

To derive the dynamics of the expansion we will apply Einstein's equations to the FRW space-time. This yields, defining the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

With the FRW metric in the form of eq. (4):

$$G_{00} = 3 \cdot \left[\left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] \quad (9)$$

$$G_{ij} = g_{ij} \cdot \left[2 \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right]$$

where the metric (4) is written as:

$$ds^2 = dt^2 - a^2(t) \cdot g_{ij} dx^i dx^j.$$

For simplification and consistency with spatial homogeneity & isotropy energy and matter is often described by an energy-momentum tensor of a perfect fluid with energy density ρ , pressure p and 4-velocity u^α :

$$T^{\alpha\beta} = (\rho + p) \cdot u^\alpha u^\beta - p \cdot g^{\alpha\beta}. \quad (10)$$

As the fluid is comoving with the expansion, it is at rest in the coordinates of the FRW metric (4):

$$u^\alpha = (1, 0, 0, 0)$$

$$\Rightarrow T^{\alpha\beta} = \begin{pmatrix} \rho & & & \\ & -p \cdot g^{ij} & & \\ & & & \end{pmatrix} \quad (11)$$

and this, together with the LHS of ³⁰ the Einstein equations (9), gives us the Friedmann equations:

$$\begin{cases} 00: & \frac{\dot{a}^2}{a^2} \equiv H^2 = \frac{8\pi G}{3} \cdot \rho - \frac{k}{a^2} \\ ij: & 2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G \cdot p \end{cases} \quad (12)$$

$$\Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi}{3} G \cdot (\rho + 3p) \quad (12')$$

"acceleration equation"

We see that the universe undergoes accelerated expansion, if:

$$p < -\frac{1}{3} \rho .$$

The Friedmann equations are to be ³¹ supplemented by energy conservation:

$$dE = d(\rho V) = -p dV \quad (13)$$

where the cosmological volume V :

$$V \sim a^3(t)$$

$$\Rightarrow d(\rho \cdot a^3) = -p \cdot d(a^3)$$

$$\Rightarrow \frac{d}{dt} (\rho a^3) = a^3 \left(\dot{\rho} + 3\rho \cdot \frac{\dot{a}}{a} \right) =$$

$$= -a^3 \cdot 3p \cdot \frac{\dot{a}}{a}$$

$$\Leftrightarrow \dot{\rho} + 3H \cdot (\rho + p) = 0 . \quad (14)$$

Finally, we need an equation of state ³²
 $p = p(\rho)$ for the description of matter
to complete, and this is often well
approximated by:

$$p = w \cdot \rho \quad (15)$$

examples are:

$w = 0$ "dust" - nonrelativistic
massive matter

$w = \frac{1}{3}$ radiation (photons...)

$w = -\frac{1}{3}$ the 3-curvature k

$w = -1$ cosmological constant
or constant scalar
potential

The case of a scalar field is particularly ³³
interesting concerning cosmological
inflation. For a scalar field ϕ
with potential $V(\phi)$ and Lagrangian:

$$\sqrt{-g} \mathcal{L} = \sqrt{-g} \cdot \left[\partial^\mu \phi \partial_\mu \phi - V(\phi) \right]$$

One can show that:

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[\frac{1}{2} \partial^\sigma \phi \partial_\sigma \phi - V(\phi) \right]$$

and from there:

$$\rho = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + V(\phi)$$

$$p = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \quad .$$

If we can arrange for a situation where for a prolonged time:

$$\partial^\mu \phi \partial_\mu \phi \ll V(\phi)$$

$$\Rightarrow p \simeq -\rho$$

and the scalar potential acts like a cosmological constant, which by definition is:

$$T_{\mu\nu} = \Lambda \cdot g_{\mu\nu} \quad (16)$$

and thus has exactly:

$$p = -\rho.$$

We can now use the equation of state $p = w\rho$ together with energy conservation (14) to determine the scaling of the energy density ρ of a given form of matter or radiation with $a(t)$ due to the expansion:

$$\rho \sim a^{-3(1+w)} \quad (17)$$

For different sources this behaves like:

$\rho \sim a^{-3}$	"dust"
$\rho \sim a^{-4}$	radiation
$\rho \sim a^{-2}$	curvature

and

$\rho \sim \text{const.}$ for a c.c.

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↳ 2 important consequences:

- i) A cosmological constant does not dilute under expansion.
- ii) Any residual spatial curvature eventually comes to dominate over dust or radiation in an expanding universe, if we wait long enough.

