Introduction to Supersymmetry and Supergravity

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Every Wednesday and Thursday 8:30-10:00am, starting 3rd November,
in room 2, building 2a, DESY

Lecture notes at www.desy.de/~simon/teaching/susy.html

Abstract

These lectures give a fairly formal developement of supersymmetry, beginning with some technical footing in symmetries (internal and external) in general in relativistic quantum mechanics, and a brief outline of the standard model and its GUT extensions. Following the Haag-Lopuszanski-Sohnius theorem, we allow for fermionic symmetry generators, and determine their properties and algebra using the restrictions of the Coleman-Mandula theorem. As an illustration we construct the supersymmetric field theory Lagrangian for the chiral supermultiplet, then discuss the more formal approach to constructing general supersymmetric field theories using superfields in superspace, including the development of supersymmetric gauge theories. Spontaneous supersymmetry breaking is discussed in some detail. Using the superfield formalism, the minimally supersymmetric standard model is developed. Next we develop supergravity, firstly in the weak field case and then to all orders. Finally, the more advanced topics of higher dimensions, extended supersymmetry and duality are discussed. The lectures will mostly follow Volume III of S. Weinberg's "The Quantum Theory of Fields".

IMPORTANT

- Lecture notes are at `www.desy.de/~simon/teaching/susy.html`. They will change!
- If you get stuck, please email me at `simon@mail.desy.de`
- If you find a mistake, please email me at `simon@mail.desy.de`
- Please let me know *by email* that you will attend this course so I can put you on the mailing list (and thus notify you at the last minute of room changes, cancellations, help with problems etc.).
- Every second Thursday I will try to allocate time for you to ask more detailed questions, for you to work through and present derivations outlined in the lectures, and (if time) for you to present solutions to interesting problems.
- There is alot of algebra to deal with in SUSY... so please think about the ratio of *algebraic explanations* to *material* that you would like!
- My suggestion: Try to derive *some* results in detail at home, and present your derivations every second Thursday.
- I am setting up an online forum where you can ask / answer SUSY questions.
- Usually in the lecture notes, I give the result first and then outline the derivation in *smaller font* below.
- I use the Einstein summation convention, i.e. and e.g. $\sum_{i=0}^{3} X_\mu X^\mu \to X_\mu X^\mu$ (and not just for spacetime indices).
- Any questions?

Contents

1 Quantum mechanics of particles

1.1 Basic principles

Physical states represented by directions of vectors (rays) $|i\rangle$ in Hilbert space of universe.

Write conjugate transpose $|i\rangle^\dagger$ as $\langle i|$, scalar product $|j\rangle^\dagger \cdot |i\rangle$ as $\langle j|i\rangle$.

Physical observable represented by Hermitian operator $A = A^\dagger$ such that $\langle A\rangle_i = \langle i|A|i\rangle$.

Functions of observables represented by same functions of their operators, $f(A)$.

Errors $\langle i|A^2|i\rangle - \langle A\rangle_i^2$ etc. vanish when $|i\rangle = |a\rangle$,

where $A|a\rangle = a|a\rangle$, i.e. $|a\rangle$ is A eigenstate, real eigenvalue a. $|a\rangle$ form complete basis.

If 2 observables A, B do not commute, $[A, B] \neq 0$, eigenstates of A do not coincide with those of B.

If basis $|X(a,b)\rangle$ are A, B eigenstates, any $|i\rangle = \sum_X C_{iX}|X\rangle$ obeys $[A,B]|i\rangle = 0 \implies [A,B] = 0$.

If A, B commute, their eigenstates coincide.

$AB|a\rangle = BA|a\rangle = aB|a\rangle$, so $B|a\rangle \propto |a\rangle$.

Completeness relation: $\boxed{\sum_a |a\rangle\langle a| = \mathbf{1}}$.

Expand $|i\rangle = \sum_a W_{ia}|a\rangle$ then act from left with $\langle a'| \longrightarrow W_{ia'} = \langle a'|i\rangle$, so $|i\rangle = \sum_a |a\rangle\langle a|i\rangle$

Probability to observe system in eigenstate $|a\rangle$ of A to be in eigenstate $|b\rangle$ of B: $P_{a\to b} = |\langle b|a\rangle|^2$.

$P_{a\to b}$ are the only physically meaningful quantities, thus $|i\rangle$ and $e^{I\alpha}|i\rangle$ for any α represent same state.

$\sum_a P_{b\to a} = 1$ for some state $|b\rangle$ as expected. $|b\rangle = \sum_a \langle a|b\rangle|a\rangle$. Act from left with $\langle b|$ gives $1 = \sum_a \langle a|b\rangle\langle b|a\rangle$.

Principle of reversibility: $P_{a\to b} = P_{b\to a}$. $\langle b|a\rangle = \langle a|b\rangle^*$

Time dependence: Time evolution of states: $|i,t\rangle = e^{-IHt}|i\rangle$, H is *Hamiltonian* with energy eigenstates.

Probability system in state $|i\rangle$ observed in state $|j\rangle$ time t later $= |M_{i\to j}|^2$, transition amplitude $M_{i\to j} = \langle j|e^{-IHt}|i\rangle$.

Average value of observable Q evolves in time as $\langle Q\rangle_i(t) = \langle i|e^{IHt}Qe^{-IHt}|i\rangle$.

Q is conserved $\iff [Q,H] = 0$ ($\langle Q\rangle(t)$ independent of t).

1.2 Fermionic and bosonic particles

Particle's eigenvalues $= \sigma$. Particle states $|\sigma, \sigma', ...\rangle$ completely span Hilbert space. Vacuum is $|0\rangle = |\rangle$.

$|\sigma, \sigma', ...\rangle = \pm|\sigma', \sigma, ...\rangle$ for bosons/fermions. *Pauli exclusion principle*: $|\sigma, \sigma, \sigma'...\rangle = 0$ if σ fermionic.

$|\sigma, \sigma', ...\rangle$ and $|\sigma', \sigma, ...\rangle$ are same state, $|\sigma, \sigma', ...\rangle = e^{I\alpha}|\sigma', \sigma, ...\rangle = e^{2I\alpha}|\sigma, \sigma', ...\rangle \implies e^{I\alpha} = \pm 1$.

Creation/annihilation operators: $a^\dagger_\sigma|\sigma', \sigma'', ...\rangle = |\sigma, \sigma', \sigma'', ...\rangle$, so $|\sigma, \sigma', ...\rangle = a^\dagger_\sigma a^\dagger_{\sigma'} \dots |0\rangle$.

$[a^\dagger_\sigma, a^\dagger_{\sigma'}]_\mp = [a_\sigma, a_{\sigma'}]_\mp = 0$ (bosons/fermions).

$|\sigma, \sigma', ...\rangle = a^\dagger_\sigma a^\dagger_{\sigma'}|\dots\rangle = \pm|\sigma', \sigma, ...\rangle = \pm a^\dagger_{\sigma'} a^\dagger_\sigma|\dots\rangle$

a_σ removes σ particle $\implies$ $a_\sigma|0\rangle = 0$.

E.g. $(a_\sigma|\sigma', \sigma''\rangle)^\dagger \cdot |\sigma'''\rangle = \langle\sigma', \sigma''|(a^\dagger_\sigma|\sigma'''\rangle) = 0$ unless $\sigma = \sigma'$, $\sigma''' = \sigma''$ or $\sigma = \sigma''$, $\sigma''' = \sigma'$. i.e. $a_\sigma|\sigma', \sigma''\rangle = \delta_{\sigma\sigma''}|\sigma'\rangle \pm \delta_{\sigma\sigma'}|\sigma''\rangle$.

$[a_\sigma, a^\dagger_{\sigma'}]_\mp = \delta_{\sigma\sigma'}$.

e.g. 2 fermions $a^\dagger_{\sigma''} a^\dagger_{\sigma'''}|0\rangle$: Operator $a^\dagger_\sigma a_{\sigma'}$ replaces any σ' with σ, must still vanish when $\sigma'' = \sigma'''$.

Check: $(a^\dagger_\sigma a_{\sigma'}) a^\dagger_{\sigma''} a^\dagger_{\sigma'''}|0\rangle = -a^\dagger_\sigma a^\dagger_{\sigma''} a_{\sigma'} a^\dagger_{\sigma'''}|0\rangle + \delta_{\sigma'\sigma''} a^\dagger_\sigma a^\dagger_{\sigma'''}|0\rangle = -\delta_{\sigma'\sigma'''} a^\dagger_\sigma a^\dagger_{\sigma''}|0\rangle + \delta_{\sigma'\sigma''} a^\dagger_\sigma a^\dagger_{\sigma'''}|0\rangle$.

Expansion of observables: $Q = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} C_{NM;\sigma'_1 \ldots \sigma'_N;\sigma_M \ldots \sigma_1} \; a^\dagger_{\sigma'_1} \ldots a^\dagger_{\sigma'_N} a_{\sigma_M} \ldots a_{\sigma_1}$.

Can always tune the C_{NM} to give any values for $\langle 0| a_{\sigma'_1} \ldots a_{\sigma'_L} Q a^\dagger_{\sigma_1} \ldots a^\dagger_{\sigma_K} |0\rangle$.

Commutations with additive observables: $[Q, a^\dagger_\sigma] = q(\sigma) a^\dagger_\sigma$ (no sum),

where Q is an observable such that for $|\sigma, \sigma', \ldots\rangle$, total $Q = q(\sigma) + q(\sigma') + \ldots$ and $Q|0\rangle = 0$ (e.g. energy).

Check for each particle state: $Q a^\dagger_\sigma |0\rangle = q(\sigma) a^\dagger_\sigma |0\rangle + a^\dagger_\sigma Q|0\rangle = q(\sigma) a^\dagger_\sigma |0\rangle$,

$Q a^\dagger_\sigma a^\dagger_{\sigma'} |0\rangle = a^\dagger_\sigma Q a^\dagger_{\sigma'} |0\rangle + q(\sigma) a^\dagger_\sigma a^\dagger_{\sigma'} |0\rangle = a^\dagger_\sigma a^\dagger_{\sigma'} Q|0\rangle + q(\sigma') a^\dagger_\sigma a^\dagger_{\sigma'} |0\rangle + q(\sigma) a^\dagger_\sigma a^\dagger_{\sigma'} |0\rangle = (q(\sigma) + q(\sigma')) a^\dagger_\sigma a^\dagger_{\sigma'} |0\rangle$ etc.

Note: Conjugate transpose is $[Q, a_\sigma] = -q(\sigma) a_\sigma$.

Number operator for particles with eigenvalues σ is $a^\dagger_\sigma a_\sigma$ (no sum). E.g. $(a^\dagger_\sigma a_\sigma) a^\dagger_\sigma a^\dagger_\sigma a^\dagger_{\sigma'} |0\rangle = 2 a^\dagger_\sigma a^\dagger_\sigma a^\dagger_{\sigma'} |0\rangle$.

Additive observable: $Q = \sum_\sigma q(\sigma) a^\dagger_\sigma a_\sigma$. E.g. $Q = H$, (free) Hamiltonian, $q(\sigma) = E_\sigma$, energy eigenvalues.

2 Symmetries in QM

2.1 Unitary operators

Symmetry is powerful tool: e.g. relates different processes.

Symmetry transformation is change in our point of view (e.g. spatial rotation / translation),

does not change experimental results. i.e. all $|i\rangle \longrightarrow U|i\rangle$ does not change any $|\langle j|i\rangle|^2$.

Continuous symmetry groups G require U unitary:

$\langle j|i\rangle \to \langle j|U^\dagger U|i\rangle = \langle j|i\rangle$, so $U^\dagger U = UU^\dagger = \mathbf{1}$, includes $U = \mathbf{1}$.

Wigner + Weinberg: General physical symmetry groups require U unitary,

or antiunitary: $\langle j|i\rangle \to \langle j|U^\dagger U|i\rangle = \langle j|i\rangle^* = \langle i|j\rangle$, e.g. (discrete) time reversal.

$\langle A\rangle$ unaffected (and $\langle f(A)\rangle$ in general), so must have $A \to UAU^\dagger$.

Transition amplitude $M_{i\to j}$ unaffected by symmetry transformation, i.e. $\langle j|U^\dagger e^{-IHt}U|i\rangle = \langle j|e^{-IHt}|i\rangle$,

which requires $[U, H] = 0$ if time translation and symmetry transformation commute.

Parameterize unitary operators as $U = U(\alpha)$, α_i real, $i = 1, ..., d(\mathrm{G})$.

$d(\mathrm{G})$ is dimension of G, minimum no. of paramenters required to distinguish elements.

Choose group identity at $\alpha = 0$, i.e. $U(0) = \mathbf{1}$.

So for small α_i, can write $U(\alpha) \approx \mathbf{1} + It_i\alpha_i$.

t_i are the linearly independent *generators* of G. Since $U^\dagger U = \mathbf{1}$, $t_i = t_i^\dagger$, i.e. t_i are Hermitian.

$[U, H] = 0 \implies [t_i, H] = 0$, so conserved observables are generators.

Can replace all $t_i \to t_i' = M_{ij}t_j$ if M invertible and real, because then $\alpha_i' = \alpha_j M_{ji}^{-1}$ real.

In general, $U(\alpha)U(\beta) = U(\gamma(\alpha, \beta))$ (up to possible phase $e^{I\rho}$, removable by enlarging group).

$U(\alpha) = \exp[It_i\alpha_i]$ if Abelian limit is obeyed: whenever $\beta_i \propto \alpha_i$, $U(\alpha)U(\beta) = U(\alpha+\beta)$

(usually true for physical symmetries, e.g. rotation about same line / translation in same direction).

Can write $U(\alpha) = [U(\alpha/N)]^N$, then for $N \to \infty$ is $[\mathbf{1} + It_i\alpha_i/N + O(1/N^2)]^N = \exp[It_i\alpha_i] + O\left(\frac{1}{N}\right)$.

Lie algebra: $[t_i, t_j] = IC_{ijk}t_k$, where C_{ijk} are the *structure constants*

appearing in $U(\alpha)U(\beta) = U(\alpha + \beta + \frac{1}{2}IC\alpha\beta + \text{cubic and higher})$

(where $(C\alpha\beta)_k = C_{ijk}\alpha_i\beta_j$, and no $O(\alpha^2)$ ensures $U(\alpha)U(0) = U(\alpha)$, likewise no $O(\beta^2)$).

LHS: $e^{I\alpha t}e^{I\beta t} \approx \left[\mathbf{1} + I\alpha t - \frac{1}{2}(\alpha t)^2\right] \times \left[\mathbf{1} + I\beta t - \frac{1}{2}(\beta t)^2\right]$

$$\approx \mathbf{1} + I(\alpha t + \beta t) - \frac{1}{2}\left[(\alpha t)^2 + (\beta t)^2 + \underline{2(\alpha t)(\beta t)}\right]$$

RHS: $e^{I(\alpha+\beta+\frac{1}{2}IC\alpha\beta)t} \approx \mathbf{1} + I(\alpha + \beta + \frac{1}{2}IC\alpha\beta)t - \frac{1}{2}\left[(\alpha + \beta + \frac{1}{2}IC\alpha\beta)t\right]^2$

$$\approx \mathbf{1} + I(\alpha t + \beta t + \underline{\tfrac{1}{2}IC\alpha\beta t}) - \frac{1}{2}\left[(\alpha t)^2 + (\beta t)^2 + \underline{(\alpha t)(\beta t) + (\beta t)(\alpha t)}\right],$$

i.e. $\frac{1}{2}IC_{ijk}\alpha_i\beta_j t_k = \frac{1}{2}[\alpha_i t_i, \beta_j t_j]$.

Now take all α_i, β_j zero except e.g. $\alpha_1 = \beta_2 = \epsilon \to IC_{12k}t_k = [t_1, t_2]$ etc., gives Lie algebra.

In fact, Lie algebra completely specifies group in *non-small* neighbourhood of identity.

This means that for $U(\alpha)U(\beta) = U(\gamma)$, $\gamma = \gamma(\alpha, \beta)$ can be found from Lie algebra.

We have shown this above in small neighbourhood of identity, i.e. to 2nd order in α, β, only.

Check to 3rd order: Write $X = \alpha_i t_i$, $Y = \beta_i t_i$, can verify Baker-Hausdorff formula

$$\exp[IX]\exp[IY] = \exp[\underbrace{I(X+Y) - \frac{1}{2}[X,Y] + \frac{I}{12}([X,[Y,X]] + [Y,[X,Y]]) + \text{quadratic and higher}}_{\text{has the form } I\gamma_i t_i,\ \gamma_i \text{ real}}].$$

Lie algebra implies: 1. $C_{ijk} = -C_{jik}$ (antisymmetric in i, j). C_{ijk} can be chosen antisymmetric in i, j, k (see later).

2. C_{ijk} real.

Conjugate of Lie Algebra is $\left[t_j^\dagger, t_i^\dagger\right] = -IC_{ijk}^* t_k^\dagger$, which is negative of Lie Algebra because $t_i^\dagger = t_i$.

Thus $C_{ijk}^* t_k = C_{ijk} t_k$, but t_k linearly independent so $C_{ijk}^* = C_{ijk}$.

3. $t^2 = t_j t_j$ is invariant (commutes with all t_i, so transforming gives $e^{I\alpha_i t_i} t^2 e^{-I\alpha_k t_k} = t^2$).

$[t^2, t_i] = t_j[t_j, t_i] + [t_j, t_i]t_j = IC_{jik}\{t_j, t_k\} = 0$ by (anti)symmetry in (j, i) j, k.

Examples: Rest mass, total angular momentum. $t_j t_j$ doesn't have to include all j, only subgroup.

2.2 (Matrix) Representations

Physically, matrix representation of any symmetry group G of nature formed by particles:

General transformation of $a_\sigma^{(\dagger)}$: $U(\alpha)a_\sigma^\dagger U^\dagger(\alpha) = D_{\sigma\sigma'}(\alpha)a_{\sigma'}^\dagger$, with **invariant vacuum**: $U(\alpha)|0\rangle = |0\rangle$.

$U(\alpha)a_\sigma^\dagger|0\rangle$ is 1 particle state, so must be linear combination of $a_{\sigma'}^\dagger|0\rangle$, i.e. $U(\alpha)a_\sigma^\dagger|0\rangle = D_{\sigma\sigma'}(\alpha)a_{\sigma'}^\dagger|0\rangle$.

Thus $U(\alpha)a_\sigma^\dagger a_{\sigma'}^\dagger|0\rangle = D_{\sigma\sigma''}(\alpha)D_{\sigma'\sigma'''}(\alpha)a_{\sigma''}^\dagger a_{\sigma'''}^\dagger|0\rangle$ etc.

$D(\alpha)$ matrices furnish a representation of G, matrix generators $(t^i)_{\sigma'\sigma}$ with same Lie algebra $[t_i, t_j]_{\sigma\sigma'} = IC_{ijk}(t_k)_{\sigma\sigma'}$.

Since $U(\alpha)U(\beta) = U(\gamma)$, must have $D_{\sigma''\sigma'}(\alpha)D_{\sigma'\sigma}(\beta) = D_{\sigma''\sigma}(\gamma)$.

If Abelian limit of page 8 is obeyed, $D_{\sigma'\sigma}(\alpha) = \left(e^{I\alpha_i t_i}\right)_{\sigma'\sigma}$.

Similarity transformation $(t_i)_{\sigma\sigma'''} \longrightarrow (t_i')_{\sigma\sigma'''} = V_{\sigma\sigma'}(t_i)_{\sigma'\sigma''}(V^{-1})_{\sigma''\sigma'''}$ also a representation.

Particle states require V unitary.

$a_\sigma^\dagger|0\rangle \to a_\sigma'^\dagger|0\rangle = V_{\sigma\sigma'}a_{\sigma'}^\dagger|0\rangle$, then othogonality $\langle 0|a_{\sigma'}' a_\sigma'^\dagger|0\rangle = \delta_{\sigma'\sigma}$ requires $V^\dagger V = \mathbf{1}$.

But in general, representations don't have to be unitary.

In "reducible" cases, can similarity transform $t_i \to V t_i V^{-1}$ such that t_i is block-diagonal matrix,

each block furnishes a representation, e.g.:

$$(t_i)_{\sigma'\sigma} = \begin{pmatrix} (t_i)_{jk} & \mathbf{0} \\ \mathbf{0} & (t_i)_{\alpha\beta} \end{pmatrix}_{\sigma'\sigma} .$$

Particles of one block don't mix with those of other — can be treated as 2 separate "species".

Each block can have different t^2, corresponds to different particle species.

Irreducible representation: Matrices $(t_i)_{\sigma'\sigma}$ not block-diagonalizable by similarity transformation.

In this sense, these particles are elementary.

Size of matrix written as $m(r) \times m(r)$, where r labels representation.

Corresponds to single species, single value of t^2: $t^2 = C_2(r)\mathbf{1}$ (consistent with $[t^2, t_i] = 0$).

$C_2(r)$ is *quadratic Casimir operator* of representation r.

Fundamental representation of G: Generators written $(t_i)_{\alpha\beta}$.

Matrices representing elements used to define group G (also called defining representation).

Adjoint representation of G ($r = A$): Generators $(t_j)_{ik} = IC_{ijk}$, satisfy Lie algebra.

Use *Jacobi identity* $[t_i, [t_j, t_k]] + [t_j, [t_k, t_i]] + [t_k, [t_i, t_j]] = 0$.

From Lie algebra, $[t_i, [t_j, t_k]] = I[t_i, C_{jkl}t_l] = -C_{jkl}C_{ilm}t_m$,

so Jacobi identity is $C_{jkl}C_{ilm} + C_{kil}C_{jlm} + C_{ijl}C_{klm} = 0$ (after removing contraction with linearly independent t_m),

or, from $C_{ijk} = -C_{jik}$, $-IC_{kjl}IC_{lim} + IC_{kil}IC_{ljm} - IC_{ijl}IC_{klm} = 0$, which from $(t_j)_{ik} = IC_{ijk}$ reads $[t_i, t_j]_{km} = IC_{ijl}(t_l)_{km}$.

Conjugate representation has generators $-t_i^* = -t_i^T$ (obey the same Lie algebra as t_i).

If $-t_i^* = Ut_iU^\dagger$ (U unitary), then $e^{-I\alpha_i t_i^*} = \left(e^{I\alpha_i t_i}\right)^* = Ue^{I\alpha_i t_i}U^\dagger$,

i.e. conjugate representation $\equiv$ original representation, $\longrightarrow$ representation is *real*.

For invariant matrix $\mathfrak{g}$ ("metric"), i.e. $e^{I\alpha_i t_i^T}\mathfrak{g}e^{I\alpha_i t_i} = \mathfrak{g}$, G transformation leaves $\phi^T\mathfrak{g}\psi$ invariant.

Semi-simple Lie algebra: no t_i that commutes with all other generators (no U(1) subgroup).

Semi-simple group's matrix generators must obey $\mathrm{tr}[t_i] = 0$.

Make all $\mathrm{tr}[t_i] = 0$ except one, $\mathrm{tr}[t_K]$, via $t'_i = M_{ij} t_i$ (this is just rotation of vector with components $\mathrm{tr}[t_i]$).

Determinant of $U(\alpha)U(\beta) = U(\gamma)$ is $e^{I\alpha_i \mathrm{tr}[t_i]} e^{I\beta_i \mathrm{tr}[t_i]} = e^{I\gamma_i \mathrm{tr}[t_i]}$, i.e. $(\alpha_i + \beta_i - \gamma_i)\mathrm{tr}[t_i] = 0$.

But only $\mathrm{tr}[t_K] \neq 0$ (we *assume*), so $\alpha_K + \beta_K - \gamma_K = 0$.

Must have $C_{ijK}\alpha_i\beta_j = 0$ (recall $\gamma_K \simeq \alpha_K + \beta_K + C_{ijK}\alpha_i\beta_j$),

or $C_{ijK} = C_{Kij} = 0$ for all i, j, so from Lie group we have $[t_K, t_i] = 0$ for all i — not possible for semi-simple group.

So assumption was wrong, must have $\mathrm{tr}[t_K] = 0$.

Normalization of generators chosen as $\mathrm{tr}[t_i t_j] = C(r)\delta_{ij}$.

N_{ij} =tr$[t_i t_j]$ becomes $M_{ik}N_{kl}(M^T)_{lj}$ after basis transformation $t_i \to M_{ij}t_j$.

N_{ij} components of real symmetric matrix N, diagonalizable via MNM^T when M real, orthogonal.

Also N is positive definite matrix $\alpha^T N\alpha = \alpha_i \mathrm{tr}[t_i t_j]\alpha_j$ =tr$[\alpha_i t_i \alpha_j t_j]$ =tr$[(\alpha_i t_i)^\dagger \alpha_j t_j] \geq 0$

(because for any matrix A, $\mathrm{tr}[A^\dagger A] = A^\dagger_{\beta\alpha}A_{\alpha\beta} = A^*_{\alpha\beta}A_{\alpha\beta} = \sum_{\alpha\beta}|A_{\alpha\beta}|^2 \geq 0$).

After diagonalization, $N_{ij} = 0$ for $i \neq j$ and above implies $N_{ii} > 0$ (no sum).

Then multiply each t_i by real number c_i, changes $N_{ii} \to c_i^2 N_{ii} > 0$.

Choose c_i such that each N_{ii} (no sum over i) all equal to positive $C(r)$.

Representation dependence of quadratic Casimir operator: $C_2(r)m(r) = C(r)d(G)$.

Definition of quadratic Casimir operator gives $\mathrm{tr}[t^2] = C_2(r)m(r)$.

Normalization of generators $\mathrm{tr}[t_i t_j] = C(r)\delta_{ij} \implies \mathrm{tr}[t^2] = C(r)d(G)$.

Example: 2 and 3 component repesentations of rotation group have different spins (i.e. $C_2(r)$) $\frac{1}{2}$ and 1.

Antisymmetric structure constants: $C_{ijk} = -\frac{I}{C(r)}\mathrm{tr}[[t_i, t_j]t_k]$.

From Lie algebra, $\mathrm{tr}[[t_i, t_j]t_l] = IC_{ijk}\mathrm{tr}[t_k t_l] = IC_{ijl}C(r)$.

Structure constants obey $C_{jki}C_{lki} = C(A)\delta_{jl}$.

In adjoint representation, quadratic Casimir operator on page 11 is $(t^2)_{jl} = C_2(A)\delta_{jl} = -C_{jik}C_{kil}$.

But $C_2(A) = C(A)$:

$C_2(A)m(A) = C(A)d(G)$ from representation dependence of quadratic Casimir operator on page 14,

and $m(A) = d(G)$.

2.3 External symmetries

2.3.1 Rotation group representations

(Spatial) rotation of vector $v \to Rv$ preserves $v^T v$, so R orthogonal ($R^T R = \mathbf{1}$).

Rotation $|\boldsymbol{\theta}|$ about $\boldsymbol{\theta}$: $U(\boldsymbol{\theta}) = e^{-I\boldsymbol{J}\cdot\boldsymbol{\theta}}$.

Lie algebra is $[J_i, J_j] = I\epsilon_{ijk}J_k$ for generators J_1, J_2, J_3 (see later).

Or use J_3 and raising/lowering operators $J_\pm = (J_1 \pm I J_2)$.

Irreducible spin j representations:

$(J_3^{(j)})_{m'm} = m\delta_{m'm}$ and $(J_\pm^{(j)})_{m'm} = [(j \mp m)(j \pm m + 1)]\delta_{m',m\pm 1}$, where $m = -j, -j+1, ..., j$,

spin $j = 0, \frac{1}{2}, 1, ...$, number of components n.o.c.$= 2j+1$ and $\boldsymbol{J}^2 = j(j+1)$.

Let $|m, j\rangle$ be $J_3 = m$ and $\boldsymbol{J}^2 = F(j)$ orthonormal eigenstates ($[J_3, \boldsymbol{J}^2] = 0$).

$J_\pm$ changes m by ± 1 because $[J_\pm, J_3] = \mp J_\pm$, so $J_\pm |m, j\rangle = C_\pm(m, j)|m \pm 1, j\rangle$.

$C_\mp(m, j) = \sqrt{F(j) - m^2 \pm m}$ (states absorb complex phase):

$|C_\mp(m, j)|^2 = \langle m, j|J_\pm J_\mp|m, j\rangle$ ($J_\pm^\dagger = J_\mp$) and $J_\pm J_\mp = \boldsymbol{J}^2 - J_3^2 \pm J_3$.

Let j be largest m value for given $F(j)$

(m bounded because $m^2 = \langle m, j|J_3^2|m, j\rangle = F(j) - \langle m, j|J_1^2 + J_2^2|m, j\rangle < F(j)$).

$F(j) = j(j+1)$ because $J_+|j, j\rangle = 0$, so $J_- J_+ |j, j\rangle = (F(j) - j^2 - j)|j, j\rangle = 0$.

Let $-j'$ be smallest m: $J_-| - j', j\rangle = 0 \implies F(j) = j'(j'+1)$, so $j' = j$ (other possibility $-j' = j + 1 > j$).

So $m = -j, -j+1, ..., j$, i.e. n.o.c.$= 2j+1$. Since n.o.c. is integer, $j = 0, \frac{1}{2}, 1, ...$

Spin decomposition of tensors: e.g. 2nd rank tensor C_{ij} (n.o.c.=9), representations are $j = 0, 1, 2$:

scalar (n.o.c.=**1**) + antisymmetric rank 2 tensor (n.o.c.=**3**) + symmetric traceless rank 2 (n.o.c.=**5**) components.

$$C_{ij} = \tfrac{1}{3}\delta_{ij}C_{kk} + \tfrac{1}{2}(C_{ij} - C_{ji}) + \tfrac{1}{2}(C_{ij} + C_{ji} - \tfrac{2}{3}\delta_{ij}C_{kk}).$$

Component irreducible representations signified by **1** + **3** + **5**.

Counting n.o.c. shows they are equivalent respectively to

the $j = 0$ $(2j+1 = \mathbf{1})$, $j = 1$ $(2j+1 = \mathbf{3})$ and $j = 2$ $(2j+1 = \mathbf{5})$ representations:

$\frac{1}{3}\delta_{ij}C_{kk} \to \frac{1}{3}\delta_{ij}C_{kk}$ is like scalar $\equiv$ spin 0,

$(\frac{1}{2}(C_{jk} - C_{kj}))|_{i\neq k,j} = \frac{1}{2}\epsilon_{ijk}C_{jk} \to R_{il}\frac{1}{2}\epsilon_{ljk}C_{jk}$ (because $\epsilon R^2 = R\epsilon$) is like vector $\equiv$ spin 1,

$\frac{1}{2}(C_{ij} + C_{ji} - \frac{2}{3}\delta_{ij}C_{ii}) \to R_{il}R_{jm}\frac{1}{2}(C_{lm} + C_{ml} - \frac{2}{3}\delta_{lm}C_{kk})$ is like rank 2 tensor $\equiv$ spin 2.

