Gauge and Lorentz covariant Schwinger-Dyson equation

for fermion propagator in arbitrary external gauge field

Qing Wang

Tsinghua University, Beijing, P.R.China

September 23, 2006, DESY, Germany

Gauge and Lorentz covariant SDE for FP in arbitrary EGF

- Motivations
- QED prototype
- QCD applications

Motivations

- We are interested in: $\langle 0|\mathbf{T}\psi(x)\overline{\psi}(y)\overline{\psi}(x_1)\Gamma_1\psi(x_1)\cdots\overline{\psi}(x_n)\Gamma_n\psi(x_n)|0\rangle$
- They satisfy constraints from symmetries: Ward-Takahashi identities
- They are determined by Schwinger-Dyson equations
- Are these requirements consistent with each other? **depending on computation procedure!**
- We hope to make process of solving SDE naturally satisfy constraints from symmetries !
- $\Rightarrow$Introduce external gauge field into fermion propagator fixed by SDE
- Taylor expansion of external field for FP will result multi-points Green's function
- Symmetry constraint is expressed through gauge covariance of external field for FP
- For FP with external gauge field: we will set up a gauge covariant SDE for it!

QED prototype

External Gauge Field Description of Fermion Propagator

Equivalence with Ward-Takahashi Identities

Gauge and Lorentz covariant SDE

External Gauge Field Description of Fermion Propagator

$$S(x,y;A)=-i\langle 0|\mathbf{T}\psi(x)\overline{\psi}(y)|0\rangle_A \xleftarrow{\text{gauge interaction}\ \ \overline{\psi}i\nabla\!\!\!\!/\,\psi} \nabla^\mu_x=\partial^\mu_x-igA^\mu(x)$$

$$S(x,y;0)=\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\ S(p)=S(i\partial_x)\delta^4(x-y)$$ **Fermion propagator**

$$\frac{\delta^n S(x,y;A)}{\delta A^{\mu_1}(x_1)\cdots\delta A^{\mu_n}(x_n)}\bigg|_{A^\mu=0}=-i\langle 0|\mathbf{T}\psi(x)\overline{\psi}(y)g\overline{\psi}(x_1)\gamma^{\mu_1}\psi(x_1)\cdots g\overline{\psi}(x_n)\gamma^{\mu_n}\psi(x_n)|0\rangle_0$$

Gauge transformation $\xRightarrow{\psi'(x)=e^{i\alpha(x)}\psi(x)}\langle 0|\mathbf{T}\psi'(x)\overline{\psi}'(y)|0\rangle=e^{i\alpha(x)}\langle 0|\mathbf{T}\psi(x)\overline{\psi}(y)|0\rangle e^{-i\alpha(y)}$

Realization of Gauge transformation $\xRightarrow{A'(x)=A(x)+\frac{1}{g}\partial_x\alpha(x)}S'(x,y;A)=S(x,y;A')=e^{i\alpha(x)}S(x,y;A)e^{-i\alpha(y)}$

Without external gauge field, it's impossible to realize above relation !

Equivalence with Ward-Takahashi Identities

$$S(x,y;A+\frac{1}{g}\partial\alpha)=e^{i\alpha(x)}S(x,y;A)e^{-i\alpha(y)}$$

$$\int d^4z\,\frac{\delta S(x,y;A)}{\delta A^\mu(z)}\frac{1}{g}\partial_\mu\alpha(z)=i[\alpha(x)-\alpha(y)]S(x,y;A)$$

$$-\frac{\partial}{\partial z^\mu}\frac{\delta S(x,y;A)}{\delta A^\mu(z)}=ig[\delta(x-z)-\delta(y-z)]S(x,y;A)$$

$$\langle 0|\mathbf{T}\partial_z^\mu[\overline{\psi}(z)\gamma_\mu\psi(z)]\psi(x)\overline{\psi}(y)|0\rangle_A=-ig[\delta(x-z)-\delta(y-z)]\langle 0|\mathbf{T}\psi(x)\overline{\psi}(y)|0\rangle_A$$

$$\langle 0|\mathbf{T}\partial_z^\mu[\overline{\psi}(z)\gamma_\mu\psi(z)]\psi(x)\overline{\psi}(y)\overline{\psi}(x_1)\gamma^{\mu_1}\psi(x_1)\cdots\overline{\psi}(x_n)\gamma^{\mu_n}\psi(x_n)|0\rangle_0$$
$$=-ig[\delta(x-z)-\delta(y-z)]\langle 0|\mathbf{T}\psi(x)\overline{\psi}(y)\overline{\psi}(x_1)\gamma^{\mu_1}\psi(x_1)\cdots\overline{\psi}(x_n)\gamma^{\mu_n}\psi(x_n)|0\rangle_0$$