Now look at the first Friedmann equation in (12):

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$$H^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} .$$

We see that a spatially flat universe, having $k=0$, corresponds to a critical density $\rho_{\text{cr.}}$:

$$k=0 \Rightarrow \rho_{\text{cr.}} \equiv \frac{3H^2}{8\pi G} \quad (18)$$

We can thus divide the first Friedmann equation by:

$$\frac{8\pi G}{3} \rho_{\text{cr.}}$$

define a cosmological density parameter :

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$$\Omega = \frac{\rho}{\rho_{cr.}}$$

$$\Omega_k = \frac{\rho_k}{\rho_{cr.}}, \rho_k = -\frac{3k}{8\pi G a^2}$$
$$= -\frac{k}{a^2 H^2}$$

$$\Rightarrow \Omega + \Omega_k = 1 \quad (19)$$

We see that :

$$\Omega_k \begin{cases} > 0, k = -1 \text{ 'open'} \\ < 0, k = +1 \text{ 'closed'} \end{cases}$$

implies that an open universe has:

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$$\Omega < 1$$

and a closed one :

$$\Omega > 1.$$

Here Ω contains all energy and matter which is not curvature, and if we divide up ρ into :

$$\rho = \rho_{rad.} + \rho_{matter} + \rho_{\Lambda}$$

then :

$$\Omega = \Omega_{\text{rad.}} + \Omega_{\text{matter}} + \Omega_{\Lambda}.$$

We can now use the results for scaling of different forms of energy density with a to determine $a(t)$ from the first Friedmann equation:

$$w=0 : \frac{\dot{a}^2}{a^2} \sim \frac{1}{a^3} \Rightarrow a(t) \sim t^{2/3} \\ \Rightarrow H(t) = \frac{2}{3t}$$

$$w = \frac{1}{3} : \frac{\dot{a}^2}{a^2} \sim \frac{1}{a^4} \Rightarrow a(t) \sim \sqrt{t} \\ \Rightarrow H(t) = \frac{1}{2t}$$

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$$\frac{41}{w} = -1 : \frac{\dot{a}}{a} = \text{const.} \Rightarrow a(t) = e^{H \cdot t} \\ \Rightarrow H = \text{const.}$$

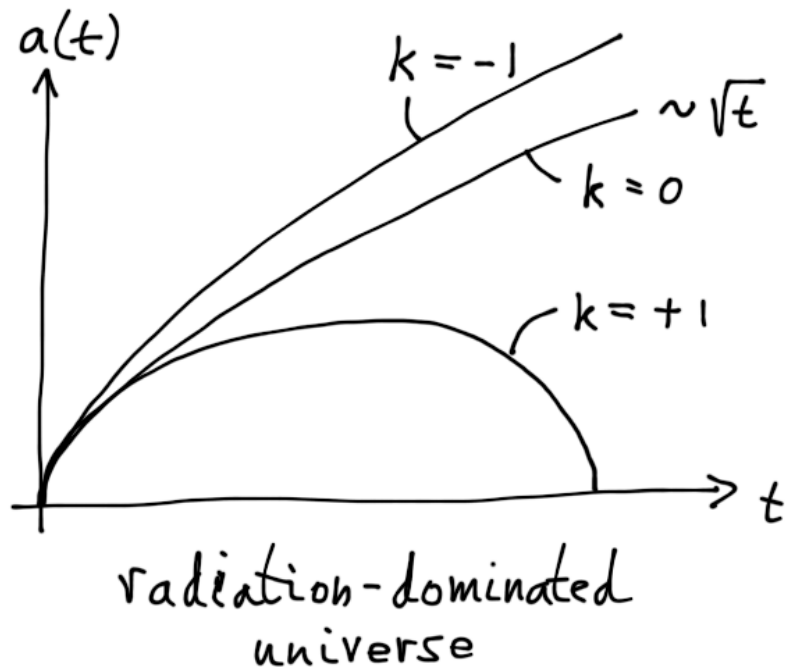
This describes a de Sitter space time dS_4 .

all in the case of a flat universe $k=0$.

In the presence of spatial curvature these results change.

In particular, a universe whose

expansion is driven by dust or radiation, re-collapses if its closed ($k = +1$), and expands forever if flat or open: ⁴²



We see, that in a matter or radiation dominated universe the Hubble parameter: ⁴³

$$H(t) \sim \frac{1}{t} \xrightarrow{t \rightarrow 0} \infty$$

diverges at a finite time in the past.

Now as the Hubble parameter:

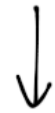
$H^2 = \frac{\dot{a}^2}{a^2} \propto \rho_{00}, \rho_{ij} \hat{=} \text{curvature}$
 is linked to the curvature of the universe, the diverging $H(t)$ with

$t \rightarrow 0$ entails diverging curvature⁴⁴ of space-time. Thus, a curvature singularity — a point of infinite curvature, energy density, and pressure — ensues at $t \rightarrow 0$, which is a point where space-time itself ceases to exist.

The Hawking - Penrose singularity theorems state for this situation, that for an expanding universe driven by energy or matter with:

$$p > -\frac{1}{3}\rho$$

Such a curvature singularity to⁴⁵ the past (and for a $k = +1$ closed universe also to the future) cannot be avoided.



Inflation with a scalar field, or a cosmological constant, with:

$$p \simeq -\rho$$

thus may provide a way out from the inevitability of a past curvature singularity.