13.4.

2. Robertson-Walker metric:

we have very good evidence for isotropy of the universe around us $\rightarrow$ best is CMB:

$$\frac{\Delta T}{T} \lesssim 10^{-4} \text{ in all directions}$$

and reasonable evidence for homogeneity ...

alternatively, can employ Copernican Principle: "we are not special"

$\rightsquigarrow$ universe should be isotropic around every point

$\Rightarrow$ via a mathematical theorem: universe must be spatially homogeneous.

<u>note:</u> this applies to space!

$\rightsquigarrow$ universe may expand (or shrink) with time ...

$\longrightarrow$ a little geometric argument:

"spatial isotropy & homogeneity implies the Friedmann-Robertson-Walker (FRW) metric" the universe is described as a space-time manifold endowed with a metric $g_{\mu\nu}$ and line element

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

under coordinate transformations:

$$x^\mu \to x^{\mu'}(x^\sigma)$$

we have:

$$g_{\mu\nu} \to g_{\rho'\sigma'} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x^{\rho'}} \cdot \frac{\partial x^\nu}{\partial x^{\sigma'}}$$

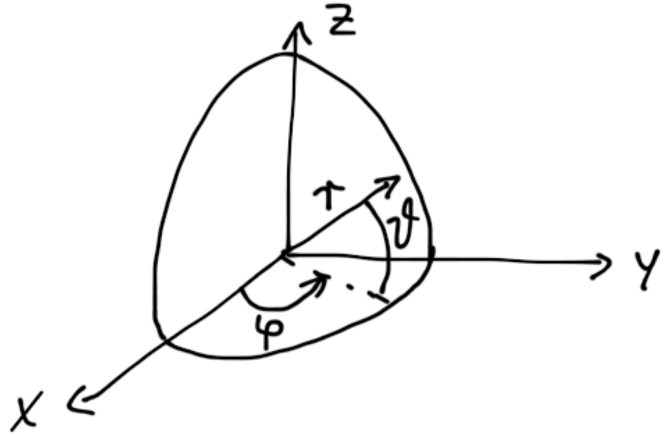
which leaves

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = g_{\mu\nu} \frac{\partial x^\mu}{\partial x^{\rho'}} \frac{\partial x^\nu}{\partial x^{\sigma'}} dx^{\rho'} dx^{\sigma'}$$

$$= g_{\rho'\sigma'}\, dx^{\rho'} dx^{\sigma'} = ds'^2$$

invariant.

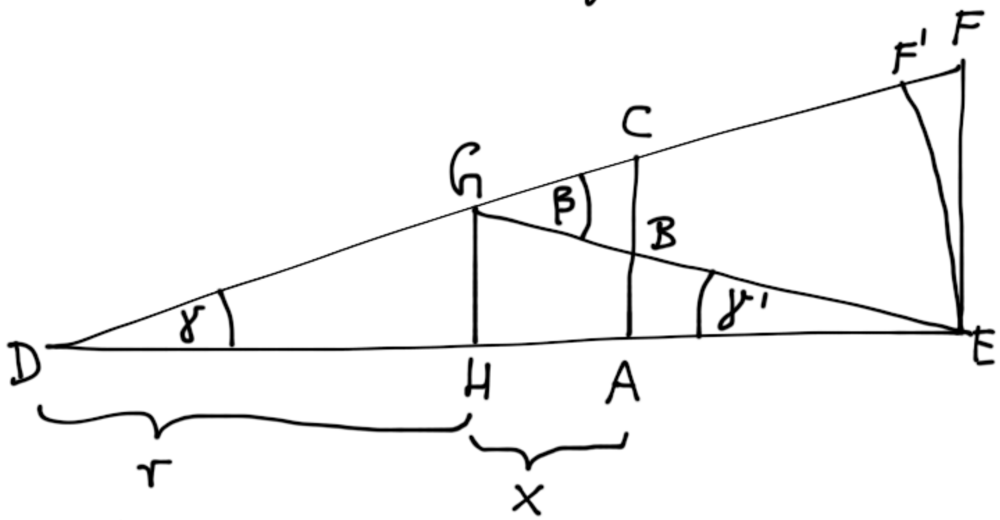
Now consider the 3D spatial part. Isotropy implies spherical symmetry:



$$ds_3^2 = dr^2 + f^2(r)\cdot\underbrace{\left[d\vartheta^2 + \sin^2(\vartheta)\,d\varphi^2\right]}_{d\Omega_2^2}$$

where $f(r) \to r$ for $r \to 0$.

