

II) The Thermal Universe

thermal plasma of mostly relativistic particles at early times...

↷ need to recall a bit of thermodynamics and cross sections of scattering

density n , ρ and p of dilute gas with g internal d.o.f.:

$$\left. \begin{aligned} n &= \frac{g}{(2\pi)^3} \int d^3p \cdot f(\vec{p}) \\ \rho &= \frac{g}{(2\pi)^3} \int d^3p \cdot E(\vec{p}) f(\vec{p}) \\ p &= \frac{g}{(2\pi)^3} \int d^3p \cdot \frac{|\vec{p}|^2}{3E(\vec{p})} f(\vec{p}) \end{aligned} \right\} (23)$$

$$\text{now: } |\vec{p}| = \sqrt{E^2 - m^2}$$

$$\Rightarrow d|\vec{p}| = \frac{E dE}{\sqrt{E^2 - m^2}}$$

$$\text{and } d^3p = 4\pi |\vec{p}|^2 d|\vec{p}|$$

$$\Rightarrow n = \frac{g}{2\pi^2} \int_m^\infty dE \cdot E \cdot \sqrt{E^2 - m^2} \cdot f(E) \quad 61$$

$$(24) \quad P = \frac{g}{2 \cdot \pi^2} \int_m^\infty dE \cdot E^2 \sqrt{E^2 - m^2} \cdot f(E)$$

$$P = \frac{g}{6\pi^2} \int_m^\infty dE \cdot (E^2 - m^2)^{3/2} \cdot f(E)$$

$f(E)$: partition function of particles

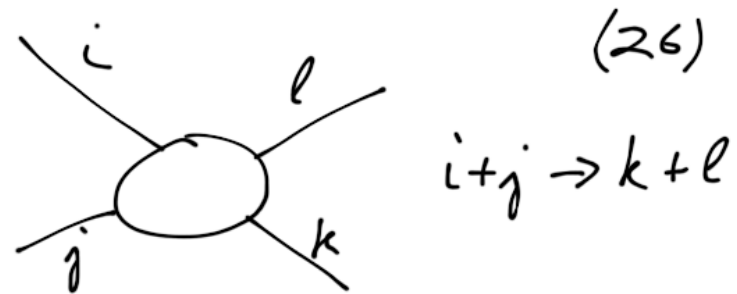
$$f(E) = \left(e^{\frac{E - \mu}{kT}} \pm 1 \right)^{-1} \quad (25)$$

+ : Fermi statistics

- : Bose statistics

μ : chemical potential 62
 $\hat{=}$ change of partition function
 by $\sim e^{-\frac{\mu}{T}}$ for a newly
 added particle

consider a reaction of particle
 species i, j, k, l :



$$\leadsto \mu_i + \mu_j = \mu_k + \mu_l \quad \text{chemical equilibrium}$$

relativistic limit $T \gg m$: 63

$$n = \begin{cases} \zeta(3)/\pi^2 \cdot g \cdot T^3 & \text{Bose} \\ \frac{3}{4} \cdot \zeta(3)/\pi^2 \cdot g T^3 & \text{Fermi} \end{cases}$$

$$\rho = \begin{cases} \frac{\pi^2}{30} \cdot g T^4 & \text{Bose} \\ \frac{7}{8} \cdot \frac{\pi^2}{30} \cdot g T^4 & \text{Fermi} \end{cases} \quad (27)$$

$$P = \frac{1}{3} \rho$$

non-relativistic limit $T \ll m$:

$$n = g \left(\frac{mT}{2\pi} \right)^{3/2} \cdot e^{-\frac{m-\mu}{T}} \quad (28)$$

$$\rho = m \cdot n, \quad P = nT \ll \rho$$

instructive application: 64

$$n_+ - n_- = n_B \quad \text{"baryon number density"} \\ \rightarrow \text{net particle over antiparticle excess}$$