Direct product: 2 particles, spins j_1, j_2, also in a representation:

$e^{I\boldsymbol{J}\cdot\boldsymbol{\theta}}|m_1, j_1; m_2, j_2\rangle = (e^{I\boldsymbol{J}^{(j_1,j_2)}\cdot\boldsymbol{\theta}})_{m_1'm_2'm_1m_2}|m_1', j_1; m_2', j_2\rangle$ where $\boldsymbol{J}^{(j_1,j_2)}_{m_1'm_2'm_1m_2} = \boldsymbol{J}^{(j_1)}_{m_1'm_1}\delta_{m_2'm_2} + \delta_{m_1'm_1}\boldsymbol{J}^{(j_2)}_{m_2'm_2}$.

From $e^{I\boldsymbol{J}\cdot\boldsymbol{\theta}}|m_1, j_1; m_2, j_2\rangle = (e^{I\boldsymbol{J}^{(j_1)}\cdot\boldsymbol{\theta}})_{m_1'm_1}(e^{I\boldsymbol{J}^{(j_2)}\cdot\boldsymbol{\theta}})_{m_2'm_2}|m_1', j_1; m_2', j_2\rangle$, i.e. total rotation is rotation of each particle in turn.

Thus $J_3|m_1, j_1; m_2, j_2\rangle = (m_1 + m_2)|m_1, j_1; m_2, j_2\rangle$,

so $|m_1, j_1; m_2, j_2\rangle$ is combination of $J_3^{(j_1,j_2)}$, $\boldsymbol{J}^{(j_1,j_2)2}$ eigenstates $|2; m_1 + m_2, j\rangle$,

with $j = m_1 + m_2, m_1 + m_2 + 1, ..., j_1 + j_2$. So $j_{\max} = j_1 + j_2$.

Triangle inequality: representation for (j_1, j_2) contains $j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2$.

Number of orthogonal eigenstates $|m_1, j_1; m_2, j_2\rangle$ = Number of orthogonal eigenstates $|2; m, j\rangle$,

i.e. $(2j_1 + 1)(2j_2 + 1) = \sum_{j=j_{\min}}^{j_1+j_2}(2j + 1)$ (now use $\sum_{j=a}^{b} j = \frac{1}{2}(b - a + 1)(b + a)$) so $j_{\min} = |j_1 - j_2|$.

Example: Representation for 2 spin 1 particles $(j_1, j_2) = (1, 1)$:

From triangle inequality, this is $\equiv$ sum of irreducible representations $j = 0, 1, 2$.

Also from tensor representation on page 18: product of 2 vectors $u_i v_j$ (2nd rank tensor) is $\mathbf{3} \times \mathbf{3} = \mathbf{1} + \mathbf{3} + \mathbf{5}$.

2.3.2 Poincaré and Lorentz groups

Poincaré (inhomogeneous Lorentz) group formed by coordinate transformations $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$,

preserving spacetime separation: $g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} dx'^\mu dx'^\nu$. Implies:

Transformation of metric tensor: $g_{\rho\sigma} = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma$ or $(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu$.

Identity: $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$, $a^\mu = 0$.

Poincaré group defined by $U(\overline{\Lambda}, \overline{a}) U(\Lambda, a) = U(\overline{\Lambda}\Lambda, \overline{\Lambda} a + \overline{a})$.

This is the double transformation $x'' = \overline{\Lambda} x' + \overline{a} = \overline{\Lambda}(\Lambda x + a) + \overline{a}$.

Poincaré group generators: $J^{\mu\nu}$ and P^μ, appearing in $U(1 + \omega, \epsilon) \simeq 1 + \frac{1}{2} I \omega_{\mu\nu} J^{\mu\nu} - I \epsilon_\mu P^\mu$.

Obtained by going close to identity, $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ and $a_\mu = \epsilon_\mu$.

Choose $J^{\mu\nu} = -J^{\nu\mu}$.

Allowed because $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$: Transformation of metric tensor reads $g_{\rho\sigma} = g_{\mu\nu}(\delta^\mu{}_\rho + \omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma) \simeq g_{\sigma\rho} + \omega_{\rho\sigma} + \omega_{\sigma\rho}$.

Transformation properties of P^μ, $J^{\mu\nu}$: $U(\Lambda,a)P^\mu U^\dagger(\Lambda,a) = \Lambda_\rho{}^\mu P^\rho$

and $U(\Lambda,a)J^{\mu\nu}U^\dagger(\Lambda,a) = \Lambda_\rho{}^\mu\Lambda_\sigma{}^\nu\left(J^{\rho\sigma} - a^\rho P^\sigma + a^\sigma P^\rho\right)$.

Apply Poincaré group to get $\underbrace{U(\Lambda,a)U(1+\omega,\epsilon)}_{=U(\Lambda(1+\omega),\Lambda\epsilon+a)}\underbrace{U^\dagger(\Lambda,a)}_{=U(\Lambda^{-1},-\Lambda^{-1}a)} = U(1+\Lambda\omega\Lambda^{-1},\Lambda(\epsilon-\omega\Lambda^{-1}a))$.

Expand both sides in ω,ϵ: $U(\Lambda,a)(1+\frac{1}{2}I\omega J - I\epsilon P)U^\dagger(\Lambda,a) = 1+\frac{1}{2}I\Lambda\omega\Lambda^{-1}J - I\Lambda(\epsilon-\omega\Lambda^{-1}a)P$, equate coefficients of ω,ϵ.

So P^μ transforms like 4-vector, J_{ij} like angular momentum.

Poincaré algebra: $I[J^{\rho\sigma},J^{\mu\nu}] = -g^{\sigma\nu}J^{\rho\mu} - g^{\rho\mu}J^{\sigma\nu} + g^{\sigma\mu}J^{\rho\nu} + g^{\rho\nu}J^{\sigma\mu}$ $\longrightarrow$ (homogeneous) Lorentz group,

$I[P^\mu,J^{\rho\sigma}] = g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho$, and $[P^\mu,P^\nu]=0$.

Obtained by taking $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$, $a_\mu = \epsilon_\mu$ in transformation properties of P^μ, $J^{\mu\nu}$, to first order in ω,ϵ gives

$$P^\mu - \tfrac{1}{2}I\omega_{\rho\sigma}[P^\mu,J^{\rho\sigma}] + I\epsilon_\nu[P^\mu,P^\nu] = P^\mu + \tfrac{1}{2}\omega_{\rho\sigma}g^{\mu\sigma}P^\rho - \tfrac{1}{2}\omega_{\rho\sigma}g^{\mu\rho}P^\sigma, \text{ and}$$

$$J^{\mu\nu} + \tfrac{1}{2}I\omega_{\rho\sigma}[J^{\rho\sigma},J^{\mu\nu}] - I\epsilon_\rho[P^\rho,J^{\mu\nu}] = J^{\mu\nu} - g^{\rho\mu}\epsilon_\rho P^\nu + g^{\rho\nu}\epsilon_\rho P^\mu + \tfrac{1}{2}\omega_{\rho\sigma}(g^{\rho\nu}J^{\sigma\mu} - g^{\sigma\nu}J^{\rho\mu}) + \tfrac{1}{2}\omega_{\rho\sigma}(g^{\sigma\mu}J^{\rho\nu} - g^{\rho\mu}J^{\sigma\nu}).$$

2.3.3 Relativistic quantum mechanical particles

Identify $H = P^0$, spatial momentum P^i, angular momentum $J_i = \frac{1}{2}\epsilon_{ijk}J^{jk}$ (i.e. $(J_1, J_2, J_3) = (J^{23}, J^{31}, J^{12})$).

$[H, \boldsymbol{P}] = [H, \boldsymbol{J}] = 0 \rightarrow \boldsymbol{P}, \boldsymbol{J}$ conserved. Rotation group $[J_i, J_j] = I\epsilon_{ijk}J_k$ is subgroup of Poincaré group.

Also define *boost generator* $\boldsymbol{K} = (J^{10}, J^{20}, J^{30})$, obeys $[J_i, K_j] = I\epsilon_{ijk}K_k$ and $[K_i, K_j] = -I\epsilon_{ijk}J_k$.

$\boldsymbol{K}$ not conserved: $[K_i, H] = IP_i$, because boost and time translation don't commute.

Explicit form of Poincaré elements: $U(\Lambda, a) = \underbrace{e^{-IP^\mu a_\mu}}_{\text{translate } a_\mu} \times \underbrace{e^{-I\boldsymbol{K}\cdot\hat{\boldsymbol{e}}\beta}}_{\text{boost along } \hat{\boldsymbol{e}} \text{ by } V=\sinh\beta} \times \underbrace{e^{I\boldsymbol{J}\cdot\boldsymbol{\theta}}}_{\text{rotate } |\boldsymbol{\theta}| \text{ about } \boldsymbol{\theta}}$

(V: magnitude of 4-velocity's spatial part.)

Lorentz transformation of 4-vectors: $(K_i)^\mu{}_\nu = I(\delta_{0\mu}\delta_{i\nu} + \delta_{i\mu}\delta_{0\nu})$ and $(J_i)^\mu{}_\nu = -I\epsilon_{0i\mu\nu}$.

Then $A_\nu \rightarrow A_\nu - \delta\omega_i(\delta_{i\nu}A_0 + \delta_{0\nu}A_i) - \delta\theta_i\epsilon_{ij\nu}A_j(1 - \delta_{0\nu})$ as required.

$\left[e^{-I\boldsymbol{K}\cdot\hat{\boldsymbol{e}}\beta}\right]^\mu{}_\nu = \left[1 - IK_i\hat{e}_i\sinh\beta - (K_i\hat{e}_i)^2(\cosh\beta - 1)\right]^\mu{}_\nu$ and $\left[e^{I\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}}\theta}\right]^\mu{}_\nu = \left[1 + IJ_i\hat{\theta}_i\sin\theta - \left(J_i\hat{\theta}_i\right)^2(1 - \cos\theta)\right]^\mu{}_\nu$.

Follows directly from $(K_i\hat{e}_i)^3 = K_i\hat{e}_i$ and $(J_i\hat{\theta}_i)^3 = J_i\hat{\theta}_i$.

Particles: Use P^μ, $P^2 = m^2$ eigenstates. Distinguish momentum $\boldsymbol{p}$ from list σ, i.e. $a_\sigma^\dagger \to a_\sigma^\dagger(\boldsymbol{p})$.

Commutation relations for $a_\sigma^{(\dagger)}(\boldsymbol{p})$: $[a_\sigma^\dagger(\boldsymbol{p}), a_{\sigma'}^\dagger(\boldsymbol{p}')]_\mp = [a_\sigma(\boldsymbol{p}), a_{\sigma'}(\boldsymbol{p}')]_\mp = 0$

and $[a_\sigma(\boldsymbol{p}), a_{\sigma'}^\dagger(\boldsymbol{p}')]_\mp = \delta_{\sigma\sigma'}(2p^0)\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')$.

As on page 4, but with different normalization (Lorentz invariant).

Application of general transformation of $a_\sigma^{(\dagger)}$ on page 10 to Lorentz transformation

complicated by mixing between σ and $\boldsymbol{p}$.

Simplify by finding 2 transformations, one that mixes σ and one that mixes $\boldsymbol{p}$, separately.

Lorentz transformation for $a_\sigma^{(\dagger)}(\boldsymbol{p})$: $U(\Lambda)a_\sigma^\dagger(\boldsymbol{p})U^\dagger(\Lambda) = D_{\sigma'\sigma}(W(\Lambda,p))a_{\sigma'}^\dagger(\boldsymbol{\Lambda p})$.

Same as general transformation on page 10, but because $\sigma \to \{\sigma, \boldsymbol{p}\}$, mixing of σ with $\boldsymbol{p}$ must be allowed.

Construction of W: Choose reference momentum k and transformation L to mix k but not σ (defines σ):

$L^\mu{}_\nu(p)k^\nu = p^\mu$ and $U(L(p))a_\sigma^\dagger(\boldsymbol{k})|0\rangle = a_\sigma^\dagger(\boldsymbol{p})|0\rangle$.

Then $W(\Lambda,p) = L^{-1}(\Lambda p)\Lambda L(p)$. W mixes σ but not k, i.e. belongs to *little group* of k: $W^\mu{}_\nu k^\nu = k^\mu$.

General transformation is $U(\Lambda)\underline{a_\sigma^\dagger(\boldsymbol{p})|0\rangle} = U(\Lambda)\underline{U(L(p))a_\sigma^\dagger(\boldsymbol{k})|0\rangle}$, using definition of L.

Multiplying by $1 = U(L(\Lambda p))U(L^{-1}(\Lambda p))$ gives

$U(\Lambda)a_\sigma^\dagger(\boldsymbol{p})|0\rangle = U(L(\Lambda p))\ \underline{U(L^{-1}(\Lambda p))\ U(\Lambda)U(L(p))}a_\sigma^\dagger(\boldsymbol{k})|0\rangle = U(L(\Lambda p))\underline{U(W(\Lambda,p))}a_\sigma^\dagger(\boldsymbol{k})|0\rangle$.

But W doesn't change k, so $U(\Lambda)a_\sigma^\dagger(\boldsymbol{p})|0\rangle = U(L(\Lambda p))D_{\sigma'\sigma}(W(\Lambda,p))a_{\sigma'}^\dagger(\boldsymbol{k})|0\rangle$.

Then $L(\Lambda p)$ changes k to Λp but doesn't change σ'.

Poincaré transformation for $a_\sigma^{(\dagger)}(\boldsymbol{p})$: $U(\Lambda,a)a_\sigma^\dagger(\boldsymbol{p})U^\dagger(\Lambda,a) = e^{-I\Lambda p\cdot a}D_{\sigma'\sigma}(W(\Lambda,p))a_{\sigma'}^\dagger(\boldsymbol{\Lambda p})$.

In Lorentz transformation for $a_\sigma^{(\dagger)}(\boldsymbol{p})$, use $U(\Lambda,a) = e^{-IP^\mu a_\mu}U(\Lambda)$ (from explit form of Poincaré elements on page 22).

2.3.4 Quantum field theory

Lorentz invariant QM: $H = \int d^3x \mathscr{H}(x)$, scalar field $\mathscr{H}(x)$ (i.e. $U(\Lambda, a)\mathscr{H}(x)U^\dagger(\Lambda, a) = \mathscr{H}(\Lambda x + a)$),

obeys *cluster decomposition principle* (two processes with large spatial separation evolve independently).

Causality: $[\mathscr{H}(x), \mathscr{H}(y)] = 0$ when $(x-y)^2 \geq 0$. Required for Lorentz invariance of S-matrix.

Intuitive reason: signal can't propagate between 2 spacelike separated events.

In QM (general): H built from $a_\sigma^{(\dagger)}$. In QFT: H built from $\mathscr{H}(x)$ built from products of

Fields $\psi_l^{-c}(x) = \int D^3p \; v_{l\sigma}(x; \boldsymbol{p}) a_\sigma^{c\dagger}(\boldsymbol{p})$ and $\psi_l^{+}(x) = \int D^3p \; u_{l\sigma}(x; \boldsymbol{p}) a_\sigma(\boldsymbol{p})$.

Lorentz invariant momentum space volume $D^3p = \frac{d^3p}{2p^0} = d^4p\delta(p^2 + m^2)\theta(p^0)$ obeys $D^3\Lambda p = D^3p$.

These fields should obey

Poincaré transformation for fields: $U(\Lambda, a)\psi_l^{(c)\pm}(x)U^\dagger(\Lambda, a) = D_{ll'}(\Lambda^{-1})\psi_{l'}^{(c)\pm}(\Lambda x + a)$.

To calculate $u_{l\sigma}(x; \boldsymbol{p})$ and $v_{l\sigma}(x; \boldsymbol{p})$, take single particle species in representation labelled j, allow $a_\sigma^{c\dagger}(\boldsymbol{p}) \neq a_\sigma^\dagger(\boldsymbol{p})$.

x **dependence of** u, v: $u_{l\ \sigma}(x;\boldsymbol{p}) = e^{Ip\cdot x} u_{l\ \sigma}(\boldsymbol{p})$, $v_{l\ \sigma}(x;\boldsymbol{p}) = e^{-Ip\cdot x} v_{l\ \sigma}(\boldsymbol{p})$.

Take $\Lambda = 1$ in Poincaré transformation for fields and for $a_\sigma^{(\dagger)}(\boldsymbol{p})$ on page 25 and 24,

e.g. $U(1,a)\psi_l^+(x)U^\dagger(1,a) = \int D^3p\ u_{l\sigma}(x;\boldsymbol{p})U(1,a)a_\sigma(\boldsymbol{p})U^\dagger(1,a) = \int D^3p\ \underline{e^{Ip\cdot a}u_{l\sigma}(x;\boldsymbol{p})}a_\sigma(\boldsymbol{p})$

$= \psi_l^+(x+a) = \int D^3p\ \underline{u_{l\sigma}(x+a;\boldsymbol{p})}a_\sigma(\boldsymbol{p})$, equate coefficients of $a_\sigma(\boldsymbol{p})$ (underlined), gives $u_{l\ \sigma}(x;\boldsymbol{p})e^{Ip\cdot a} = u_{l\sigma}(x+a;\boldsymbol{p})$.

Klein-Gordon equation: $(\partial^2 - m^2)\psi_l^{\pm(c)}(x) = 0$.

Act on e.g. $\psi_l^{-c}(x) = \int D^3p\ e^{-Ip\cdot x}v_{l\ \sigma}(\boldsymbol{p})a_\sigma^{c\dagger}(\boldsymbol{p})$ with $(\partial^2 - m^2)$, use $p^2 = -m^2$.

Transformation of u, v: $D_{ll'}(\Lambda)u_{l'\sigma}(\boldsymbol{p}) = D^{(j)}_{\sigma'\sigma}(W(\Lambda,p))u_{l\sigma'}(\boldsymbol{\Lambda p})$, $D_{ll'}(\Lambda)v_{l'\sigma}(\boldsymbol{p}) = D^{(j)}_{\sigma\sigma'}(W^{-1}(\Lambda,p))v_{l\sigma'}(\boldsymbol{\Lambda p})$.

E.g. consider v, use Poincaré transformation for fields and for $a_\sigma^{(\dagger)}(\boldsymbol{p})$ on page 25 and 24.

LHS: $U(\Lambda,a)\psi_l^{-c}(x)U^\dagger(\Lambda,a) = \int D^3p\ v_{l\sigma}(x;\boldsymbol{p})U(\Lambda,a)a_\sigma^{c\dagger}(\boldsymbol{p})U^\dagger(\Lambda,a) = \int D^3p\ \underline{e^{-I\Lambda p\cdot a}v_{l\sigma}(x;\boldsymbol{p})D^{(j)}_{\sigma'\sigma}(W(\Lambda,p))}a_{\sigma'}^{c\dagger}(\boldsymbol{\Lambda p})$, and

RHS: $D_{ll'}(\Lambda^{-1})\psi_{l'}^{-c}(\Lambda x + a) = \int D^3p\ D_{ll'}(\Lambda^{-1})v_{l\sigma}(\Lambda x + a;\boldsymbol{p})a_\sigma^{c\dagger}(\boldsymbol{p}) = \int D^3\Lambda p\ \underline{D_{ll'}(\Lambda^{-1})v_{l\sigma}(\Lambda x + a;\boldsymbol{\Lambda p})}a_\sigma^{c\dagger}(\boldsymbol{\Lambda p})$ $(p \to \Lambda p)$.

Use $D^3\Lambda p = D^3p$, equate coefficient of $a_\sigma^{c\dagger}(\boldsymbol{\Lambda p})$ (underlined), multiply by $D_{l''l}(\Lambda)D^{(j)}_{\sigma''\sigma}(W^{-1}(\Lambda,p))$.

$\boldsymbol{p}$ **dependence of** u, v: $u_{l\sigma}(\boldsymbol{p}) = D_{ll'}(L(p))u_{l'\sigma}(\boldsymbol{k})$. ($\boldsymbol{p}$ dependence of v is the same.)

In transformation of u, v, take $p = k$ so $L(p) = 1$, $\Lambda = L(q)$ so $\Lambda p = L(q)k = q$, then $W(\Lambda,p) = L^{-1}(\Lambda p)L(q) = 1$.

2.3.5 Causal field theory

Since $[\psi_l^+(x), \psi_{l'}^{-c}(x')]_\mp \neq 0$, causality on page 25 only gauranteed by taking $\mathscr{H}(x)$ to be functional of

complete field $\psi_l(x) = \kappa\psi_l^+(x) + \lambda\psi_l^{-c}(x)$ (so representations $D_{ll'}(\Lambda^{-1})$ for $\psi_l^+(x)$ and $\psi_l^{-c}(x)$ the same), with

Causality: $[\psi_l(x), \psi_{l'}(y)]_\mp = [\psi_l(x), \psi_{l'}^\dagger(y)]_\mp = 0$ when $(x-y)^2 > 0$ by suitable choice of κ, λ.

Now $\langle 0|H|0\rangle = \infty$, i.e. consistency with gravity not gauranteed by QFT.

In each term in H, last operator on right hand side is not always an annihilation operator.

Complete field: $\psi_l(x) = \int D^3p \left[\kappa e^{Ip\cdot x} u_{l\sigma}(\boldsymbol{p}) a_\sigma(\boldsymbol{p}) + \lambda e^{-Ip\cdot x} v_{l\sigma}(\boldsymbol{p}) a_\sigma^{c\dagger}(\boldsymbol{p})\right]$.

2.3.6 Antiparticles

$\mathscr{H}(x)$ **commutes with conserved additive** Q: $[Q, \mathscr{H}(x)] = 0$.

Imposed in order to satisfy $[Q, H] = 0$.

This is achieved as follows:

Commutation of fields with conserved additive Q: $[Q, \psi_l(x)] = -q_l\psi_l(x)$, and

Field construction of $\mathscr{H}(x)$: $\mathscr{H} = \sum \psi_{l_1}^{L_1}\psi_{l_2}^{L_2}\ldots\psi_{m_1}^{M_1\dagger}\psi_{m_2}^{M_2\dagger}\ldots$ with $q_{l_1}^{L_1} + q_{l_2}^{L_2} + \ldots - q_{m_1}^{M_1} - q_{m_2}^{M_2} - \ldots = 0$

(M_i, L_j label particle species).

Antiparticles: For every particle species there is another species with opposite conserved quantum numbers.

Commutation of fields with conserved additive Q implies $[Q, \psi_l^{-c}(x)] = -q_l\psi_l^{-c}(x)$ and $[Q, \psi_l^{+}(x)] = -q_l\psi_l^{+}(x)$,

but since $[Q, a_\sigma^{c\dagger}] = q_\sigma^c a_\sigma^{c\dagger}$ and $[Q, a_\sigma] = -q_\sigma a_\sigma$ (no sum) from page 5, $q_l = q_\sigma^c$ and $q_l = -q_\sigma$, i.e. $q_\sigma^c = -q_\sigma$.

2.3.7 Spin in relativistic quantum mechanics

Lorentz group algebra simplified by choosing generators $A_i = \frac{1}{2}(J_i - IK_i)$ and $B_i = \frac{1}{2}(J_i + IK_i)$,

behaves like 2 independent rotations: $[A_i, A_j] = I\epsilon_{ijk}A_k$, $[B_i, B_j] = I\epsilon_{ijk}B_k$ and $[A_i, B_j] = 0$,

i.e. relativistic particle of type (A, B) (i.e. in eigenstate of $\boldsymbol{A}^2 = A(A+1)$, $\boldsymbol{B}^2 = B(B+1)$)

$\equiv$ 2 particles at rest, ordinary spins A, B (in representation sense).

In terms of degrees of freedom, $(A, B) = (\mathbf{2}\boldsymbol{A}\mathbf{+1}) \times (\mathbf{2}\boldsymbol{B}\mathbf{+1})$.

Triangle inequality: ordinary spin $\boldsymbol{J} = \boldsymbol{A} + \boldsymbol{B}$, so $j = |A - B|, |A - B| + 1, ..., A + B$.

Derived on page 19.

Can have eigenstates of $\boldsymbol{A}^2 = A(A+1)$, $\boldsymbol{B}^2 = B(B+1)$ and $\boldsymbol{J}^2 = j(j+1)$ *simultaneously*,

because $[\boldsymbol{J}^2, A_i] = [\boldsymbol{J}^2, B_i] = 0$. Use $\boldsymbol{J}^2 = (\boldsymbol{A} + \boldsymbol{B})^2$ and $\boldsymbol{A}$, $\boldsymbol{B}$ commutation relations.

Simultaneous eigenstate also with any $\boldsymbol{K} = I(\boldsymbol{A} - \boldsymbol{B})$ or $\boldsymbol{K}^2$ not possible.

Example: $(A,B)=(\frac{1}{2},\frac{1}{2})$ is representation of 4-vector: $(A_3)^\mu{}_\nu$, $(B_3)^\mu{}_\nu$ can have eigenvalues $\pm\frac{1}{2}$ only.

From triangle inequality on page 29, $j=0,1$.

Also follows from tensor representation on page 18: $\mathbf{2}\times\mathbf{2}=\mathbf{1}+\mathbf{3}$.

More generally, rank N tensor is $(\frac{1}{2},\frac{1}{2})^N=\sum_{A=0}^{\frac{N}{2}}\sum_{B=0}^{\frac{N}{2}}(A,B)$,

i.e. $(\frac{1}{2},\frac{1}{2})^N=\left(\frac{N}{2},\frac{N}{2}\right)$ +lower spins, where $\left(\frac{N}{2},\frac{N}{2}\right)\equiv$ traceless symmetric rank N tensor, $j=0,...,N$.

$(N,0)$ and $(0,N)$ are purely spin $j=N$.

2.3.8 Irreducible representation for fields

If particles created by $a_\sigma^{(c)\dagger}(\boldsymbol{p})$ have spin j (i.e. $[\boldsymbol{J}^2, a_\sigma^{(c)\dagger}(\boldsymbol{p})] = j(j+1)a_\sigma^{(c)\dagger}(\boldsymbol{p})$),

must take field with same spin j but any (A, B) consistent with triangle inequality on page 29:

$\psi_{ab}(x) = \int D^3p \left[\kappa\, e^{Ip\cdot x} u_{ab\ \sigma}(\boldsymbol{p}) a_\sigma(\boldsymbol{p}) + \lambda\, e^{-Ip\cdot x} v_{ab\ \sigma}(\boldsymbol{p}) a_\sigma^{c\dagger}(\boldsymbol{p})\right]$ $(l = ab,\ l' = a'b')$,

Lorentz transformation uses generators $\boldsymbol{A}_{a'b'ab} = \boldsymbol{J}^{(A)}_{a'a}\delta_{b'b}$, $\boldsymbol{B}_{a'b'ab} = \delta_{a'a}\boldsymbol{J}^{(B)}_{b'b}$

where $a = -A, -A+1, ..., A$ and $b = -B, -B+1, ..., B$.

2.3.9 Massive particles

Choose $k = (0, 0, 0, m)$ (momentum of particle at rest) Then W is an element of the spatial rotation group.

$L(p)$ **in terms of** $\boldsymbol{K}$: $L(p) = \exp[-I\hat{\boldsymbol{p}} \cdot \boldsymbol{K}\beta]$.

Then $W(R, p) = R$, i.e. rotation group apparatus of subsubsection 2.3.1 applies to relativistic particles too.

Conditions on u, v: $\boldsymbol{J}^{(j)}_{\sigma'\sigma} u_{ab\ \sigma'}(0) = \boldsymbol{J}^{(A)}_{aa'} u_{a'b\ \sigma}(0) + \boldsymbol{J}^{(B)}_{bb'} u_{ab'\ \sigma}(0)$, $-\boldsymbol{J}^{(j)*}_{\sigma'\sigma} v_{ab\ \sigma'}(0) = \boldsymbol{J}^{(A)}_{aa'} v_{a'b\ \sigma}(0) + \boldsymbol{J}^{(B)}_{bb'} v_{ab'\ \sigma}(0)$.

In transformation of u, v on page 26, take $\Lambda = R$ and $\boldsymbol{p} = 0$ (i.e. $p = k$)

(so $\boldsymbol{p} = \boldsymbol{k}$, $Rp = p$, $L(p) = 1$, $W(R, p) = L^{-1}(Rp)RL(p) = L^{-1}(p)R = R$), so e.g. $D_{ll'}(R)v_{l'\sigma}(0) = D^{(j)}_{\sigma\sigma'}(R^{-1})v_{l\sigma'}(0)$,

and use $D^{(j)}_{\sigma\sigma'}(R^{-1}) = D^{(j)*}_{\sigma'\sigma}(R)$ because irreducible representations of R are unitary (see form of $(J^{(j)}_i)_{\sigma'\sigma}$ on page 17).

Take $l = ab$, so $D_{aba'b'}(R)v_{a'b'\sigma}(0) = D^{(j)*}_{\sigma'\sigma}(R)v_{ab\sigma'}(0)$. Generators of $D^{(j)*}(R)$, $D(R)$ are respectively $-\boldsymbol{J}^{(j)*}$, $\boldsymbol{A} + \boldsymbol{B}$.

u, v **relation**: $v_{ab\ \sigma}(0) = (-1)^{j+\sigma} u_{ab\ -\sigma}(0)$ up to normalization.

Conditions on u, v with (no sum over σ, σ') $-\boldsymbol{J}^{(j)*}_{\sigma\sigma'} = (-1)^{\sigma-\sigma'} \boldsymbol{J}^{(j)}_{-\sigma,-\sigma'}$ from page 17

gives $v_{ab\ \sigma}(0) \propto (-1)^{\sigma} u_{ab\ -\sigma}(0)$, absorb proportionality constant into u, v.

The $\boldsymbol{p}$ dependence of u, v are the same, so $v_{ab\ \sigma}(\boldsymbol{p}) = (-1)^{j+\sigma} u_{ab\ -\sigma}(\boldsymbol{p})$.

2.3.10 Massless particles

Take reference vector $k = (0, 0, 1, 1)$.

Little group transformation: $W(\theta, \mu, \nu) \simeq \mathbf{1} + I\theta J_3 + I\mu M + I\nu N$ with $M = J_2 + K_1$, $N = -J_1 + K_2$.

W has 3 degrees of freedom: For $(t_i)^\mu{}_\nu k^\nu = 0$, take $t_i = (J_3, M, N)$, check with Lorentz transformation of 4-vectors on page 22.

Choice of states: $(J_3, M, N)|k, \sigma\rangle = (\sigma, 0, 0)|k, \sigma\rangle$.

Since $[M, N] = 0$, try eigenstates for which $M|k, m, n\rangle = m|k, m, n\rangle$, $N|k, m, n\rangle = n|k, m, n\rangle$.

Then m, n continuous degrees of freedom, unobserved: $[J_3, M] = IN$, so $M(1 - I\theta J_3)|k, m, n\rangle = (m - n\theta)(1 - I\theta J_3)|k, m, n\rangle$,

i.e. $(1 - I\theta J_3)|k, m, n\rangle$ is eigenvector of M, eigenvalue $m - n\theta$.

Similarly, $[J_3, N] = -IM$, so $(1 - I\theta J_3)|k, m, n\rangle$ is eigenvector of N, eigenvalue $n + m\theta$.

