## Toolkit/checklist

$$I = \sqrt{-1}.$$

Analytic function of complex number $z$ must be of form $f(z)$, i.e. excludes $f(z, z^*)$.

Kronecker delta: $\delta_{ij} = 1$ if $i = j$, $0$ otherwise.

Completely antisymmetric tensor:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123 \text{ or cyclic permutations thereof} \\ -1 & \text{if } ijk = 321 \text{ or cyclic permutations thereof} \\ 0 & \text{otherwise.} \end{cases}$$

Summation convention: $\sum_i A_i B_i = A_i B_i$, i.e. if same index ($i$) appears twice in term, sum over it, don't bother writing $\sum$.

Note e.g. $A_{ij}\delta_{jk} = A_{ik}$.

If $A_{ij} = A_{ji}$ (symmetric in $i, j$), $\epsilon_{ijk} A_{jk} = 0$.

Scalar product: $A_i B_i$. (For complex $A_i, B_i$, scalar product defined as $A_i^* B_i$.)

Contraction of index $k$ in $A_{ijk}$ with $B_i$: $A_{ijk} B_k = C_{ij}$, i.e. $k$ becomes dummy index, rank of $A$ reduced from 3 to 2 (a matrix).

A vector $V$ is an $n$-tuple "down the page"

$$V = \begin{pmatrix} 2 + 3I \\ 1 - 4I \\ \vdots \end{pmatrix}.$$

Its *transpose* is $V^T = (2 + 3I, 1 - 4I, \ldots)$.

Its *conjugate transpose* is $V^\dagger = V^{T*} = (2 - 3I, 1 + 4I, \ldots)$

Scalar product: $A^{(\dagger)} \cdot B$.

Matrix $A$ acting on vector $v$: $A_{ij} v_j = (Av)_i$ (and $v_j A_{ji} = (v^T A)_i$).

Two matrices $A, B$: $(AB)^\dagger = B^\dagger A^\dagger$ (likewise for simple transpose $^T$).

Unit matrix $\mathbf{1}$ (not 1): $\mathbf{1}_{ij} = \delta_{ij}$.

Hermitian matrix $H$: $H^\dagger = H$.

Unitary matrix $U$: $U^\dagger U = \mathbf{1}$.

Orthogonal matrix $M$: $M^T M = \mathbf{1}$.

Definition of "exponentiation" of a matrix $A$: $e^A = \mathbf{1} + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + ...$ (this series is always convergent).

Note $e^{VAV^{-1}} = Ve^A V^{-1}$ for any $V$ (because $VA^nV^{-1} = (VAV^{-1})^n$).

$e^{IA}$ unitary if $A$ Hermitian (because $(e^{IA})^\dagger e^{IA} = e^{-IA^\dagger} e^{IA} = e^{-IA} e^{IA} = \mathbf{1}$).

Trace of matrix $A$: $\mathrm{tr}[A] = A_{ii}$.

Determinant of matrix $n \times n$ $A$: $\det(A) = \epsilon_{ijk...} A_{i1} A_{j2} A_{k3} \ldots = \frac{1}{n!} \epsilon_{ijk...} \epsilon_{abc...} A_{ia} A_{jb} A_{kc} \ldots$.

Note $\det(AB) = \det(A)\det(B)$

(because $\det(AB) = \frac{1}{n!} \epsilon_{ijk...} \epsilon_{abc...} (A_{im} B_{ma})(A_{jn} B_{nb})(A_{kp} B_{pc}) \ldots = \frac{1}{n!} n! \frac{1}{n!} \epsilon_{ijk...} \epsilon_{def...} A_{id} A_{je} A_{kf} \ldots \frac{1}{n!} \epsilon_{qrs...} \epsilon_{abc...} B_{qa} B_{rb} B_{sc} \ldots$).

If $Av = 0$, then either $v = 0$, or $v \neq 0$ and $\det(A) = 0$

$$(v_m \det(A) = \epsilon_{ijk...} (v_m A_{im}) A_{j2} A_{k3} \ldots = \epsilon_{ijk...} \underbrace{\left(\sum_n v_n A_{in}\right)}_{=0} A_{j2} A_{k3} \ldots = 0).$$

Linearly independent vectors $v_i$: $\sum_i \alpha_i v_i = 0 \implies \alpha_i = 0$.

$N$ linearly independent vectors $v_i$ form *complete basis* in some $N$ dimensional vector space. (i.e. any $V = \sum_i \alpha_i v_i$).

Any other basis of $N$ linearly independent vectors $v'_i$ in this space is complete, i.e. any $V = \sum_i \alpha'_i v'_i$:

Can write $v'_i = \sum_j C_{ij} v_j$, so $\alpha'_i = \sum_j \alpha_j (C^{-1})_{ji}$,

and $C$ must be invertible for $v'_i$ to be linearly independent ($\alpha_i = 0 \implies \alpha'_i = 0$).

Eigenvalues $\lambda_k$ and eigenvectors $w^{\{k\}}$ of matrix $A$ from *secular equation*: $Aw^{\{k\}} = \lambda_k w^{\{k\}}$.