\sim reaction $\bullet_+ + \bar{\bullet}_- \rightarrow 2\gamma$

if fast $\Rightarrow \mu_+ = -\mu_- \equiv \mu$

$$\Rightarrow n_+ - n_- = \frac{g}{2\pi^2} \int_m^\infty dE \cdot E \sqrt{E^2 - m^2} \quad (29)$$

$$\times \left[\frac{1}{1 + e^{\frac{E-\mu}{T}}} - \frac{1}{1 + e^{\frac{E+\mu}{T}}} \right]$$

$$\approx \begin{cases} \frac{gT^3}{6\pi^2} \left[\pi^2 \cdot \frac{\mu}{T} + \mathcal{O}\left(\frac{\mu^2}{T^2}\right) \right], & T \gg m \\ 2g \left(\frac{mT}{2\pi} \right)^{3/2} \sinh(\mu T) \cdot e^{-\frac{m}{T}}, & T \ll m \end{cases}$$

measurable quantity:

64'

$$\frac{n_B}{n_\gamma} \equiv \zeta_B \quad \text{"baryon asymmetry"}$$

in the non-relativistic limit:

$$\zeta_B = \frac{n_B}{n_\gamma} \sim \frac{e^{-m/T}}{T^3} \rightarrow 0$$

for $T \ll m$ exponentially

fast ...

$$\Rightarrow \zeta_B(T_0) \sim \frac{e^{-\frac{m}{T_0}}}{T_0^3} \ll \zeta_B^{\text{obs.}}$$

$$\text{for } \zeta_B^{\text{obs.}} \sim 10^{-9}, T_0 \approx 3\text{K}, m \sim m_p \approx 1\text{GeV}$$

\sim need B-violation to generate matter-antimatter asymmetry!

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total energy density of all species:

\rightarrow species i with temperature T_i

$$\Rightarrow \rho_R = \sum_i \frac{g_i}{2\pi^2} \int_{m_i}^{\infty} dE \cdot E^2 \frac{\sqrt{E^2 - m_i^2}}{1 + e^{\frac{E - \mu_i}{T_i}}}$$

$$E \rightarrow u_i = \frac{E}{T_i}, \quad m_i \rightarrow x_i = \frac{m_i}{T_i}$$

$$\mu_i \rightarrow y_i = \frac{\mu_i}{T_i}$$

$$= T^4 \sum_i \left(\frac{T_i}{T}\right)^4 \frac{g_i}{2\pi^2} \int_{x_i}^{\infty} du_i u_i^2 \frac{\sqrt{u_i^2 - x_i^2}}{1 + e^{u_i - y_i}}$$

$\sim \rho_R$ analogously.

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non-relat. species exponentially damped in ρ (eq. (28)) ... 66

\Rightarrow good approximation:

ρ_R only from relativistic species

$$\Rightarrow \rho_R = \frac{\pi^2}{30} g_* \cdot T^4 \quad (31)$$

with:

$$g_* = \sum_{i=\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{j=\text{fermions}} g_j \cdot \left(\frac{T_j}{T}\right)^4 \quad (32)$$

species count: 67

i) $T \ll m_e \Rightarrow$ only γ, ν 's

$$\Rightarrow g_* = 1 \cdot 2 + \frac{7}{8} \cdot 3 \cdot 2 \cdot \left(\frac{4}{11}\right)^{4/3} \approx 3.36$$

γ spin

righthanded ν 's
3 families

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}$$

from neutrino decoupling
(see below)

ii) $100 \text{ MeV} \gtrsim T \gtrsim m_e \approx \text{MeV} \Rightarrow \gamma, \nu$'s, e^\pm

$$\Rightarrow g_* = 1 \cdot 2 + \frac{7}{8} (2 \cdot 2 + 3 \cdot 2) = 10.75$$

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 Annotations: γ spin; $e^- \& e^+$ spin; ν 's 3 families; spin

$\sim g_*$ increases with $T \dots$

iii) $T \geq 300 \text{ GeV} \Rightarrow$ the whole SM

$$\Rightarrow g_* = (8 + 3 + 1) \cdot 2 + 4 + \frac{7}{8} (3 + 3 \cdot 2) + 3 \cdot 3 \cdot 2 = 106.75$$

