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Does QCD describe all strong interaction phenomena?

Complementary properties: *Confinement & Asymptotic freedom*

Uncovering the connection between the low- and high-energy regimes of QCD: Demands *non*-perturbative treatment of the theory (NP)

Jochen Heitger  
Fundamental QCD parameters from the lattice
Does \textbf{QCD} describe \textit{all} strong interaction phenomena?

**Complementary properties: \textit{Confinement} \& \textit{Asymptotic freedom}**

Uncovering the connection between the low- and high-energy regimes of QCD:
Demands \textit{non}-perturbative treatment of the theory (NP)

**Scope of lattice QCD computations:**

- Determination of fundamental parameters
  \[ \rightarrow \Lambda_{\text{QCD}}, \text{quark masses} \]

- Stringent tests of QCD
  \[ \rightarrow \text{e.g. } F_\pi/\Lambda_{\text{QCD}}, \text{(light) hadron spectrum} \]

- Predict experimentally inaccessible physical quantities
  \[ \rightarrow \text{e.g. hadron masses, decay constants, mixing amplitudes} \]

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Fundamental QCD parameters from the lattice
Lattice QCD

‘Ab initio’ approach to determine phenomenologically relevant key parameters

\[ \mathcal{L}_{\text{QCD}} [g_0, m_f] = -\frac{1}{2g_0^2} \text{Tr} \{ F_{\mu\nu} F_{\mu\nu} \} + \sum_{f=u,d,s,...} \bar{\psi}_f \{ \gamma_\mu (\partial_\mu + g_0 A_\mu) + m_f \} \psi_f \]

\[
\begin{bmatrix}
F_\pi \\
m_\pi \\
m_K \\
m_D \\
m_B
\end{bmatrix}
\xrightarrow{\mathcal{L}_{\text{QCD}} [g_0, m_f]}
\begin{bmatrix}
\Lambda_{\text{QCD}} \\
\frac{1}{2}(M_u + M_d) \\
M_s \\
M_c \\
M_b
\end{bmatrix}
+ \begin{bmatrix}
F_D \\
F_B \\
B_B \\
\xi \\
\vdots
\end{bmatrix}
\]

\( \mathcal{L}_{\text{QCD}} [g_0, m_f] \) means formulation of QCD on a Euclidean lattice with:

- Gauge invariance
- Locality
- Unitarity
- Applicable for all scales
- Technical issues/obstacles
  - Continuum limit & Renormalization
  - Computing resources of > 1 TFlop/s
Outline

1. Lattice QCD basics

2. Determining fundamental QCD parameters
   - The running of the coupling in two-flavour QCD
   - Quenched versus unquenched: The strange quark’s mass

3. New perspectives for B-physics
   - Non-perturbative Heavy Quark Effective Theory
   - Application: The b-quark’s mass

4. Summary & Outlook

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Lattice QCD basics
Lattice QCD basics

- Lattice cutoff: $a^{-1} \sim \Lambda_{\text{UV}}$
- Finite volume: $L^3 \times T$
- Lattice action

$$S[U, \bar{\psi}, \psi] = S_G[U] + S_F[U, \bar{\psi}, \psi]$$

$$S_G = \frac{1}{g_0^2} \sum_p \text{Tr}\{1 - U(p)\}$$

$$S_F = a^4 \sum_x \bar{\psi}(x) D[U] \psi(x)$$

**EVs:** represented as path integrals

$$Z = \int \mathcal{D}[U] \mathcal{D}[\bar{\psi}, \psi] e^{-S[U, \bar{\psi}, \psi]} = \int \mathcal{D}[U] \prod_f \det (\mathcal{D} + m_f) e^{-S_G[U]}$$

$$\langle O \rangle = \frac{1}{Z} \int \prod_{x, \mu} dU_{\mu}(x) O \prod_f \det (\mathcal{D} + m_f) e^{-S_G[U]} \equiv \text{thermal average}$$

Stochastic evaluation with *Monte Carlo methods*

→ Observables: $\langle O \rangle = \frac{1}{N} \sum_{n=1}^{N} O_n \pm \Delta_O$ from numerical simulations
Towards realistic QCD simulations — Systematics

(1) Effects of *dynamical* fermions

- Valence quark approximation (quenching):
  \[
  \det (\mathcal{D} + m_f) = 1
  \]
  → neglection of quark loops

- Costs: \( \frac{\text{‘full’ QCD}}{\text{quenched QCD}} \gtrsim 100 - 1000 \)
Towards *realistic* QCD simulations — Systematics

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(2) Lattice artefacts cause discretization errors

\[ \langle O \rangle_{\text{lattice}} = \langle O \rangle_{\text{continuum}} + O(a^p) \]

- removable by extrapolating to the continuum limit: \( a \rightarrow 0 \)
- accelerated convergence for \( p' > p \) \( \rightarrow O(a^p) \) improvement
Towards realistic QCD simulations — Systematics

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(3) Quark mass restrictions

\[ a \ll m_q^{-1} \ll L \]

→ Extrapolations to \( m_{u,d} \) and \( m_b \), guided by ChPT and HQET
Determining fundamental QCD parameters
Renormalization group (RG) equations, $\bar{g} \equiv \bar{g}(\mu)$

$$\mu \frac{\partial \bar{g}}{\partial \mu} = \beta(\bar{g}) \quad \bar{g} \to 0 \sim -\bar{g}^3 \left\{ b_0 + b_1 \bar{g}^2 + b_2 \bar{g}^4 + \ldots \right\}$$

$$\mu \frac{\partial m}{\partial \mu} = \tau(\bar{g}) \bar{m} \quad \tau(\bar{g}) \to 0 \sim -\bar{g}^2 \left\{ d_0 + d_1 \bar{g}^2 + \ldots \right\}$$
Determining fundamental QCD parameters

Scale dependence of QCD parameters

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\]

- Solution leads to exact equations in a mass-independent scheme

\[
\Lambda \equiv \mu \left( b_0 \bar{g}^2 \right)^{-\frac{b_1}{2b_0}} e^{-\frac{1}{2b_0 \bar{g}^2}} \exp \left\{ - \int_0^{\bar{g}} d\bar{g} \left[ \frac{1}{\beta(\bar{g})} + \frac{1}{b_0 \bar{g}^3} - \frac{b_1}{b_0 \bar{g}} \right] \right\}
\]

\[
\mathcal{M} \equiv \bar{m}(\mu) \left( 2b_0 \bar{g}^2 \right)^{-\frac{d_0}{2b_0}} \exp \left\{ - \int_0^{\bar{g}} d\bar{g} \left[ \frac{\tau(\bar{g})}{\beta(\bar{g})} - \frac{d_0}{b_0 \bar{g}} \right] \right\} \quad \text{RG invariant quark mass}
\]
Determining fundamental QCD parameters