Requirement of $(A - \lambda_k \mathbf{1})w^{\{k\}} = 0$ and $w^{\{k\}} \neq 0$: *Characteristic equation* $|A - \lambda_k \mathbf{1}| = 0$.

If $A$ is $N \times N$, this is polynomial in $\lambda_k$ with $N$ solutions $\longrightarrow$ $N$ eigenvectors.

Hermitian $A$:

$\lambda_k$ real $(w^{\{k\}\dagger} \cdot Aw^{\{k\}} = \lambda_k w^{\{k\}\dagger} \cdot w^{\{k\}} = (w^{\{k\}\dagger} \cdot Aw^{\{k\}})^\dagger = \lambda_k^* w^{\{k\}\dagger} \cdot w^{\{k\}})$.

$w^{\{k\}}$ orthogonal $(w^{\{i\}\dagger} \cdot Aw^{\{j\}} = \lambda_i w^{\{i\}\dagger} \cdot w^{\{j\}} = \lambda_j w^{\{i\}\dagger} \cdot w^{\{j\}}$, so $i \neq j$, $\lambda_i \neq \lambda_j \implies w^{\{i\}\dagger} \cdot w^{\{j\}} = 0$ (choose this for $\lambda_i = \lambda_j$)).

Normalize $w^{\{k\}}$ to be orthonormal: $w^{\{i\}\dagger} \cdot w^{\{j\}} = \delta_{ij}$. $w^{\{k\}}$ form complete basis $(\sum_i \alpha_i w^{\{i\}} = 0 \implies 0 = \sum_i \alpha_i w^{\{j\}\dagger} \cdot w^{\{i\}} = \alpha_j)$.

Diagonalization of any matrix $A$: $(P^{-1}AP) = \Lambda,$ where $\Lambda_{ij} = \lambda_i \delta_{ij}$ (no sum)

(equivalent to the secular equation above with $P_{ij} = w_i^{\{j\}}$).

Hermitian $A$: $P$ is unitary $((P^\dagger P)_{ij} = w_k^{\{i\}*} w_k^{\{j\}} = w^{\{i\}\dagger} \cdot w^{\{j\}} = \delta_{ij})$.

Symmetric $A$: $P$ is orthogonal.

For any $A$, $\det(e^A) = e^{\mathrm{tr}[A]}$

$$(\det(e^A) = \det(P^{-1}e^A P) = \det(e^{P^{-1}AP}) = \det \underbrace{e^\Lambda}_{\text{diagonal}} = (e^\Lambda)_{11}(e^\Lambda)_{22}\ldots = e^{\mathrm{tr}[\Lambda]} = e^{\mathrm{tr}[P\Lambda P^{-1}]} = e^{\mathrm{tr}[A]}).$$

Positive definite matrix $A$: $v^\dagger A v \geq 0$ for any vector $v$, with equality only when $v = 0$.

If $A$ Hermitian, eigenvalues $\lambda_k > 0$: $0 \leq \underbrace{v^\dagger P^\dagger}_{=v'^\dagger} \underbrace{P^\dagger A P}_{\text{diagonal, elements } \lambda_k} \underbrace{Pv}_{=v'} = \sum_k \lambda_k |v'_k|^2$.

Use natural units where $c = \hbar = 1$.

Spacetime indices are Greek letters, e.g. $\mu, \nu = 0, 1, 2, 3$
(spatial indices Roman, e.g. $i, j = 1, 2, 3$).

4-vector $A \equiv A^\mu = (\mathbf{A}, A^0)$, where $\mathbf{A} = (A^1, A^2, A^3)$ is spatial 3-vector.

Metric tensor of special relativity:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu}$$

Raising and lowering of indices: $g_{\mu\nu} A^{\mu\rho} = A_\nu{}^\rho$ etc.
Note $g_{\mu\nu} g^{\nu\rho} = g_\mu{}^\rho = \delta_\mu{}^\rho$.
Define $x^0 = +t$, so $x_0 = -t$. Likewise $p^0 = E$, $p_0 = -E$ etc.
Scalar product of 4-vectors: $A \cdot B = A^\mu B_\mu = A_\mu B^\mu = g^{\mu\nu} A_\mu B_\nu = g_{\mu\nu} A^\mu B^\nu = A^1 B^1 + A^2 B^2 + A^3 B^3 - A^0 B^0$.
Shorthand for spacetime differentiation: $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\partial^\mu = \frac{\partial}{\partial x_\mu}$.
d'Alembertian: $\partial^2 = \partial_\mu \partial^\mu$.

Commutator of operators $O_i, O_j$ (e.g. $O_i$ can be matrices): $[O_i, O_j] = [O_i, O_j]_- = O_i O_j - O_j O_i$.
Anticommutator: $\{O_i, O_j\} = [O_i, O_j]_+ = O_i O_j + O_j O_i$.

Dirac delta function $\delta(x - y) = \int \frac{dp}{2\pi} e^{Ip(x-y)}$:
$\int_a^b dx\, f(x) \delta(x - y) = f(y)$ if $a < y$, $b > y$ (otherwise $= 0$).
(Continuous version of $f_i \delta_{ij} = f_j$.)
$\delta^3(\mathbf{x}) = \delta(x_1)\delta(x_2)\delta(x_3)$.