Gauge and Lorentz covariant SDE

$$S(x,y;A+\frac{1}{g}\partial\alpha)=e^{i\alpha(x)}S(x,y;A)e^{-i\alpha(y)}$$ $\Leftrightarrow$ **Ward-Takahashi Identity**

ladder approx SDE: $S^{-1}(x,y;A)-i\nabla\!\!\!/_x\delta^4(x-y)=g^2D^{\mu\nu}(x,y)\gamma_\mu S(x,y;A)\gamma_\nu$

$\nabla_x^\mu=\partial_x^\mu-igA^\mu(x)$ **photon propagator**

find method solve SDE consistent with WTI

$\Rightarrow$ **keep covariance of $S(x,y;A)$** **A dependence** **when solving SDE with EGF**

$$S(x,y;0)=\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}S(p)=S(i\partial_x)\delta^4(x-y)\;\Rightarrow\;S(x,y;A)=S(i\nabla_x)\delta(x-y)$$

$$A'_\mu=A_\mu+\frac{1}{g}\partial_\mu\alpha\;\Rightarrow\;(\nabla_x^\mu)'=e^{i\alpha(x)}\nabla_x^\mu e^{-i\alpha(x)}\;\Rightarrow\;S(i\nabla'_x)=e^{i\alpha(x)}S(i\nabla_x)e^{-i\alpha(x)}$$

$$S(x,y;A')=e^{i\alpha(x)}S(i\nabla_x)e^{-i\alpha(x)}\delta(x-y)=e^{i\alpha(x)}S(i\nabla_x)\delta(x-y)e^{-i\alpha(y)}=e^{i\alpha(x)}S(x,y;A)e^{-i\alpha(y)}$$

Gauge and Lorentz covariant SDE

$$[S^{-1}(i\nabla_x) - i\not\nabla_x]\delta^4(x-y) = g^2 D^{\mu\nu}(x,y)\gamma_\mu S(i\nabla_x)\delta^4(x-y)\gamma_\nu$$

$S(x,y;A) \equiv S(i\nabla_x)\delta(x-y)$ **involving differentials in coordinate space, not convenient !**

$$S(i\nabla_x)\delta(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} S(i\nabla_x + p)1 \qquad S(i\nabla'_x + p)1 \neq e^{i\alpha(x)}[S(i\nabla_x + p)1]e^{-i\alpha(y)}$$

Momentum Space: replace ∇^μ_x **with** $[\nabla^\mu_x, \nabla^\nu_x] = -iF^{\mu\nu}_x$ $\quad S(x,y;A') = e^{i\alpha(x)}S(x,y;A)e^{-i\alpha(y)}$

$$i\nabla_x + p = e^{i\nabla_x\cdot\frac{\partial}{\partial p}}(p - \tilde{F}_{p,x})e^{-i\nabla_x\cdot\frac{\partial}{\partial p}}$$

Y.An, H.Yang, Q.Wang, EPJC 29(1)65(2003)

$$\tilde{F}^\mu_{p,x} = -\frac{1}{2}[\nabla^\nu_x, \nabla^\mu_x]\frac{\partial}{\partial p^\nu} + \frac{i}{3}\partial^\lambda_x[\nabla^\nu_x, \nabla^\mu_x]\frac{\partial^2}{\partial p^\lambda \partial p^\nu} + \frac{1}{8}\partial^{\mu'}_x\partial^{\nu'}_x[\nabla^\nu_x, \nabla^\mu_x]\frac{\partial^3}{\partial p^{\mu'}\partial p^{\nu'}\partial p^\nu} + \cdots$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} e^{i\nabla_x\cdot\frac{\partial}{\partial p}} S(p - \tilde{F}_{p,x}) e^{-i\nabla_x\cdot\frac{\partial}{\partial p}} 1 = e^{-(x-y)\cdot\nabla_x 1} \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} S(p - \tilde{F}_{p,x})1$$

$\tilde{F}'_{p,x} = \tilde{F}_{p,x}$ $\quad S(p - \tilde{F}'_{p,x})1 = S(p - \tilde{F}_{p,x})1$ $\quad e^{-(x-y)\cdot\nabla'_x 1} = e^{i\alpha(x)}[e^{-(x-y)\cdot\nabla_x 1}]e^{-i\alpha(y)}$

$$S^{-1}(p - \tilde{F}_{p,x})1 - \not p = g^2 \int \frac{d^4q}{(2\pi)^4} D^{\mu\nu}(p-q)\gamma_\mu S(q - \tilde{F}_{q,x})1\, \gamma_\nu$$