Scale dependence of QCD parameters

- **Renormalization group (RG) equations, \( \bar{g} \equiv \bar{g}(\mu) \)**

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  \]

  \[
  M \equiv \bar{m}(\mu) \left( 2b_0 \bar{g}^2 \right)^{-d_0/(2b_0)} \exp \left\{ - \int_0^{\bar{g}} dg \left[ \frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0 g} \right] \right\} \text{ RG invariant quark mass}
  \]

- **Simple relations between different renormalization schemes:**

  \[
  \Lambda'/\Lambda = \exp \left\{ c/(4\pi b_0) \right\}, \quad \alpha' = \alpha + c \alpha^2 + O(\alpha^3) \quad \text{and} \quad M' = M
  \]

  \( \Rightarrow \) **Choose suitable physical scheme/coupling to compute \( \Lambda \) and \( M \)**
Physical couplings

- are defined for all energies $\mu$ and fix a renormalization scheme
- should be independent of the regularization procedure
- in a lattice computation: Problem of a hierarchy of disparate scales

$$L^{-1} \ll 0.2 \text{ GeV} \ll \mu \simeq 10 \text{ GeV} \ll a^{-1}$$
Determining fundamental QCD parameters

The running of the coupling in two-flavour QCD

Physical couplings

- are defined for all energies $\mu$ and fix a renormalization scheme
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\[
\frac{1}{L} \ll 0.2 \text{ GeV} \ll \mu \approx 10 \text{ GeV} \ll a^{-1}
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Solution


Identify $\mu = 1/L \implies$ One is left with $L/a \gg 1 \implies$ Finite-size scaling
Determining fundamental QCD parameters

The running of the coupling in two-flavour QCD

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Realization: QCD Schrödinger functional (SF) as an intermediate scheme

$T \times L^3$ QCD with Dirichlet boundary conditions in time

- Take finite-size effect as physical observable:
  $$ 1/\bar{g}_\text{SF}^2(L) \equiv \delta(\text{free energy})/\delta(\text{boundary conditions}) $$
  $$ \sigma(u) \equiv \bar{g}^2(2L) \bigg| \bar{g}^2(L) = u, \ m(L) = 0 $$

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Fundamental QCD parameters from the lattice
Fix $\bar{g}^2(L_0 \equiv L_{\text{max}})$, compute $\bar{g}^2(L)$ for lattices alternating in $L = 2^{-k}L_0$ resp. $a = 2^{-k}a_0$

\[ L = L_0, \ a = a_0 \]

\[ L = \frac{1}{2}L_0 \quad a = \frac{1}{2}a_0 \]

\[ \ldots \]

$\alpha \to 0$ limit of $\Sigma(u, a/L) = \sigma(u) + O(a/L)$

Comparison to PT and NP fit of $\sigma(u)$ to
\[ \sigma(u) = u + s_0 u^2 + s_1 u^3 + s_2 u^4 + \ldots \]
$N_f = 2$ $\beta$–function in the SF scheme

Comparison to PT and $N_f = 2$:

- Non-perturbative deviations from 3–loop $\beta$ for $\alpha > 0.25$
- Perturbative series by itself doesn’t show signs of failure at e.g. $\alpha \approx 0.4$
Determining fundamental QCD parameters

\( N_f = 2 \) β–function in the SF scheme

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\( N_f = 0 \) (SF scheme) & Experiment + PT (\( \overline{\text{MS}} \) scheme, [Bethke, 2004])

Jochen Heitger  Fundamental QCD parameters from the lattice
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Determining fundamental QCD parameters

\[ \beta_{\text{Nf}=2} \text{ function in the SF scheme} \]

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Non-perturbative running of \( \alpha_{\text{SF}} \) and the \( \Lambda \)–parameter for \( \text{Nf} = 2 \)

- Start from \( \bar{g}^2(L_{\text{max}}) \equiv 5.5 \) and evolve \( k \) steps with NP step scaling function:
  \[ \bar{g}^2(L_{\text{max}}/2^k) = \sigma(\bar{g}^2(L_{\text{max}}/2^{k+1})) \]
  - \( \bar{g}^2 \leq u_k \) small: continue with pert. \( \beta \)
  - Result: \( -\ln(\Lambda L_{\text{max}}) = 1.09(7) \)
Determining fundamental QCD parameters

The $N_f = 2$ running coupling

$N_f = 2 \beta$–function in the SF scheme

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Non-perturbative running of $\alpha_{SF}$ and the $\Lambda$–parameter for $N_f = 2$

- Start from $\bar{g}^2(L_{\text{max}}) \equiv 5.5$ and evolve $k$ steps with NP step scaling function:
  
  $$\bar{g}^2(L_{\text{max}}/2^k) = \sigma(\bar{g}^2(L_{\text{max}}/2^{k+1}))$$

  - $\bar{g}^2 \leq u_\kappa$ small: continue with pert. $\beta$
  - Result: $-\ln(\Lambda L_{\text{max}}) = 1.09(7)$

- To set the scale for $\Lambda$ in MeV, use
  
  - ideally: large volume computation of $F_K$
  - here: $r_0 = 0.5 \text{ fm}$, $F_{Q\bar{Q}}(r_0)r_0^2 = 1.65$
Determining fundamental QCD parameters

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$\Lambda_{\overline{\text{MS}}}r_0 = 0.62(8) \Rightarrow \Lambda_{\overline{\text{MS}}} = 245(32)$ MeV

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Fundamental QCD parameters from the lattice
Determining fundamental QCD parameters

Discussion

▶ Running close to PT below $\alpha_{SF} \approx 0.2$, non-perturbative above 0.25
▶ $\Lambda_{MS}^r 0$: $N_f$ dependence & Comparison to phenomenology

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<th>2</th>
<th>4</th>
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<td>[ALPHA Collaboration]</td>
<td>0.60(5)</td>
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<td>[Bethke, Loops&amp;Legs04] ‘experiment’</td>
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<td>0.74(10)</td>
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Irregular $N_f$ dependence? But errors relatively large . . .
## Discussion