Avoid this problem by taking $m = n = 0$, so left with states $J_3|k, \sigma\rangle = \sigma|k, \sigma\rangle$.

Since $J_3 = \boldsymbol{J} \cdot \hat{\boldsymbol{k}}$, σ is *helicity*, component of spin in direction of motion.

Representation for massless particles: $D_{\sigma'\sigma}(W) = e^{I\theta\sigma}\delta_{\sigma'\sigma}$.

$U(W)|k, \sigma\rangle = (1 + I\theta J_3 + I\mu M + I\nu N)|k, \sigma\rangle = (1 + I\theta\sigma)|k, \sigma\rangle$. For finite θ, $U(W)|k, \sigma\rangle = e^{I\theta\sigma}|k, \sigma\rangle$.

As on page 26.

p **dependence of** u: $u_{l\ \sigma}(\boldsymbol{p}) = D_{ll'}(L(p))u_{l'\ \sigma}(\boldsymbol{k})$. ($\boldsymbol{p}$ dependence of v is the same.)

Little group transformation of u, v: $u_{l\ \sigma}(\boldsymbol{k})e^{I\theta(W,k)\sigma} = D_{ll'}(W)u_{l'\ \sigma}(\boldsymbol{k})$, $v_{l\ \sigma}(\boldsymbol{k})e^{-I\theta(W,k)\sigma} = D_{ll'}(W)v_{l'\ \sigma}(\boldsymbol{k})$.

Transformation of u, v on page 26 reads $u_{l\ \sigma}(\boldsymbol{\Lambda p})e^{I\theta(\Lambda,p)\sigma} = D_{ll'}(\Lambda)u_{l'\ \sigma}(\boldsymbol{p})$. Take $\Lambda = W$, $p = k$.

Rotation of u, v: $(J_3)_{ll'}u_{l'\ \sigma}(\boldsymbol{k}) = \sigma u_{l\ \sigma}(\boldsymbol{k})$, $(J_3)_{ll'}v_{l'\ \sigma}(\boldsymbol{k}) = -\sigma v_{l\ \sigma}(\boldsymbol{k})$.

Take W to be just rotation about 3-axis ($W(\theta,\mu,\nu) \simeq \mathbf{1} + I\theta J_3$, i.e. $\mu = \nu = 0$) in little group transformation of u, v.

M, N **transformation of** u, v: $M_{ll'}u_{l'\ \sigma}(\boldsymbol{k}) = N_{ll'}u_{l'\ \sigma}(\boldsymbol{k}) = 0$. Same for v.

Take $W = (1 + I\mu M + I\nu N)$ in little group transformation of u, v.

u, v **relation**: $v_{l\ \sigma}(\boldsymbol{p}) = u^*_{l\ \sigma}(\boldsymbol{p})$.

Implied (up to proportionality constant) by rotation (note $(J_3)_{ll'}$ is imaginary) and M, N transformation of u, v.

Allowed helicities for fields in given (A, B) **representation**: $\sigma = \pm(B - A)$ for particle/antiparticle.

$\boldsymbol{J} = \boldsymbol{A} + \boldsymbol{B}$, and e.g. $A^3_{aba'b'} = a\delta_{aa'}\delta_{bb'}$, so rotation of u is $\sigma u_{ab\ \sigma}(\boldsymbol{k}) = (a+b)u_{ab\ \sigma}(\boldsymbol{k})$.

M, N transformation of u is $M_{aba'b'}u_{a'b'\ \sigma} = N_{aba'b'}u_{a'b'\ \sigma} = 0$. Using $M = IA_- - IB_+$ and $N = -A_- - B_+$,

where $A_\pm = A_1 \pm IA_2$, $B_\pm = B_1 \pm IB_2$ are usual raising/lowering operators, $(A_-)_{aba'b'}u_{a'b'\ \sigma} = (B_+)_{aba'b'}u_{a'b'\ \sigma} = 0$,

i.e. must have $a = -A$, $b = B$ or $u_{ab\ \sigma} = 0$. So $\sigma = B - A$. Similar for v, gives $\sigma = A - B$.

2.3.11 Spin-statistics connection

Determine which of $\mp$ for given j is possible for causality on page 27 to hold. Demand more general condition

$$[\psi_{ab}(x), \tilde{\psi}^\dagger_{\tilde{a}\tilde{b}}(y)]_\mp = \int D^3p\, \pi_{ab,\tilde{a}\tilde{b}}(\boldsymbol{p}) \left[\kappa\tilde{\kappa}^* e^{Ip\cdot(x-y)} \mp \lambda\tilde{\lambda}^* e^{-Ip\cdot(x-y)}\right] = 0 \text{ for } (x-y)^2 > 0,$$

where $\psi_{ab}, \tilde{\psi}_{\tilde{a}\tilde{b}}$ for same particle species and $\pi_{ab,\tilde{a}\tilde{b}}(\boldsymbol{p}) = u_{ab\ \sigma}(\boldsymbol{p})\tilde{u}^*_{\tilde{a}\tilde{b}\ \sigma}(\boldsymbol{p}) = v_{ab\ \sigma}(\boldsymbol{p})\tilde{v}^*_{\tilde{a}\tilde{b}\ \sigma}(\boldsymbol{p})$.

In field on page 31, use commutation relations for $a^{(\dagger)}_\sigma(\boldsymbol{p})$ on page 23.

$u_{ab\ \sigma}(\boldsymbol{p})\tilde{u}^*_{\tilde{a}\tilde{b}\ \sigma}(\boldsymbol{p}) = v_{ab\ \sigma}(\boldsymbol{p})\tilde{v}^*_{\tilde{a}\tilde{b}\ \sigma}(\boldsymbol{p})$ holds for massive particles from u, v relation on page 32.

($u_{ab\ \sigma}(\boldsymbol{p})\tilde{u}^*_{\tilde{a}\tilde{b}\ \sigma}(\boldsymbol{p}) = [v_{ab\ \sigma}(\boldsymbol{p})\tilde{v}^*_{\tilde{a}\tilde{b}\ \sigma}(\boldsymbol{p})]^*$ in massless case from u, v relation on page 34).

Relation between κ, λ of different massive fields: $\boxed{\kappa\tilde{\kappa}^* = \pm(-1)^{2\tilde{A}+2B}\lambda\tilde{\lambda}^*}$.

Explicit calculation shows $\pi_{ab,\tilde{a}\tilde{b}}(\boldsymbol{p}) = P_{ab,\tilde{a}\tilde{b}}(\boldsymbol{p}) + 2\sqrt{\boldsymbol{p}^2 + m^2}Q_{ab,\tilde{a}\tilde{b}}(\boldsymbol{p})$,

where $(P,Q)(\boldsymbol{p})$ are polynomial in $\boldsymbol{p}$, obey $(P,Q)(-\boldsymbol{p}) = (-1)^{2\tilde{A}+2B}(P,-Q)(\boldsymbol{p})$.

Take $(x-y)^2 > 0$, use frame $x^0 = y^0$, write $\Delta_+(x) = \int D^3p e^{Ip\cdot x}$:

$$[\psi_{ab}, \tilde{\psi}_{\tilde{a}\tilde{b}}]_\mp = \left[\kappa\tilde{\kappa}^* \mp (-1)^{2\tilde{A}+2B}\lambda\tilde{\lambda}^*\right] P_{ab,\tilde{a}\tilde{b}}(-I\boldsymbol{\nabla})\Delta_+(\boldsymbol{x}-\boldsymbol{y},0) + \left[\kappa\tilde{\kappa}^* \pm (-1)^{2\tilde{A}+2B}\lambda\tilde{\lambda}^*\right] Q_{ab,\tilde{a}\tilde{b}}(-I\boldsymbol{\nabla})\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}).$$

Commutator must vanish for $\boldsymbol{x} \neq \boldsymbol{y}$, so coefficient of P zero.

Relations between κ, λ of single field: $|\kappa|^2 = |\lambda|^2$ and $\pm(-1)^{2A+2B} = 1$.

For $A = \tilde{A}$, $B = \tilde{B}$, relation between κ, λ of different fields reads $|\kappa|^2 = \pm(-1)^{2A+2B}|\lambda|^2$.

Spin-statistics: Bosons (fermions) have even (odd) $2j$ and vice versa.

From triangle inequality on page 29,

$j - (A+B)$ is integer, so $\pm(-1)^{2j} = 1$, i.e. in $[\psi_{ab}, \tilde{\psi}_{\tilde{a}\tilde{b}}]_{\mp}$, must have $-$ $(+)$ for even (odd) $2j$.

Relation between κ, λ of single massive field: $\lambda = (-1)^{2A} e^{Ic}\kappa$, c is the same for all fields.

Divide relation between κ, λ of different massive fields on page 35 by $|\tilde{\kappa}|^2 = |\tilde{\lambda}|^2$: $\frac{\kappa}{\tilde{\kappa}} = \pm(-1)^{2\tilde{A}+2B}\frac{\lambda}{\tilde{\lambda}} = (-1)^{2A+2\tilde{A}}\frac{\lambda}{\tilde{\lambda}}$.

Absorb κ into field, e^{Ic} into $a^{c\dagger}_{\sigma}(\boldsymbol{p})$ (does not affect commutation relations on page 23).

$(-1)^{2A}$ can't be absorbed into $a^{c\dagger}_{\sigma}(\boldsymbol{p})$ since this is independent of A,

nor absorbed into v since this is already chosen. So

Massive irreducible field: $\psi_{ab}(x) = \int D^3p \, \left[e^{Ip\cdot x} u_{ab\ \sigma}(\boldsymbol{p}) a_{\sigma}(\boldsymbol{p}) + (-1)^{2A} e^{-Ip\cdot x} v_{ab\ \sigma}(\boldsymbol{p}) a^{c\dagger}_{\sigma}(\boldsymbol{p})\right]$,

or more fully as $\psi_{ab}(x) = \int D^3p \, D^{(j)}_{aba'b'}(L(p)) \left[e^{Ip\cdot x} u_{a'b'\ \sigma}(0) a_{\sigma}(\boldsymbol{p}) + (-1)^{2A+j+\sigma} e^{-Ip\cdot x} u_{a'b'\ -\sigma}(0) a^{c\dagger}_{\sigma}(\boldsymbol{p})\right]$.

Use u, v relation on page 32.

2.4 External symmetries: fermions

2.4.1 Spin $\frac{1}{2}$ fields

$2j$ is odd $\longrightarrow$ particles are fermions. $j = \frac{1}{2}$ representations include $(A, B) = \left(\frac{1}{2}, 0\right)$ and $(A, B) = \left(0, \frac{1}{2}\right)$.

In each case, group element acts on 2 component *spinor* $X^{(A,B)}$ ($X^{(A,B)}$ is e.g. a field operator),

with 2 components $X_{ab}^{(A,B)}$:

$X^{\left(\frac{1}{2},0\right)} \equiv X_L = \left(X_{\frac{1}{2},0}^{\left(\frac{1}{2},0\right)}, X_{-\frac{1}{2},0}^{\left(\frac{1}{2},0\right)}\right)$ ("left-handed") and

$X^{\left(0,\frac{1}{2}\right)} \equiv X_R = \left(X_{0,\frac{1}{2}}^{\left(0,\frac{1}{2}\right)}, X_{0,-\frac{1}{2}}^{\left(0,\frac{1}{2}\right)}\right)$ ("right-handed").

"Handedness"/chirality refers to eigenstates of helicity for *massless* particles (see later).

Lorentz transformation of spinors: $U(\Lambda)X_{L/R}U^\dagger(\Lambda) = h_{L/R}(\Lambda)X_{L/R}$ (D acts on the 2 a, b components).

From explicit form of Lorentz group elements on page 22, $h_{L/R}(\Lambda) = e^{I\boldsymbol{J}^{\left(\frac{1}{2},0\right)/\left(0,\frac{1}{2}\right)}\cdot\boldsymbol{\theta}}e^{-I\boldsymbol{K}^{\left(\frac{1}{2},0\right)/\left(0,\frac{1}{2}\right)}\cdot\hat{\boldsymbol{e}}\beta}$, where

Lorentz group generators for spinors: $J_i^{\left(\frac{1}{2},0\right)} = \frac{1}{2}\sigma_i,\ K_i^{\left(\frac{1}{2},0\right)} = I\frac{1}{2}\sigma_i$, $J_i^{\left(0,\frac{1}{2}\right)} = \frac{1}{2}\sigma_i,\ K_i^{\left(0,\frac{1}{2}\right)} = -I\frac{1}{2}\sigma_i$,

where σ_i are the *Pauli σ matrices* $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which obey

Rotation group algebra: $\left[\frac{1}{2}\sigma_i, \frac{1}{2}\sigma_j\right] = I\epsilon_{ijk}\frac{1}{2}\sigma_k$.

Follows from $J_i = J_i^{(A)} + J_i^{(B)}$ and $K_i = I(J_i^{(A)} - J_i^{(B)})$ on page 29,

and $J_i^{(0)} = 0$ and $J_i^{\left(\frac{1}{2}\right)} = \frac{1}{2}\sigma_i$ from irreducible representation for spin j on page 17.

Explicit form of $h_{L/R}$: $h_{L/R} = e^{I\frac{1}{2}\sigma_i\theta_i}e^{\mp\frac{1}{2}\sigma_i\hat{e}_i\beta}$. Note σ matrices are Hermitian.

Product of σ matrices: $\sigma_i\sigma_j = \delta_{ij} + I\epsilon_{ijk}\sigma_k$. Follows by explicit calculation.

Direct calculation of $h_{L/R}$: $h_{L/R} = (\cos\frac{\theta}{2} + I\sigma_i\hat{\theta}_i\sin\frac{\theta}{2})(\cosh\frac{\beta}{2} \mp \sigma_i\hat{e}_i\sinh\frac{\beta}{2})$.

From product of σ matrices, $T^2 = \mathbf{1}$ where $T = \sigma_i\hat{e}_i$ ($T = \sigma_i\hat{\theta}_i$). Then $e^{xT} = \cosh x + T\sinh x$ ($e^{IxT} = \cos x + IT\sin x$).

Write $h_L = h$ and $X_L = X$ which has components $X_a = (X_1, X_2)$, which transforms as $\boxed{1.\ X'_a = h_a{}^b X_b}$.

Can also define spinor transforming with h^*: Use dotted indices for h^*, so $\boxed{2.\ X'^{\dagger}_{\dot{a}} = h^*_{\dot{a}}{}^{\dot{b}} X^{\dagger}_{\dot{b}}}$.

Then $h_R = h^{*-1T}$. From explicit form of $h_{L/R}$ on page 38.

Use upper indices for h^{-1}, and X_R has components $X^{\dagger\dot{a}} = (X^{\dagger\dot{1}}, X^{\dagger\dot{2}})$, Dotted indices because h^* is used.

transformation is $\boxed{4.\ X'^{\dagger\dot{a}} = \left(h^{*-1T}\right)^{\dot{a}}{}_{\dot{b}} X^{\dagger\dot{b}}}$, where we define $(h^T)^a{}_b = h_b{}^a$.

Conjugate of this turns dotted indices into undotted indices, so $\boxed{3.\ X'^a = \left(h^{-1T}\right)^a{}_b X^b}$.

Conjugate of spinors: $(X^a)^\dagger = X^{\dagger\dot{a}}$, $(X_a)^\dagger = X^\dagger_{\dot{a}}$. Definition of $X^{\dagger\dot{a}}$ in terms of X^a, $X^\dagger_{\dot{a}}$ in terms of X_a.

Check that if X_a transforms as transformation 1., $X^\dagger_{\dot{a}}$ transforms as transformation 2.: $X'^{\dagger}_{\dot{a}} = (X'_a)^\dagger = (h_a{}^b X_b)^\dagger = h^{*\dot{b}}_{\dot{a}} X^\dagger_{\dot{b}}$.

Spinor metric: $\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\epsilon_{ab} = -\epsilon^{ab}$. Note $\epsilon_{ab}\epsilon^{bc} = \delta_a{}^c$. ϵ is unitary matrix.

Pseudo reality: $\epsilon^{ac}(\sigma_i)_c{}^d\epsilon_{db} = -(\sigma_i^T)^a{}_b$.

Follows by explicit calculation. Since $\sigma_i^* = \sigma_i^T$, shows rotation group representation by σ matrices is real (see page 12).

$X^\dagger_{\dot{a}}$ is $\left(0, \frac{1}{2}\right)$, i.e. right-handed, like $X^{\dagger\dot{a}}$.

Pseudo reality above implies $\epsilon^{\dot{a}\dot{c}}h^{*\dot{d}}_{\dot{c}}\epsilon_{\dot{d}\dot{b}} = (h^{*-1T})^{\dot{a}}{}_{\dot{b}}$, i.e. h^*, h_R same up to unitary similarity transformation.

For unitary representations, follows because $A = B^\dagger$ from definition on page 29, so conjugation makes $\left(\frac{1}{2}, 0\right) \to \left(0, \frac{1}{2}\right)$.

This means dotted indices are for right-handed ($\left(0, \frac{1}{2}\right)$) fields.

(Similarly, undotted indices are for left-handed ($\left(\frac{1}{2}, 0\right)$) fields.)

Metric condition: $h^*_{\dot{e}}{}^{\dot{d}}\epsilon_{\dot{d}\dot{b}}(h^{*T})^{\dot{b}}{}_{\dot{f}} = \epsilon_{\dot{e}\dot{f}}$.

Follows from $\epsilon^{\dot{a}\dot{c}}h^{*\dot{d}}_{\dot{c}}\epsilon_{\dot{d}\dot{b}} = (h^{*-1T})^{\dot{a}}{}_{\dot{b}}$ result above. Thus ϵ is the group's invariant matrix $\mathfrak{g}$ ("metric") on page 12.

Raising and lowering of spinor indices: $X^a = \epsilon^{ab} X_b$. This is definition of X^a in terms of X_a.

Follows that $X_a = \epsilon_{ab} X^b$. Same definition / behaviour for dotted indices.

Check that if X_a transforms as transformation 1. on page 39, X^a transforms as transformation 3.: $X'^a = \epsilon^{ab} X'_b = \epsilon^{ab} h_b{}^c X_c$.

From pseudo reality, $\epsilon^{ab} h_b{}^c \epsilon_{cd} = \left(h^{-1T}\right)^a{}_d$, or $\epsilon^{ab} h_b{}^c = \left(h^{-1T}\right)^a{}_b \epsilon^{bc}$, so $X'^a = \left(h^{-1T}\right)^a{}_b \epsilon^{bc} X_c = \left(h^{-1T}\right)^a{}_b X^b$.

Right-handed from left-handed fields: $X^{\dagger\dot{a}} = \left(\epsilon^{ab} X_b\right)^\dagger$.

So all fields can be expressed in terms of left-handed fields.

Scalar from 2 spinors: $XY = X^a Y_a = -Y_a X^a = Y^a X_a = -X_a Y^a = YX$.

$X^{a\prime} Y'_a = \left(h^{-1T}\right)^a{}_c X^c h_a{}^b Y_b = \left(h^{-1}\right)_c{}^a h_a{}^b X^c Y_b$. X, Y anticommute (spinor operators). $X^a Y_a = -X_a Y^a$ because $\epsilon^{ab} \epsilon_{ac} = -\delta^b{}_c$.

Hermitian conjugate of scalar: $(XY)^\dagger = (X^a Y_a)^\dagger = (Y_a)^\dagger (X^a)^\dagger = Y^\dagger_{\dot{a}} X^{\dagger\dot{a}} = Y^\dagger X^\dagger$.

4-vector σ matrices $\sigma^\mu_{a\dot{b}}$: $\sigma^i_{a\dot{b}} = (\sigma_i)_{a\dot{b}}$, $\sigma^0_{a\dot{b}} = (\sigma_0)_{a\dot{b}}$ and $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

4-vector σ matrices with raised indices: $\overline{\sigma}^{\mu\ \dot{a}b} = \epsilon^{bc}\sigma^\mu_{c\dot{d}}\epsilon^{\dot{a}\dot{d}}$, so $\overline{\sigma}^{i\ \dot{a}b} = -(\sigma_i)^{\dot{a}b}$, $\overline{\sigma}^{0\ \dot{a}b} = (\sigma_0)^{\dot{a}b}$.

Second result follows from definition by explicit calculation.

Inner product of 4-vector σ matrices: $g_{\rho\omega}\sigma^\omega_{e\dot{f}}\overline{\sigma}^{\rho\ \dot{b}a} = -2\delta_e^{\ a}\delta_{\dot{f}}^{\ \dot{b}}$.

Outer product of 4-vector σ matrices: $\sigma^\nu_{a\dot{b}}\overline{\sigma}^{\rho\ \dot{b}a} = -2g^{\nu\rho}$.

$X^a\sigma^\mu_{a\dot{b}}Y^{\dagger\dot{b}}$ is a 4-vector.

Need to show $X'^a\sigma^\mu_{a\dot{b}}Y'^{\dagger\dot{b}} = \Lambda^\mu_{\ \nu}X^a\sigma^\nu_{a\dot{b}}Y^{\dagger\dot{b}}$. Since $X'^a\sigma^\mu_{a\dot{b}}Y'^{\dagger\dot{b}} = \left(h^{-1T}\right)^a_{\ c}X^c\sigma^\mu_{a\dot{b}}\left(h^{*-1T}\right)^{\dot{b}}_{\ \dot{d}}Y^{\dagger\dot{d}}$,

need to show $\left(h^{-1}\right)_a^{\ c}\sigma^\mu_{c\dot{d}}\left(h^{*-1T}\right)^{\dot{d}}_{\ \dot{b}} = \Lambda^\mu_{\ \nu}\sigma^\nu_{a\dot{b}}$. Contracting with $\overline{\sigma}^{\rho\ \dot{b}a}$ and using outer product of 4-vector σ matrices

gives $\Lambda^{\mu\rho} = -\frac{1}{2}\overline{\sigma}^{\rho\ \dot{b}a}\left(h^{-1}\right)_a^{\ c}\sigma^\mu_{c\dot{d}}\left(h^{*-1T}\right)^{\dot{d}}_{\ \dot{b}} = -\frac{1}{2}\mathrm{tr}\left[\overline{\sigma}^\rho h^{-1}\sigma^\mu h^{*-1T}\right]$, which is equivalent because σ matrices linearly independent

(or multiply this by $g_{\rho\omega}\sigma^\omega_{e\dot{f}}$ and use inner product of 4-vector σ matrices). Check group property:

$\Lambda^\mu_{\ \nu}\Lambda'^\nu_{\ \ \rho} = \frac{1}{2}\mathrm{tr}\left[\overline{\sigma}^\mu h^{-1}(\Lambda)\sigma^\nu h^{*-1T}(\Lambda)\right]\frac{1}{2}\mathrm{tr}\left[\overline{\sigma}^\nu h^{-1}(\Lambda')\sigma^\rho h^{*-1T}(\Lambda')\right] = \frac{1}{2}\overline{\sigma}^\mu_{ij}h_{jk}(\Lambda)\sigma_{\mu kl}h^\dagger_{li}(\Lambda)\overline{\sigma}^\nu_{mn}h_{np}(\Lambda')\sigma_{\rho pq}h^\dagger_{qm}(\Lambda')$. From

inner product of 4-vector σ matrices, $\overline{\sigma}^\nu_{mn}\sigma_{\mu kl} = -2\delta_{ml}\delta_{kn}$, so $\Lambda^\mu_{\ \nu}\Lambda'^\nu_{\ \ \rho} = -\frac{1}{2}\overline{\sigma}^{\mu\ \dot{b}a}\left(h^{-1}(\Lambda')h^{-1}(\Lambda)\right)_a^{\ c}\sigma_{\rho c\dot{d}}\left(h^{*-1T}(\Lambda)h^{*-1T}(\Lambda')\right)^{\dot{d}}_{\ \dot{b}}$

Convenient to put left- (X) and right-handed (Z) fields together as 4 component spinor:

$$\psi^T = \left(C_1 X^{\left(\frac{1}{2},0\right)}_{\frac{1}{2},0}, C_1 X^{\left(\frac{1}{2},0\right)}_{-\frac{1}{2},0}, C_2 Z^{\left(0,\frac{1}{2}\right)}_{0,\frac{1}{2}}, C_2 Z^{\left(0,\frac{1}{2}\right)}_{0,-\frac{1}{2}} \right)$$, where C_1, C_2 scalar constants.

Lorentz group generators: $J_i = \begin{pmatrix} \frac{1}{2}\sigma_i & 0 \\ 0 & \frac{1}{2}\sigma_i \end{pmatrix}$ and $K_i = \begin{pmatrix} I\frac{1}{2}\sigma_i & 0 \\ 0 & -I\frac{1}{2}\sigma_i \end{pmatrix}$.

Follows from Lorentz group generators for spinors on page 38.

This is the *chiral* (*Weyl*) representation,

others representations from similarity transformation: $J_i' = V J_i V^{-1}$, $K_i' = V K_i V^{-1}$, $X' = VX$ etc.

Preferable to express in terms of left-handed fields only:

$$\psi = \begin{pmatrix} X_a \\ (\epsilon^{bc} Y_c)^\dagger \end{pmatrix} = \begin{pmatrix} X_a \\ Y^{\dagger \dot{b}} \end{pmatrix}$$, where $Y = Z^\dagger$ is left-handed like X.

2.4.2 Spin $\frac{1}{2}$ in general representations

Any 4×4 matrix can be constructed from sums/products of gamma matrices (next page):

Gamma matrices (chiral representation): $\gamma^0 = -I \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$ and $\gamma^i = -I \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$, or $\gamma^\mu = -I \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}$.

Gamma matrix in different representations related by similarity transformation $\gamma'^\mu = V \gamma^\mu V^{-1}$.

Anticommutation relations for γ^μ: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. This is representation independent.

Check by explicit calculation in chiral representation, use rotation group algebra on page 38.

Define $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -I\gamma^0\gamma^1\gamma^2\gamma^3$. Check last representation-independent equality explicitly in chiral representation.

Then $P_L = \frac{1}{2}(1 + \gamma_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ projects out left-handed spinor: $P_L \begin{pmatrix} X_a \\ (\epsilon^{bc} Y_c)^\dagger \end{pmatrix} = \begin{pmatrix} X_a \\ 0 \end{pmatrix}$,

similarly $P_R = \frac{1}{2}(1 - \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ projects out right-handed spinor: $P_R \begin{pmatrix} X_a \\ (\epsilon^{bc} Y_c)^\dagger \end{pmatrix} = \begin{pmatrix} 0 \\ (\epsilon^{bc} Y_c)^\dagger \end{pmatrix}$.

Note $P_{L/R}$ are projection operators: $P_{L/R}^2 = P_{L/R}$, $P_{L/R} P_{R/L} = 0$.

Any 4×4 matrix is linear combination of $\mathbf{1}$, γ^μ, $[\gamma^\mu, \gamma^\nu]$, $\gamma^\mu\gamma_5$, γ_5.

Because these are 16 non-zero linearly independent 4×4 matrices.

Non-zero because their squares, calculated from anticommutation relations on page 44, are non-zero.

Linearly independent because they are orthogonal if we define scalar product of any two to be trace of their matrix product:

$\mathrm{tr}[\mathbf{1}\gamma^\mu] = 0$, because $\mathrm{tr}[\gamma^\mu] = 0$ in chiral representation, therefore in any other representation.

$\mathrm{tr}[\mathbf{1}[\gamma^\mu, \gamma^\nu]] = 0$ by (anti)symmetry. From anticommutation relations, $\mathrm{tr}[\mathbf{1}\gamma^\mu\gamma_5] = -\mathrm{tr}[\gamma_5\gamma^\mu] = -\mathrm{tr}[\gamma^\mu\gamma_5] = 0$

and also $\mathrm{tr}[\mathbf{1}\gamma_5] = \mathrm{tr}[\gamma_5] = 0$ by commuting γ^0 from left to right.

Next, $\mathrm{tr}[\gamma^\rho[\gamma^\mu, \gamma^\nu]] = 0$: From anticommutation relations, $\gamma_5^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^{0\ 2}\gamma^{1\ 2}\gamma^{2\ 2}\gamma^{3\ 2} = \mathbf{1}$.

So $\mathrm{tr}[\gamma^\rho[\gamma^\mu, \gamma^\nu]] = \mathrm{tr}[\gamma_5^2\gamma^\rho[\gamma^\mu, \gamma^\nu]] = -\ \mathrm{tr}[\gamma_5\gamma^\rho[\gamma^\mu, \gamma^\nu]\gamma_5] = \mathrm{tr}[\gamma_5\gamma^\rho[\gamma^\mu, \gamma^\nu]\gamma_5]$.

To show $\mathrm{tr}[\gamma^\rho\gamma^\mu\gamma_5] = 0$, first consider case $\rho = \mu$. Then $\gamma^\rho\gamma^\mu = g^{\rho\rho}$, and result is $\propto \mathrm{tr}[\gamma_5] = 0$.

If e.g. $\rho = 1$, $\mu = 2$, $\mathrm{tr}[\gamma^\rho\gamma^\mu\gamma_5] = I\mathrm{tr}[\gamma^0\gamma^3] = 0$. $\mathrm{tr}[\gamma^\mu\gamma_5] = 0$ already shown.

$\mathrm{tr}[[\gamma^\mu, \gamma^\nu]\gamma^\rho\gamma_5] = 0$ from anticommutation relations. $\mathrm{tr}[[\gamma^\mu, \gamma^\nu]\gamma_5] = 0$ because $\mathrm{tr}[\gamma^\rho\gamma^\mu\gamma_5] = 0$. Finally $\mathrm{tr}[\gamma^\mu\gamma_5\gamma_5] = \mathrm{tr}[\gamma^\mu] = 0$.

So all spinorial observables expressible as representation independent sums/products of gamma matrices.

Lorentz group generators from γ^μ: $\boxed{J^{\mu\nu} = -\frac{I}{4}[\gamma^\mu, \gamma^\nu]}$.

Explicit calculation in chiral representation, $J_i = \frac{1}{2}\epsilon_{ijk}J_{jk}$, $\boldsymbol{K} = (J^{10}, J^{20}, J^{30})$,

Lorentz group generators on page 43, Gamma matrices on page 44.

Similarity transformation on page 43 equivalent to similarity transformation on page 44.

Infinitesimal Lorentz transformation of γ^μ: $\boxed{I[J^{\mu\nu}, \gamma^\rho] = g^{\nu\rho}\gamma^\mu - g^{\mu\rho}\gamma^\nu}$,

Follows from anticommutation relations for γ^μ on page 44 and Lorentz group generators from γ^μ above.

Lorentz transformation of γ^μ: $\boxed{D(\Lambda)\gamma^\mu D^{-1}(\Lambda) = \Lambda_\nu{}^\mu\gamma^\nu}$, i.e. γ^μ transforms like a vector.

Agrees with infinitesimal Lorentz transformation of γ^μ, because for infinitesimal case,

LHS is $(1 + \frac{1}{2}I\omega_{\rho\sigma}J^{\rho\sigma})\gamma^\mu(1 - \frac{1}{2}I\omega_{\omega\eta}J^{\omega\eta}) = \gamma^\mu - \frac{1}{2}I\omega_{\rho\sigma}[\gamma^\mu, J^{\rho\sigma}]$, and RHS is $(\delta_\nu{}^\mu + \omega_{\nu\sigma}g^{\mu\sigma})\gamma^\nu = \gamma^\mu - \frac{1}{2}\omega_{\rho\sigma}(g^{\mu\rho}\gamma^\sigma - g^{\mu\sigma}\gamma^\rho)$.