Annotations: gluons; Z, W, γ spin; Higgs; ν 's 3 families; colors quarks particles 3 families & antipart.; e, μ, τ

early on: radiation domination 69

$$\rho \approx \rho_R = \frac{\pi^2}{30} g_* T^4$$

$$\Rightarrow H = \frac{\pi}{\sqrt{90}} g_*^{1/2} \cdot \frac{T^2}{M_p} \quad (33)$$

With: $M_p = \sqrt{\frac{\hbar c^5}{8\pi G}} \approx 2.4 \cdot 10^{18} \frac{\text{GeV}}{c^2}$

and: $H = \frac{1}{2t}$

$$\Rightarrow t = \frac{\sqrt{90}}{2\pi} \cdot g_*^{-1/2} \cdot \frac{M_p}{T^2} \sim \frac{1}{T^2} \quad (34)$$

Thermodynamics of expansion: 70

2nd law always holds.

$$\Rightarrow TdS = dE + PdV, S: \text{entropy}$$

$$= d(PV) + PdV$$

$$= d[(P+P)V] - VdP \quad (35)$$

Maxwell relations:

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T}$$

and apply to 2nd law ...

$$dS = \frac{\partial S}{\partial V} dV + \frac{\partial S}{\partial T} dT \quad 71$$

$$= \frac{1}{T} d[(P+P)V] - \frac{V}{T} dP$$

$$= \frac{P+P}{T} dV + \frac{V}{T} dP$$

We know: $P = P(T), P = P(T)$

$$\Rightarrow dS = \frac{P+P}{T} dV + \frac{V}{T} \frac{dP}{dT} \cdot dT$$

$$\Rightarrow \frac{\partial S}{\partial V} = \frac{P+P}{T}, \quad \frac{\partial S}{\partial T} = \frac{V}{T} \frac{dP}{dT}$$

$$\Rightarrow \frac{\partial}{\partial V} \left(\frac{\partial S}{\partial T} \right) = \frac{1}{T} \frac{dP}{dT}$$

$$\frac{\partial}{\partial T} \left(\frac{\partial S}{\partial V} \right) = -\frac{P+P}{T^2} + \frac{1}{T} \frac{dP}{dT} \quad 72$$

$$+ \frac{1}{T} \cdot \frac{dS}{dT}$$

thus:

$$0 = \frac{\partial^2 S}{\partial T \partial V} - \frac{\partial^2 S}{\partial V \partial T} = -\frac{P+P}{T^2} + \frac{1}{T} \frac{dP}{dT}$$

$$\Leftrightarrow \frac{dP}{dT} = \frac{P+P}{T} \quad (36)$$

$$(35) \Rightarrow dS = \frac{1}{T} d[(P+P)V] - \frac{P+P}{T^2} V dT$$

$$= d \left[\frac{(P+P)V}{T} \right]$$

$$\Rightarrow S = \frac{P+P}{T} \cdot a^3(t) \quad (36).$$

first law: 73

$dE = -pdV$ in a closed system
 \Leftrightarrow total universe

$$\Rightarrow \boxed{dS = 0}$$

Entropy is conserved during expansion!
(37)

$$\Rightarrow S = \frac{P+P}{T} \cdot a^3 = \text{const.}$$

$$\Rightarrow \rho = \frac{S}{V} = \frac{S}{a^3} = \frac{P+P}{T} \sim \frac{1}{a^3} \quad (38)$$

for relativistic species thus: 74

$$P_R = \frac{1}{3} \rho_R$$

$$\Rightarrow \rho = \frac{\rho_R + P_R}{T} = \frac{4}{3} \cdot \frac{\rho_R}{T}$$

$$= \frac{2\pi^2}{45} \cdot g_{*S} \cdot T^3 \quad (39)$$

$$g_{*S} = \sum_{i=\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{j=\text{fermions}} g_j \left(\frac{T_j}{T}\right)^3$$

note: (40)

$$\rho \sim \frac{1}{a^3} \Rightarrow \boxed{N = V \cdot n = \frac{n}{\rho}} \quad \begin{array}{l} \text{particle} \\ \text{number} \end{array}$$

and:

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per comoving volume

$$S = \rho a^3 = \text{const.} \Rightarrow \boxed{T \sim \frac{1}{a}} \quad (42)$$