- Running close to PT below $\alpha_{SF} \approx 0.2$, *non*-perturbative above 0.25
- $\Lambda_{\overline{MS}} r_0$: $N_f$ dependence & Comparison to phenomenology

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- **Improvement of lattice results**: Via replacing $L_{\text{max}}/r_0$ by $F_K \times L_{\text{max}}$
  - with small quark masses and small lattice spacings
  - Also: $N_f = 3, 4$ ($N_f = 3$ on agenda of JLQCD and CP-PACS)
  - Lattice will yield best controlled and most precise results for $\Lambda$ (minimal assumptions !)
A precision calculation of $m_s$

**Strategy**

Combination of lattice QCD with chiral perturbation theory

→ relate $\mathcal{L}_{\text{QCD}}$ with experimentally known (spectroscopic) quantity, here:
Determining fundamental QCD parameters

The strange quark's mass

A precision calculation of \( m_s \)

Garden, H., Sommer & Wittig, NPB571(2000)237

Strategy

Combination of lattice QCD with chiral perturbation theory

→ relate \( \mathcal{L}_{\text{QCD}} \) with experimentally known (spectroscopic) quantity, here:

\[
m_{PS}^2(M_{\text{ref}}) r_0^2 = (m_K r_0)^2 \bigg|_{\exp} \quad \left\{ \begin{array}{l}
\text{ChPT:} \\
2M_{\text{ref}} \simeq M_s + M_\ell, \quad \frac{M_s}{M_\ell} = 24.4(1.5)
\end{array} \right.
\]

where \( M_\ell = \frac{1}{2}(M_u + M_d) \) and \( M \equiv \lim_{\mu \to \infty} \left\{ \left[ 2b_0 \bar{g}^2(\mu) \right]^{-d_0/(2b_0)} \bar{m}(\mu) \right\} \): RGI quark mass
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**Determination of $M_{\text{ref}}$ through numerical simulations**

1. **PCAC relation**: $F_K m_K^2 = (\bar{m}_u + \bar{m}_s) \left\langle 0 | \bar{u} \gamma_5 s | K \right\rangle$

$$\left( \bar{u} \gamma_5 s \right)_{\text{MS}} = Z_P(g_0, a\mu) \left( \bar{u} \gamma_5 s \right)_{\text{lattice}}$$

2. $Z_P(g_0, a\mu)$ is poorly convergent in PT

→ non-perturbative values for $Z_P$ & $M/\bar{m}(\mu)$ needed [ALPHA Collaboration 1999]

3. $M_{\text{ref}}$ fixed by $K$-meson matrix elements to be taken from the lattice

$$\frac{M_s + M_\ell}{F_K} = \frac{M}{\bar{m}(\mu)} \frac{1}{Z_P(g_0, a\mu)} \times \frac{m_{PS}^2}{\left\langle 0 | \bar{\ell} \gamma_5 s | PS \right\rangle} \bigg|_{m_{PS} = m_K} + O(a^2)$$
Hadron physics from the Schrödinger Functional

Volume: $L \gtrsim 1.5 \text{ fm}, \ T = 2L$

$\text{SF} = \text{QCD partition function with Dirichlet BCs in time}$

(finite-V renormalization scheme, but has also some practical benefits for large-V physics)
Volume: $L \gtrsim 1.5 \text{ fm}, \ T = 2L$  


$\text{SF} = \text{QCD partition function with Dirichlet BCs in time}$

(finite-$V$ renormalization scheme, but has also some practical benefits for large-$V$ physics)

**Fermionic correlation functions**

$$f_X(x_0) \propto \left\langle \sum_{y,z} X(x) \bar{\zeta}(y) \gamma_5 \zeta(z) \right\rangle$$

$$X = \{ A_\mu = \bar{\psi}_i \gamma_\mu \gamma_5 \psi_j : \text{axial vector current} \}
\{ P = \bar{\psi}_i \gamma_5 \psi_j : \text{pseudoscalar density} \}$$

$$f_1 \propto \left\langle \sum_{u,v,y,z} \bar{\zeta}_i'(u) \gamma_5 \zeta_j'(v) \bar{\zeta}_j(y) \gamma_5 \zeta_i(z) \right\rangle$$

- $\bar{\zeta}, \zeta$: boundary (anti-)quark fields
- construct ratios with $f_1$ to eliminate $Z\{\zeta, \bar{\zeta}\}$
Fermionic correlation functions

\[ f_X(x_0) \propto \left\langle \sum_{y,z} X(x) \bar{\zeta}(y)\gamma_5\zeta(z) \right\rangle \]

\[ X = \begin{cases} A_\mu = \bar{\psi}_i\gamma_\mu\gamma_5\psi_j : & \text{axial vector current} \\ P = \bar{\psi}_i\gamma_5\psi_j : & \text{pseudoscalar density} \end{cases} \]

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- \( \bar{\zeta}, \zeta \): boundary (anti-)quark fields
- construct ratios with \( f_1 \) to eliminate \( Z_{\{\zeta, \bar{\zeta}\}} \)

\[ \Rightarrow \text{for } 0 \ll x_0 \ll T : \]

\[ \frac{f_A}{\sqrt{f_1}} \approx \frac{\langle 0|A|PS \rangle e^{-\left(x_0 - T/2\right)m_{PS}}}{\langle 0|P|PS \rangle} \quad \frac{f_A}{f_P} \approx \frac{\langle 0|A|PS \rangle}{\langle 0|P|PS \rangle} \]
The strange quark’s mass

\[ m_{s}^{\overline{MS}}(2 \text{ GeV}) = 97 \pm 4 \text{ MeV} \]

- NP renormalization & Continuum limit
  - Scale set by \( F_K = 160(2) \text{ MeV} \)
  - \( \exists \) all errors except quenching (\( \sim 15\% \)?)
Result in quenched QCD

\[ (M_s + M_l)/F_K \]

\[ m_{s}^{\text{MS}}(2 \text{ GeV}) = 97 \pm 4 \text{ MeV} \]

- **NP renormalization & Continuum limit**
  - scale set by \( F_K = 160(2) \text{ MeV} \)
  - \( \exists \) all errors except quenching (~15%?)