Lorentz transformation of vector: $\boxed{\gamma^\mu(\Lambda p)_\mu = D(\Lambda)\ \gamma^\mu p_\mu\ D^{-1}(\Lambda)}$.

From Lorentz transformation of γ^μ above.

Reference boost: $\boxed{\frac{\gamma^\mu p_\mu}{m} = -D(L(p))\gamma^0 D^{-1}(L(p))}$. Equivalent of $p = L(p)k$ on page 24, $-\gamma^0 m = \gamma^\mu k_\mu$.

In Lorentz transformation of vector, take $p = k$, $\Lambda = L(q)$. Then $\gamma^\mu q_\mu = D(L(q))\ \gamma^\mu k_\mu\ D^{-1}(L(q)) = -D(L(q))\ \gamma^0 m\ D^{-1}(L(q))$.

Parity transformation matrix: $\boxed{\beta = I\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}}$. Note $\beta^2 = \mathbf{1}$.

Pseudo-unitarity of Lorentz transformation : $\boxed{J^{\mu\nu\dagger} = \beta J^{\mu\nu}\beta}$, $\boxed{D^\dagger = \beta D^{-1}\beta}$.

$\beta\gamma^0\beta = \gamma^0 = -\gamma^{0\dagger}$ and $\beta\gamma^i\beta = -\gamma^i = -\gamma^{i\dagger}$, or $\beta\gamma^\mu\beta = -\gamma^{\mu\dagger}$. Then use Lorentz group generators from γ^μ on page 46.

Infinitesimal $D^\dagger$ is $1 - \frac{1}{2}I\omega_{mu\nu}J^{\mu\nu\dagger} = \beta(1 - \frac{1}{2}I\omega_{mu\nu}J^{\mu\nu})\beta$.

Adjoint spinor: $\boxed{\overline{X} = X^\dagger\beta}$. Allows construction of scalars, vectors etc. from spinors:

Covariant products: $\boxed{\overline{X}Y \text{ is scalar}}$, $\boxed{\overline{X}\gamma^\mu Y \text{ is vector}}$.

First case: $\overline{X'}Y' = X'^\dagger\beta Y' = X^\dagger D^\dagger\beta DY$.

From pseudo-unitarity of Lorentz transformation, $D^\dagger\beta = \beta D^{-1}\beta^2 = \beta D^{-1}$, so $\overline{X'}Y' = X^\dagger\beta D^{-1}DY = X^\dagger\beta Y = \overline{X}Y$.

Second case: $\overline{X'}\gamma^\mu Y' = X^\dagger D^\dagger\beta\gamma^\mu DY = X^\dagger\beta D^{-1}\gamma^\mu DY$. Lorentz transformation of γ^μ on page 46 can be rewritten

$D^{-1}\gamma^\mu D = \left(\Lambda^{-1}\right)_\nu^{\ \mu}\gamma^\nu = \Lambda^\mu{}_\nu\gamma^\nu$, so $\overline{X'}\gamma^\mu Y' = \Lambda^\mu{}_\nu X^\dagger\beta\gamma^\nu Y = \Lambda^\mu{}_\nu\overline{X}\beta\gamma^\nu Y$.

Vanishing products: $\boxed{\overline{\psi}_{R/L}\psi_{L/R}}$ and $\boxed{\overline{\psi}_{L/R}\gamma^\mu\psi_{L/R}}$, where $\psi_{L/R} = P_{L/R}\psi$ (see page 44 for $P_{L/R}$).

Using $\gamma_5^\dagger = \gamma_5$ in chiral representation and $P_{L/R}\beta = \beta P_{R/L}$ gives $\overline{\psi}_R\psi_L = \psi^\dagger P_L\beta P_L\psi = \psi^\dagger\beta P_R P_L\psi = 0$ (because $P_{L/R}P_{R/L} = 0$).

Similarly, using $\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu$, one has $\overline{\psi}_L\gamma^\mu\psi_L = \psi^\dagger P_R\beta\gamma^\mu P_L\psi = \psi^\dagger\beta\gamma^\mu P_R P_L\psi = 0$.

Problems 1

1: If the parameterization α_i of group elements $U(\boldsymbol{\alpha})$, chosen such that $U(\mathbf{0}) = 1$, can be chosen such that the Abelian limit $U(\boldsymbol{\alpha})U(\boldsymbol{\beta}) = U(\boldsymbol{\alpha}+\boldsymbol{\beta})$ is obeyed for $\beta_i = c\alpha_i$, show that $U(\alpha) = \exp\left[It_i\alpha_i\right]$. Hint: Use the Abelian limit to write $U(\boldsymbol{\alpha}) = \left[U\left(\frac{\boldsymbol{\alpha}}{N}\right)\right]^N$, take $N \to \infty$, expand $U\left(\frac{\boldsymbol{\alpha}}{N}\right) = 1 + It_i\frac{\alpha_i}{N} + O\left(\frac{1}{N^2}\right)$ and use $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$ and the binomial theorem $(x+y)^N = \sum_{k=0}^{N}\frac{N!}{k!(N-k)!}x^k y^{N-k}$.

2: Show that the symmetry transformation $U(\boldsymbol{\alpha})a^\dagger_\sigma a^\dagger_{\sigma'}\ldots|0\rangle = D_{\sigma\sigma''}(\boldsymbol{\alpha})D_{\sigma'\sigma'''}(\boldsymbol{\alpha})a^\dagger_{\sigma''}a^\dagger_{\sigma'''}\ldots|0\rangle$ and $U(\boldsymbol{\alpha})|0\rangle = |0\rangle$ is *uniquely* satisfied by the condition $U(\boldsymbol{\alpha})a^\dagger_\sigma U^\dagger(\boldsymbol{\alpha}) = D_{\sigma\sigma'}(\boldsymbol{\alpha})a^\dagger_{\sigma'}$. Show that the matrices D furnish a representation of the group defined by U (namely $\boldsymbol{\gamma}(\boldsymbol{\alpha},\boldsymbol{\beta})$ in $U(\boldsymbol{\gamma}(\boldsymbol{\alpha},\boldsymbol{\beta})) = U(\boldsymbol{\alpha})U(\boldsymbol{\beta})$).

3: Show that a semi-simple group's matrix generators t_i all obey $\mathrm{tr}[t_i] = 0$.

4: Use the Lorentz group algebra $I[J^{\rho\sigma}, J^{\mu\nu}] = -g^{\sigma\nu}J^{\rho\mu} - g^{\rho\mu}J^{\sigma\nu} + g^{\sigma\mu}J^{\rho\nu} + g^{\rho\nu}J^{\sigma\mu}$ to derive the rotation group algebra $[J_i, J_j] = I\epsilon_{ijk}J_k$, where $(J_1, J_2, J_3) = (J^{23}, J^{31}, J^{12})$.

5: Using the explicit result for the generator of 4-vector boosts, namely $(K_i)^\mu{}_\nu = I(\delta_{0\mu}\delta_{i\nu} + \delta_{i\mu}\delta_{0\nu})$, show that the explicit calculation of a general boost is $\left[e^{-I\boldsymbol{K}\cdot\hat{\boldsymbol{e}}\beta}\right]^\mu{}_\nu = \left[1 - IK_i\hat{e}_i\sinh\beta - (K_i\hat{e}_i)^2(\cosh\beta - 1)\right]^\mu{}_\nu$. Hint: Show by explicit calculation the result $(K_i\hat{e}_i)^3 = K_i\hat{e}_i$ and then use it.

6: Define a field $\psi^{c-}_l(x) = \int D^3p\ v_{l\sigma}(x;\boldsymbol{p})a^{c\dagger}_\sigma(\boldsymbol{p})$, where $v_{l\sigma}(x;\boldsymbol{p})$ is such that $\psi^{c-}_l(x)$ has the transformation property $U(\Lambda,a)\psi^{c-}_l(x)U^\dagger(\Lambda,a) = D_{ll'}(\Lambda^{-1})\psi^{c-}_{l'}(\Lambda x + a)$. Show that $D_{ll'}(\Lambda)v_{l'\sigma}(\boldsymbol{p}) = D^{(j)}_{\sigma\sigma'}(W^{-1}(\Lambda,p))v_{l\sigma'}(\boldsymbol{\Lambda p})$, where $v_{l\sigma}(\boldsymbol{p}) = v_{l\sigma}(0;\boldsymbol{p})$. For a massive particle, for reference momentum $k = (0,0,0,1)$ and for L a pure boost, show that this implies that $-\boldsymbol{J}^{(j)*}_{\sigma'\sigma}v_{ab\ \sigma'}(0) = \boldsymbol{J}^{(A)}_{aa'}v_{a'b\ \sigma}(0) + \boldsymbol{J}^{(B)}_{bb'}v_{ab'\ \sigma}(0)$. Hint: Take $\Lambda = R$ and $p = k$.

7: Show that $J_i^{\left(\frac{1}{2},0\right)} = \frac{1}{2}\sigma_i$ and $K_i^{\left(\frac{1}{2},0\right)} = I\frac{1}{2}\sigma_i$. Hint: Use $\boldsymbol{J} = \boldsymbol{A}+\boldsymbol{B}$ and $\boldsymbol{K} = I(\boldsymbol{A}-\boldsymbol{B})$, and the irreducible representation for $A = \frac{1}{2}$ and $B = 0$.

8: A left-handed 2-component spinor operator X transforms according to $U(\Lambda)XU^\dagger(\Lambda) = h(\Lambda)X$, where the 2×2 matrix $h = e^{I\frac{1}{2}\sigma_i\theta_i}e^{-\frac{1}{2}\sigma_i\hat{e}_i\beta}$. Show that $h^{*-1T} = e^{I\frac{1}{2}\sigma_i\theta_i}e^{\frac{1}{2}\sigma_i\hat{e}_i\beta}$ (the equivalent matrix for right-handed spinors). Hint: The Pauli matrices are Hermitian.

9: Show that $\gamma^\mu(\Lambda p)_\mu = D(\Lambda)\ \gamma^\mu p_\mu\ D^{-1}(\Lambda)$. Hint: Consider the infinitessimal case, which follows if $I[J^{\mu\nu}, \gamma^\rho] = g^{\nu\rho}\gamma^\mu - g^{\mu\rho}\gamma^\nu$. To prove the latter, use $J^{\mu\nu} = -\frac{I}{4}[\gamma^\mu, \gamma^\nu]$ and $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

10: Show that $\overline{X}\gamma^\mu Y$ is a 4-vector, where $\overline{X} = X^\dagger\beta$ with $\beta = I\gamma^0$.

2.4.3 The Dirac field

Group 4 possibilities for $u^{(A,B)}_{ab\ \sigma}$ together as 4 component spinor: $u^T_\sigma = \left(u^{(\frac{1}{2},0)}_{\frac{1}{2},0,\sigma}, u^{(\frac{1}{2},0)}_{-\frac{1}{2},0,\sigma}, u^{(0,\frac{1}{2})}_{0,\frac{1}{2},\sigma}, u^{(0,\frac{1}{2})}_{0,-\frac{1}{2},\sigma} \right)$.

Likewise, $v^T_\sigma = \left(-v^{(\frac{1}{2},0)}_{\frac{1}{2},0,\sigma}, -v^{(\frac{1}{2},0)}_{-\frac{1}{2},0,\sigma}, v^{(0,\frac{1}{2})}_{0,\frac{1}{2},\sigma}, v^{(0,\frac{1}{2})}_{0,-\frac{1}{2},\sigma} \right)$.

First 2 v components multiplied by $(-1)^{2A} = -1$ to remove it from massive irreducible field on page 36.

Dirac field: $\psi(x) = \begin{pmatrix} X_a \\ (\epsilon^{bc} Y_c)^\dagger \end{pmatrix} = \int \frac{d^3p}{(2\pi)^3} \left[e^{Ip\cdot x} u_\sigma(\boldsymbol{p}) a_\sigma(\boldsymbol{p}) + e^{-Ip\cdot x} v_\sigma(\boldsymbol{p}) a^{c\dagger}_\sigma(\boldsymbol{p}) \right]$.

(Note $D^3p \to d^3p$ for convention.)

Anticommutation relations for spin $\frac{1}{2}$: $[a_\sigma(\boldsymbol{p}), a^\dagger_{\sigma'}(\boldsymbol{p}')]_+ = (2\pi)^3 \delta_{\sigma\sigma'} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')$ (i.e. no $2p^0$ factor).

$\boldsymbol{p}$ dependence of spin $\frac{1}{2}$ u, v: $u_\sigma(\boldsymbol{p}) = \sqrt{\frac{m}{p^0}} D(L(p)) u_\sigma(0)$ and $v_\sigma(\boldsymbol{p}) = \sqrt{\frac{m}{p^0}} D(L(p)) v_\sigma(0)$.

This is just the $\boldsymbol{p}$ dependence of u, v on page 26.

Condition on spin $\frac{1}{2}$ u, v: $-\frac{1}{2}\sigma^*_{i\ \sigma'\sigma} v^{(0,\frac{1}{2})}_{0b\ \sigma'}(0) = \frac{1}{2}\sigma_{i\ bb'} v^{(0,\frac{1}{2})}_{0b'\ \sigma}(0)$, $\frac{1}{2}\sigma_{i\ \sigma'\sigma} u^{(0,\frac{1}{2})}_{0b\ \sigma'}(0) = \frac{1}{2}\sigma_{i\ bb'} u^{(0,\frac{1}{2})}_{0b'\ \sigma}(0)$.

From conditions on u, v on page 32 and Lorentz group generators on page 43. Recall $A_i = \frac{1}{2}(J_i - IK_i)$ and $B_i = \frac{1}{2}(J_i + IK_i)$.

Spin $\frac{1}{2}$ u, v relation: $v_{1\text{ or }2\ \sigma}(0) = -(-1)^{\frac{1}{2}+\sigma} u_{1\text{ or }2\ -\sigma}(0)$ and $v_{3\text{ or }4\ \sigma}(0) = (-1)^{\frac{1}{2}+\sigma} u_{3\text{ or }4\ -\sigma}(0)$.

From u, v relation on page 32, $v^{(\frac{1}{2},0)}_{a0\ \sigma}(0) = (-1)^{\frac{1}{2}+\sigma} u^{(\frac{1}{2},0)}_{a0\ -\sigma}(0)$ and $v^{(0,\frac{1}{2})}_{0b\ \sigma}(0) = (-1)^{\frac{1}{2}+\sigma} u^{(0,\frac{1}{2})}_{0b\ -\sigma}(0)$.

Then multiply $v^{(\frac{1}{2},0)}_{a0\ \sigma}(0)$ by $(-1)^{2A} = -1$ as discussed on page 49.

Form of u, v: $u^T_{\sigma=\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(1\ 0\ 1\ 0)$, $u^T_{\sigma=-\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(0\ 1\ 0\ 1)$, $v^T_{\sigma=\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(0\ 1\ 0\ -1)$, $v^T_{\sigma=-\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(-1\ 0\ 1\ 0)$.

Solution to condition on spin $\frac{1}{2}$ v is $v^{(0,\frac{1}{2})}_{0,-\frac{1}{2},\frac{1}{2}} = -v^{(0,\frac{1}{2})}_{0,\frac{1}{2},-\frac{1}{2}}$ and $v^{(0,\frac{1}{2})}_{0,\frac{1}{2},\frac{1}{2}} = v^{(0,\frac{1}{2})}_{0,-\frac{1}{2},-\frac{1}{2}} = 0$. Components for u constrained similarly.

Use spin $\frac{1}{2}$ u, v relation, and adjust normalizations of $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$ parts individually.

Massless u, v: $u^T_{\sigma=\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(0\ 0\ 1\ 0)$, $u^T_{\sigma=-\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(0\ 1\ 0\ 0)$, $v^T_{\sigma=\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(0\ 1\ 0\ 0)$, $v^T_{\sigma=-\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(0\ 0\ 1\ 0)$.

From allowed helicities for fields in given (A, B) representation on page 34, e.g. for $\left(\frac{1}{2},0\right)$ field such as neutrino, only

allowed $\sigma = 0 - \frac{1}{2}$ for particle and $\sigma = \frac{1}{2} - 0$ for antiparticle, i.e. $\psi_L(x) = \int \frac{d^3p}{(2\pi)^3}\left[e^{Ip\cdot x} u^{(\frac{1}{2},0)}_{-\frac{1}{2}}(\boldsymbol{p}) a_{-\frac{1}{2}}(\boldsymbol{p}) + e^{-Ip\cdot x} v^{(\frac{1}{2},0)}_{\frac{1}{2}}(\boldsymbol{p}) a^{c\dagger}_{\frac{1}{2}}(\boldsymbol{p})\right]$.

Majorana particle = antiparticle: $a^{c\dagger}_\sigma(\boldsymbol{p}) = a^\dagger_\sigma(\boldsymbol{p})$, so $\psi^T_M = \left(X_a, X^{\dagger\dot{b}}\right)$ (i.e. $Y_a = X_a$ on page 49).

2.4.4 The Dirac equation

Representation independent definition of spin $\frac{1}{2}$ u, v: $(I\gamma^\mu p_\mu + m)u_\sigma(\boldsymbol{p}) = 0$ and $(-I\gamma^\mu p_\mu + m)v_\sigma(\boldsymbol{p}) = 0$.

For u and v, reference boost on page 46 gives $-I\frac{\gamma^\mu p_\mu}{m}u_\sigma(\boldsymbol{p}) = D(L(p))\beta D^{-1}(L(p))u_\sigma(\boldsymbol{p}) = \sqrt{\frac{m}{p^0}}D(L(p))\beta u_\sigma(0)$,

last step from $\boldsymbol{p}$ dependence of spin $\frac{1}{2}$ u, v on page 49.

In the chiral representation, and therefore any other representation, $\beta u_\sigma(0) = u_\sigma(0)$ and $\beta v_\sigma(0) = -v_\sigma(0)$,

so $-I\frac{\gamma^\mu p_\mu}{m}u_\sigma(\boldsymbol{p}) = \sqrt{\frac{m}{p^0}}D(L(p))u_\sigma(0) = u_\sigma(\boldsymbol{p})$, last step from $\boldsymbol{p}$ dependence of spin $\frac{1}{2}$ u, v again, likewise $-I\frac{\gamma^\mu p_\mu}{m}v_\sigma(\boldsymbol{p}) = -v_\sigma(\boldsymbol{p})$.

Dirac equation: $(\gamma^\mu\partial_\mu + m)\,\psi_l^{\pm(c)}(x) = 0$.

Act on Dirac field on page 49 with $(\gamma^\mu\partial_\mu + m)$, then use representation independent definition of spin $\frac{1}{2}$ u, v.

Consistent with Klein-Gordon equation $\left(\partial^2 - m^2\right)\psi^{\pm(c)}(x) = 0$.

From page 26. To check, act on Dirac equation from left with $(\gamma^\nu\partial_\nu - m)$: $0 = (\gamma^\nu\partial_\nu - m)\,(\gamma^\mu\partial_\mu + m)\,\psi^{\pm(c)}$

$= \left(\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu - m^2\right)\psi^{\pm(c)} = \left(\frac{1}{2}\{\gamma^\nu,\gamma^\mu\}\partial_\nu\partial_\mu - m^2\right)\psi^{\pm(c)} = \left(g^{\mu\nu}\partial_\nu\partial_\mu - m^2\right)\psi^{\pm(c)}$

using anticommutation relations for γ^μ on page 44.

2.4.5 Dirac field equal time anticommutation relations

Projection operators from u, v: $u_{l\ \sigma}(\boldsymbol{p})\overline{u}_{l'\ \sigma}(\boldsymbol{p}) = \frac{1}{2p^0}\left(-I\gamma^\mu p_\mu + m\right)_{ll'}$, $v_{l\ \sigma}(\boldsymbol{p})\overline{v}_{l'\ \sigma}(\boldsymbol{p}) = \frac{1}{2p^0}\left(-I\gamma^\mu p_\mu - m\right)_{ll'}$.

Define $N_{ll'}(p) = u_{l\ \sigma}(\boldsymbol{p})\ \overline{u}_{l'\ \sigma}(\boldsymbol{p}) = \frac{m}{p^0}[D(L(p))u_\sigma(0)]_l\ [u^\dagger_\sigma(0)D^\dagger(L(p))\beta]_{l'}$ from $\boldsymbol{p}$ dependence of spin $\frac{1}{2}$ u, v on page 49.

So from pseudo-unitarity of Lorentz transformation on page 47, $N(p) = \frac{m}{p^0}D(L(p))N(0)D^{-1}(L(p))$.

Explicit calculation from the form of u, v on page 50 gives $N(0) = \frac{1}{2}(\beta + \mathbf{1})$ which is true in any representation.

So $N(p) = \frac{1}{2p^0}D(L(p))\left(I\gamma^0 m + m\right)D^{-1}(L(p))$, then use reference boost on page 46.

Equal time anticommutation relations: $[\psi_l(\boldsymbol{x},t), \psi^\dagger_{l'}(\boldsymbol{y},t)]_+ = \delta_{ll'}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$.

Define $R_{ll'} = [\psi_l(\boldsymbol{x},t), \psi^\dagger_{l'}(\boldsymbol{y},t)]_+ = \int \frac{d^3p}{(2\pi)^3}e^{I\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\left[u_{l\ \sigma}(\boldsymbol{p})[\overline{u}(\boldsymbol{p})\beta]_{l'\ \sigma} + v_{l\ \sigma}(-\boldsymbol{p})[\overline{v}(-\boldsymbol{p})\beta]_{l'\ \sigma}\right]$,

using Dirac field and anticommutation relations for spin $\frac{1}{2}$ on page 49. Then in "v" term, take $\boldsymbol{p} \to -\boldsymbol{p}$.

From projection operators from u, v, $R = \int \frac{d^3p}{(2\pi)^3 2p^0}e^{I\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\left[\left(I\gamma^0p^0 - I\boldsymbol{\gamma}\cdot\boldsymbol{p} + m\right) + \left(I\gamma^0p^0 + I\boldsymbol{\gamma}\cdot\boldsymbol{p} - m\right)\right]\beta = \mathbf{1}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$.

2.5 External symmetries: bosons

Scalar boson field: $\psi(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}(2p^0)^{\frac{1}{2}}} \left[e^{Ip\cdot x} a(\boldsymbol{p}) + e^{-Ip\cdot x} a^{c\dagger}(\boldsymbol{p})\right]$.

From general form of irreducible field on page 36, where $u(\boldsymbol{p}) = u(0)$ and $v(\boldsymbol{p}) = u(\boldsymbol{p})$, absorb overall $u(0)$ into field.

Vector boson field $(\frac{1}{2}, \frac{1}{2})$, **spin** 1: $\psi^\mu(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}(2p^0)^{\frac{1}{2}}} \left[e^{Ip\cdot x} u^\mu_\sigma(\boldsymbol{p}) a_\sigma(\boldsymbol{p}) + e^{-Ip\cdot x} v^\mu_\sigma(\boldsymbol{p}) a^{c\dagger}_\sigma(\boldsymbol{p})\right]$, $u^0_\sigma(0) = v^0_\sigma(0) = 0$.

Conditions on u, v on page 32 (using $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{J}$) give e.g. $(J^{(j)}_k)_{\sigma'\sigma} u^\mu_{\sigma'} = (J_k)^\mu{}_\nu u^\nu_\sigma$, so $(J^{(j)}_k J^{(j)}_k)_{\sigma'\sigma} u^\mu_{\sigma'} = (J_k J_k)^\mu{}_\nu u^\nu_\sigma$.

But $(J^{(j)}_k J^{(j)}_k)_{\sigma'\sigma} = j(j+1)\delta_{\sigma'\sigma}$ and $(J_k J_k)^i{}_j = 2\delta^i{}_j$, $(J_k J_k)^0{}_\mu = 0$ from Lorentz transformation of 4-vectors on page 22,

so $j(j+1)u^i_\sigma(0) = 2u^i_\sigma(0)$, $j(j+1)u^0_\sigma(0) = 0$, i.e. $j = 0$ and $u^i_\sigma(0) = 0$ or $j = 1$ and $u^0_\sigma(0) = 0$.

Projection operator for vector boson $(\frac{1}{2},\frac{1}{2})$**, spin** 1: $u^\mu_\sigma(\boldsymbol{p})u^{\nu*}_\sigma(\boldsymbol{p}) = v^\mu_\sigma(\boldsymbol{p})v^{\nu*}_\sigma(\boldsymbol{p}) = g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}$.

From conditions on u, v, page 32, $(J_3)^j{}_i\, u^i_0(0) = \left(J^{(1)}_3\right)_{\sigma' 0} u^j_{\sigma'}(0)$. But $\left(J^{(1)}_3\right)_{\sigma' 0} = 0$, $(J_3)^j{}_i = -I\epsilon_{3ji}$, so $u^1_0(0) = u^2_0(0) = 0$,

i.e. $u_0(0) = (2m)^{-1/2}(0,0,1,0)$ by choice of normalization. $J^{(1)}_1 \pm I J^{(1)}_2$ gives $u_{\pm 1}(0) = (2m)^{-1/2}(\pm 1, -I, 0, 0)/\sqrt{2}$.

Similarly, $v_\sigma(0) = u^*_\sigma(0)$. Explicitly show that $u^\mu_\sigma(0)u^{\nu*}_\sigma(0)$ projects onto space orthogonal to time direction,

i.e. $u^\mu_\sigma(0)u^{\nu*}_\sigma(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. But $u^\mu_\sigma(\boldsymbol{p}) = L^\mu{}_\nu(\boldsymbol{p})u^\nu_\sigma(0)$ ($\boldsymbol{p}$ **dependence of** u, v, page 26),

so $\Pi^\mu{}_\nu(\boldsymbol{p}) = u^\mu_\sigma(\boldsymbol{p})(u^*_\nu)_\sigma(\boldsymbol{p}) = L^\nu{}_\rho(p)L_\nu{}^\alpha(p)\Pi^\rho{}_\alpha(\mathbf{0})$ projects onto space orthogonal to p,

i.e. $\Pi^2(\boldsymbol{p}) = \Pi(\boldsymbol{p})$ because $\Pi^2(\mathbf{0}) = \Pi(\mathbf{0})$ and $p\Pi(\boldsymbol{p})q = 0$ for any q:

$p_\mu \Pi^\mu{}_\nu(\boldsymbol{p})q^\nu = \underline{p_\nu L^\nu{}_\rho(p)}\Pi^\rho{}_\alpha(\mathbf{0})L_\nu{}^\alpha(p)q^\nu$ But $\underline{L^\nu{}_\rho(p)p_\nu} = (L^{-1})_\rho{}^\nu(p)p_\nu = k_\rho$, and $k_\rho\Pi^\rho{}_\alpha(\mathbf{0}) = 0$ because $k_i = 0$.

Finally, write $q = \alpha p + \beta\Pi^\mu{}_\nu(p)q^\nu$ and show $\beta = 1$:

$q^2 = \alpha^2p^2 + \beta^2\Pi^\mu{}_\nu(p)q^\nu\Pi_\mu{}^\rho(p)q_\rho$, i.e. $\Pi^\mu{}_\nu(p)q^\nu\Pi_\mu{}^\rho(p)q_\rho = \underline{\frac{1}{\beta^2}(q^2-\alpha^2p^2)}$.

But $\Pi^\mu{}_\nu(p)q^\nu\Pi_\mu{}^\rho(p)q_\rho = q^\nu\Pi_\nu{}^\rho(p)q_\rho = q^\nu(q_\nu - \alpha p_\nu)\frac{1}{\beta} = \underline{\frac{1}{\beta}(q^2-\alpha^2p^2)}$, i.e. $\frac{1}{\beta^2} = \frac{1}{\beta}$.

Projection operator shows there is problem for $m \to 0$. From allowed helicities for fields in given

(A, B) representation on page 34, can't construct $(\frac{1}{2}, \frac{1}{2})$ 4-vector field, where helicity $\sigma = 0$,

from massless helicity $\sigma = \pm 1$ particle. But can construct 4-component field:

Massless helicity ± 1 field: $A^\mu(x) = \int D^3p \left[e^{Ip\cdot x} u^\mu_\sigma(\boldsymbol{p}) a_\sigma(\boldsymbol{p}) + e^{-Ip\cdot x} v^\mu_\sigma(\boldsymbol{p}) a^{c\dagger}_\sigma(\boldsymbol{p})\right]$, where $\sigma = \pm 1$.

Lorentz transformation of massless helicity ± 1 polarization vector: $e^{-I\theta\sigma} u_\sigma(\boldsymbol{p}) = \Lambda^{-1} u_\sigma(\boldsymbol{\Lambda p}) + \omega(W, \boldsymbol{k}) p$.

Simplest approach: take p dependence and rotation of u on page 34 to be true, gives $u^\mu_\sigma(\boldsymbol{k}) \propto (1, I\sigma, 0, 0)$.

Then M, N transformation on page 34 cannot be true, in fact $M u_\sigma \propto (0, 0, 1, 1) \propto k$ and likewise for N.

Then $D(W) u_\sigma(\boldsymbol{k}) = (1 + I\theta J_3 + I\mu M + I\nu N) u_\sigma(\boldsymbol{k}) = u_\sigma(\boldsymbol{k})(1 + I\sigma\theta) + (\mu + I\sigma\nu)(0, 0, 1, 1) = \ldots = e^{I\theta\sigma}[u_\sigma(\boldsymbol{k}) + \omega(W, \boldsymbol{k}) k]$.

Since $D(\Lambda) = \Lambda$, multiplying this from the left by $e^{-I\theta\sigma} \Lambda^{-1} D(\Lambda L(p)) D(W^{-1}) = e^{-I\theta\sigma} \Lambda^{-1} L(\Lambda p)$ gives result.

Lorentz transformation of massless helicity ± 1 field: $U(\Lambda) A^\mu(x) U^\dagger(\Lambda) = \Lambda_\nu{}^\mu A^\nu(\Lambda x) + \partial^\mu \alpha(x)$.

Use Lorentz transformation of $u_{\mu\ \sigma}$ above in $U(\Lambda) A^\mu(x) U^\dagger(\Lambda) = \int D^3p \left[e^{Ip\cdot x} u^\mu_\sigma(\boldsymbol{p}) e^{-I\theta(\Lambda,p)\sigma} a_\sigma(\boldsymbol{\Lambda p}) + e^{-Ip\cdot x} v^\mu_\sigma(\boldsymbol{p}) e^{I\theta(\Lambda,p)\sigma} a^{c\dagger}_\sigma(\boldsymbol{\Lambda p})\right]$.

As for Poincaré transformation for fields on page 25, up to gauge transformation.

This implies $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is Lorentz covariant, as expected:

it is antisymmetric, i.e. is $(1, 0)$ or $(0, 1)$ if $F^{\mu\nu} = \pm\frac{I}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$, so from $\sigma = \pm(A - B)$, can have $\sigma = \pm 1$.