now look at this triangle:



in the plane given by $\vartheta = \frac{\pi}{2}$.

We have: $DH = HE \equiv r$, $HA \equiv x$

and we work in the limit where DE, γ, γ' and β are small.

isotropy and homogeneity

$$\Rightarrow \gamma = \gamma'$$

Then we get:

$$EF \simeq EF' \Rightarrow f(2r)\gamma \simeq f(r)\beta$$

and:

$$AC = \gamma \cdot f(r+x) = AB + AC = f(r-x)\gamma + f(x)\beta$$

$$\Rightarrow \frac{df}{dr} = \lim_{x \to 0} \frac{f(r+x) - f(r-x)}{2x}$$

$$= \underbrace{\frac{f(x)}{x}}_{\longrightarrow 1} \cdot \frac{f(2r)}{2f(r)} \qquad (1)$$

solutions are:

$$f(r) = \begin{cases} \sin r \\ r \\ \sinh r \end{cases} \qquad (2)$$

and one show that with $f(r)$ also $f(r/\alpha)$, $\alpha \in \mathbb{R}$ solves (1). Writing $f(r)$ as a power series and using

analyticity, one show (2) to be all solutions up to scaling.

$\Rightarrow$ The space-time of a spatially homogeneous & isotropic universe necessarily has a metric of FRW form:

$$ds^2 = dt^2 - a^2(t) \cdot \left[d\rho^2 + f^2(\rho) \cdot d\Omega_2^2 \right]$$

$$\text{with } f(\rho) = \begin{cases} \sin \rho \\ \rho \\ \sinh \rho \end{cases} \qquad (3)$$

$a(t)$: scale factor of the universe – "scale of its spatial size"

2 other useful coordinate choices:

$\rightsquigarrow$ change of radial variable

$$ds^2 = dt^2 - a^2(t)\cdot\left[\frac{dr^2}{1-k\cdot r^2} + r^2\cdot d\Omega_2^2\right] \tag{4}$$

calculate 3-curvature R_{ij} for $[\ldots]$ in (4):

$$R_{ij} \sim k\cdot g_{ij} \Rightarrow {}^3R = R^i{}_i \sim k$$

$\Rightarrow$ 3-slices of constant time are Einstein spaces of constant curvature k.

$$k = \begin{cases} 1 & , f(\rho) = \sin\rho \text{ , "closed universe"} \\ 0 & , f(\rho) = \rho \text{ , "flat universe"} \\ -1 & , f(\rho) = \sinh\rho \text{ , "open universe"} \end{cases} \tag{4'}$$

sometimes, conformal time η is useful instead of comoving time t:

$$\eta(t) = \int^t \frac{dt'}{a(t')}$$

$$ds^2 = a^2(t(\eta))\cdot\left[d\eta^2 - \frac{dr^2}{1-k\cdot r^2} - r^2\cdot d\Omega_2^2\right]$$

the Hubble parameter H then is: (5)

$$H \equiv \frac{\dot a}{a} \quad , \quad (\dot{\ }) = \frac{d}{dt}$$

$\Rightarrow \quad v(r) \simeq \dot a = H\cdot a \simeq H\cdot r$

for distances not too large.

comoving coordinates:

a coordinate system, in which all massive matter eventually comes to rest by the dilution of its kinetic energy with the expansion as $\sim a^{-3}(t)$, and this means: to rest with the expanding frame of reference given by the comoving coordinates.