- **Computation in similar spirit:**
  - \( N_f = 0 \) \( m_{\text{charm}} \), input: \( r_0 = 0.5 \text{ fm}, m_{D_s} \)
  - \( \overline{m}_{c}^{\text{MS}}(\overline{m}_{c}) = 1.30(3) \text{ GeV} \) [Rolf&Sint, '02]
**Result in quenched QCD**

\[ \frac{(M_s + M_l)}{F_K} \]

- Continuum extrapolation in 0(a) improved QCD

- \( N_f = 0 \)

\[ (a/r_0)^2 \]

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**Crucial for precise predictions:**

- Inclusion of dynamical (sea) quark effects
Determining fundamental QCD parameters

The strange quark’s mass

\[ m_{s}^{\overline{MS}}(2 \text{ GeV}) = 97 \pm 4 \text{ MeV} \]

- NP renormalization & Continuum limit
  - scale set by \( F_{K} = 160(2) \text{ MeV} \)
  - \( \exists \) all errors except quenching (~15%?)

- Computation in similar spirit:
  - \( N_{f} = 0 \) \( m_{\text{charm}} \), input: \( r_{0} = 0.5 \text{ fm}, m_{D_{s}} \)
  - \( m_{c}^{\overline{MS}}(m_{c}) = 1.30(3) \text{ GeV} \) [Rolf&Sint, ’02]

Crucial for precise predictions: Inclusion of dynamical (sea) quark effects

Non-perturbative running of \( \overline{m} \) for \( N_{f} = 2 \)

- \( \overline{m}_{i}(\mu) = Z_{m}(g_{0}, a_{\mu}) \times m_{i}^{\text{bare}}(g_{0}) \)
  - whereby \( Z_{m} \) is flavour independent

- Scale evolution & RGI mass \( M \) via the same recursive strategy as for \( \alpha(\mu) \):
  - exp. input: \( r_{0} = 0.5 \text{ fm}, m_{K} = 495 \text{ MeV} \)
  - \( m_{s}^{\overline{MS}}(2 \text{ GeV}) = 97 \pm 22 \text{ MeV} \)
Determining fundamental QCD parameters

The strange quark’s mass

Unquenched $m_{\text{strange}}$ ($N_f = 2, 2 + 1$)

[Status November 2005]

Unquenched estimates have not yet stabilized

- Large cutoff effects at $a \approx 0.1 \text{ fm}$
- Better control over continuum limit required

Impact of perturbative versus non-perturbative renormalization?

- Results with pert. renormalization tend to be smaller than NP ones

Presently: Hard to claim a definitive $N_f$ dependence of $m_{\text{strange}}$

- Note: QCD sum rules in agreement with range of lattice results

Jochen Heitger  
Fundamental QCD parameters from the lattice
New perspectives for B-physics
B-physics & Lattice QCD

- Precision CKM-physics involves hadronic matrix elements:
  - Determination of CKM matrix parameters, tests of its unitarity
  - CP violation
- b-quark mass, spectrum and lifetimes of beauty-hadrons, . . .
B-physics & Lattice QCD

- Precision CKM-physics involves hadronic matrix elements:
  - Determination of CKM matrix parameters, tests of its unitarity
  - CP violation

- b-quark mass, spectrum and lifetimes of beauty-hadrons, . . .

Illustration: Unitarity triangle analysis ('CKM fit')

\[
\langle \overline{M} | 0_{\Delta M=2} | M \rangle = \frac{4}{3} m_M F_M^2 B_M \\
\langle 0 | \overline{b} \gamma_\mu \gamma_5 q | B_q \rangle = i p_\mu F_{B_q} , q = d, s
\]

\[
\Delta m_d \propto F_{B_d}^2 \hat{B}_{B_d} |V_{td} V_{tb}^*|^2 \\
\Delta m_s \propto \frac{F_{B_s}^2 \hat{B}_{B_s}}{F_{B_d}^2 \hat{B}_{B_d}} \frac{|V_{ts}|^2}{|V_{td}|^2} = \xi^2 \frac{|V_{ts}|^2}{|V_{td}|^2}
\]

UT with lattice input from \( B_K, F_{B_d}, B_{B_d}, \xi \): Today versus expected (5-3)% accuracy
Consider a large lattice as possible in the quenched approximation

\[ \lambda_\pi = \frac{1}{m_\pi} \approx L \]

\[ \lambda_B \sim \frac{1}{m_B} < a \]

- **Light quarks: too light**
  - Widely spread objects
  - Finite-volume errors due to light pions

- **b-quark: too heavy**
  - Extremely localized object
  - B-mesons need very fine resolutions, otherwise:
    - Discretization errors
    - ‘Fall through the lattice’
New perspectives for B-physics

Non-perturbative HQET

Lattice HQET — Why?

Consider a large lattice as possible in the quenched approximation

\[ \lambda_\pi = \frac{1}{m_\pi} \approx L \]

\[ \lambda_B \sim \frac{1}{m_B} < a \]

- Light quarks: too light
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  - Finite-volume errors due to light pions

- b-quark: too heavy
  - Extremely localized object
  - B-mesons need very fine resolutions, otherwise:
    - Discretization errors
    - ‘Fall through the lattice’

⇒ Propagating b on the lattice beyond today’s computing resources
⇒ Recourse to an effective theory for the b-quark:

Heavy Quark Effective Theory

[ Eichten, 1988; Eichten & Hill, 1990 ]
HQET — An asymptotic expansion of QCD

Physical picture

Momentum scales in heavy-light ($Q\bar{q}$) mesons
- $Q$ almost at rest at bound state’s center, surrounded by the light DOFs
- Motion of the heavy quark is suppressed by $\Lambda_{QCD}/m_Q$

Formally, in case of the B-system:
\[ \mathcal{L}_{HQET} = \frac{1}{m_b} - \text{expansion of continuum QCD} \]

- $\overline{\psi}_b \{ \gamma_\mu D_\mu + m_b \} \psi_b \rightarrow \mathcal{L}_{stat} + O(1/m_b)$
- $\mathcal{L}_{stat}(x) = \overline{\psi}_h(x) \{ D_0 + m_b \} \psi_h(x)$
- $P\psi_h = \psi_h, \ P_+ = \frac{1}{2} (1 + \gamma_0) \Rightarrow 2 \text{ d.o.f.}$
- Systematic & accurate for $m_h/\Lambda_{QCD} \gg 1$
Non-perturbative formulation of HQET