Brehmstrahlung: Adding emission of massless helicity $\pm j$ boson with momentum $q \simeq 0$

to process with particles n with momenta p_n modifies amplitude by factor $\propto u_{\mu_1\mu_2\ldots\mu_j\ \sigma}(\mathbf{q}) \sum_n \eta_n \frac{g_n p_n^{\mu_1} p_n^{\mu_2} \ldots p_n^{\mu_j}}{p_n \cdot q}$,

where g_n is coupling of these bosons to fermion n, and $\eta_n = \pm 1$ for outgoing / incoming particles.

Lorentz invariance condition: $q_{\mu_1} \sum_n \eta_n \frac{g_n p_n^{\mu_1} p_n^{\mu_2} \ldots p_n^{\mu_j}}{p_n \cdot q} = 0$.

Can show this for massless helicity ± 1 boson: Lorentz transformation of polarization vector on page 55

implies amplitude not Lorentz invariant unless this is true.

Einstein's principle of equivalence: Helicity ± 2 bosons have identical coupling to all fermions.

For soft emssion of graviton from process involving multiple fermions of momentum p_n,

Lorentz invariance condition reads $\sum_n \eta_n g_n p_n^{\mu_2} = 0$. But momentum conservation is $\sum_n \eta_n p_n^{\mu_2} = 0$, so g_n same for all particles.

Constraint on particle spins: Massless particles must have helicity ≤ 2 and ≥ -2.

Lorentz invariance condition can be written $\sum_n \eta_n g_n p_n^{\mu_2} \ldots p_n^{\mu_j} = 0$.

For $j > 2$ this overconstrains $2 \to 2$ processes, since momentum conservation alone $\Longrightarrow$ it depends on scattering angle only.

2.6 The Lagrangian Formalism

2.6.1 Generic quantum mechanics

Lagrangian formalism is natural framework for QM implementation of symmetry principles.

Can be applied to *canonical* fields (e.g. Standard Model):

Fields $\psi_l(\boldsymbol{x}, t)$ behave as canonical coordinates, i.e. with conjugate momenta $p_l(\boldsymbol{x}, t)$ such that

$[\psi_l(\boldsymbol{x}, t), p_{l'}(\boldsymbol{y}, t)]_\mp = I\delta^3(\boldsymbol{x} - \boldsymbol{y})\delta_{ll'}$ and $[\psi_l(\boldsymbol{x}, t), \psi_{l'}(\boldsymbol{y}, t)]_\mp = [p_l(\boldsymbol{x}, t), p_{l'}(\boldsymbol{y}, t)]_\mp = 0$ as usual in QM.

In practice, find suitable $p_l(\boldsymbol{x}, t)$ by explicit calculation of $[\psi_l(x), \psi^\dagger_{l'}(y)]_\mp$.

Lagrangian formalism: Define *action* $\mathscr{A}[\psi, \dot{\psi}] = \int_{-\infty}^{\infty} dt L[\psi(t), \dot{\psi}(t)]$, where L is *Lagrangian*.

Laws of physics obeyed when action is stationary, i.e. coordinates obey field equations $\frac{d}{dt}\frac{\partial L}{d\dot{\psi}_l} = \frac{\partial L}{\partial \psi_l}$.

Alternatively, define $p_l = \frac{\delta L}{\delta \dot{\psi}_l}$ and *Hamiltonian* $H[\psi(t), p(t)] = \int d^3x \; p_l(\boldsymbol{x}, t)\dot{\psi}_l(\boldsymbol{x}, t) - L[\psi(t), \dot{\psi}(t)]$.

Then action stationary when coordinates obey field equations $\dot{p}_l = -\frac{\delta H}{\delta \psi_l}$.

2.6.2 Relativistic quantum mechanics

Take $L[\psi(t), \dot{\psi}(t)] = \int d^3x \mathscr{L}(\psi(x), \partial_\mu \psi(x))$,

where *Lagrangian density* $\mathscr{L}(x)$ is scalar so $\mathscr{A} = \int d^4x \mathscr{L}(x)$ is Lorentz invariant.

In practice, determine $\mathscr{L}$ from classical field theory, e.g. electrodynamics,

then $\mathscr{L}(\psi(x), \partial_\mu \psi(x)) = p_l(x)\dot{\psi}_l(x) - \mathscr{H}(\psi(x), p(x))$ and p_l from $\mathscr{L}$: $p_l(x) = \frac{\partial \mathscr{L}(x)}{\partial \dot{\psi}_l(x)}$.

Stationary $\mathscr{A}$ requirement gives field equations $\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi_l)} = \frac{\partial \mathscr{L}}{\partial \psi_l}$ (*Euler-Lagrange equations*),

e.g. Klein-Gordon equation for free spin 0 field, Dirac equation for free spin $\frac{1}{2}$ field etc.

(some definition of derivative with respect to operator field ψ_l must be given here).

$\mathscr{A}$ must be real. Let $\mathscr{A}$ depend on N real fields. Stationary real and imaginary parts of $\mathscr{A} \to 2N$ field equations.

Noether's theorem: Symmetries imply conservation:

$\mathscr{A}$ invariant under $\psi_l(x) \to \psi_l(x) + I\alpha\mathscr{F}_l[\psi; x] \to$ conserved current $J^\mu(x)$, $\partial_\mu J^\mu = 0$, for stationary $\mathscr{A}$.

If α made dependent on x, $\mathscr{A}$ no longer invariant. But change must be $\underline{\delta\mathscr{A} = \int d^4x J^\mu \partial_\mu \alpha}$

so that $\delta\mathscr{A} = 0$ when α constant: Ignore l label, use discrete lattice in 1-D of N points so e.g. $\psi(x) \to \psi_i$ and $\delta\mathscr{A} = \sum_i \frac{\partial\mathscr{A}}{\partial\psi_i} F_i \alpha_i$.

and write $\frac{\partial\mathscr{A}}{\partial\psi_i} F_i = (J_i - J_{i+1})$ (no sum) for $i = 1, \ldots, N-1$, and $\frac{\partial\mathscr{A}}{\partial\psi_N} F_N = (J_N - K)$.

But symmetry of action gives $\sum_i \frac{\partial\mathscr{A}}{\partial\psi_i} F_i = 0$, so $K = J_{N+1}$. (Indices are cyclical, i.e. $X_{N+1} = X_1$, $X_0 = X_N$ etc.)

Thus $\delta\mathscr{A} = \sum_i \frac{\partial\mathscr{A}}{\partial\psi_i} F_i \alpha_i = \sum_i (J_i - J_{i+1})\alpha_i = \sum_i J_i(\alpha_i - \alpha_{i-1})$. Returning to continuous case, $\delta\mathscr{A} = \int dx \frac{\delta\mathscr{A}}{\delta\psi(x)} F(x) = \int dx J(x) \frac{d\alpha}{dx}$.

Thus $\underline{\delta\mathscr{A} = -\int d^4x \alpha \partial_\mu J^\mu}$. Now take $\mathscr{A}$ stationary: $\delta\mathscr{A} = 0$ even though α depends on x, so $\partial_\mu J^\mu = 0$.

Explicit result for Noether current when $\mathscr{L}$ is invariant: $\boxed{J^\mu = \frac{\partial\mathscr{L}}{\partial\partial_\mu\psi_l}\mathscr{F}_l}$.

$\mathscr{A} = \int d^4x \mathscr{L}$ so, for x-dependent α, $\delta\mathscr{A} = I \int d^4x \left[\frac{\partial\mathscr{L}}{\partial\psi_l}\mathscr{F}_l\alpha + \frac{\partial\mathscr{L}}{\partial\partial_\mu\psi_l}\partial_\mu(\mathscr{F}_l\alpha)\right]$. But $0 = \delta\mathscr{L} = \left(\frac{\partial\mathscr{L}}{\partial\psi_l}\mathscr{F}_l + \frac{\partial\mathscr{L}}{\partial\partial_\mu\psi_l}\partial_\mu\mathscr{F}_l\right) I\alpha$,

so $\delta\mathscr{A} = \int d^4x \underline{I\frac{\partial\mathscr{L}}{\partial\partial_\mu\psi_l}\mathscr{F}_l}\partial_\mu\alpha$, then compare with $\delta\mathscr{A} = \int d^4x \underline{J^\mu}\partial_\mu\alpha$ (from above).

Conserved charge: $\boxed{F = \int d^3x J^0}$ obeys $\frac{dF}{dt} = 0$.

Use $\partial_\mu J^\mu = 0$ and $\int d^3x \nabla\mathbf{J} = 0$ because $\mathbf{J}$ vanishes at infinity.

Scalar boson $(0,0)$: $\mathscr{L}_{\text{scalar}} = -\frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}m^2\psi^2$.

Scalar boson field on page 53 implies $\left[\psi(\boldsymbol{x},t),\dot{\psi}(\boldsymbol{y},t)\right]_- = \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$, i.e. $p=\dot{\psi}$, consistent with p_l from $\mathscr{L}$ on page 58.

Field equations on page 58 give Klein-Gordon equation $\left(\partial^2 - m^2\right)\psi = 0$ as required. Note ψ is a single operator.

Dirac fermion $(\frac{1}{2},0)+(0,\frac{1}{2})$: $\mathscr{L}_{\text{Dirac}} = -\overline{\psi}\left(\gamma^\mu\partial_\mu + m\right)\psi$.

Recall ψ is a column of 4 operators, and covariant quantities on page 47.

Recall equal time anticommutation relations on page 52, $[\psi_l(\boldsymbol{x},t),\psi^\dagger_{l'}(\boldsymbol{x},t)]_+ = \delta_{ll'}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$, so $p=\psi^\dagger$,

consistent with p_l from $\mathscr{L}$ on page 58. Field equations give Dirac equation $\left(\gamma^\mu\partial_\mu + m\right)\psi = 0$ as required.

Vector boson $(\frac{1}{2},\frac{1}{2})$, **spin** 1: $\mathscr{L}_{\text{spin 1 vector}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2\psi_\mu\psi^\mu$, where $F_{\mu\nu} = \partial_\mu\psi_\nu - \partial_\nu\psi_\mu$.

From vector boson field and projection operator for vector boson $(\frac{1}{2},\frac{1}{2})$, spin 1, on pages 53 and 54,

$\left[\psi_i(\boldsymbol{x},t),\dot{\psi}_j(\boldsymbol{y},t)+\partial_j\psi^0(\boldsymbol{y},t)\right]_- = \delta_{ij}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$, i.e. conjugate momentum to ψ_i is $p_i = \dot{\psi}_i + \partial_i\psi^0 = F^{i0}$,

consistent with p_l from $\mathscr{L}$ on page 58. ψ_0 is auxilliary field because $p_0 = \frac{\partial\mathscr{L}}{\partial\dot{\psi}^0} = F_{00} = 0$.

Also $\partial_\mu\psi^\mu = 0$ and Klein-Gordon equation $\left(\partial^2 - m^2\right)\psi^\mu = 0$, which is found from field equations on page 58.

2.7 Path-Integral Methods

Follows from Lagrangian formalism. Assume H is quadratic in the p_l.

Gives direct route from Lagrangian to calculations, all symmetries manifestly preserved along the way.

Can work in simpler classical limit then return to QM later.

Result is that bosons described by ordinary numbers, fermions by Grassmann variables.

LSZ reduction gives S-matrix from vacuum matrix elements of-time ordered product of functions of fields,

given by path integral as $\frac{\langle 0,\ \mathrm{out}|T\left\{\psi_{l_A}(x_A),\psi_{l_B}(x_B),\ldots\right\}|0,\ \mathrm{in}\rangle}{\langle 0,\ \mathrm{out}|0,\ \mathrm{in}\rangle} = \frac{\int\prod_{x,l} d\psi_l(x)\psi_{l_A}(x_A)\psi_{l_B}(x_B)\ldots e^{I\mathscr{A}[\psi]}}{\int\prod_{x,l} d\psi_l(x)e^{I\mathscr{A}[\psi]}}$.

Contribution mostly from field configurations for which $\mathscr{A}$ is minimal, i.e. fluctuation around classical result.

Noether's theorem again (see page 59): $\mathscr{A}$ invariant under $\psi_l(x) \to \psi_l(x) = \psi_l(x) + I\alpha\mathscr{F}_l[\psi; x]$.

So if α dependent on x, $\int\prod_{x,l} d\psi_l(x)\exp[I\mathscr{A}] \to \int\prod_{x,l} d\psi_l(x)\exp\left[I\left(\mathscr{A} - \int d^4x\partial_\mu J^\mu(x)\alpha(x)\right)\right]$

assuming measure $\prod_{x,l} d\psi_l(x)$ invariant. This is just change of variables, so $\langle\partial_\mu J^\mu(x)\rangle = 0$.

2.8 Internal symmetries

Consider unitary group representations.

Unitary U(N): elements can be represented by $N \times N$ unitary matrices U ($U^\dagger U = \mathbf{1}$).

Dimension d(U(N))$= N^2$.

$2N^2$ degrees of freedom in complex $N \times N$ matrix, $U^\dagger U = \mathbf{1}$ is N^2 conditions

or N^2 Hermitian $N \times N$ matrices: N diagonal reals, $N^2 - N$ off-diagonal complexes but lower half conjugate to upper.

Special unitary SU(N): same as U(N) but U's have unit determinant ($\det(U) = 1$).

Thus $\mathrm{tr}[t_i] = 0$, i.e. group is semi-simple.

d(SU(N))$= N^2 - 1$.

Fundamental representation denoted $\boldsymbol{N}$.

Normalization of fundamental representation: $\mathrm{tr}[t_i t_j] = \frac{1}{2}\delta_{ij}$ (i.e. $C(\boldsymbol{N}) = \frac{1}{2}$).

U(1) (Abelian group): elements can be represented by phase $e^{Iq\alpha}$.

One generator: the real number q (the *charge*).

SU(2): Fundamental representation denoted **2**, spin $\frac{1}{2}$ representation of rotation group.

SU(2) is actually the *universal covering group* of rotation group. 3 generators $t_i = \frac{\sigma_i}{2}$, $[t_i, t_j] = I\epsilon_{ijk}t_k$.

Adjoint representation denoted **3**. $C(\mathbf{3}) = 2$.

$$\epsilon_{jki}\epsilon_{lki} = (d(\mathrm{SU}(2)) - 1)\delta_{jl} = 2\delta_{jl}.$$

2 representation is real, $\mathbf{2} = \overline{\mathbf{2}}$ (i.e. $-\frac{\sigma_i^*}{2} = U\frac{\sigma_i}{2}U^\dagger$, pseudoreal), and $\mathfrak{g}_{\alpha\beta} = \epsilon_{\alpha\beta}$ and $\delta_{\alpha\beta}$.

SU(3): 8 generators $\frac{\lambda_i}{2}$, structure constants f_{ijk}.

Fundamental representation **3**: λ_i are 3×3 *Gell-Mann matrices*. Adjoint representation **8**. $C(\mathbf{8}) = 3$.

3 representation is complex, $\mathbf{3} \neq \overline{\mathbf{3}}$.

Example: $\mathbf{3} \times \overline{\mathbf{3}} = \mathbf{8} + \mathbf{1}$, i.e. quark and antiquark can be combined to behave like gluon or colour singlet.

2.8.1 Abelian gauge invariance

Global gauge invariance: Consider complex fermion / boson field $\psi_l(x)$, arbitrary spin.

Each operator in $\mathscr{L}_{\text{free}}$ is product of $(\partial_\mu)\psi_l$ with $(\partial^\mu)\psi_l^\dagger$,

invariant under U(1) transformation $\psi_l \to e^{Iq\alpha}\psi_l$ (whence $\partial_\mu\psi_l \to e^{Iq\alpha}\partial_\mu\psi_l$)

if α independent of spacetime coords.

Local gauge invariance: Find $\mathscr{L}$ invariant when $\alpha = \alpha(x)$. Leads to renormalizable interacting theory.

In $\mathscr{L}_{\text{free}}$, $\partial_\mu\psi_l \to \partial_\mu e^{Iq\alpha}\psi_l = e^{iq\alpha}\left[\partial_\mu\psi_l + \underline{Iq(\partial_\mu\alpha)\psi_l}\right] \neq e^{Iq\alpha}\partial_\mu\psi_l$, $\therefore$ replace ∂_μ by 4-vector "derivative" D_μ,

such that D_μ gauge transformation $D_\mu\psi_l \to e^{Iq\alpha}D_\mu\psi_l$. Then $D_\mu\psi_l(D^\mu\psi_l)^\dagger$ invariant.

Simplest choice: $D_\mu - \partial_\mu$ is 4-component field:

Covariant derivative: $D_\mu = \partial_\mu - IqA_\mu(x)$.

Gauge transformation: $A_\mu \to A_\mu + \partial_\mu\alpha$ whenever $\psi_l \to e^{Iq\alpha}\psi_l$.

Write transformation as $A_\mu \to A'_\mu(\alpha)$. Require $D_\mu\psi_l \to D'_\mu e^{iq\alpha}\psi_l = e^{iq\alpha}D_\mu\psi_l$,

i.e. $e^{Iq\alpha}\left[\partial_\mu\psi_l - Iq(\partial_\mu\alpha)\psi_l - IqA'_\mu\psi_l\right] = e^{Iq\alpha}\left[\partial_\mu\psi_l - IqA_\mu\psi_l\right]$, so $A'_\mu = A_\mu + \partial_\mu\alpha$.

Use D_ν to find invariant (free) Lagrangian for A_μ, quadratic in $(\partial_\nu)A_\mu$:

From D_μ gauge transformation, $D_\mu D_\nu \ldots \psi_l \to e^{Iq\alpha} D_\mu D_\nu \ldots \psi_l$. Products $D_\mu D_\nu \ldots$ contain spurious ∂_ρs, but

$F_{\mu\nu}$ **from** D_μ: $qF_{\mu\nu} = I\,[D_\mu, D_\nu]$, where electromagnetic field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

$$[D_\mu, D_\nu]\,\psi_l = \big(\underbrace{[\partial_\mu, \partial_\nu]}_{=0} + Iq([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu]) - q^2\underbrace{[A_\mu, A_\nu]}_{=0}\big)\psi_l.$$

$F_{\mu\nu}$ is gauge invariant.

$F_{\mu\nu}\psi_l \to F'_{\mu\nu} e^{Iq\alpha}\psi_l = e^{Iq\alpha} F_{\mu\nu}\psi_l$, i.e. $F'_{\mu\nu} = F_{\mu\nu}$, or $F_{\mu\nu} \to F_{\mu\nu}$. Also check explicitly from $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Conversely, choose A^μ to be massless helicity ± 1 field, whose Lorentz transformation on page 55

implies Lorentz invariant free Lagrangian for A^μ must be gauge invariant.

$F_{\mu\nu}$ in representation of U(1): Since $F_{\mu\nu} \to F_{\mu\nu}$, $F_{\mu\nu}$ transforms in adjoint representation of U(1).

Example: QED Lagrangian for fermions: $\mathscr{L}_{\text{Dirac, QED}} = \underbrace{-\overline{\psi}\,(\gamma^\mu D_\mu + m)\,\psi}_{\mathscr{L}_{\text{Dirac, free}} + Iq\overline{\psi}\gamma^\mu A_\mu\psi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

Interactions due to $Iq\overline{\psi}\gamma^\mu A_\mu\psi$. Most general Lagrangian locally gauge invariant under U(1) ($\psi \to e^{Iq\alpha}\psi$, $A_\mu \to A_\mu + \partial_\mu\alpha$),

assuming P, T invariance and no mass dimension > 4 terms (Wilson: no contribution).

2.8.2 Non-Abelian gauge invariance

Global gauge invariance: Each term in $\mathscr{L}_{\text{free}}$ proportional to $(\partial^\mu)\psi_{l\ \gamma}(\partial_\mu)\psi^\dagger_{l'\ \gamma}$, $\gamma = 1, \ldots, N$.

Then $\mathscr{L}_{\text{free}}$ invariant under $\psi_{l\ \gamma} \to U_{\gamma\delta}\psi_{l\ \delta}$, where $U = \exp[I\alpha_i t_i]$, α_i spacetime independent.

So ψ_γ is in fundamental representation of group G =SU(N) formed by matrices $U_{\gamma\delta}$, $i = 1, \ldots, d(G)$.

Local gauge invariance: Spacetime derivatives in Lagrangian appear as

Covariant derivative: $D_\mu = \partial_\mu - IA_\mu(x)$ with $A^\mu = A^\mu_i t_i$,

t_i contain couplings, A^μ_i for $i = 1, \ldots, d(G)$ are (for) massless helicity ± 1 gauge fields.

To achieve $D_\mu\psi \to UD_\mu\psi$, require

Transformation of covariant derivative: $D_\mu \to UD_\mu U^\dagger$, which requires

Transformation of gauge fields: $A_\mu \to UA_\mu U^\dagger - I\,(\partial_\mu U)\,U^\dagger$.

Non-Abelian field strength: $T_i F^i_{\mu\nu} = F_{\mu\nu} = I[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - I[A_\mu, A_\nu]$.

$F_{\mu\nu}$ in adjoint representation: $F_{\mu\nu} \to U F_{\mu\nu} U^\dagger$.

In infinitesimal case, $F^i t_{i\ \alpha\beta} \to F^i \left[(1 + I\alpha_k t_k) t_i (1 - I\alpha_k t_k)\right]_{\alpha\beta} = F^i \left[t_i + I\alpha_k \left[t_k, t_i\right]\right]_{\alpha\beta}$

$= F^i \left[t_i + I\alpha_k (IC_{kij}) t_j\right]_{\alpha\beta} = F^i \left[t_{i\ \alpha\beta} + I\alpha_k (t^k_{ji}) t_{j\ \alpha\beta}\right] = F^i t_{j\ \alpha\beta} \left[\delta_{ji} + I\alpha_k (t^k_{ji})\right] = F^i t_{j\ \alpha\beta} U_{ji} = U_{ij} F_j t_{i\ \alpha\beta}$, i.e. $F^i \to U_{ij} F^j$.

Example: QCD Lagrangian for fermions: $\mathscr{L}_{\text{Dirac, QCD}} = -\overline{\psi}_\alpha \left(\gamma^\mu D_{\alpha\beta\ \mu} + m\right) \psi_\beta - \frac{1}{4} F^i_{\mu\nu} F^{i\ \mu\nu}$.

More general result is $-\frac{1}{4} g_{ij} F^i_{\mu\nu} F^{j\ \mu\nu}$, but can always diagonalize and rescale so $g_{ij} \to \delta_{ij}$.

2.9 The Standard Model

Symmetry of vacuum is G=SU(3)$_{\text{colour}}\times$U(1)$_{\text{e.m.}}$ gauge group.

SM: At today's collider energies, some "hidden" (broken) symmetries become apparent:

G=SU(3)$_{\text{colour}}\times$SU(2)$_{\text{weak isospin}}\times$U(1)$_{\text{weak hypercharge}}$.

Table 2.9.1: SM fermions and their SU(3)$_C\times$SU(2)$_L\times$U(1)$_Y$ representations, written as (SU(3)$_C$ rep.,SU(2)$_L$ rep.,U(1)$_Y$ hypercharge = generator / [coupling $\equiv Y$]). The SU(3)$_C$ charges (3 for quarks, none for leptons) are not shown but, since SU(2)$_L$ is broken, particles differing only in T_3 (component of weak isospin SU(2)$_L$) are shown explicitly, namely u_L / ν_e ($T_3 = 1/2$) and d_L / e_L ($T_3 = -1/2$). Recall $\psi_{L/R} = \frac{1}{2}(1 \pm \gamma_5)\psi$. Note e.g. u_L annihilates u_L^- and creates u_R^+, and ν_e is left-handed.

Names	Label	Representation under SU(3)$_C\times$SU(2)$_L\times$U(1)$_Y$
Quarks	$\mathscr{Q}_L = (u_L, d_L)$	$(\mathbf{3}, \mathbf{2}, \frac{1}{6})$
	$u_R^\dagger$	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})$
	$d_R^\dagger$	$(\overline{\mathbf{3}}, \mathbf{1}, \frac{1}{3})$
Leptons	$\mathscr{E}_L = (\nu_e, e_L)$	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$
	$e_R^\dagger$	$(\mathbf{1}, \mathbf{1}, 1)$

Gauge fields are written as

$$\text{SU(3):}\quad \boxed{\widehat{g}_\mu = g_{i\mu}\; g_s \tfrac{\lambda_i}{2}}$$

$$\text{SU(2):}\quad \boxed{\widehat{A}_\mu = A_{i\mu}\; g \tfrac{\sigma_i}{2}}$$

$$\text{U(1):}\quad \boxed{\widehat{B}_\mu = B_\mu\; g' Y}\,.$$

We have only discussed the "1st generation" of fermions, in fact $(\mathscr{E}_L, e_R^\dagger, \mathscr{Q}_L, u_R^\dagger, d_R^\dagger)^K$, $K = 1, 2, 3$.

where $(e^1, e^2, e^3) = (e, \mu, \tau)$, $(\nu_e^1, \nu_e^2, \nu_e^3) = (\nu_e, \nu_\mu, \nu_\tau)$, $(u^1, u^2, u^3) = (u, c, t)$ and $(d^1, d^2, d^3) = (d, s, b)$.

Allow mixing between particles of different generations with same transformation properties.

From Table 2.9.1, can construct the full Lagrangian by including all renormalizable invariant $(\mathbf{1}, \mathbf{1}, 0)$ terms.

These are all possible terms of form $\overline{\psi}^K \gamma^\mu D_\mu \psi^K$:

$$\mathscr{L}_{\text{quark}} = \overline{\mathscr{Q}}_L^K \gamma^\mu [\partial_\mu - I(\widehat{g}_\mu + \widehat{A}_\mu + \widehat{B}_\mu)] \mathscr{Q}_L^K \quad + \quad \overline{u}_R^K \gamma^\mu [\partial_\mu - I(\widehat{g}_\mu + \widehat{B}_\mu)] u_R^K \quad + \quad \overline{d}_R^K \gamma^\mu [\partial_\mu - I(\widehat{g}_\mu + \widehat{B}_\mu)] d_R^K.$$

$$\mathscr{L}_{\text{lepton}} = \overline{\mathscr{E}}_L^K \gamma^\mu [\partial_\mu - I(\widehat{A}_\mu + \widehat{B}_\mu)] \mathscr{E}_L^K \quad + \quad \overline{e}_R^K \gamma^\mu [\partial_\mu - I \widehat{B}_\mu] e_R^K.$$

More general $\overline{\psi}^K \gamma^\mu D_\mu R^{KM} \psi^M$ for some constant matrix R not allowed, gives terms $\overline{\psi}^K \gamma^\mu \partial_\mu \psi^M$ for $K \neq M$.

$$\mathscr{L}_{\text{spin 1}} = -\tfrac{1}{4} F^i_{\mu\nu}(\widehat{g}) F^{i\ \mu\nu}(\widehat{g}) - \tfrac{1}{4} F^i_{\mu\nu}(\widehat{A}) F^{i\ \mu\nu}(\widehat{A}) - \tfrac{1}{4} F_{\mu\nu}(\widehat{B}) F^{\mu\nu}(\widehat{B}).$$

Recall only real part of $\mathscr{L}$ to be taken.

2.9.1 Higgs mechanism

Mass terms $m\overline{\psi}_{L/R}\psi_{R/L}$ are all $(\mathbf{1},\mathbf{2},\pm\frac{1}{2}) \to$ violate gauge symmetry and thus renormalizability.

Instead introduce Yukawa coupling $\lambda\phi_H\overline{\psi}_{L/R}\psi_{R/L}$ which is $(\mathbf{1},\mathbf{1},0)$, i.e. invariant (thus renormalizable),

so ϕ_H is $(\mathbf{1},\overline{\mathbf{2}}=\mathbf{2},\frac{1}{2})$ scalar field, then hide ("break") symmetry so that $\lambda\langle 0|\phi_H|0\rangle = m$.

$\mathscr{L}_{\text{Higgs}} = \mathscr{L}_{\text{pure Higgs}} + \mathscr{L}_{\text{Higgs}-\text{fermion}}$.

Writing $D_\mu = \partial_\mu - I(\widehat{A}_\mu + \widehat{B}_\mu)$ and $\mathrm{SU}(2)_L$ components $\phi_H^T = (\phi_{H\ 1}, \phi_{H\ 2}) = (\phi_H^+, \phi_H^0)$, $(\epsilon\phi_H^\dagger)^T = (\phi_H^{0\dagger}, -\phi_H^{+\dagger})$,

$\mathscr{L}_{\text{pure Higgs}} = -\frac{1}{2}(D_\mu\phi_H)^\dagger D^\mu\phi_H - V(\phi_H)$, *Higgs potential* $V(\phi_H) = \frac{m_H^2}{2}\phi_H^\dagger\phi_H + \frac{\lambda}{4}(\phi_H^\dagger\phi_H)^2$

$\mathscr{L}_{\text{Higgs}-\text{fermion}} = -G_e^{KM}\overline{\mathscr{E}}^K_{L\ a}\phi_{H\ a}e_R^M - G_u^{KM}\overline{\mathscr{Q}}^K_{L\ a}(\epsilon\phi_H)^\dagger_a u_R^M - G_d^{KM}\overline{\mathscr{Q}}^K_{L\ a}\phi_{H\ a}d_R^M$.

All three terms are $(\mathbf{1},\mathbf{1},0)$, i.e. invariant. Consider e.g. second term: From table 2.9.1, $Y = -\frac{1}{6} - \frac{1}{2} + \frac{2}{3} = 0$.

Write $\mathrm{SU}(2)_L$ transformation of ϕ_H as $\phi'_{H\ a} = U_{ab}\phi_{H\ b}$ ($U = e^{I\frac{1}{2}\sigma_i\alpha_i}$ from page 63),

so $\underline{(\epsilon\phi'_H)^\dagger_a = (\epsilon\phi_H)^\dagger_b U^{-1*}_{ba}}$ from $\epsilon_{ac}\sigma_{cd}\epsilon^T_{db} = -\sigma^*_{ab}$. (Same transformation as ϕ_H^T: $\phi'_{H\ a} = \phi_{H\ b}U^T_{ba} = \phi_{H\ b}U^{-1*}_{ba}$.)

Also $\mathscr{Q}'_{L\ a} = U_{ab}\mathscr{Q}_{L\ b}$, so $\underline{\mathscr{Q}'^\dagger_{L\ a} = U^*_{ac}\mathscr{Q}^\dagger_{L\ c}}$, so $\mathscr{Q}'^\dagger_{L\ a}(\epsilon\phi'_H)^\dagger_a = \mathscr{Q}^\dagger_{L\ a}(\epsilon\phi_H)^\dagger_a$. Note u_R is an $\mathrm{SU}(2)_L$ singlet.

$m_H^2 > 0$: Stationary $\mathscr{L}$ (vacuum) occurs when all fields vanish.