Comoving coordinates co-move with the matter receding by just the expansion itself.

Thus, in comoving coordinates matter stays at rest, if not sped otherwise, because the coordinate frame stretches with the expansion itself, and thus is 'comoving' with the locally-at-rest matter stretching the same way with the expansion.

Thus comoving time t measures cosmological age as seen by an observer swept along by the expansion, and agrees with 'redshift-time' as inferred via distances from red shifts.

3. Dynamics of expansion

The expansion of the universe as a whole is governed by Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \cdot T_{\mu\nu} \qquad (6)$$

applied to the FRW metric of a spatially homogeneous and isotropic expanding space-time

Here we denote:

$$\boxed{R_{\mu\nu} = g^{\rho\sigma} R^{\rho}{}_{\mu\sigma\nu}}$$ Ricci tensor

$$\boxed{R = g^{\mu\nu} R_{\mu\nu}}$$ Ricci scalar

$$\boxed{R^{\rho}{}_{\mu\sigma\nu} = \partial_\mu \Gamma^{\rho}{}_{\sigma\nu} - \partial_\nu \Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}}$$ Riemann curvature tensor

$$\boxed{\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} g^{\mu\delta}\left(\frac{\partial g_{\mu\delta}}{\partial x^{\nu}} + \frac{\partial g_{\nu\delta}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\delta}}\right)}$$ Christoffel symbols

The field equations follow from the Einstein-Hilbert action:

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x \sqrt{-g}\,(R + 2\Lambda) + \int d^4x \sqrt{-g}\,\mathcal{L}_{matter}$$

give $T_{\mu\nu}$

if we define:

$$\sqrt{-g}\,\mathcal{L}_{EH} = \sqrt{-g}\cdot R$$

$$\sqrt{-g}\,\mathcal{L}_{\Lambda+\text{matter}} = \sqrt{-g}\left(-\frac{\Lambda}{16\pi G} + \mathcal{L}_{\text{matter}}\right)$$

$$\Rightarrow \begin{cases} \dfrac{\delta\left(\sqrt{-g}\,\mathcal{L}_{EH}\right)}{\delta g^{\mu\nu}} \rightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\[2em] T_{\mu\nu} = \dfrac{2}{\sqrt{-g}} \cdot \dfrac{\delta\left(\sqrt{-g}\,\mathcal{L}_{\Lambda+\text{matter}}\right)}{\delta g^{\mu\nu}} \end{cases}$$

via the Euler-Lagrange equations:

$$(7) \qquad \frac{\delta\left(\sqrt{-g}\,\mathcal{L}\right)}{\delta g^{\mu\nu}} = 0\,.$$

note: the partial derivative $\frac{\partial}{\partial x^\mu} = \Lambda_\mu^{\ \nu'} \frac{\partial}{\partial x^{\nu'}}$ transforms as a vector (covariant 1-tensor) under Lorentz transformations on Minkowski space-time.

On curved space-time there is a generalized covariant derivative D_μ transforming as a covariant 1-tensor under general coordinate transformations.

Acting on a covariant vector A_ν it reads:

$$D_\mu A_\nu = \frac{\partial A_\nu}{\partial x^\mu} - \Gamma^\rho_{\mu\nu} A_\rho$$

and on a contravariant 2-tensor $C^{\mu\nu}$, for example:

$$D_\mu C^{\rho\sigma} = \frac{\partial C^{\rho\sigma}}{\partial x^\mu} + \Gamma^\rho_{\mu\delta} C^{\delta\sigma} + \Gamma^\sigma_{\mu\delta} C^{\delta\rho}$$

One can show that:

$$D_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0$$

where as usual:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = g^{\mu\rho} g^{\nu\sigma} \left(R_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma} R \right)$$

and by Einstein's equations this implies:

$$D_\mu T^{\mu\nu} = 0$$

which is the covariant conservation of energy and momentum.

Thus, energy-momentum conservation is a consequence of the gravitational field equations in GR.