Beyond the static approximation

\[ S_{\text{HQET}} = a^4 \sum_{x} \left\{ L_{\text{stat}}(x) + \sum_{\nu=1}^{n} L^{(\nu)}(x) \right\}, \quad L^{(\nu)}(x) = \sum_{i} \omega^{(\nu)}_{i} L^{(\nu)}_{i}(x) \]

\[ L_{\text{stat}} = \bar{\psi}_{h} \left[ \nabla^{*}_{0} + \delta m \right] \psi_{h} \]

\[ L^{(1)}_{1} = \bar{\psi}_{h} \left( -\frac{1}{2} \sigma \cdot B \right) \psi_{h} \equiv \mathcal{O}_{\text{spin}} \rightarrow \text{chromomagnetic interaction with the gluon field} \]

\[ L^{(1)}_{2} = \bar{\psi}_{h} \left( -\frac{1}{2} D^{2} \right) \psi_{h} \equiv \mathcal{O}_{\text{kin}} \rightarrow \text{kinetic energy from heavy quark's residual motion} \]

H. & Sommer, JHEP0402(2004)022
Non-perturbative formulation of HQET

Beyond the static approximation

\[ S_{\text{HQET}} = a^4 \sum_x \left\{ \mathcal{L}_{\text{stat}}(x) + \sum_{\nu=1}^n \mathcal{L}^{(\nu)}(x) \right\} , \quad \mathcal{L}^{(\nu)}(x) = \sum_i \omega_i^{(\nu)} \mathcal{L}_i^{(\nu)}(x) \]

\[ \mathcal{L}_{\text{stat}} = \bar{\psi}_h \left[ \nabla_0^* + \delta m \right] \psi_h \rightarrow \text{Eichten-Hill action} \]

\[ \mathcal{L}_1^{(1)} = \bar{\psi}_h \left( -\frac{1}{2} \sigma \cdot B \right) \psi_h \equiv \mathcal{O}_{\text{spin}} \rightarrow \{ \text{chromomagnetic interaction with the gluon field} \} \]

\[ \mathcal{L}_2^{(1)} = \bar{\psi}_h \left( -\frac{1}{2} D^2 \right) \psi_h \equiv \mathcal{O}_{\text{kin}} \rightarrow \{ \text{kinetic energy from heavy quark's residual motion} \} \]

- Effective theory regularized on a space-time lattice
- \( \delta m, \omega = \omega(g_0, m) \) to be determined such that HQET matches QCD
- At classical level: \( \omega_{\text{spin}} = \omega_{\text{kin}} = 1/m + O(g_0^2) , \delta m = 0 + O(g_0^2) \)
Non-perturbative formulation of HQET

Beyond the static approximation

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- Effective theory regularized on a space-time lattice
- \( \delta m, \omega = \omega(g_0, m) \) to be determined such that HQET matches QCD
  - At classical level: \( \omega_{\text{spin}} = \omega_{\text{kin}} = 1/m + O(g_0^2) \), \( \delta m = 0 + O(g_0^2) \)

- Analogously: Composite fields in the effective theory, e.g.

\[ A_0^{\text{HQET}}(x) = \underbrace{Z_A^{\text{HQET}}}_{1+O(g_0^2)} \bar{\psi}_l(x) \gamma_0 \gamma_5 \psi_h(x) + \underbrace{c_A^{\text{HQET}}}_{\propto 1/m} \bar{\psi}_l(x) \gamma_j \gamma_5 \vec{D}_j \psi_h(x) + \ldots \]
**Expectation values**

Path integral representation at the quantum level

\[
\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi] \, O[\varphi] \, e^{-(S_{\text{rel}} + S_{\text{HQET}})} \quad \mathcal{Z} = \int \mathcal{D}[\varphi] \, e^{-(S_{\text{rel}} + S_{\text{HQET}})}
\]

Now the *integrand* is expanded in a *power series* in \(1/m\)

\[
\exp\{-S_{\text{HQET}}\} = \exp\left\{-a^4 \sum_x \mathcal{L}_{\text{stat}}(x)\right\} \left\{1 - a^4 \sum_x \mathcal{L}^{(1)}(x) + \frac{1}{2} \left[a^4 \sum_x \mathcal{L}^{(1)}(x)\right]^2 - a^4 \sum_x \mathcal{L}^{(2)}(x) + \ldots\right\}
\]

\[
\Rightarrow \quad \langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi] \, e^{-S_{\text{rel}} - a^4 \sum_x \mathcal{L}_{\text{stat}}(x)} \, O \left\{1 - a^4 \sum_x \mathcal{L}^{(1)}(x) + \ldots\right\}
\]

**Important (but not automatic) implications of this definition of HQET**

- \(1/m\) – terms appear only as *insertions* of local operators
  - Power counting: *Renormalizability* at any given order in \(1/m\)
- \(\Leftrightarrow\) Existence of the *continuum limit* with *universality*
- Effective theory = *Continuum asymptotic* expansion in \(1/m\)
Renormalization

We assume massless quarks except the $b$ and no $1/m$–term in $\mathcal{O}$

\[
(f_A)_R \left( M_b, \Lambda, x_0 \Lambda \right) =
\]

\[
Z_{A}^{\text{HQET}} e^{-m_{\text{bare}} x_0} \left\langle A_0^{\text{stat}}(x) \mathcal{O} \left[ 1 + \omega_{\text{kin}} \sum_y \mathcal{O}_{\text{kin}}(y) + \omega_{\text{spin}} \sum_y \mathcal{O}_{\text{spin}}(y) \right] \right\rangle_{\text{stat}} + c_{A}^{\text{HQET}} \left\langle \delta A_0^{\text{stat}} \mathcal{O} \right\rangle_{\text{stat}} =
\]

\[
Z_{A}^{\text{HQET}} e^{-m_{\text{bare}} x_0} \left\{ f_{A}^{\text{stat}} + \omega_{\text{kin}} f_{A}^{\text{kin}} + \omega_{\text{spin}} f_{A}^{\text{spin}} + c_{A}^{\text{HQET}} f_{A}^{\delta A} \right\}
\]