Spontaneous symmetry breaking (SSB): $m_H^2 < 0$: tree level vacuum obeys $\frac{\partial V(\phi_H)}{\partial \phi}\Big|_{\phi_H = \phi_{H0}} = 0$

$\implies |\phi_{H0}^2| = v^2 = \frac{|m_H^2|}{\lambda}$. From Higgs potential on page 70.

Infinite number of choices for $\phi_{0H} = \langle 0|\phi_H|0\rangle$. Take general $\phi_H = e^{Iv\xi_i(x)\frac{\sigma_i}{2}} \begin{pmatrix} 0 \\ v + \eta(x) \end{pmatrix}$.

Vacuum taken as $\xi_i = \eta = 0$, no longer invariant under symmetry transformations.

$\boxed{\mathscr{L}_{\text{gauge mass}} = -m_W^2 W_\mu^\dagger W^\mu - \frac{1}{2} m_Z^2 Z_\mu Z^\mu}$, where $m_Z = \frac{v}{2}\sqrt{g^2 + g'^2}$, $m_W = \frac{v}{2} g = m_Z \cos\theta_W$, $\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}$,

$W_\mu = \frac{1}{\sqrt{2}}(A_{1\ \mu} - IA_{2\ \mu})$, $Z_\mu = \cos\theta_W A_{3\ \mu} - \sin\theta_W B_\mu$.

From $\mathscr{L}_{\text{pure Higgs}}$. Start with $\mathscr{L}_{\text{gauge mass}} = -\frac{1}{2}\left| \left(gA_{i\ \mu}\frac{\sigma_i}{2} + g'B_\mu \frac{1}{2}\right)\begin{pmatrix} 0 \\ v \end{pmatrix}\right|^2 = -\frac{1}{2}\left|\begin{pmatrix} g(A_{1\ \mu} + IA_{2\ \mu}) \\ -gA_{3\ \mu} + g'B_\mu \end{pmatrix}\frac{v}{2}\right|^2$.

$\boxed{\mathscr{L}_{\text{gauge dynamic}} = -\frac{1}{2}|\partial^\mu W^\nu - \partial^\nu W^\mu|^2 - \frac{1}{4}|\partial^\mu Z^\nu - \partial^\nu Z^\mu|^2 - \frac{1}{4}|\partial^\mu A^\nu - \partial^\nu A^\mu|^2}$,

where $A_\mu = \sin\theta_W A_{3\ \mu} + \cos\theta_W B_\mu$ must be massless photon.

From $\mathscr{L}_{\text{spin 1}}$. Start with $\mathscr{L}_{\text{gauge dynamic}} = -\frac{1}{4}|\partial^\mu A_i^\nu - \partial^\nu A_i^\mu|^2 - \frac{1}{4}|\partial^\mu B^\nu - \partial^\nu B^\mu|^2$.

In $\mathscr{L}_{\text{quark}}$ and $\mathscr{L}_{\text{lepton}}$ on page 69, between left-handed fermions:

$$(\widehat{A}_\mu + \widehat{B}_\mu)_L = \begin{pmatrix} e\left(\frac{1}{2}+Y\right)A_\mu + \left(\frac{g}{2}\cos\theta_W - g'Y\sin\theta_W\right)Z_\mu & \frac{g}{\sqrt{2}}W_\mu^\dagger \\ \frac{g}{\sqrt{2}}W_\mu & e\left(-\frac{1}{2}+Y\right)A_\mu + \left(-\frac{g}{2}\cos\theta_W - g'Y\sin\theta_W\right)Z_\mu \end{pmatrix},$$

between right-handed: $(\widehat{A}_\mu + \widehat{B}_\mu)_R = eYA_\mu + (-g'\sin\theta_W Y)Z_\mu$, where $e = g\sin\theta_W = g'\cos\theta_W$.

So charge/e is $Q = T_3 + Y$, i.e. charge of u_L, d_L, ν_e, e_L is $\frac{2}{3}$, $-\frac{1}{3}$, 0, -1 and of u_R, d_R, e_R is $\frac{2}{3}$, $-\frac{1}{3}$, -1.

$\boxed{\mathscr{L}_{\text{fermion mass}} = -\overline{e}_L^K m_e^{KM} e_R^M - \overline{u}_L^K m_u^{KM} u_R^M - \overline{d}_L^K m_d^{KM} d_R^M}$, where $m_\psi^{KM} = G_\psi^{KM} v$.

From $\mathscr{L}_{\text{Higgs}-\text{fermion}}$ on page 70.

Can always transform $u_R^{K\prime} = A_{u_R}^{KM} u_R^M$, likewise for u_L, d_L, d_R, ν_e, e_L, e_R.

A matrices must be unitary so that kinetic terms retain their previous forms, $\overline{u}_R^{K\prime} \gamma^\mu \partial_\mu u_R^{K\prime}$ etc.

Choose A matrices such that new mass matrices $m_u' = A_{u_L} m_u A_{u_R}^\dagger$ etc. diagonal, entries $m_u^{K\prime}$:

$\mathscr{L}_{\text{fermion mass}} = \sum_K -\overline{e}_L^{K\prime} m_e^K e_R^{K\prime} - \overline{u}_L^{K\prime} m_u^{K\prime} u_R^{K\prime} - \overline{d}_L^{K\prime} m_d^{K\prime} d_R^{K\prime}$.

Then $\mathscr{L}_{W\ -\text{fermion}} \propto \overline{d}_L^K \gamma^\mu W_\mu u_L^K + \overline{e}_L^K \gamma^\mu W_\mu \nu_e^K = \overline{d}_L^{K\prime} \gamma^\mu W_\mu (V^\dagger)^{KN} u_L^{N\prime} + \overline{e}_L^{K\prime} \gamma^\mu W_\mu \nu_e^{K\prime\prime}$

(proportionality constant is $-\frac{Ig}{\sqrt{2}}$), where *CKM matrix* $V = A_{u_L} A_{d_L}^{-1}$.

Analogous leptonic matrix absorbed into $\nu_e^{K\prime\prime} = \left(A_{e_L} A_{\nu_e}^{-1}\right)^{KN} \nu_e^{N\prime}$.

(In contrast to u_L^K, any combination of ν_e^K is mass eigenstate because mass matrix is zero.)

2.9.2 Some remaining features

Neutrino mass by adding to $\mathscr{L}_{\text{Higgs}-\text{fermion}}$ a term $-G_{\nu_e}\overline{\mathscr{E}}^K_a(\epsilon\phi_H)^\dagger_a\nu_{eR} \to -\overline{\nu}_e m_{\nu_e}\nu_{eR}$, where ν_{eR} is $(\mathbf{1},\mathbf{1},0)$.

Expect $m_{\nu_e} \sim v$ to be similar order of magnitude to quark and charged lepton masses.

Also allowed SM invariant term $-\frac{1}{2}\nu^T_{eR}M_R\nu_{eR}$, can only come from higher scale symmetry breaking,

so $M_R \gg v \sim m_{\nu_e}$, i.e. no right handed neutrinos at low energy.

Gives $\mathscr{L}_{\text{neutrino mass}} = -\frac{1}{2}(\overline{\nu}_e \;\; \nu^T_{eR})\begin{pmatrix} 0 & m_{\nu_e} \\ m_{\nu_e} & M_R \end{pmatrix}\begin{pmatrix} \overline{\nu}^T_e \\ \nu_{eR} \end{pmatrix} \simeq -\frac{1}{2}(\overline{\nu}'_e \;\; \nu'^T_{eR})\begin{pmatrix} -\frac{m^2_{\nu_e}}{M_R} & 0 \\ 0 & M_R \end{pmatrix}\begin{pmatrix} \overline{\nu}'^T_e \\ \nu'_{eR} \end{pmatrix}$,

i.e. *seesaw mechanism*: mass $\frac{m^2_{\nu_e}}{M_R}$ of (almost) left handed ν'_e (i.e. m_{ν_e} suppressed by $\frac{m_{\nu_e}}{M_R}$).

Invariance with respect to parity P, charge conjugation C and time reversal T transformations.

CPT conserved, but CP-violation due to phases in CKM matrix.

CP and P violating terms $\frac{\theta}{64\pi^2}\epsilon^{\kappa\lambda\rho\sigma}F^i_{\kappa\lambda}F^i_{\rho\sigma}$ allowed, but are total derivatives and therefore non-perturbative.

Current observation suggest θ consistent with zero (no CP violation in QCD).

Cancelled by (harmless) anomaly (subsubsection 2.9.4) of global symmetry $\psi_f \to e^{I\gamma_5\alpha_f}\psi_f$ when $\sum\alpha_f = -\frac{1}{2}\theta$,

but this introduces unobserved CP violating phase $e^{-I\theta}$ on quark masses.

Peccei-Quinn mechanism: where $(\epsilon\phi_H)^\dagger$ in $\mathscr{L}_{\text{Higgs}-\text{fermion}}$ on page 70 is replaced with second Higgs,

which transforms differently to first Higgs and can soak up this phase at least at some GUT scale.

2.9.3 Grand unification

Suppose SM unifies to single group G at scale M_X, then $t^{\mathrm{U}(1)_Y}$ (diagonal), $t_i^{\mathrm{SU}(2)_L}$, $t_j^{\mathrm{SU}(3)_C}$ are generators of G.

Tracelessness requires sum of Y values (=elements of $t^{\mathrm{U}(1)_Y}$) to vanish, which is the case from Table 2.9.1.

Normalization of generators as on page 14, so $\mathrm{tr}[t^{\mathrm{U}(1)_Y 2}]$ $=\mathrm{tr}[t_i^{\mathrm{SU}(2)_L 2}]$ $=\mathrm{tr}[t_j^{\mathrm{SU}(3)_c 2}]$, so

$g_s^2(M_X) = g^2(M_X) = \frac{5}{3} g'^2(M_X)$ (after dividing by 2×no. generations). Implies $\sin^2\theta_W(M_X) = \frac{3}{8}$ from page 71.

Within couplings' exp. errors, unification occurs (provided $N = 1$ SUSY is included) at $M_X = 2 \times 10^{16}$ GeV.

To find simplest unification with no new particles, note SM particles are *chiral*, require complex representations:

In general, define all particles f_L to be left-handed, then antiparticles $f_R = f_L^\dagger$ are right-handed.

Then if f_L in representation $\boldsymbol{R}$ of some group G, f_R is in representation $\overline{\boldsymbol{R}}$.

If f_L, f_R equivalent (have same transformation properties), then $\boldsymbol{R} = \overline{\boldsymbol{R}}$, i.e. pseudoreal representation.

SM particles $f_L = (\mathscr{E}_L, e_R^\dagger, \mathscr{Q}_L, u_R^\dagger, d_R^\dagger)$ require complex representation because $f_R = f_L^\dagger$ inequivalent, e.g. $\overline{\mathbf{3}} \neq \mathbf{3}$.

Pseudoreal representation possible if particle content enlarged to $f_L \to F_L$ so that F_L, $F_R = F_L^\dagger$ equivalent.

e.g. in SO(10), can fit 15 particles of each generation into real **16** representation, requires adding 1 ν_{eR}.

SU(5) is simplest unification.

Since SU(3)×SU(2)×U(1) ⊂ SU(5), all internal symmetries accounted for by fermions ψ_α with $\alpha = 1, ..., 5$.

$$\text{Choose } t_i^{\mathrm{SU(3)}} = g_s \begin{pmatrix} & & & 0 & 0 \\ & \frac{\lambda_i}{2} & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = t_i^{\mathrm{SU(3)}_C}, \; i = 1, ..., 8, \qquad t_i^{\mathrm{SU(2)}} = g \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & \multicolumn{2}{c}{\frac{\sigma_i}{2}} \end{pmatrix} = t_i^{\mathrm{SU(2)}_L}, \; i = 1, ..., 3.$$

U(1) generator must commute with generators above and be traceless. Tentatively take

$$t^{\mathrm{U(1)}} = 2g' \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} = 2t^{\mathrm{U(1)}_Y}.$$

Then fermions form $\overline{\mathbf{5}}$ (fundamental) and $\mathbf{10}$ representations of SU(5):

$$\begin{pmatrix} \overline{d}_R^1 \\ \overline{d}_R^2 \\ \overline{d}_R^3 \\ e_L \\ -\nu_{eL} \end{pmatrix}, \begin{pmatrix} 0 & \overline{u}_R^3 & -\overline{u}_R^2 & u_L^1 & d_L^1 \\ & 0 & \overline{u}_R^1 & u_L^2 & d_L^2 \\ & & 0 & u_L^3 & d_L^3 \\ & \text{—“} & & 0 & \overline{e}_R \\ & & & & 0 \end{pmatrix}.$$

(Note: all particles left-handed, 1,2,3 superscripts are colour indices, $\mathbf{10}$ matrix is antisymmetric).

In SO(10), 1 generation fits into $\mathbf{16} = \mathbf{1} + \overline{\mathbf{5}} + \mathbf{10}$, and $\mathbf{1}$ is identified with right-handed neutrino.

2.9.4 Anomalies

Gauge anomalies modify symmetry relations (Ward identities), spoils renormalizability and maybe unitarity.

Anomaly occurs because $\mathscr{A} = \int d^4x \mathscr{L}$ respects symmetry, but not measure $\int \prod_{x,l} d\psi_l(x) = d[\psi]d[\overline{\psi}]d[A]$.

Relevant example: Let $\mathscr{A}$ be invariant under $\psi_\alpha = U_{k\ \alpha\beta}\psi_\beta$,

where $U_k = \exp[I\gamma_5\alpha t_k]$ is *chiral symmetry* (global) and each ψ_α is a Dirac field.

Problem: although $\mathscr{A}$ is invariant, measure is not: Resulting change in path-integral $\int d[\psi]d[\overline{\psi}]d[A]\exp[I\mathscr{A}]$

is "as if" $\mathscr{L}$ changes by $\alpha\,\mathscr{J}_k[A]$, where $\mathscr{J}_k = -\frac{1}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F_i^{\mu\nu}F_j^{\rho\sigma}\mathrm{tr}[\{t_i,t_j\}t_k]$.

Noether's theorem on page 61: $\int d[\psi]d[\overline{\psi}]d[A]\exp[I\mathscr{A}] \to \int d[\psi]d[\overline{\psi}]d[A]\exp[I\left(\mathscr{A} + \int d^4x\alpha(x)\left[\mathscr{J}_k - \partial_\mu J_k^\mu(x)\right]\right)]$,

i.e. conservation violation: $\langle\partial_\mu J_k^\mu(x)\rangle_A = -\frac{1}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F_i^{\mu\nu}F_j^{\rho\sigma}\mathrm{tr}[\{t_i,t_j\}t_k]$ ($\langle\rangle_A$ means no A integration).

Anomalous (non-classical) triangle diagrams between J_k^μ, $F_i^{\mu\nu}$ and $F_j^{\rho\sigma}$ modify Ward identities.

Physical theories must be *anomaly free* (i.e. $\mathrm{tr}[\{t_i,t_j\}t_k]$ cancel), e.g. real representations: $\mathrm{tr}[\{t_i,t_j\}t_k] = 0$.

$\mathrm{tr}[\{t_i^*,t_j^*\}t_k^*] = \mathrm{tr}[\{(-Ut_iU^\dagger),(-Ut_jU^\dagger)\}(-Ut_kU^\dagger)] = -\mathrm{tr}[\{t_i,t_j\}t_k]$. But $\mathrm{tr}[\{t_i^*,t_j^*\}t_k^*] = \mathrm{tr}[\{t_i^T,t_j^T\}t_k^T] = \mathrm{tr}[\{t_i,t_j\}t_k]$.

SM is anomaly free. SM in $\mathbf{10}+\overline{\mathbf{5}}$ of SU(5), in real representation $\mathbf{16}$ of SO(10) (thus $\mathrm{tr}[\{t_i,t_j\}t_k]_{\overline{\mathbf{5}}} = -\mathrm{tr}[\{t_i,t_j\}t_k]_{\mathbf{10}}$).

3 Supersymmetry: development

3.1 Why SUSY?

Attractive features of SUSY:

1. Eliminates fine tuning in Higgs mass.

2. Gauge coupling unification.

3. Radiative electroweak symmetry breaking: SUSY $\Longrightarrow$ Higgs potential on page 70 with $m_H^2 < 0$.

4. Excess of matter over anti-matter (large CP violation, not in SM) possible from SUSY breaking terms.

5. Cold dark matter may be stable neutral lightest SUSY particle (LSP) = gravitino / lightest neutralino.

6. Gravity may be described by local SUSY = *supergravity*.

SM is accurately verified but incomplete — e.g. does not + cannot contain gravity,

so must break down at / before energies around Planck scale $M_P = (8\pi G)^{-1/2} = 2.4 \times 10^{18}$ GeV.

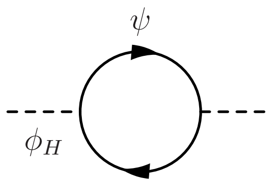
In fact, SM cannot hold without modification much above 1 TeV, otherwise we have

Gauge hierarchy problem: Since $v = |\langle 0|\phi_H|0\rangle| = 246\text{GeV}$ and $\lambda = O(1)$, $|m_H| = |\sqrt{\lambda}v| = O(100)$ GeV.

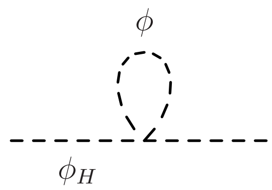
If $\Lambda_{\text{UV}} > O(1)\text{TeV}$, fine tuning between

Δm_H^2 from quantum loop corrections (Fig. 3.1) and tree level (bare) m_H^2:

$$\Delta m_H^2 = \frac{\lambda_\phi}{8\pi^2}\Lambda_{\text{UV}}^2 - \underbrace{3}_{\text{colour "3"}} \frac{|\kappa_t|^2}{8\pi^2}\Lambda_{\text{UV}}^2 + \underbrace{\cdots}_{\text{smaller terms}} .$$



(a) Fermion field ψ, Lagrangian term $-\kappa_\psi \phi_H \overline{\psi}\psi$, giving 1-loop contribution to Higgs mass of $-\frac{|\kappa_\psi|^2}{8\pi^2}\Lambda_{\text{UV}}^2$.



(b) Boson field ϕ, Lagrangian term $\lambda_\phi |\phi_H|^2|\phi|^2$, giving 1-loop contribution to Higgs mass of $\frac{\lambda_\phi}{8\pi^2}\Lambda_{\text{UV}}^2$.

Figure 3.1: Fermion and boson contributions to Higgs mass parameter m_H^2.

(Largest from $-\frac{\lambda_\phi}{4}|\phi_H|^4$ and top quark ($\kappa_t \simeq 1$ because $m_t \simeq v$).)

No similar problem for fermion and gauge boson masses, but these masses affected by m_H^2.

Avoid fine tuning by taking $\Lambda_{\text{UV}} \sim 1$ TeV, i.e. modify SM above this scale.

One solution: Higgs is composite of new fermions bound by new strong force at $\Lambda_{\text{UV}} \simeq 1$ TeV $\longrightarrow$ difficult.

Alternatively, forbid bare $m_H^2|\phi_H|^2$ term by some new symmetry $\delta\phi_H = \epsilon \times$ something.

Various choices for "something" bosonic (leads to "little Higgs" models, extra dimensions).

For a standard symmetry, "something" would be $I\,[Q_a, \phi_H]$, i.e. $\phi_H \to e^{I\epsilon Q_a}\phi_H e^{-I\epsilon Q_a}$.

Try "fermionic" generator Q_a, which must be a $\left(\frac{1}{2}, 0\right)$ spinor (so ϵ a spinor of Grassmann variables).

Relation with momentum: $\boxed{\{Q_a, Q^\dagger_{\dot{b}}\} = 2\sigma^\mu_{a\dot{b}} P_\mu}$.

$\{Q_a, Q^\dagger_{\dot{b}}\}$ is $\left(\frac{1}{2}, 0\right) \times \left(0, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$ from triangle inequality. Only candidate is P^μ (see Coleman-Mandula theorem later).

Lorentz invariance requires combination $\sigma^\mu_{a\dot{b}} P_\mu$, factor 2 comes from suitable normalization of Q_a.

Note $\{Q_a, Q^\dagger_{\dot{b}}\} \neq 0$ because for any state $|X\rangle$,

$\langle X|\{Q_a, (Q_a)^\dagger\}|X\rangle = \langle X|Q_a (Q_a)^\dagger |X\rangle + \langle X| (Q_a)^\dagger Q_a|X\rangle = \left| (Q_a)^\dagger |X\rangle\right|^2 + \left|Q_a|X\rangle\right|^2 \geq 0$. If equality holds for all $|X\rangle$, $Q_a = 0$.

At least one of P_μ non-zero on every state, so Q_a affects every state, not just Higgs,

i.e. <u>every</u> particle has a *superpartner* with opposite statistics and spin difference of 1/2,

together called a *supermultiplet*. This fermion-boson symmetry is *supersymmetry*.

For every fermion field (component) ψ_f with $-\kappa_f \phi_H \overline{\psi}_f \psi_f$, introduce boson field ϕ_f with $-\lambda_f |\phi_H|^2 |\phi_f|^2$.

From Fig. 3.1, contribution of this supermultiplet (component) to Δm_H^2 is $\Delta m_H^2 = \underbrace{\frac{\lambda_f}{8\pi^2}\Lambda_{\mathrm{UV}}^2}_{\text{boson}} \underbrace{-\frac{|\kappa_f|^2}{8\pi^2}\Lambda_{\mathrm{UV}}^2}_{\text{fermion}}$.

Just requiring fermion-boson symmetry guarantees $\lambda_f = |\kappa_f|^2$, and $\Delta m_H^2 = 0$ (+ finite terms) to all orders.

3.2 Haag-Lopuszanski-Sohnius theorem and SUSY algebra

Reconsider symmetries: So far assumed generators are bosonic. Now generalize to include fermionic ones.

Generalize additive observable on page 5 to $Q = Q_{\sigma\sigma'} a_\sigma^\dagger a_{\sigma'}$.

(If $Q_{\sigma\sigma'}$ are components of Hermitian matrix, unitary transformation of particle states gives back original result.)

Since Q is bosonic, $a_\sigma^\dagger$, $a_{\sigma'}^\dagger$ both bosons or both fermions, i.e. $Q_{\sigma\sigma'} = 0$ if $a_\sigma^\dagger$ bosonic, $a_{\sigma'}^\dagger$ fermionic, or vice versa.

SUSY: Allow for Q's containing fermionic parts to also be generators of symmetries that commute with S-matrix.

For convenience, distinguish between fermionic and bosonic parts of any Q.

Fermionic $Q = Q_{\sigma\rho} a_\sigma^\dagger a_\rho + R_{\sigma\rho} a_\rho^\dagger a_\sigma$, where σ sums over bosonic particles, ρ over fermionic particles.

Such a generator converts bosons into fermions and vice versa.

E.g. action of Q on 1 fermion + 1 boson state using (anti) commutation relations on page 4 gives

$Q a_{\rho'}^\dagger a_{\sigma'}^\dagger |0\rangle = Q_{\sigma\rho'} a_\sigma^\dagger a_{\sigma'}^\dagger |0\rangle + R_{\sigma'\rho} a_\rho^\dagger a_{\rho'}^\dagger |0\rangle$.

But as a symmetry implies there are fermions and bosons with similar properties.

Identify symmetry generators t_i also with fermionic Q.

Graded parameters α_i, β_j obey $\alpha_i\beta_j = (-1)^{\eta_i\eta_j}\beta_j\alpha_i$, where *grading* $\eta_i = 0(1)$ for complex (Grassmann) α_i.

Graded generator t_j obeys $\alpha_i t_j = (-1)^{\eta_i\eta_j} t_j\alpha_i$, where $\eta_i = 0(1)$ for bosonic (fermionic) generator t_i.

For transformations $\mathscr{O} \to e^{I\alpha_i t_i}\mathscr{O}e^{-I\alpha_i t_i}$ to preserve grading of any operator $\mathscr{O}$, α_i has same grading as t_i.

Graded Lie algebra: $[(-1)^{\eta_i\eta_j} t_i t_j - t_j t_i] = IC_{ijk}t_k$.

Repeating steps in derivation of Lie algebra on page 8 gives

$\frac{1}{2}IC_{ijk}\alpha_i\beta_j t_k = \frac{1}{2}[\alpha_i t_i \beta_j t_j - \beta_j t_j \alpha_i t_i] = \frac{1}{2}[(-1)^{\eta_i\eta_j}\alpha_i\beta_j t_i t_j - \alpha_i\beta_j t_j t_i]$.

Fermionic generators Q_i: $U(\Lambda)Q_i U^\dagger(\Lambda) = C_{ij}(\Lambda)Q_j$, so Q_i furnishes representation of Lorentz group.

Choose $Q_i = Q_{ab}^{(A,B)}$ in (A,B) representation: $[\boldsymbol{A}, Q_{ab}^{(A,B)}] = -\boldsymbol{J}_{aa'}^{(A)}Q_{a'b}^{(A,B)}$ and $[\boldsymbol{B}, Q_{ab}^{(A,B)}] = -\boldsymbol{J}_{bb'}^{(B)}Q_{ab'}^{(A,B)}$.

Anticommutators of fermionic generators can be used to build bosonic generators of various (A,B).

Coleman-Mandula theorem puts limits on allowed bosonic generators, and hence allowed (A,B) for the $Q_{ab}^{(A,B)}$.

Coleman-Mandula theorem: Only bosonic generators are of internal + Poincaré group symmetries.

Simple argument: Additional conserved additive rank ≥ 1 tensors constrain scattering amplitude too much.

Only 1 4-vector, P^μ: Consider 2→2 scattering, c.m. frame.

Conservation of momentum and angular momentum → amplitude depends on scattering angle θ.

Second conserved additive 4-vector R^μ gives additional constraints unless $R^\mu \propto P^\mu$.

Only 1 2nd rank tensor, $J^{\mu\nu}$: Assume rank 2 conserved additive tensor $\Sigma_{\mu\nu}$.

Additive property means $[\Sigma_{\mu\nu}, a^\dagger_\sigma(\boldsymbol{p})] = C_{\mu\nu\ \sigma}(\boldsymbol{p}) a^\dagger_\sigma(\boldsymbol{p})$. Then $C_{\mu\nu}(\boldsymbol{p},\sigma) = \alpha_\sigma(m^2) p_\mu p_\nu + \beta_\sigma(m^2) g_{\mu\nu}$.

Lorentz transformation of RHS is $U(\Lambda)[\Sigma_{\mu\nu}, a^\dagger_\sigma(\boldsymbol{p})]U^\dagger(\Lambda) = [U(\Lambda)\Sigma_{\mu\nu}U^\dagger(\Lambda), U(\Lambda)a^\dagger_\sigma(\boldsymbol{p})U^\dagger(\Lambda)]$

$= [\Lambda^\rho{}_\mu \Lambda^\delta{}_\nu \Sigma_{\rho\delta}, D_{\sigma'\sigma}(W(\Lambda,p)) a^\dagger_{\sigma'}(\boldsymbol{\Lambda p})] = D_{\sigma'\sigma}(W(\Lambda,p)) \underline{\Lambda^\rho{}_\mu \Lambda^\delta{}_\nu C_{\rho\delta}(\boldsymbol{\Lambda p},\sigma)} a^\dagger_{\sigma'}(\boldsymbol{\Lambda p})$,

and of LHS is $\underline{C_{\mu\nu}(\boldsymbol{p},\sigma)} D_{\sigma'\sigma}(W(\Lambda,p)) a^\dagger_{\sigma'}(\boldsymbol{\Lambda p})$, so $\Lambda_\mu{}^\rho \Lambda_\nu{}^\delta C_{\rho\delta}(\boldsymbol{p},\sigma) = C_{\mu\nu}(\boldsymbol{\Lambda p},\sigma)$. Only candidates are $p_\mu p_\nu$ and $g_{\mu\nu}$.

In 2→2 scattering, conservation of $\Sigma_{\mu\nu}$ implies $\alpha_{\sigma_1}(m_1^2) p_1^\mu p_1^\nu + \alpha_{\sigma_2}(m_2^2) p_2^\mu p_2^\nu = \alpha_{\sigma_1}(m_1^2) p_1'^\mu p_1'^\nu + \alpha_{\sigma_2}(m_2^2) p_2'^\mu p_2'^\nu$.

P^μ conservation: $p_1^\mu + p_2^\mu = p_1'^\mu + p_2'^\mu \implies p_{1,2}^\mu = p_{1,2}'^\mu$, i.e. no scattering (allowed $p_{1,2}^\mu = p_{2,1}'^\mu$ if $\alpha_{\sigma_1}(m_1^2) = \alpha_{\sigma_2}(m_2^2)$).

No higher rank tensors: Generalize last argument to higher rank tensors.

Allowed representations for fermionic generators $Q_{ab}^{(A,B)}$: $A+B=\frac{1}{2}$, i.e. $\left(\frac{1}{2},0\right)$ or $\left(0,\frac{1}{2}\right)$, and $j=\frac{1}{2}$.

$\{Q_{C,-D}^{(C,D)}, Q_{C,-D}^{(C,D)\dagger}\} = X_{C+D,-C-D}^{(C+D,C+D)}$: Firstly, $Q^{(C,D)\dagger}$ is of type $(A,B)=(D,C)$ because $\boldsymbol{B}^\dagger = \boldsymbol{A}$.

Writing $Q_{C,-D}^{(C,D)\dagger} = \widetilde{Q}_{a,b}^{(D,C)}$, we find $b=-C$ because $[B_-, \widetilde{Q}_{a,b}^{(D,C)}] = -[A_+, Q_{C,-D}^{(C,D)}]^\dagger = 0$ using $B_- = A_+^\dagger$. Similarly $a=D$.

Now $\{Q_{C,-D}^{(C,D)}, Q_{C,-D}^{(C,D)\dagger}\} = \{Q_{C,-D}^{(C,D)}, \widetilde{Q}_{D,-C}^{(D,C)}\}$, must have $A_3 = C+D$ and $B_3 = -C-D$, i.e. $A, B \geq C+D$.

But since A must be $\leq (C+D)$ (from triangle inequality), it must be $= C+D$. Similarly for B.

Since $X^{(C+D,C+D)}$ is bosonic, CM theorem means it must be P^μ $\left(\left(\frac{1}{2},\frac{1}{2}\right)\right)$, or internal symmetry generator $((0,0))$.

Latter implies $C=D=0$, not possible by spin-statistics connection. Final result is relation with momentum on page 82.

Take Q_a to be $\left(\frac{1}{2},0\right)$ spinor, i.e. $[\boldsymbol{A}, Q_a] = -\frac{1}{2}\sigma_{ab}Q_b$, $[\boldsymbol{B}, Q_a] = 0$ ($Q_a^\dagger$ will be $\left(0,\frac{1}{2}\right)$).

Q_a not ruled out by reasoning of CM theorem, because no similar conservation law:

Take $|i\rangle$, $|j\rangle$ to have definite particle number. $\langle j|i\rangle \neq 0 \implies$ even difference in fermion numbers, so $\langle j|Q_a|i\rangle = 0$.

Can have multiple generators Q_{ar}, $r=1,\ldots,N \longrightarrow$ *simple* SUSY is $N=1$, *extended* SUSY is $N \geq 2$.

