langle \sigma_{x\nu} \rangle}$$

now we have:

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$$T_0 \sim 10^{-4} \text{ eV}, \quad M_P \sim 2 \cdot 10^{18} \text{ GeV}$$

$$\frac{H_0}{c} \approx 3 \cdot 10^{-9} \frac{\text{s}}{\text{m}} \cdot \frac{72 \cdot 10^3 \text{ m}}{5 \cdot \text{Mpc}}$$

$$\approx \frac{2 \cdot 10^{-4}}{\text{Mpc}} \approx \frac{2 \cdot 10^{-4}}{3 \cdot 10^6 \cdot 10 \cdot 10^{12} \text{ km}}$$

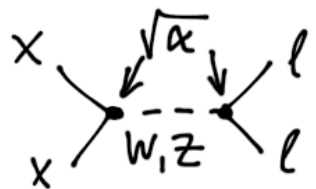
$$\approx 6 \cdot 10^{-29} \frac{1}{\text{cm}}$$

plug into (80):

$$\Omega_{x,0} \sim 10^{-94} \cdot \frac{10^{58}}{36} \cdot \frac{\text{cm}^2}{\langle \sigma_{x\nu} \rangle}$$

$$\Rightarrow \Omega_{X,0} \sim \frac{3 \cdot 10^{-38} \text{ cm}^2}{\langle \sigma_X v \rangle} \quad (81) \quad 114$$

for $\langle \sigma_X v \rangle$ we get:

σ_X :  $\alpha \sim 0.01$
(EW interaction!)

$$\Rightarrow \mu \sim \frac{\alpha}{M_W^2} \sim \alpha G_F$$

$$\Rightarrow \langle \sigma_X v \rangle \sim \alpha^2 G_F^2 m_X^2$$

$$\begin{aligned} & 10^{-4} \cdot \left(\frac{1}{(300 \text{ GeV})^2} \right)^2 \cdot (100 \text{ GeV})^2 \cdot \left(\frac{m_X}{100 \text{ GeV}} \right)^2 \\ & \approx 10^{-10} \text{ GeV}^{-2} \cdot \left(\frac{m_X}{100 \text{ GeV}} \right)^2 \quad (82) \end{aligned}$$

little conversion help: 115

$$\text{GeV}^{-1} \sim 0.2 \text{ fm} = 2 \cdot 10^{-14} \text{ cm}$$

$$\Rightarrow \langle \sigma_X v \rangle \sim \left(\frac{m_X}{100 \text{ GeV}} \right)^2 \cdot 10^{-37} \text{ cm}^2$$

plug this into (81):

$$\Rightarrow \Omega_{X,0} \sim 0.3 \cdot \left(\frac{100 \text{ GeV}}{m_X} \right)^2 \quad (83)$$

\leadsto A WIMP with mass of $\sim 100 \text{ GeV}$, natural in EW-scale SUSY, is a perfect DM candidate!

III Structure formation, 116
density perturbations & CMB

We will discuss the Newtonian theory of small perturbations, as the relevant length scales will be always be $\ll H^{-1}$...

Relevant equations of motion:
 'fluid of non-relativistic dust'

$$\Rightarrow \begin{cases} \dot{\rho} + \vec{\nabla}(\rho \vec{v}) = \dot{\rho} + \rho \cdot \vec{\nabla} \vec{v} = 0, \rho = \rho(t) \\ \text{continuity eq.} \\ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{\rho} \vec{\nabla} P + \vec{g} \end{cases} \quad (84)$$

Euler fluid e.o.m.

supplemented by the gravitational field equation: 117

$$\begin{cases} \vec{\nabla} \times \vec{g} = 0 \\ \vec{\nabla} \cdot \vec{g} = \Delta \phi = -4\pi G \rho \end{cases} \quad (85)$$

using a few little tricks, like:

$$\vec{\nabla} \cdot \vec{r} = 3 \quad \text{and} \quad (\vec{r} \cdot \vec{\nabla}) \cdot \vec{r} = \vec{r}$$

one can show starting with an ansatz for \vec{v} (Hubble's law):

$$\vec{v}_0(\vec{r}, t) = \vec{r} \cdot H(t), \quad H(t) = \frac{\dot{a}}{a} \quad (86)$$

... that the continuity eq. 118
is solved by:

$$\rho_0(t) = \rho_0 \cdot \left(\frac{a_0}{a(t)} \right)^3 \quad (87)$$

as befits non-relativistic matter.