⇒ Effective theory receives renormalizations in

\[
m_{\text{bare}} = m_b + \delta m
\]

\[
Z_{A}^{\text{HQET}} c_{A}^{\text{HQET}} \omega_{\text{kin}} \omega_{\text{spin}}
\]
Renormalization
We assume massless quarks except the $b$ and no $1/m$ term in $\mathcal{O}$

\[
(f_A)_R (\mathcal{M}_b/\Lambda, x_0 \Lambda) =
\]

\[
Z_{A}^{\text{HQET}} e^{-m_{\text{bare}} x_0} \left\langle A_0^{\text{stat}}(x) \mathcal{O} \left[ 1 + \omega_{\text{kin}} \sum_y \mathcal{O}_{\text{kin}}(y) + \omega_{\text{spin}} \sum_y \mathcal{O}_{\text{spin}}(y) \right] \right\rangle_{\text{stat}}
\]

\[
+ c_A^{\text{HQET}} \left\langle \delta A_0^{\text{stat}} (x) \mathcal{O} \right\rangle_{\text{stat}} =
\]

\[
Z_{A}^{\text{HQET}} e^{-m_{\text{bare}} x_0} \left( f_{A}^{\text{stat}} + \omega_{\text{kin}} f_{A}^{\text{kin}} + \omega_{\text{spin}} f_{A}^{\text{spin}} + c_A^{\text{HQET}} f_{\delta A}^{\text{stat}} \right)
\]

\[
\Rightarrow \text{Effective theory receives renormalizations in}
\]

\[
m_{\text{bare}} = m_b + \delta m \quad Z_{A}^{\text{HQET}} \quad c_A^{\text{HQET}} \quad \omega_{\text{kin}} \quad \omega_{\text{spin}}
\]

The Problem: *Power-law divergences* due to operator mixing

\[
\mathcal{O}_R^{d=5} = Z_{\mathcal{O}} \left\{ \mathcal{O}^{d=5} + \sum_k c_k \mathcal{O}_k^{d=4} \right\}, \quad c_k = a^{-1} \left\{ c_k^{(0)} + c_k^{(1)} g_0^2 + \ldots \right\}
\]

\[
\Rightarrow \text{for } a \to 0 \text{ divergent remainder, if } c_k \text{ are only perturbatively calculated}
\]

Example at the static level:
Linearly divergent counterterm $\delta m \propto 1/a$ originates from mixing of $\overline{\psi}_h D_0 \psi_h$ with $\overline{\psi}_h \psi_h$

\[
\Rightarrow \text{Non-pert. renormalization of HQET needed to have a continuum limit}
\]
Effective theory approximates QCD, if the coefficients $\{c_k\}$ are chosen correctly such that

$$\Phi^{\text{HQET}}(M) = \Phi^{\text{QCD}}(M) + O\left(1/\left[ r_0 M \right]^{n+1}\right)$$

$M = \text{RG-invariant (heavy) quark mass}$ (free of renormalization scheme dependence)

Strategy to guarantee this equivalence of HQET and QCD

Fix the coefficients of regularized HQET by imposing matching conditions

$$\Phi^{\text{HQET}}_k(M) = \Phi^{\text{QCD}}_k(M) \quad k = 1, \ldots, N_{\text{HQET}}$$

$\star$ determines the set $\{c_k\}$ for any lattice spacing (bare coupling)

$\Phi^{\text{QCD}}_k = \text{Renormalized quantities, computable for } \alpha \to 0 \text{ in QCD}$
Effective theory approximates QCD, if the coefficients \( \{ c_k \} \) are chosen correctly such that

\[
\Phi^{\text{HQET}}(M) = \Phi^{\text{QCD}}(M) + O \left( \frac{1}{r_0 M} \right)^{n+1}
\]

\( M = \text{RG-invariant (heavy) quark mass} \)

(free of renormalization scheme dependence)

**Strategy to guarantee this equivalence of HQET and QCD**

Fix the coefficients of regularized HQET by imposing *matching conditions*

\[
\Phi_k^{\text{HQET}}(M) = \Phi_k^{\text{QCD}}(M) \quad k = 1, \ldots, N_{\text{HQET}} \quad (\star)
\]

- \((\star)\) determines the set \( \{ c_k \} \) for any lattice spacing (bare coupling)
- \( \Phi_k^{\text{QCD}} \) = Renormalized quantities, computable for \( \alpha \to 0 \) in QCD

**Question**

How to treat the heavy quark as particle with finite mass on the lattice?
Idea: Employing *finite* volumes

Goal: *Non-*perturbative matching of HQET & QCD

\[ \frac{1}{m_b} \gg a \]

Matching conditions

\[ \Phi_{QCD}^k = \Phi_{HQET}^k \]

for observables \( \Phi_k \)
(e.g. matrix elements)

\[ \frac{1}{m_b} \ll L \]

- HQET parameters fixed by relating them to QCD observables in small \( V \)
- **Legitimate:** Underlying Lagrangian does not know about the finite \( V \)!
Idea: Employing *finite* volumes

**Goal:** Non-perturbative matching of HQET & QCD

**Objection:** Requires to simulate the b-quark as a relativistic particle

**Matching conditions**

\[
\Phi_{QCD}^k = \Phi_{HQET}^k \quad \text{for observables } \Phi_k
\]

(e.g. matrix elements)

- HQET parameters fixed by relating them to QCD observables in small V
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Idea: Employing *finite* volumes

**Goal:** *Non*-perturbative matching of HQET & QCD

**Objection:** Requires to simulate the b-quark as a *relativistic* particle

⇒ **Trick:** Start with QCD in *small* volume $V = L^4$, $L \equiv L_0 \simeq 0.2 \text{ fm}$

**QCD**

\[
\frac{1}{m_b} \gg a
\]

**HQET**

Matching conditions

\[
\Phi_{QCD}^k = \Phi_{HQET}^k
\]

for observables $\Phi_k$

(e.g. matrix elements)

\[
\frac{1}{m_b} \ll L
\]

- HQET parameters fixed by relating them to QCD observables in small $V$
- **Legitimate:** Underlying Lagrangian does not know about the finite $V$!
Connecting small and large volumes

(\star) in finite \(V\) : \(\Phi^{HQET}_k(L, M) = \Phi^{QCD}_k(L, M)\quad k = 1, \ldots, N_{HQET}\)

- **Typical choice:** \(L = L_0 \simeq 0.2 - 0.4\) fm, very small lattice spacings
- **Large volumes required to extract physical observables** (e.g. \(m_B, F_{B_s}\))
  \(\rightarrow\) How can we bridge the gap to practicable lattice spacings?
Connecting small and large volumes

(\star) \text{ in finite } V: \quad \Phi_{k}^{\text{HQET}}(L, M) = \Phi_{k}^{\text{QCD}}(L, M) \quad k = 1, \ldots, N_{\text{HQET}}