Summary:

CM theorem: only bosonic generators are $(0,0)$ (internal symmetry), $\left(\frac{1}{2},\frac{1}{2}\right)$ (P^μ), and $(1,0)$ and $(0,1)$ ($J^{\mu\nu}$).

HLS theorem: only fermionic generators are $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$ (Q_{ar}).

Relation with momentum for any N: $\{Q_{ar}, Q^{\dagger}_{\dot{b}s}\} = 2\delta_{rs}\sigma^{\mu}_{a\dot{b}}P_{\mu}$.

From allowed representations for fermionic generators on page 86, $\{Q_{ar}, Q^{\dagger}_{\dot{b}s}\} = 2N_{rs}\sigma^{\mu}_{a\dot{b}}P_{\mu}$. N_{rs} is Hermitian,

because $\{Q_{ar}, Q^{\dagger}_{\dot{b}s}\}^{\dagger} = \{Q_{bs}, Q^{\dagger}_{\dot{a}r}\} = 2N^{*}_{rs}\sigma^{\mu}_{b\dot{a}}P_{\mu}$, but $\{Q_{bs}, Q^{\dagger}_{\dot{a}r}\} = 2N_{sr}\sigma^{\mu}_{b\dot{a}}P_{\mu}$. So N diagonalized by unitary matrix W.

Writing $Q'_{ar'} = W_{r'r}Q_{ar}$ gives $\{Q'_{ar}, Q'^{\dagger}_{\dot{b}s}\} = 2n_r\delta_{rs}\sigma^{\mu}_{a\dot{b}}P_{\mu}$ (no sum over r on RHS), where n_r are eigenvalues of N_{rs}.

Writing $Q_{ar} = Q'_{ar}/\sqrt{n_r}$ gives result if $n_r > 0$ (otherwise we have a factor -1):

Taking $Q'^{\dagger}_{\dot{b}s} = (Q'_{ar})^{\dagger}$ and operating from right and left with $|X(p)\rangle$ and $\langle X(p)|$ where $Q'_{ar}|X(p)\rangle \neq 0$ gives

on LHS: $\langle X(p)|\{Q'_{ar}, (Q'_{ar})^{\dagger}\}|X(p)\rangle = |\,(Q'_{ar})^{\dagger}\}|X(p)\rangle|^2 + |Q'_{ar}\}|X(p)\rangle|^2 > 0$, and

on RHS: $2n_r(p^0 \pm p^3)$, where $\pm$ for $a = 1, 2$. If $p^0 \geq \mp p^3$, then $n_r > 0$ as required.

SUSY implies $\langle 0|H|0\rangle = 0$ for supersymmetric vacuum ($Q_{ar}|0\rangle = Q^{\dagger}_{\dot{a}r}|0\rangle = 0$).

Commutation with momentum: $[Q_a, P^{\mu}] = 0$.

$[Q_a, P^{\mu}]$ is $\left(\frac{1}{2}, 0\right) \times \left(\frac{1}{2}, \frac{1}{2}\right) = \left(1, \frac{1}{2}\right) + \left(0, \frac{1}{2}\right)$. No $\left(1, \frac{1}{2}\right)$ generator, but $Q^{\dagger\dot{a}}$ is $\left(0, \frac{1}{2}\right)$.

So $[Q_b, P^{\mu}] = k\sigma^{\mu}_{b\dot{a}}Q^{\dagger\dot{a}}$ and therefore $\left[Q^{\dagger}_{\dot{a}}, P^{\mu}\right] = -k^{*}Q^{b}\sigma^{\mu}_{b\dot{a}}$, or, using ϵ matrix, $\left[Q^{\dagger\dot{a}}, P^{\mu}\right] = k^{*}\overline{\sigma}^{\mu\ \dot{a}b}Q_b$.

Jacobi identity: $0 = [[Q_a, P^{\mu}], P^{\nu}] + [[P^{\mu}, P^{\nu}], Q_a] + [[P^{\nu}, Q_a], P^{\mu}] = k\sigma^{\mu}_{a\dot{b}}[Q^{\dagger\dot{b}}, P^{\nu}] - k\sigma^{\nu}_{a\dot{b}}[Q^{\dagger\dot{b}}, P^{\mu}] = |k|^2[\sigma^{\mu}, \overline{\sigma}^{\nu}]_a{}^b Q_b$.

Since $[\sigma^{\mu}, \overline{\sigma}^{\nu}]_a{}^b \neq 0$ for all μ, ν, must have $|k|^2 = k = 0$.

Anticommuting generators: $\{Q_{ar}, Q_{bs}\} = \epsilon_{ab} Z_{rs}$, with $(0,0)$ generators $Z_{rs} = -Z_{sr}$.

$\{Q_{ar}, Q_{bs}\}$ is $(1,0) + (0,0)$. From CM theorem, only $(1,0)$ generator is $J^{\mu\nu}$. But $[\{Q_{ar}, Q_{bs}\}, P^\mu] = 0$ (commutation with momentum, page 87), while (linear combinations of) $J^{\mu\nu}$ doesn't commute with P^μ (see Poincare algebra, page 21).

So only possibility is $(0,0)$ generators, which in general commute with P^μ from CM theorem.

Lorentz invariance requires $\epsilon_{ab}(= -\epsilon_{ba})$, but whole expression symmetric under $ar \leftrightarrow bs$ so $Z_{rs} = -Z_{sr}$.

Antisymmetry of $Z_{rs} \rightarrow$ vanish for $N = 1$. Z_{rs} are *central charges* due to following commutation relations:

Commutation with central charges: $[Z_{rs}, Q_{at}] = [Z_{rs}, Q^\dagger_{\dot{a}t}] = 0$.

Jacobi identity $0 = [\{Q_{ar}, Q_{bs}\}, Q^\dagger_{\dot{c}t}] + [\{Q_{bs}, Q^\dagger_{\dot{c}t}\}, Q_{ar}] + [\{Q^\dagger_{\dot{c}t}, Q_{ar}\}, Q_{bs}]$.

2nd, 3rd terms vanish from commutation with momentum on page 87. Thus $[Z_{rs}, Q^\dagger_{\dot{a}t}] = 0$.

Jacobi identity $0 = -[Z_{rs}, \{Q_{at}, Q^\dagger_{\dot{b}u}\}] + \{Q^\dagger_{\dot{b}u}, [Z_{rs}, Q_{at}]\} - \{Q_{at}, [Q^\dagger_{\dot{b}u}, Z_{rs}]\}$.

1st, 3rd terms vanish because Z_{rs} commutes with P^μ and $Q^\dagger_{\dot{b}u}$.

Commutator in 2nd term must be $[Z_{rs}, Q_{at}] = M_{rstv} Q_{av}$, so $0 = \{Q^\dagger_{\dot{b}u}, [Z_{rs}, Q_{at}]\} = 2M_{rstu} \sigma^\mu_{a\dot{b}} P_\mu$, i.e. $M_{rstu} = 0$ so $[Z_{rs}, Q_{at}] = 0$.

Commuting central charges: $[Z_{rs}, Z_{tu}] = [Z_{rs}, Z^\dagger_{tu}] = 0$. $[Z_{rs}, Z_{tu}] = [\{Q_{1r}, Q_{2s}\}, Z_{tu}] = 0$, etc.

***R*-symmetry**: $Z_{rs} = 0$ gives U(N) symmetry $Q_{ar} \rightarrow V_{rs} Q_{as}$ ($N = 1$ case: $Q_a \rightarrow e^{I\phi} Q_a$, always true).

CM theorem: $[t_i, P^\mu] = [t_i, J^{\mu\nu}] = 0$ for generators t_i of group G.

Since t_i is $(0,0)$, must have $[t_i, Q_{\frac{1}{2}r}] = -(a_i)_{rs} Q_{\frac{1}{2}s}$. Matrices a_i represent G.

Jacobi identity $0 = [[t_i, t_j], Q_{\frac{1}{2}r}] + [[Q_{\frac{1}{2}r}, t_i], t_j] + [[t_j, Q_{\frac{1}{2}r}], t_i] = IC_{ijk}[t_k, Q_{\frac{1}{2}r}] + (a_i)_{rt}[Q_{\frac{1}{2}t}, t_j] - (a_j)_{rt}[Q_{\frac{1}{2}t}, t_i]$

$= -IC_{ijk}(a_k)_{rs} Q_{\frac{1}{2}s} + (a_i)_{rt}(a_j)_{ts} Q_{\frac{1}{2}s} - (a_j)_{rt}(a_i)_{ts} Q_{\frac{1}{2}s}$, i.e. $[a_i, a_j]_{rs} = IC_{ijk}(a_k)_{rs}$.

Simple SUSY and internal symmetry generators commute. If N=1, numbers a_i cannot represent G unless $a_i = 0$.

Commutation of central charges and internal symmetry generators: $[t_i, Z_{rs}] = 0$.

From anticommuting generators on page 88, $[t_i, Z_{rs}] = [t_i, \{Q_{1r}, Q_{2s}\}] = -(a_i)_{rr'} Z_{rs} - (a_i)_{ss'} Z_{rs'}$.

This has the form $[t_i, Z_\alpha] = A_{i\alpha\beta} Z_\beta$, i.e. Z_α form invariant subalgebra of G if any $A_{i\alpha\beta} \neq 0$ or are U(1) if all $A_{i\alpha\beta} = 0$.

But from CM theorem, generators of invariant subalgebra not allowed, only U(1) generators. So all $A_{i\alpha\beta} = 0$.

3.3 Supermultiplets

Supermultiplets: Particles that mix under SUSY transformations, furnish representation of SUSY.

Irreducible representations of SUSY: doesn't reduce to 2+ supermultiplets separately mixing under SUSY.

Action of Q_{ar} or $Q^\dagger_{\dot{a}r}$ converts one particle into another of the same irreducible supermulitplet (superpartners).

Since $[Q_{ar}, P^\mu] = [Q^\dagger_{\dot{a}r}, P^\mu] = 0$, all particles in supermultiplet have same P^μ (and hence same mass P^2).

Equal number of bosonic and fermionic degrees of freedom in supermultiplet: $n_B = n_F$.

Convention: $\sum_X$ over supermultiplet states, $\sum_{\text{all } X}$ over complete basis. Choose given r for $Q_{ar} \equiv Q_a$, $Q^\dagger_{\dot{a}r} \equiv Q^\dagger_{\dot{a}}$ below.

Supermultiplet's states $|X\rangle$ have same p^μ, and $(-1)^{2s}|X\rangle = \pm 1|X\rangle$ for spin s bosonic/fermionic $|X\rangle$,

so $P'_\mu \equiv \sum_X \langle X|(-1)^{2s} P_\mu |X\rangle = p_\mu(n_B - n_F)$. We show $P'_\mu = 0$: From relation with momentum for any N on page 87,

$$2\sigma^\mu_{a\dot{b}} P'_\mu = \sum_X \langle X|(-1)^{2s} Q_a Q^\dagger_{\dot{b}}|X\rangle + \underline{\sum_X \langle X|(-1)^{2s} Q^\dagger_{\dot{b}} Q_a|X\rangle} = \sum_X \langle X|(-1)^{2s} Q_a Q^\dagger_{\dot{b}}|X\rangle + \underline{\sum_{X,\text{all } Y} \langle X|(-1)^{2s} Q^\dagger_{\dot{b}}|Y\rangle\langle Y|Q_a|X\rangle}.$$

Since $Q_a|X\rangle$, $|X\rangle$ in same supermultiplet, limit $\sum_{\text{all } Y} \to \sum_Y$. Conversely, extend $\sum_X \to \sum_{\text{all } X}$ (so $\sum_{\text{all } X} |X\rangle\langle X| = 1$).

$$2\sigma^\mu_{a\dot{b}} P'_\mu = \langle X|(-1)^{2s} Q_a Q^\dagger_{\dot{b}}|X\rangle + \underline{\sum_{\text{all } X,Y} \langle Y|Q_a|X\rangle\langle X|(-1)^{2s} Q^\dagger_{\dot{b}}|Y\rangle} = \sum_X \langle X|(-1)^{2s} Q_a Q^\dagger_{\dot{b}}|X\rangle + \underline{\sum_Y \langle Y|Q_a (-1)^{2s} Q^\dagger_{\dot{b}}|Y\rangle}.$$

But $Q_a(-1)^{2s}|Y\rangle = -(-1)^{2s} Q_a|Y\rangle$, so $2\sigma^\mu_{a\dot{b}} P'_\mu = \sum_X \langle X|(-1)^{2s} Q_a Q^\dagger_{\dot{b}}|X\rangle + \underline{-\sum_Y \langle Y|(-1)^{2s} Q_a Q^\dagger_{\dot{b}}|Y\rangle} = 0.$

Massless supermultiplets: In frame where $p^1 = p^2 = 0, p^3 = p^0$, $\begin{pmatrix} \{Q_{\frac{1}{2}r}, Q^\dagger_{\frac{1}{2}s}\} = 4p^0\delta_{rs} & \{Q_{\frac{1}{2}r}, Q^\dagger_{-\frac{1}{2}s}\} = 0 \\ \{Q_{-\frac{1}{2}r}, Q^\dagger_{\frac{1}{2}s}\} = 0 & \{Q_{-\frac{1}{2}r}, Q^\dagger_{-\frac{1}{2}s}\} = 0 \end{pmatrix}$.

$Q_{-\frac{1}{2}r}$, $Q^\dagger_{-\frac{1}{2}r}$ annihilate supermultiplet states, so Z_{rs} annihilate supermultiplet states.

For any $|X\rangle$ in supermultiplet, $0 = \langle X|\{Q_{-\frac{1}{2}r}, Q^\dagger_{-\frac{1}{2}r}\}|X\rangle = |\left(Q_{-\frac{1}{2}r}\right)^\dagger |X\rangle|^2 + |Q_{-\frac{1}{2}r}|X\rangle|^2$ (no sum over r).

All supermultiplet states reached by acting on maximum helicity state $|\lambda_{\text{max}}\rangle$ with the $Q_{\frac{1}{2}r}$.

$Q^\dagger_{\frac{1}{2}r}$ give no new states: Consider $|X\rangle$ not containing $Q_{\frac{1}{2}r}$. Then $Q^\dagger_{\frac{1}{2}r}|X\rangle = 0$ ($Q^\dagger_{\frac{1}{2}r}$ commutes across to act directly on $|\lambda_{\text{max}}\rangle$).

Then $Q^\dagger_{\frac{1}{2}r} Q_{\frac{1}{2}r}|X\rangle = \{Q_{\frac{1}{2}r}, Q^\dagger_{\frac{1}{2}r}\}|X\rangle = 4p^0|X\rangle$ (no sum over r), i.e. $Q^\dagger_{\frac{1}{2}r}$ just removes $Q_{\frac{1}{2}r}$.

$Q^\dagger_{\frac{1}{2}r}|X\rangle$ (or $Q_{\frac{1}{2}r}|X\rangle$) has helicity greater (or less) than $|X\rangle$ by $\frac{1}{2}$. Recall $[J_3, Q^\dagger_{\frac{1}{2}r}] = \frac{1}{2}Q^\dagger_{\frac{1}{2}r}$.

Range of helicities in supermultiplet: $\frac{N!}{n!(N-n)!}$ helicity $\lambda_{\text{max}} - \frac{n}{2}$ states, $\lambda_{\text{min}} = \lambda_{\text{max}} - \frac{N}{2}$.

Obtain supermultiplet states $|X\rangle$ by acting on maximum helicity state $|\lambda_{\text{max}}\rangle$ with any n of $Q_{\frac{1}{2}1}$, $Q_{\frac{1}{2}2}$, ..., $Q_{\frac{1}{2}N}$.

Order doesn't matter since $Q_{\frac{1}{2}a}$ anticommute, each generator cannot appear more than once since $Q^2_{\frac{1}{2}r} = 0$.

Constraint on particle spins on page 56 implies $\lambda_{\text{min}} \geq -2$ and $\lambda_{\text{max}} \leq 2$, i.e. $N \leq 8$.

SM particles probably belong to $\sim$ massless supermultiplets.

Superpartners (masses $\sim M$) of SM particles ($\sim m$) not seen, i.e. $M \gg m$, so SUSY is broken, at energy $< m_{\text{SUSY}}$.

SUSY restored at m_{SUSY} $\therefore$ $m_{\text{SUSY}} \gg M - m \sim M$ (superpartner has same mass) $\therefore$ supermultiplets $\sim$ massless.

Most likely scenario is simple SUSY: quark, lepton (spin $\frac{1}{2}$) superpartners are scalars (0): squarks, sleptons.

Higgs (0), gauge bosons (1), graviton (2) superpartners fermionic: Higgsino + gauginos ($\frac{1}{2}$), gravitino ($\frac{3}{2}$).

No alternatives: In simple SUSY, SM gauge bosons cannot be superpartners of SM fermions.

SM fermions, gauge bosons in different representations (recall simple SUSY and SM SU(3)×SU(2) commute from page 89).

Quarks, leptons cannot be in same supermultiplet as any beyond-SM vector (gauge) bosons.

Gauge bosons are in adjoint representation of a group. If e.g. helicity $+\frac{1}{2}$ fermion, $+1$ gauge boson in same supermultiplet,

fermion in adjoint = real representation. But SM is chiral, i.e. helicity $+\frac{1}{2}$ fermions belong to complex representations.

Superpartners of gauge bosons must be helicity $\pm\frac{1}{2}$, not $\pm\frac{3}{2}$, fermions.

Helicity $\pm\frac{3}{2}$ particle couples only to Q_a (principle of equivalence on page 56: ± 2 only couples to P^μ).

Extended SUSY probably cannot be realised in nature.

In extended SUSY, helicity $\pm\frac{1}{2}$ fermions either in same supermultiplet as gauge bosons ($N \geq 3$), or each other ($N = 2$).

Massive supermultiplets: For particles with masses $M \gg m_{\mathrm{SUSY}}$, e.g. heavy gauge bosons in SU(5).

In frame where $p^1 = p^2 = p^3 = 0$, $p_0 = M$, $\begin{pmatrix} \{Q_{\frac{1}{2}r}, Q^\dagger_{\frac{1}{2}s}\} = 2M\delta_{rs} & \{Q_{\frac{1}{2}r}, Q^\dagger_{-\frac{1}{2}s}\} = 0 \\ \{Q_{-\frac{1}{2}r}, Q^\dagger_{\frac{1}{2}s}\} = 0 & \{Q_{-\frac{1}{2}r}, Q^\dagger_{-\frac{1}{2}s}\} = 2M\delta_{rs} \end{pmatrix}$,

so $Q_{\frac{1}{2}r}$, $Q^\dagger_{-\frac{1}{2}r}$ (or $Q_{-\frac{1}{2}r}$, $Q^\dagger_{\frac{1}{2}r}$) lower (or raise) spin 3 component by $\frac{1}{2}$.

$a_{A(a,r)} = \frac{1}{\sqrt{2M}} Q_{ar}$ are fermionic annihilation / creation operators: $\{a_A, a^\dagger_B\} = \delta_{AB}$, $\{a_A, a_B\} = \{a^\dagger_A, a^\dagger_B\} = 0$.

Define Clifford "vacuum" $|\Omega\rangle$: $a_A|\Omega\rangle = 0$. $|\Omega\rangle$ has given spin j, range of spin 3 is $-j \leq \sigma \leq j$.

Supermultiplet is all states $a^\dagger_{A_1} \ldots a^\dagger_{A_n}|\Omega\rangle$, spins ranging from $\mathrm{Max}(j - \frac{N}{2}, 0)$, …, $j + \frac{N}{2}$.

Simple SUSY: States with spin $j \pm \frac{1}{2}$, and 2 sets of states with spin j.

If $j = 0$, there are two bosonic spin 0 states and two fermionic states of spin $\frac{1}{2}$ (i.e. spin 3 is $\pm\frac{1}{2}$).

Lower mass bound in extended SUSY: $M \geq \frac{1}{2N}\mathrm{Tr}\sqrt{Z^\dagger Z}$.

Using anticommuting generators on page 88 and relation with momentum for any N on page 87,

$0 \leq \{\underline{Q_{ar} - \epsilon_{ab}U_{rs}Q^*_{bs}}, Q^*_{ar} - \epsilon_{ac}U^*_{rt}Q_{ct}\} = 8NP^0 - 2\mathrm{Tr}(ZU^\dagger + UZ^\dagger)$ for any unitary U. Act on supermultiplet at rest.

Polar decomposition theorem means any $Z = HV$, where H is positive Hermitian and V unitary.

Let $U = V$. Then use $M \geq \frac{1}{2N}\mathrm{Tr}H$ and $Z^\dagger Z = H^2$.

When equality obeyed, supermultiplets are smaller ("short") and similar to massless supermultiplets.

For states obeying $M = \frac{1}{2N}\mathrm{Tr}\sqrt{Z^\dagger Z}$, underlined quantity above must be zero on these states.

Then there are just N independent helicity-lowering $Q_{\frac{1}{2}r}$ and N independent helicity-raising $Q_{-\frac{1}{2}r}$.

3.3.1 Field supermultiplets (the left-chiral supermultiplet)

Simplest case: $N=1$, scalar $\phi(x)$ obeying $[Q^\dagger_{\dot{b}}, \phi(x)] = 0$: Write $[Q_a, \phi(x)] = -I\zeta_a(x)$, which is a $\left(\frac{1}{2}, 0\right)$ field.

$\{Q^\dagger_{\dot{a}}, \zeta_b(x)\} = 2\sigma^\mu_{b\dot{a}}\partial_\mu\phi(x)$. Thus $[Q,\phi] \sim \zeta$ and $\{Q^\dagger, \zeta\} \sim \phi$, i.e. ϕ and ζ are each others' superpartners.

$\{Q^\dagger_{\dot{a}}, -I\zeta_b\} = \{Q^\dagger_{\dot{a}}, [Q_b, \phi]\} = [\{Q^\dagger_{\dot{a}}, Q_b\}, \phi] = 2\sigma^\mu_{b\dot{a}}[P_\mu, \phi]$. Poincaré transformation for fields on page 25: $[P_\mu, \phi] = I\partial_\mu\phi$.

$\{Q_a, \zeta_b(x)\} = 2I\epsilon_{ab}\mathscr{F}(x)$.

$\{Q_a, -I\zeta_b\} = \{Q_a, [Q_b, \phi]\} = -\{Q_b, [Q_a, \phi]\} = -\{Q_b, I\zeta_a\} \propto \epsilon_{ab}$. $\mathscr{F}$ is $(0,0)$ in $\left(\frac{1}{2}, 0\right) \times \left(\frac{1}{2}, 0\right) = (0,0) + (1,0)$.

$[Q_a, \mathscr{F}(x)] = 0$. $\mathscr{F}$ has no superpartner, it is *auxiliary field*.

$2I\epsilon_{ab}[Q_c, \mathscr{F}] = [Q_c, \{Q_a, \zeta_b\}] = -[Q_a, \{Q_c, \zeta_b\}] = -2I\epsilon_{cb}[Q_a, \mathscr{F}]$. For $a = c \neq b$, $2I\epsilon_{cb}[Q_c, \mathscr{F}] = -2I\epsilon_{cb}[Q_c, \mathscr{F}]$, so $[Q_c, \mathscr{F}] = 0$.

$[Q^{\dagger\dot{a}}, \mathscr{F}(x)] = -\overline{\sigma}^{\mu\ \dot{a}b}\partial_\mu\zeta_b(x)$.

$2I\epsilon_{ab}[Q^\dagger_{\dot{c}}, \mathscr{F}] = [Q^\dagger_{\dot{c}}, \{Q_a, \zeta_b\}] = [\{Q^\dagger_{\dot{c}}, Q_a\}, \zeta_b] - [Q_a, \{Q^\dagger_{\dot{c}}, \zeta_b\}] = [2\sigma^\mu_{a\dot{c}}P_\mu, \zeta_b] - [Q_a, 2\sigma^\mu_{b\dot{c}}\partial_\mu\phi] = 2I\sigma^\mu_{a\dot{c}}\partial_\mu\zeta_b - 2I\sigma^\mu_{b\dot{c}}\partial_\mu\zeta_a$.

Then $\epsilon^{da}\epsilon_{ab}[Q^\dagger_{\dot{c}}, \mathscr{F}] = \delta^d{}_b[Q^\dagger_{\dot{c}}, \mathscr{F}] = \epsilon^{da}\sigma^\mu_{a\dot{c}}\partial_\mu\zeta_b - \epsilon^{da}\sigma^\mu_{b\dot{c}}\partial_\mu\zeta_a = \epsilon^{da}\sigma^\mu_{a\dot{c}}\partial_\mu\zeta_b - \sigma^\mu_{b\dot{c}}\partial_\mu\zeta^d$. Sum $d = b$: $\delta^d{}_d[Q^\dagger_{\dot{c}}, \mathscr{F}] = 2[Q^\dagger_{\dot{c}}, \mathscr{F}] = -2\sigma^\mu_{b\dot{c}}\partial_\mu\zeta^b$.

Then $[Q^{\dagger\dot{a}}, \mathscr{F}] = \epsilon^{\dot{a}\dot{c}}[Q^\dagger_{\dot{c}}, \mathscr{F}] = -\epsilon^{\dot{a}\dot{c}}\sigma^\mu_{b\dot{c}}\epsilon^{bd}\zeta_d = -\overline{\sigma}^{\mu\ \dot{a}d}\zeta_d$, using 4-vector σ matrices with raised indices on page 42.

Conjugation of everything above gives right-chiral supermulitplet.

To simplify algebra, use 4-component Majorana spinors, $\psi = \begin{pmatrix} X_a \\ X^{\dagger\dot{b}} \end{pmatrix}$, with notation of page 43.

Majorana conjugation: $\overline{\psi} = \psi^T \gamma_5 E$ where $E = \begin{pmatrix} \epsilon^{ab} & 0 \\ 0 & -\epsilon_{\dot{a}\dot{b}}(=\epsilon^{ab}) \end{pmatrix}$. From $\overline{\psi} = \psi^\dagger \beta = (X^a, X^\dagger_{\dot{b}})$.

(Anti)commutation relations: $\{Q, \overline{Q}\} = -2I\gamma^\mu P_\mu$ and $[Q, P^\mu] = 0$, where Majorana $Q = \begin{pmatrix} Q_a \\ Q^{\dagger\dot{b}} = (\epsilon^{bc} Q_c)^\dagger \end{pmatrix}$.

Directly from relations on page 87.

Infinitesimal transformation of operator $\mathscr{O}$: $\delta\mathscr{O} = [I\overline{\alpha}Q, \mathscr{O}]$,

where Grassmann spinor $\alpha = \begin{pmatrix} \alpha_a \\ \alpha^{\dagger\dot{b}} \end{pmatrix}$, so $\overline{\alpha}Q = \alpha^a Q_a + \alpha^\dagger_{\dot{b}} Q^{\dagger\dot{b}}$. Implies $\delta\mathscr{O}^\dagger = [I\overline{\alpha}Q, \mathscr{O}^\dagger]$.

$(\delta\mathscr{O})^\dagger = [I\overline{\alpha}Q, \mathscr{O}]^\dagger = [I\overline{\alpha}Q, \mathscr{O}^\dagger] = \delta(\mathscr{O}^\dagger)$, because, as for scalar from 2 spinors and Hermitian conjugate of scalar on page 41,

$(\overline{\alpha}Q)^\dagger = (\alpha^a Q_a + \alpha^\dagger_{\dot{b}} Q^{\dagger\dot{b}})^\dagger = (Q^\dagger_{\dot{a}} \alpha^{\dagger\dot{a}} + Q^b \alpha_b) = (-\alpha^{\dagger\dot{a}} Q^\dagger_{\dot{a}} - \alpha_b Q^b) = (\alpha^\dagger_{\dot{a}} Q^{\dagger\dot{a}} + \alpha^b Q_b) = \overline{\alpha}Q$.

Product rule for δ: $\delta(AB) = (\delta A)B + A\delta B$, where A, B can be fermionic / bosonic.

$[I\overline{\alpha}Q, AB] = I(\overline{\alpha}QAB - AB\overline{\alpha}Q) = I(\overline{\alpha}QAB - A\overline{\alpha}QB + A\overline{\alpha}QB - AB\overline{\alpha}Q) = [I\overline{\alpha}Q, A]B + A[I\overline{\alpha}Q, B]$.

If Abelian limit on page 8 obeyed, finite transformation is $\mathscr{O} \to e^{I\overline{\alpha}Q} \mathscr{O} e^{-I\overline{\alpha}Q}$.

(Anti)commutation relations on page 95 can be simplified with 4-component spinor notation

and definitions $\phi = \frac{1}{\sqrt{2}}(A + IB)$, $\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \zeta_a \\ \zeta^{\dagger\dot{b}} \end{pmatrix}$ and $\mathscr{F} = \frac{1}{\sqrt{2}}(F - IG)$:

$\boxed{1.\ \delta A = \overline{\alpha}\psi}$, $\boxed{2.\ \delta B = -I\overline{\alpha}\gamma_5\psi}$.

$$[Q,\phi] = \begin{pmatrix} [Q_a,\phi] = -I\zeta_a \\ [Q^{\dagger\dot{b}},\phi] = \epsilon^{\dot{b}\dot{a}}[Q^\dagger_{\dot{a}},\phi] = 0 \end{pmatrix} \text{ and } [Q,\phi^\dagger] = \begin{pmatrix} [Q_a,\phi^\dagger] = -[Q^\dagger_{\dot{a}},\phi]^\dagger = 0 \\ [Q^{\dagger\dot{b}},\phi^\dagger] = -\epsilon^{\dot{b}\dot{a}}[Q_a,\phi]^\dagger = -I\zeta_{\dagger\dot{b}} \end{pmatrix},$$

so $[Q,A] = [Q,\frac{1}{\sqrt{2}}(\phi+\phi^\dagger)] = -I\psi$, and $[Q,B] = [Q,\frac{I}{\sqrt{2}}(\phi^\dagger - \phi)] = -\gamma_5\psi$ from definition of γ_5 on page 44..