QCD applications

QCD with Flavor External Field and Local Chiral Symmetry

Low Energy Chiral Lagrangian

Gauge and Lorentz covariant SDE for FP with Flavor External Field

QCD with Flavor External Field and Local Chiral Symmetry

$$\mathcal{L}_J=\mathcal{L}_{\rm QCD}(\psi,\bar\psi,\Psi,\bar\Psi,A_\mu)+\bar\psi J\psi \qquad J(x)=-s(x)+ip(x)\gamma_5+\not v(x)+\not a(x)\gamma_5$$

Local Chiral Symmetry: $\psi'(x)=[R(x)P_R+L(x)P_L]\psi(x)$

$$J'(x)=[R(x)P_L+L(x)P_R][i\not\partial+J(x)][R^\dagger(x)P_R+L^\dagger(x)P_L]$$

$$Z[J]=\int\mathcal{D}\psi\mathcal{D}\bar\psi\mathcal{D}\Psi\mathcal{D}\bar\Psi\mathcal{D}A_\mu\; e^{i\int d^4x\;\mathcal{L}_J}$$ **generate current-current Green's functions**

$$J(x)\equiv J_\Gamma(x)\Gamma \qquad \frac{\delta^n Z[J]}{\delta J_{\Gamma_1}(x_1)\cdots\delta J_{\Gamma_n}(x_n)}=i^n\langle 0|\mathbf{T}\;\overline{\psi}(x_1)\Gamma_1\psi(x_1)\cdots\overline{\psi}(x_n)\Gamma_n\psi(x_n)|0\rangle$$

$Z[J']=Z[J]+$ **anomaly**

Low Energy Chiral Lagrangian

Gasser-Leutwyler: $$Z[J]=\int \mathcal{D}U\ \delta(U^\dagger U-1)\delta(\det U-1)\ e^{iS_{\rm eff}[U,J]}$$

$$S_{\rm eff}[U,J]=S_{\rm normal}[U,J]+S_{\rm anomaly}[U,J]\qquad S_{\rm normal}[U,J]=S_{\rm normal}[1,J_\Omega]$$ **many coefficients**

$$J_\Omega(x)=[\Omega(x)P_R+\Omega^\dagger(x)P_L][J(x)+i\partial\!\!\!/][\Omega(x)P_R+\Omega^\dagger(x)P_L]\qquad U(x)\equiv\Omega^2(x)$$

First Principle Calculation:

- **Coefficients are determined by QCD Green's functions !** Q.Wang,Y.P.Kuang,et al, PRD61,54011(2000)
- **Green's functions are determined by SDE with external flavor fields !**
- **Local chiral symmetry relate various Green's functions !**

Sino-German Workshop in Beijing, P.R.China; Sep 6-9, 2004

QCD at High Energies/Frontiers in QCD

Chiral Lagrangian from QCD

- Why we need QCD investigations?
  Theoretically; Phenomenologically; Technically; Applications
- Relation between CL and QCD
  Generating functional; Formal derivation of chiral Lagrangian from QCD; Different approaches
- First principle calculation of LECs
  Anomaly approach; Dynamical quark approach **H.Yang, Q.Wang, et al, PLB532, 240(2002); PRD66, 014019(2002)**
- Summary

Gauge and Lorentz covariant SDE for FP with Flavor External Field

$S_{\rm eff}[U,J]=S_{\rm normal}[U,J]+S_{\rm anomaly}[U,J]$ $\quad S_{\rm normal}[U,J]=S_{\rm normal}[1,J_\Omega]$ **many coefficients**

$$J_\Omega(x)=-s_\Omega(x)+ip_\Omega(x)\gamma_5+\not{v}_\Omega(x)+\not{a}_\Omega(x)\gamma_5$$

Hidden local symmetry: $s'_\Omega=h^\dagger s_\Omega h,\ p'_\Omega=h^\dagger p_\Omega h,\ a^{\mu\prime}_\Omega=h^\dagger a^\mu_\Omega h,\ v^{\mu\prime}_\Omega=h^\dagger v^\mu_\Omega h+h^\dagger i\partial^\mu h$

Fermion Propagator : $\Phi^T_{\Omega c}(x,y)=\dfrac{-1}{N_c}\langle 0|\mathbf{T}\psi_\Omega(x)\overline{\psi}_\Omega(y)|0\rangle_{J_\Omega}$ $\Rightarrow$ **fix all coefficients** $\ {\rm tr}[\gamma_5\Phi^T_{\Omega c}(x,x)]=0$