- Typical choice: \( L = L_{0} \approx 0.2 - 0.4 \, \text{fm} \), very small lattice spacings
- Large volumes required to extract physical observables (e.g. \( m_{B}, F_{B_{s}} \))
  \rightarrow \text{ How can we bridge the gap to practicable lattice spacings? }

Recursive finite-size scaling

\[ \Phi_{k}^{\text{HQET}}(L_{0}) \rightarrow \Phi_{k}^{\text{HQET}}(8L_{0}) \rightarrow \]

\[ \Phi_{k}^{\text{HQET}}(2L) = \sigma_{k}\left\{ \Phi_{j}^{\text{HQET}}(L), j = 1, \ldots, N \right\} \]

\rightarrow \text{ Large } V, \text{ where the } B \text{ fits comfortably}

\Rightarrow \text{ First fully non-perturbative formulation of HQET, including matching}


- Non-perturbative, continuum limit exists
- Use of the \textit{QCD Schrödinger Functional}
- Change \( L \rightarrow 2L \) via step scaling functions
**Application: Computation of $M_b$ in lowest-order HQET**

Introduce $L$–dependent energies from correlators in the B-channel (and with SF boundary conditions):

\[
\begin{align*}
\Gamma(L, M) \quad & = \quad \{ \text{B-meson mass in a } \textit{finite volume} \text{ of extent } L^4 \\
\Gamma_{\text{stat}}(L) \quad & = \quad \{ \text{energy of a state with B-meson quantum numbers in } L^4 \}
\end{align*}
\]

\[
C(x_0, M) : \quad \begin{cases} 
\zeta_{l_1} \\
\bar{\zeta}_{l_2}
\end{cases} \quad \xrightarrow{x_0 = 0} \quad A_0
\]

\[
\rightarrow \quad \Gamma(L, M) \equiv - \frac{d}{dx_0} \ln [C(x_0, M)] \bigg|_{x_0 = \frac{L}{2}}
\]

\[
C_{\text{stat}}(x_0) : \quad \begin{cases} 
\bar{\zeta}_l \\
\zeta_h
\end{cases} \quad \xrightarrow{x_0 = L} \quad A_0^{\text{stat}}
\]

\[
\rightarrow \quad \Gamma_{\text{stat}}(L) \equiv - \frac{d}{dx_0} \ln [C_{\text{stat}}(x_0)] \bigg|_{x_0 = \frac{L}{2}}
\]
⇒ Matching condition by equating in small volume with linear extent $L_0$:

$$\Gamma_{\text{stat}}(L_0) + m_{\text{bare}} = \Gamma(L_0, M_b) \quad \text{[Recall: } m_{\text{bare}} = m + \frac{1}{\alpha} \ln(1 + \alpha \delta m)\text{]}
$$

As $C(x_0) \xrightarrow{x_0 \to \infty} e^{-m_B x_0}$ and $C_{\text{stat}}(x_0) \xrightarrow{x_0 \to \infty} e^{-E_{\text{stat}} x_0}$ in the large-$L$ limit, we have to connect this condition (by finite-size scaling) to

$$E_{\text{stat}} + m_{\text{bare}} = m_B$$
Matching condition by equating in small volume with linear extent $L_0$:

$$\Gamma_{\text{stat}}(L_0) + m_{\text{bare}} = \Gamma(L_0, M_B) \quad \text{[Recall: } m_{\text{bare}} = m + \frac{1}{\alpha} \ln(1 + \alpha \delta m)]$$

As $C(x_0) \sim e^{-m_B x_0}$ and $C_{\text{stat}}(x_0) \sim e^{-E_{\text{stat}} x_0}$ in the large-L limit, we have to connect this condition (by finite-size scaling) to

$$E_{\text{stat}} + m_{\text{bare}} = m_B$$

**Sketch of the method**

- **Experiment**
  - $m_B = 5.4$ GeV
  - $\Gamma_{\text{stat}}(L_2)$
  - $\sigma_m(u_1)$
  - $u_i = \bar{g}^2(L_i)$
  - $\Gamma_{\text{stat}}(L_1)$
  - $\sigma_m(u_0)$
  - $\Gamma_{\text{stat}}(L_0)$

- **Lattice with $am_b \ll 1$**
  - $\Gamma(L_0, M)$

A step scaling function $\sigma_m(u) \equiv 2L \left[ \Gamma_{\text{stat}}(2L) - \Gamma_{\text{stat}}(L) \right]$ bridges between small & larger volumes by evolving $L_0 \rightarrow L_2 = 2^2L_0 \approx 1 \text{ fm}$

For $L \approx 2 \text{ fm} @$ same resolution:
Physical situation to accommodate (& calculate) B-meson properties

**Fundamental QCD parameters from the lattice**
⇒ Equation to solve for the b-quark mass:

\[ L_0 \Gamma(L_0, M) = L_0 m_B^{(exp)} - L_0 \left\{ [\Gamma_{stat}(L_2) - \Gamma_{stat}(L_0)] - [E_{stat} - \Gamma_{stat}(L_2)] \right\} \]

\[ \alpha \to 0 \implies \Omega_{QCD} \text{ in QCD } (L_0^4) \]

\[ \alpha \to 0 \implies \Omega_{HQET} \text{ in HQET (contains } \sigma_m) \]

- Divergent static quark’s self-energy \( \delta m \) cancels in differences!
- \( \Gamma(L_0, M) \) carries entire (relativistic) heavy quark mass dependence
⇒ Equation to solve for the b-quark mass:

\[
L_0 \Gamma(L_0, M) = L_0 m_B^{(\text{exp})} - L_0 \left\{ \left[ \Gamma_{\text{stat}}(L_2) - \Gamma_{\text{stat}}(L_0) \right] - \left[ E_{\text{stat}} - \Gamma_{\text{stat}}(L_2) \right] \right\}
\]

\( a \to 0 \equiv \Omega_{\text{QCD}} \) in QCD \((L_0^4)\)

\( a \to 0 \equiv \Omega_{\text{HQET}} \) in HQET (contains \( \sigma_m \))

- Divergent static quark’s self-energy \( \delta m \) cancels in differences!
- \( \Gamma(L_0, M) \) carries entire (relativistic) heavy quark mass dependence