For such ρ then (85) has the solution:

$$\bar{g}_0(t) = -\bar{\gamma} \cdot \frac{4\pi G}{3} \cdot \rho_0(t) \quad (88)$$

and if we plug this into the Euler equation, we get the Friedmann eq.:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \rho_0(t) \quad (89)$$

which is solved by: 119

$$a(t) \sim t^{2/3} \quad (90)$$

again befitting non-relativistic matter.

Now perturb this background solution (86) - (88):

$$\left. \begin{aligned} P(\bar{r}, t) &= P_0(t) + P_1(\bar{r}, t) \\ v(\bar{r}, t) &= \bar{v}_0(\bar{r}, t) + \bar{v}_1(\bar{r}, t) \\ \bar{g}(\bar{r}, t) &= \bar{g}_0(t) + g_1(\bar{r}, t) \end{aligned} \right\} (91)$$

and write the perturbation in a spatial Fourier decomposition:

$$\begin{aligned}
 p_1(\vec{r}, t) &= p_1(t) \cdot e^{i\vec{k}_{\text{phys.}} \cdot \vec{r}} && |20 \\
 &= p_1(t) \cdot e^{i\frac{\vec{k}}{a} \cdot \vec{r}} && (92)
 \end{aligned}$$

\vec{k} : comoving
wave number

etc. ...

and plug this into the e.o.m. (84)

& (85):

$$\Rightarrow \begin{cases} \dot{p}_1 + 3H p_1 + i p_0 \frac{\vec{k} \cdot \vec{v}_1}{a(t)} = 0 \\ \dot{\vec{v}}_1 + H \vec{v}_1 = -i \frac{c_s^2}{p_0(t)} \frac{\vec{k}}{a(t)} \cdot p_1 + \vec{g}_1 \\ \vec{g}_1 = i \cdot 4\pi G p_1 \cdot a(t) \cdot \frac{\vec{k}}{k^2} \end{cases} \quad (93)$$

here: |21

$$c_s = \sqrt{\frac{dp}{d\rho}} \quad \text{'speed of sound'} \quad (94)$$

We see that:

$$\vec{v}_{1\perp} : \vec{v}_{1\perp} \cdot \vec{k} = 0, \vec{v}_{1\perp} \perp \vec{k}$$

falls as: $\sim \frac{1}{a(t)}$

For the other perturbations define:

$$\delta(t) \equiv \frac{p_1(t)}{p_0(t)} \quad \text{'density contrast'} \quad (95)$$

$$\epsilon(t) = -i \cdot a(t) \cdot \frac{\vec{k} \cdot \vec{v}_1}{k^2} \sim \vec{v}_{1\parallel}$$

in terms of these:

$$\left. \begin{aligned} \dot{\delta} &= \frac{\bar{k}^2}{a^2(t)} \cdot \epsilon \\ \dot{\epsilon} &= \left(-c_s^2 + \frac{4\pi G \rho_0(t)}{\bar{k}^2} \right) \cdot \delta \end{aligned} \right\} \begin{array}{l} 122 \\ (96) \end{array}$$

can be combined:

$$\ddot{\delta} + 2H \cdot \dot{\delta} + \left[c_s^2 \cdot \frac{\bar{k}^2}{a^2} - 4\pi G \cdot \rho_0(t) \right] \cdot \delta = 0 \quad (97)$$

The bracket [...] in (97) defines a critical "Jeans wave number" k_J :

$$\frac{k_J}{a(t)} = \sqrt{\frac{4\pi G \rho_0(t)}{c_s^2}} \quad (98)$$

its meaning:

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i) $\delta \sim \cos\left(c_s \frac{k}{a} t\right)$ 'oscillatory'
for $k \gg k_J$

ii) $\delta \sim t^p, p > 0$ 'growing and non-oscillatory'
for $k \ll k_J$

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exist only in matter- or radiation-dominated regime, because for ρ_{Λ} or ρ_{curv} we have the damping term:

$2H \cdot \dot{\delta} \sim 2H^2 \delta \sim \rho_{\text{tot}} \cdot \delta$ falling