Coefficients of CL are determined by coincidence limit of FP with external flavor fields

$$\Phi^{T\prime}_{\Omega c}(x,y)=h^\dagger(x)\Phi^T_{\Omega c}(x,y)h(y)$$

$$[i\not{\partial}+i\Phi^{T,-1}_{\Omega c}+\not{v}_\Omega+\not{a}_\Omega\gamma_5-s_\Omega+ip_\Omega\gamma_5+\tilde{\Xi}](x,y)+\frac{g^2N_c}{2}G_{\mu\nu}(x,y)\gamma^\mu\Phi^T_{\Omega c}(x,y)\gamma^\nu=0$$

gluon propagator

QCD applications

- Coefficients of CL are determined by coincidence limit of FP with external flavor fields
- FP satisfy SDE and must be covariant under hidden local symmetry transformations
- Keeping HLST covariance is equivalent to keep local chiral symmetry of the theory

$$\Phi^T_{\Omega c}(x,y) = \tilde{\Phi}[\overline{\nabla}^\mu_x, a^\nu_\Omega(x), s_\Omega(x), p_\Omega(x)]\delta(x-y) \qquad \overline{\nabla}^\mu_x \equiv \partial^\mu_x - iv^\mu_\Omega(x)$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)}\tilde{\Phi}[-ip^\mu + \overline{\nabla}^\mu_x, a^\nu_\Omega(x), s_\Omega(x), p_\Omega(x)]$$

$$= e^{-(x-y)\cdot\overline{\nabla}_x}1\int\frac{d^4p}{(2\pi)^4}\ e^{-ip\cdot(x-y)}\ \tilde{\Phi}[-i(p^\mu - \tilde{F}^\mu_{p,x}), \tilde{a}^\nu_{p,x}, \tilde{s}_{p,x}, \tilde{p}_{p,x}]1 \qquad \int\frac{d^4p}{(2\pi)^4}\mathrm{tr}[\tilde{\Phi}_{p,x}1\gamma_5] = 0$$

$$-ip^\mu + \overline{\nabla}^\mu_x \equiv -ie^{i\overline{\nabla}_x\cdot\frac{\partial}{\partial p}}(p^\mu - \tilde{F}^\mu_{p,x})e^{-i\overline{\nabla}_x\cdot\frac{\partial}{\partial p}} \qquad \phi_\Omega(x) \equiv e^{i\overline{\nabla}_x\cdot\frac{\partial}{\partial p}}\tilde{\phi}_{p,x}e^{-i\overline{\nabla}_x\cdot\frac{\partial}{\partial p}} \qquad \phi = a^\mu;\ s;\ p;\ \Xi$$

$$\tilde{F}^\mu_x = -\frac{1}{2}[\overline{\nabla}^\nu_x, \overline{\nabla}^\mu_x]\frac{\partial}{\partial p^\nu} + \frac{i}{3}[\overline{\nabla}^\lambda_x, [\overline{\nabla}^\nu_x, \overline{\nabla}^\mu_x]]\frac{\partial^2}{\partial p^\lambda \partial p^\nu} + \cdots \qquad \tilde{\phi}_x = \phi_\Omega(x) - i[\overline{\nabla}^\nu_x, \phi_\Omega(x)]\frac{\partial}{\partial p^\mu} - \frac{1}{2}[\overline{\nabla}^\lambda_x, [\overline{\nabla}^\nu_x, \phi_\Omega(x)]]\frac{\partial^2}{\partial p^\lambda \partial p^\mu} + \cdots$$

$$\not{p} - \tilde{\not{F}}_{p,x} + \tilde{\not{a}}\,\gamma_5 - \tilde{s}_{p,x} + i\tilde{p}_{p,x}\gamma_5 + i\tilde{\Phi}^{-1}_{p,x}1 + \tilde{\Xi}_{p,x} = -\frac{N_c g^2}{2}\int\frac{d^4q}{(2\pi)^4}G_{\mu\nu}(p-q)\gamma^\mu\tilde{\Phi}_{q,x}1\gamma^\nu$$

Summary

- We are interested in Green's functions: $\langle 0|\mathbf{T}\psi(x)\overline{\psi}(y)\overline{\psi}(x_1)\Gamma_1\psi(x_1)\cdots\overline{\psi}(x_n)\Gamma_n\psi(x_n)|0\rangle$
- They are generated by fermion propagator with external fields
- Schwinger-Dyson equation for fermion propagator with external field fully determine them
- They satisfy Ward-Takahashi identities
- Effects of WTI in terms of FP is replaced by its external gauge field covariance
- External gauge field transformation covariant SDE is built up !