**QCD \((z \equiv L_0 M \gg 1, L_0 \approx 0.2 \text{ fm})\)**

**Quenched static result**

\[ \bar{m}_b^{\text{MS}}(\bar{m}_b^{\text{MS}}) = 4.12(8) \text{ GeV} + O\left(\frac{1}{M_b}\right) \]
Inclusion of $1/m$–terms in the computation of $M_b$


$m_B$ at next-to-leading order of HQET

\[
m_B = E_{\text{stat}} + m_{\text{bare}} + \omega_{\text{kin}} E_{\text{kin}} + \omega_{\text{spin}} E_{\text{spin}}
\]

- $E_{\text{kin}}, E_{\text{spin}}$ associated with $\bar{\psi}_h(-\frac{1}{2}D^2)\psi_h$ and $\bar{\psi}_h(-\frac{1}{2}\sigma \cdot B)\psi_h$ in $\mathcal{L}^{(1)}$

\rightarrow Three observables $\Phi_1, \Phi_2, \Phi_3$ required in the matching step

- Considering the spin-averaged $B$-meson instead, $\omega_{\text{spin}}$ cancels:

\[
m_{B}^{(\text{av})} = \frac{1}{4} m_B + \frac{3}{4} m_B^* = E_{\text{stat}} + m_{\text{bare}} + \omega_{\text{kin}} E_{\text{kin}}
\]

\rightarrow Only two observables $\Phi_1, \Phi_2$ necessary

The previous strategy now extends to

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Lattice with $\alpha m_B \ll 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_B = 5.4$ GeV</td>
<td>$L_1 \simeq 0.4$ fm, $L_2 = 2L_1$</td>
</tr>
<tr>
<td>$\Phi_1^{HQET}(L_2), \Phi_2^{HQET}(L_2)$</td>
<td>$u_1 = \bar{g}^2(L_1)$</td>
</tr>
<tr>
<td>$\sigma_m(u_1), \sigma_{1\text{kin}}(u_1), \sigma_{2\text{kin}}(u_1)$</td>
<td>$\Phi_1^{HQET}(L_1), \Phi_2^{HQET}(L_1)$</td>
</tr>
</tbody>
</table>
Matching formula = Static part + \( \frac{1}{m} \) – correction :

\[
L_2 m_B^{(av)} = L_2 m_B^{stat} \quad \left\{ \begin{array}{l}
L_2 \left[ E_{stat} - \Gamma_1^{stat}(L_2) \right] + \sigma_m(u_1) + 2\Phi_2(L_1, M) \\
L_2 m_B^{(1)} \end{array} \right.
\]

= \sigma_2^{kin}(u_1) \Phi_1(L_1, M) + L_2 \left[ \hat{E}_{kin} - \Gamma_1^{kin}(L_2) \right] \omega_{kin}

Implementation

As in the static case, but technical details more involved (& omitted here)

- **SF boundary conditions**, i.e. \( T \times L^3 \), Dirichlet at \( x_0 = 0, T \), fermion fields periodic in space modulo a phase: \( \psi(x + L \hat{k}) = e^{i\theta} \psi(x) \)

- Avoid extra term \( \left( \frac{1}{m} f_{stat}^{\delta_A} \right) \) in boundary-to-\( \Lambda_0 \) correlator by boundary-to-boundary ones with pseudoscalar or vector quantum numbers

\[
(f_1)_R (T) = Z_{boundary} e^{-m_{bare} T} \left\{ f_{stat}^{\delta_A} + \omega_{kin} f_{kin}^{\delta_A} + \omega_{spin} f_{spin}^{\delta_A} \right\}
\]

- **\( \Phi_1 \)**: Suitable ratios of \( f_1(\theta), f_1(\theta') \) such that \( Z \)–factors drop out

- **\( \Phi_2 \)**: *Spin-averaged* energy \( \Gamma_1 = -\partial_T \ln f_1^{(av)} = m_{bare} + \Gamma_1^{stat} + \omega_{kin} \Gamma_1^{kin} \) plus corresponding step scaling functions \( \sigma_1^{kin}(u), \sigma_2^{kin}(u) \)
Examples for continuum limit extrapolations (quenched)

Note: $1/m$–terms approach $a = 0$ only with a rate $\propto a$

- **Left:**
  Dimensionless spin-averaged finite-volume energy from $f_1$ in QCD

- **Right:**
  Cancellation of $1/a^2$ power-divergence in $\sigma_{2}^{\text{kin}} = \lim_{a/L \to 0} \Sigma_{2}^{\text{kin}}$
New perspectives for B-physics

The b-quark’s mass

- **Left: Leading order (\(\approx\) static)**
  - \(m_{\text{light}} = 0 \& L_1 \approx 0.4 \text{ fm}\)
  - Input: \(r_0 m_B^{(\text{exp})}, r_0 = 0.5 \text{ fm}\)
  - Error dominated by that on \(Z_M\) in \(LM = Z_M Z (1 + b_m a m_q) \times L m_q\) (NP known incl. \(O(a)\) [\(\text{\textit{ALPHA}}\) Collaboration])

- Quenched b-quark’s mass to order \(\Lambda^2/m_b\) in HQET
  - Result in the \(\overline{\text{MS}}\) scheme (reported @ Lattice Conference 2005):
    \[
    m_b (m_b) = m_b^{\text{stat}} + m_b^{(1)} \quad m_b^{\text{stat}} = 4.35(6) \text{ GeV} , m_b^{(1)} = -0.05(3) \text{ GeV} \\
    = 4.30(7)
    \]
  - Though quenched approximation, well within range quoted by PDG
  - Difficult piece: Large-volume HQET matrix element \([\hat{E}_{\text{kin}} - \Gamma_{1}^{\text{kin}}(L_2)]]\)
Summary & Outlook

- Predictive power of lattice QCD demands high precision:
  - Overcoming the error from the quenched approximation is under way
  - Several & fine enough resolutions required to quantify cutoff effects

- New quality of the computations employing lattice HQET:
  - Non-perturbative formulation including the matching
  - Continuum limit at large quark masses (small-volume setup!)
  - Renormalization of $1/m$ terms can be performed non-perturbatively
  - Physics results still quenched, but extension of the methods to QCD with dynamical quarks ($N_f > 0$) straightforward and already started
    [first steps towards $F_{B_s}$ for $N_f = 2$: Della Morte, Fritzsch & H.]

- Next steps, resp. related work in progress:
  - Decay constant $F_B$
  - Spin-splitting $m_{B^*} - m_B$
  - NP matching for $N_f = 2$