$\boxed{3.\ \delta F = \overline{\alpha}\gamma^\mu\partial_\mu\psi}$, $\boxed{4.\ \delta G = -I\overline{\alpha}\gamma_5\gamma^\mu\partial_\mu\psi}$.

$$[Q,\mathscr{F}] = \begin{pmatrix} [Q_a,\mathscr{F}] = 0 \\ [Q^{\dagger\dot{b}},\mathscr{F}] = -\overline{\sigma}^{\mu\ \dot{b}a}\partial_\mu\zeta_a \end{pmatrix} \text{ and, using } \overline{\sigma}^{\mu\ \dot{b}c*} = \overline{\sigma}^{\mu\ \dot{c}b},$$

$$[Q,\mathscr{F}^\dagger] = \begin{pmatrix} [Q_a,\mathscr{F}^\dagger] = -\epsilon_{ab}[Q^{\dagger\dot{b}},\mathscr{F}]^\dagger = \epsilon_{ab}\overline{\sigma}^{\mu\ \dot{b}c*}\partial_\mu\zeta^\dagger_{\dot{c}} = \epsilon_{ab}\overline{\sigma}^{\mu\ \dot{c}b}\partial_\mu\zeta^\dagger_{\dot{c}} = \epsilon_{ab}\overline{\sigma}^{\mu\ \dot{c}b}\epsilon_{\dot{c}\dot{d}}\partial_\mu\zeta^{\dagger\dot{d}} = -\sigma^\mu_{a\dot{d}}\partial_\mu\zeta^{\dagger\dot{d}} \\ [Q^{\dagger\dot{b}},\mathscr{F}^\dagger] = -\epsilon^{ba}[Q_a,\mathscr{F}]^\dagger = 0 \end{pmatrix}, \text{ so}$$

$$[Q,F] = [Q,\tfrac{1}{\sqrt{2}}(\mathscr{F}+\mathscr{F}^\dagger)] = -\frac{I}{\sqrt{2}}\begin{pmatrix} -I\sigma^\mu_{a\dot{b}}\partial_\mu\zeta^{\dagger\dot{b}} \\ -I\overline{\sigma}^{\mu\ \dot{b}d}\partial_\mu\zeta_d \end{pmatrix} = -I\gamma^\mu\partial_\mu\psi \text{ and}$$

$$[Q,G] = [Q,\tfrac{I}{\sqrt{2}}(\mathscr{F}-\mathscr{F}^\dagger)] = \frac{1}{\sqrt{2}}\begin{pmatrix} I\sigma^\mu_{a\dot{b}}\partial_\mu\zeta^{\dagger\dot{b}} \\ -I\overline{\sigma}^{\mu\ \dot{b}d}\partial_\mu\zeta_d \end{pmatrix} = -\gamma_5\gamma^\mu\partial_\mu\psi.$$

$\boxed{5.\ \delta\psi = \partial_\mu(A + I\gamma_5 B)\gamma^\mu\alpha + (F - I\gamma_5 G)\alpha}$.

$$[I\overline{\alpha}Q,\psi] = \frac{I}{\sqrt{2}}\begin{pmatrix} \alpha^c\{Q_c,\zeta_a\} + \alpha^\dagger_{\dot{d}}\{Q^{\dagger\dot{d}},\zeta_a\} \\ \alpha^c\{Q_c,\zeta^{\dagger\dot{b}}\} + \alpha^\dagger_{\dot{d}}\{Q^{\dagger\dot{d}},\zeta^{\dagger\dot{b}}\} \end{pmatrix} = \frac{I}{\sqrt{2}}\begin{pmatrix} \alpha^c\{Q_c,\zeta_a\} + \alpha^\dagger_{\dot{d}}\epsilon^{\dot{d}\dot{c}}\{Q^\dagger_{\dot{c}},\zeta_a\} \\ \alpha^c\{Q^\dagger_{\dot{c}},\zeta_d\}^\dagger\epsilon^{\dot{b}\dot{d}} + \alpha^\dagger_{\dot{d}}\epsilon^{\dot{d}\dot{c}}\epsilon^{\dot{b}\dot{a}}\{Q_c,\zeta_a\}^\dagger \end{pmatrix}$$

$$= \frac{I}{\sqrt{2}}\begin{pmatrix} \alpha^c(2I\epsilon_{ca}\mathscr{F}) + \alpha^\dagger_{\dot{d}}\epsilon^{\dot{d}\dot{c}}(2\sigma^\mu_{a\dot{c}}\partial_\mu\phi) \\ \epsilon^{ca}\alpha_a(2\sigma^\mu_{c\dot{d}}\partial_\mu\phi^\dagger)\epsilon^{\dot{b}\dot{d}} - \alpha^\dagger_{\dot{d}}\epsilon^{\dot{d}\dot{c}}\epsilon^{\dot{b}\dot{a}}(2I\epsilon_{\dot{c}\dot{a}}\mathscr{F}^\dagger) \end{pmatrix}$$

$$= I\sqrt{2}\begin{pmatrix} -\alpha_a(I\mathscr{F}) - \alpha^{\dagger\dot{c}}(\sigma^\mu_{a\dot{c}}\partial_\mu\phi) \\ -\alpha_a(\overline{\sigma}^{\mu\ \dot{b}a}\partial_\mu\phi^\dagger) - \alpha^{\dagger\dot{b}}(I\mathscr{F}^\dagger) \end{pmatrix} = \begin{pmatrix} \alpha_a(F - IG) - I\sigma^\mu_{a\dot{c}}\alpha^{\dagger\dot{c}}\partial_\mu(A + IB) \\ \alpha^{\dagger\dot{b}}(F + IG) - I\overline{\sigma}^{\mu\ \dot{b}a}\alpha_a\partial_\mu(A - IB) \end{pmatrix}.$$

For $\mathscr{L} = -\frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}\partial_\mu B\partial^\mu B - \frac{1}{2}\overline{\psi}\gamma^\mu\partial_\mu\psi + \frac{1}{2}(F^2 + G^2) + m(FA + GB - \frac{1}{2}\overline{\psi}\psi)$

$+g[F(A^2 + B^2) + 2GAB - \overline{\psi}(A + I\gamma_5 B)\psi]$, above transformations 1 — 5 leave action $\mathscr{A} = \int d^4x\mathscr{L}$ invariant.

For example, for $m = g = 0$, $\delta\mathscr{L} = -\partial_\mu\delta A\partial^\mu A - \partial_\mu\delta B\partial^\mu B - (\delta\overline{\psi})\gamma^\mu\partial_\mu\psi + F\delta F + G\delta G$.

We have replaced $-\frac{1}{2}\overline{\psi}\gamma^\mu\partial_\mu\delta\psi \to -\frac{1}{2}(\delta\overline{\psi})\gamma^\mu\partial_\mu\psi$, since difference with analogous term in hermitian conjugate (h.c.) of $\mathscr{L}$

(must be added to make $\mathscr{L}$ real) is total derivative $\partial_\mu f$ which doesn't contribute to $\mathscr{A}$:

$(\overline{X}\gamma^\mu\partial_\mu Z)^\dagger = (X^\dagger\beta\gamma^\mu\partial_\mu Z)^\dagger = (\partial_\mu Z^\dagger)\gamma^{\mu\dagger}\beta^\dagger X = -(\partial_\mu Z^\dagger)\beta\gamma^\mu X = -(\partial_\mu\overline{Z})\gamma^\mu X = \overline{Z}\gamma^\mu\partial_\mu X - \partial_\mu(\overline{Z}\gamma^\mu X)$.

No derivatives of F, G (or $\mathscr{F}$) appear, so they are *auxiliary fields*,

i.e. can be expressed in terms of the other supermultiplet fields by solving equations of motion $\frac{\partial\mathscr{L}}{\partial F} = \frac{\partial\mathscr{L}}{\partial G} = 0$.

Problems 2

1: Derive the equal time anticommutation relations $[\psi_l(\boldsymbol{x},t),\psi^\dagger_{l'}(\boldsymbol{y},t)]_+ = \delta_{ll'}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$ for a $\left(\frac{1}{2},0\right)+\left(0,\frac{1}{2}\right)$ field $\psi(x) = \int \frac{d^3p}{(2\pi)^3}\left[e^{Ip\cdot x}u_\sigma(\boldsymbol{p})a_\sigma(\boldsymbol{p}) + e^{-Ip\cdot x}v_\sigma(\boldsymbol{p})a^{c\dagger}_\sigma(\boldsymbol{p})\right]$ where u and v obey the projection operator properties $u_{l\ \sigma}(\boldsymbol{p})\overline{u}_{l'\ \sigma}(\boldsymbol{p}) = \frac{1}{2p^0}\left(-I\gamma^\mu p_\mu + m\right)_{ll'}$ and $v_{l\ \sigma}(\boldsymbol{p})\overline{v}_{l'\ \sigma}(\boldsymbol{p}) = \frac{1}{2p^0}\left(-I\gamma^\mu p_\mu - m\right)_{ll'}$.

2: Show that the canonically conjugate momenta to the $\left(\frac{1}{2},0\right)+\left(0,\frac{1}{2}\right)$ field $\psi(x)$ for the Lagrangian density $\mathscr{L}_{\text{Dirac}} = -\overline{\psi}\left(\gamma^\mu\partial_\mu + m\right)\psi$ is consistent with the equal time anticommutation relations $[\psi_l(\boldsymbol{x},t),\psi^\dagger_{l'}(\boldsymbol{y},t)]_+ = \delta_{ll'}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$.

3: Under a local gauge transformation of a group G, the field strength $F_{\mu\nu} = F^i_{\mu\nu}t_i$ transforms as $F_{\mu\nu} \to UF_{\mu\nu}U^\dagger$, where U and t_i are respectively any element of G and the generators of G in any representation. By considering infinitessimal transformations, show that this implies that $F_{\mu\nu}$ transforms in the adjoint representation of G, i.e. $F^i \to U_{ij}F^j$, where the generators of U_{ij} are $t^k_{ji} = IC_{kij}$.

4: How must the right-handed neutrino field ν_{eR} transform under the SM group in order for the Lagrangian term proportional to $\overline{\mathscr{E}}^K_a(\epsilon\phi_H)^\dagger_a\nu_{eR}$ to be invariant?

5: Suppose that there exists a two-component *fermionic* additive symmetry generator Q_a which transforms in the $\left(\frac{1}{2},0\right)$ representation. Using the Coleman-Mandula theorem, show that $\{Q_a, Q^\dagger_{\dot{b}}\} = 2\sigma^\mu_{a\dot{b}}P_\mu$. What does this tell you about any particle? (Note that, as an additive symmetry generator, $Q = Q_{\sigma\rho}a^\dagger_\sigma a_\rho + R_{\sigma\rho}a^\dagger_\rho a_\sigma$, where σ sums over bosonic particles, ρ over fermionic particles.)

6: For more than one spinor of such generators Q_{ar} distinguished by the label r, show that $\{Q_{ar}, Q^\dagger_{\dot{b}s}\} = 2\delta_{rs}\sigma^\mu_{a\dot{b}}P_\mu$.

7: How many different states are there in a massless supermultiplet for a given N? (Hint: working in a frame for which the particles of a massless supermultiplet have a momentum proportional to $p = (0,0,1,1)$, show that the $Q_{\frac{1}{2}r}$, $r = 1,\ldots,N$, are the only generators that convert the maximum helicity state $|\lambda_{\max}\rangle$ in this supermultiplet into any other state in this supermultiplet, because the other fermionic generators either annihilate $|\lambda_{\max}\rangle$ or just remove any $Q_{\frac{1}{2}r}$ generator.)

8: Starting with a field $\phi(x)$, the generators Q_a and the constraint $[Q^\dagger_{\dot{b}},\phi(x)] = 0$, determine all the other fields in the same supermulitplet as $\phi(x)$.

9: The adjoint of a $\left(\frac{1}{2},0\right)+\left(0,\frac{1}{2}\right)$ spinor ψ is defined as $\overline{\psi} = \psi^\dagger\beta$. If ψ is a Majorana spinor, show that $\overline{\psi} = \psi^T\gamma_5 E$. (Hint: Write $\psi^T = \left(X_a, X^{\dagger\dot{b}} = (\epsilon^{bc}X_c)^\dagger\right)$.)

10: Defining $Q = \left(Q_a, Q^{\dagger\dot{b}}\right)$, show that the anticommutation relations above read $\{Q,\overline{Q}\} = -2I\gamma^\mu P_\mu$.

3.4 Grassman variables

N independent Grassman variables θ_α are defined to obey $\{\theta_\alpha, \theta_\beta\} = 0$.

Thus $\theta_\alpha^2 = 0$ (no sum), so for e.g. $N = 3$, *any* function $f(\boldsymbol{\theta})$ has the Grassman variable expansion

$$f(\boldsymbol{\theta}) = A + B\theta_1 + C\theta_2 + D\theta_3 + E\theta_1\theta_2 + F\theta_2\theta_3 + G\theta_3\theta_1 + H\theta_1\theta_2\theta_3.$$

Define (no sums, $\beta \neq \alpha$) $\frac{\partial}{\partial\theta_\alpha}\theta_\alpha = 1$, $\frac{\partial}{\partial\theta_\alpha}\theta_\beta = 0$, $\frac{\partial}{\partial\theta_\alpha}\theta_\alpha\theta_\beta = \theta_\beta$ and $\frac{\partial}{\partial\theta_\alpha}\theta_\beta\theta_\alpha = -\theta_\beta$.

Note (β can be equal to α now) that $\left\{\frac{\partial}{\partial\theta_\alpha}, \frac{\partial}{\partial\theta_\beta}\right\} = 0$ and $\left\{\frac{\partial}{\partial\theta_\alpha}, \theta_\beta\right\} = \delta_{\alpha\beta}$.

By anticommuting Grassman variables $\theta_\alpha, \theta_\beta$ to the left, all terms in the Grassman variable expansion of any function f will be of the form g, $\theta_\alpha g$, $\theta_\beta g$ and $\theta_\alpha\theta_\beta g$, where g is independent of $\theta_\alpha, \theta_\beta$ but can depend on the other θ_γ.

Then show $\begin{pmatrix} \left[\frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta_\beta} + \frac{\partial}{\partial\theta_\beta}\frac{\partial}{\partial\theta_\alpha}\right] \\ \left[\frac{\partial}{\partial\theta_\alpha}\theta_\beta + \theta_\beta\frac{\partial}{\partial\theta_\alpha}\right]\ (\beta \neq \alpha) \end{pmatrix} \begin{pmatrix} g = 0 \\ \theta_\alpha g = 0 \\ \theta_\beta g = 0 \\ \theta_\alpha\theta_\beta g = 0 \end{pmatrix}$ and (no sum, $\beta \neq \alpha$) $\left[\frac{\partial}{\partial\theta_\alpha}\theta_\alpha + \theta_\alpha\frac{\partial}{\partial\theta_\alpha}\right] \begin{pmatrix} g = g \\ \theta_\alpha g = \theta_\alpha g \\ \theta_\beta g = \theta_\beta g \\ \theta_\alpha\theta_\beta g = \theta_\alpha\theta_\beta g \end{pmatrix}$.

Let Q generate translations in *superspace* $(x, \theta = \left(\theta_a, \theta^{\dagger \dot{b}}\right)^T)$ (θ_a are Grassman) on *superfield* $S(x,\theta)$ via

$[I\overline{\alpha}Q, S(x,\theta)] = \overline{\alpha}\mathscr{Q}S(x,\theta)$, as P^μ generates spacetime translations on field $\phi_l(x)$ via $[P^\mu, \phi_l(x)] = I\partial^\mu\phi_l(x)$.

Definition of superfield: $\delta S = \overline{\alpha}\mathscr{Q}S$. From infinitesimal transformation of operator $\mathscr{O}$ on page 96.

Condition on $\mathscr{Q}$: $\{\mathscr{Q}_\alpha, \overline{\mathscr{Q}}_\beta\} = 2\gamma^\mu_{\alpha\beta}\partial_\mu$, where α, β run over the 4 spinor indices.

From (anti)commutation relations on page 96, $[\{Q_\alpha, \overline{Q}_\beta\}, S] = -2I\gamma^\mu_{\alpha\beta}[P_\mu, S]$, which from above reads $\{\mathscr{Q}_\alpha, \overline{\mathscr{Q}}_\beta\}S = 2\gamma^\mu_{\alpha\beta}\partial_\mu S$.

Definition of $\mathscr{Q}$: $\mathscr{Q} = -\frac{\partial}{\partial\overline{\theta}} + \gamma^\mu\theta\partial_\mu$ (explicitly, $\mathscr{Q}_\alpha = -\frac{\partial}{\partial\theta_\beta}(\gamma_5 E)_{\beta\alpha} + \gamma^\mu_{\alpha\beta}\theta_\beta\partial_\mu = (\gamma_5 E)_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} + \gamma^\mu_{\alpha\beta}\theta_\beta\partial_\mu$).

For a (and b) Grassmann, $\frac{\partial}{\partial a}$ is *left derivative*: anticommute a left then remove it, e.g. $\frac{\partial}{\partial a}ba = -\frac{\partial}{\partial a}ab = -b$.

This choice satisfies condition on $\mathscr{Q}$: We use $(\gamma_5 E)^2 = -\mathbf{1}$, $(\gamma_5 E)^T = -\gamma_5 E$ (direct calculation), and $\left(\frac{\partial}{\partial\overline{\theta}}\right)_\alpha = \left(\frac{\partial}{\partial\theta}\right)_\beta(\gamma_5 E)_{\beta\alpha}$

from Majorana conjugation on page 96. Now $\overline{\mathscr{Q}}_\delta = \mathscr{Q}^\dagger_\alpha\beta_{\alpha\delta} = -\left(\frac{\partial}{\partial\theta^\dagger}\right)_\beta(\gamma_5 E)_{\beta\alpha}\beta_{\alpha\delta} + \gamma^{\mu*}_{\alpha\beta}\theta^\dagger_\beta\beta_{\alpha\delta}\partial_\mu$. But $\gamma^{\mu*} = -\beta\gamma^{\mu T}\beta$,

and $\beta^2 = \mathbf{1}$ so that $\gamma^{\mu*}_{\alpha\beta}\theta^\dagger_\beta\beta_{\alpha\delta} = -\overline{\theta}_\alpha\gamma^\mu_{\alpha\delta}$. Thus, using $\{\gamma_5 E, \beta\} = 0$, $\overline{\mathscr{Q}}_\delta = -\left(\frac{\partial}{\partial\theta}\right)_\alpha(\gamma_5 E)_{\alpha\beta}(\gamma_5 E)_{\beta\delta} - \theta_\alpha(\gamma_5 E)_{\alpha\beta}\gamma^\mu_{\beta\delta}\partial_\mu$.

So $\overline{\mathscr{Q}}_\delta = \left(\frac{\partial}{\partial\theta}\right)_\delta - \theta_\alpha(\gamma_5 E\gamma^\mu)_{\alpha\delta}\partial_\mu$. Then from $\left\{\frac{\partial}{\partial\theta_\alpha}, \theta_\beta\right\} = \delta_{\alpha\beta}$, $\{\mathscr{Q}_\alpha, \overline{\mathscr{Q}}_\beta\} = \delta_{\sigma\rho}(\gamma_5 E)_{\rho\alpha}(\gamma_5 E\gamma^\mu)_{\sigma\beta}\partial_\mu + \gamma^\mu_{\alpha\beta}\partial_\mu = 2\gamma^\mu_{\alpha\beta}\partial_\mu$.

Superfields from superfields I: $S = S_1 + S_2$, $S = S_1 S_2$, etc. are superfields when S_1, S_2 are superfields.

1st case clearly obeys SUSY transformation using $\mathscr{Q}$ above.

2nd case: From product rule for δ on page 96, $\delta S_1 S_2 = (\delta S_1)S_2 + S_1 \delta S_2 = (\overline{\alpha}\mathscr{Q}S_1)S_2 + S_1\overline{\alpha}\mathscr{Q}S_2 = \overline{\alpha}\mathscr{Q}(S_1S_2)$.

Superfields from superfields II: $S' = \mathscr{D}_\beta S$, $S' = \overline{\mathscr{D}}_\beta S$,

where *superderivative* $\mathscr{D} = -\frac{\partial}{\partial\overline{\theta}} - \gamma^\mu \theta \partial_\mu$ obeys $\{\mathscr{D}_\beta, \mathscr{Q}_\gamma\} = \{\overline{\mathscr{D}}_\beta, \mathscr{Q}_\gamma\} = 0$.

First case: $\delta S' = \delta \mathscr{D}_\beta S = [I\overline{\alpha}Q, \mathscr{D}_\beta S] = \mathscr{D}_\beta[I\overline{\alpha}Q, S]$ since $\overline{\alpha}Q$ is a commuting object. Then $\delta S' = \mathscr{D}_\beta \overline{\alpha}_\gamma \mathscr{Q}_\gamma S = -\overline{\alpha}_\gamma \mathscr{D}_\beta \mathscr{Q}_\gamma S$

since $\overline{\alpha}_\gamma$, $\mathscr{D}_\beta$ anticommute. Then $\delta S' = \overline{\alpha}_\gamma \mathscr{Q}_\gamma \mathscr{D}_\beta S = \overline{\alpha}\mathscr{Q}S'$. To show e.g. $\{\mathscr{D}_\beta, \mathscr{Q}_\gamma\} = 0$,

first obtain $\{\mathscr{D}_\beta, \mathscr{Q}_\gamma\} = \left(\gamma^\mu_{\alpha\gamma}\{\theta_\gamma, \left(\frac{\partial}{\partial\theta}\right)_\delta\}(\gamma_5 E)_{\delta\beta} - \alpha \leftrightarrow \beta\right)\partial_\mu f$ and use $(\gamma^\mu\gamma_5 E)^T = \gamma^\mu\gamma_5 E$.

Summary: Function of superfields and their superderivatives is a superfield.

R **quantum number assignments for superspace**: $\theta_{L/R}$ has R quantum number $\mathscr{R} = \pm 1$.

Recall definition of R-symmetry on page 88: $Q_a \to e^{I\mathscr{R}_L\phi}Q_a$ and $Q^\dagger_a \to e^{I\mathscr{R}_R\phi}Q^\dagger_a$, where $\mathscr{R} = \mathscr{R}_{L/R} = \pm 1$.

Assignments for superspace follow from definition of superfield (i.e. $[Q, S(x,\theta)] = -I\mathscr{Q}S(x,\theta)$) and definition of $\mathscr{Q}$ on page 101.

Note $\theta_R \sim \theta^*_L$.

From now on, write e.g. $A_\mu\gamma^\mu = \not{A}$.

General form of superfield: $S(x,\theta) = C(x) - I[\overline{\theta}\gamma_5]\omega(x) - \frac{I}{2}\left[\overline{\theta}\gamma_5\theta\right] M(x) - \frac{1}{2}[\overline{\theta}\theta]N(x) + \frac{I}{2}[\overline{\theta}\gamma_5\gamma_\mu\theta]V^\mu(x)$

$$-I[(\overline{\theta}\gamma_5\theta)\overline{\theta}]\left(\lambda(x) + \tfrac{1}{2}\not{\partial}\omega(x)\right) - \tfrac{1}{4}[\overline{\theta}\gamma_5\theta]^2\left(D(x) + \tfrac{1}{2}\partial^2 C(x)\right),$$

4 (Pseudo)scalar fields: C, M, N, D, 2 Spinor fields: ω, λ, 1 Vector field: V^μ

(i.e. 8 bosonic and 8 fermionic degrees of freedom).

This is most general Taylor series in $\theta^T = (\theta_1, \theta_2, \theta_3, \theta_4)$, or $\overline{\theta} = \theta^T\gamma_5 E = (-\theta_2, \theta_1, \theta_4, -\theta_3)$, in manifestly Lorentz invariant form:

2nd term $-I\overline{\theta}\gamma_5\omega(x)$ most general quantity linear in θ_α.

Next 3 bilinears in θ_α is expansion for any bilinear $B = B_{12}\theta_1\theta_2 + B_{13}\theta_1\theta_3 + B_{14}\theta_1\theta_4 + B_{23}\theta_2\theta_3 + B_{24}\theta_2\theta_4 + B_{34}\theta_3\theta_4$:

$\overline{\theta}\gamma_5(\frac{1}{I}\gamma_0 \text{ or } \frac{1}{I}\gamma_3)\theta = \underline{2\theta_2\theta_3 \pm 2\theta_4\theta_1}$, $\overline{\theta}\gamma_5(\frac{1}{I}\gamma_1 \text{ or } \frac{1}{I}\gamma_2)\theta = \underline{2\theta_2\theta_4 \pm 2\theta_3\theta_1}$ (gamma matrices on page 44), $\overline{\theta}(\mathbf{1} \text{ or } \gamma_5)\theta = \underline{2\theta_1\theta_2 \pm 2\theta_4\theta_3}$.

6th term has all 4 possible cubics in θ_α: $(\overline{\theta}\gamma_5\theta)\overline{\theta} = 2(\theta_4\theta_3\theta_2, -\theta_4\theta_3\theta_1, \theta_1\theta_2\theta_4, -\theta_1\theta_2\theta_3)$. Last term $\propto [\overline{\theta}\gamma_5\theta]^2 = 8\theta_1\theta_2\theta_3\theta_4$.

Any higher products of θ_α vanish, since $\theta_\alpha^2 = 0$ (no sum). E.g. $\theta_1\theta_2\theta_3\theta_4\theta_3 = -\theta_1\theta_2\theta_3^2\theta_4 = 0$.

Superfield transformation: $\delta S = I\overline{\alpha}\gamma_5\omega + \overline{\theta}(-\partial\!\!\!/ C + I\gamma_5 M + N - I\gamma_5 V\!\!\!/)\alpha + \frac{I}{2}[\overline{\theta}\gamma_5\theta]\overline{\alpha}(\lambda + \partial\!\!\!/\omega)$

$$-\tfrac{I}{2}[\overline{\theta}\theta]\overline{\alpha}\gamma_5(\lambda + \partial\!\!\!/\omega) + \tfrac{I}{2}[\overline{\theta}\gamma_5\gamma^\mu\theta]\overline{\alpha}\gamma_\mu\lambda + \tfrac{I}{2}[\overline{\theta}\gamma_5\gamma^\nu\theta]\overline{\alpha}\partial_\nu\omega$$

$$+\tfrac{1}{2}[(\overline{\theta}\gamma_5\theta)\overline{\theta}]\left(I\partial\!\!\!/ M - \gamma_5\partial\!\!\!/ N - I\partial_\mu V\!\!\!/\gamma^\mu + \gamma_5(D + \tfrac{1}{2}\partial^2 C)\right)\alpha - \tfrac{I}{4}[\overline{\theta}\gamma_5\theta]^2\overline{\alpha}\gamma_5(\partial\!\!\!/\lambda + \tfrac{1}{2}\partial^2\omega).$$

From definition of superfield and of $\mathscr{Q}$, on page 101, i.e. $\delta S = \overline{\alpha}\left(-\frac{\partial}{\partial\overline{\theta}} + \gamma^\mu\theta\partial_\mu\right)S$.

Note for $M = \mathbf{1}$, $M = \gamma_5\gamma_\mu$ and $M = \gamma_5$, we have $\frac{\partial}{\partial\overline{\theta}}(\overline{\theta}M\theta) = 2M\theta$. See also Weinberg III (hardback), page 63.

Component field transformations: $\delta C = I(\overline{\alpha}\gamma_5\omega)$, $\delta\omega = (-I\gamma_5\partial\!\!\!/ C - M + I\gamma_5 N + V\!\!\!/)\alpha$,

$\delta M = -\overline{\alpha}(\lambda + \partial\!\!\!/\omega)$, $\delta N = I\overline{\alpha}\gamma_5(\lambda + \partial\!\!\!/\omega)$, $\delta V_\mu = \overline{\alpha}(\gamma_\mu\lambda + \partial_\mu\omega)$,

$\delta\lambda = (\frac{1}{2}[\partial_\mu V\!\!\!/, \gamma^\mu] + I\gamma_5 D)\alpha$ and $\delta D = I\overline{\alpha}\gamma_5\partial\!\!\!/\lambda$.

Compare superfield transformation above with general form of superfield on page 103.

D-component of general superfield (coefficient of $[\overline{\theta}\gamma_5\theta]^2$) is candidate for SUSY Lagrangian.

If $\mathscr{L} \propto [S]_D \propto D + \frac{1}{2}\partial^2 C$, then $\delta\mathscr{L} \propto \delta D + \frac{1}{2}\partial^2\delta C = \partial_\mu(I\overline{\alpha}\gamma_5\gamma^\mu\lambda + \frac{1}{2}\partial^\mu\delta C)$, i.e. a derivative,

so action $\mathscr{A} = \int d^4x\mathscr{L}$ obeys $\delta\mathscr{A} = 0$.

D-component of superfield cannot be used to form kinematic Lagrangian.

Kinematic action must be quadratic in elementary fields and therefore in elementary superfields, so can only try

$[S^*S]_D = -\partial_\mu C^*\partial^\mu C - \frac{1}{2}\left(\overline{\omega}\gamma^\mu\partial_\mu\omega\right) + \frac{1}{2}(\partial_\mu\overline{\omega})\gamma^\mu\omega + C^*D + D^*C - \overline{\omega}\lambda - \overline{\lambda}\omega + M^*M + N^*N - V_\mu^*V^\mu$.

Terms with D and λ constrain C and ω to vanish.

So introduce constraints on elementary fields.

3.5.1 Chiral superfield

Definition of chiral superfield $X(x,\theta)$: $\lambda = D = 0$, $V_\mu = \partial_\mu B$, B is scalar field with $\delta B = \overline{\alpha}\omega$.

This is superfield because these conditions are preserved by SUSY transformations: $\delta D = I\overline{\alpha}\gamma_5\not{\partial}\lambda = 0$,

and $\delta\lambda = (\frac{1}{2}[\partial_\mu\not{V}, \gamma^\mu] + I\gamma_5 D)\alpha = \frac{1}{2}[\partial_\mu\not{V}, \gamma^\mu]\alpha = 0$ because $V_\mu = \partial_\mu B$: $[\partial_\mu\not{V}, \gamma^\mu] = \partial_\mu\partial_\nu B[\gamma^\nu, \gamma^\mu] = 0$.

Finally, $\delta V^\mu = \overline{\alpha}(\gamma_\mu\lambda + \partial_\mu\omega) = \partial_\mu\overline{\alpha}\omega$, i.e. $\delta B = \overline{\alpha}\omega$, i.e. $V_\mu = \partial_\mu B$ condition is preserved.

Chiral superfield decomposition: $X(x,\theta) = \frac{1}{\sqrt{2}}[\Phi_+(x,\theta) + \Phi_-(x,\theta)]$ with left / right-chiral superfields

$$\Phi_\pm(x,\theta) = \phi_\pm(x) - \sqrt{2}\,\overline{\theta}\psi_{L/R}(x) + [\overline{\theta}P_{L/R}\theta]\mathscr{F}_\pm(x) \pm \frac{1}{2}[\overline{\theta}\gamma_5\gamma_\mu\theta]\partial^\mu\phi_\pm(x) \mp \frac{1}{\sqrt{2}}(\overline{\theta}\gamma_5\theta)\overline{\theta}\not{\partial}\psi_{L/R}(x) - \frac{1}{8}[\overline{\theta}\gamma_5\theta]^2\partial^2\phi_\pm(x),$$

with $\psi_{L/R} = P_{L/R}\psi$, $\phi_\pm = \frac{1}{\sqrt{2}}(A \pm IB)$, $\mathscr{F}_\pm = \frac{1}{\sqrt{2}}(F \mp IG)$.

To make contact with notation on page 97, write $C = A$, $\omega = -I\gamma_5\psi$, $M = G$, $N = -F$, for later convenience

in general form of superfield on page 103, so $X = A - \overline{\theta}\psi + \frac{1}{2}\overline{\theta}\theta F - \frac{I}{2}\overline{\theta}\gamma_5\theta G + \frac{I}{2}\overline{\theta}\gamma_5\gamma_\mu\theta\partial^\mu B + \frac{1}{2}(\overline{\theta}\gamma_5\theta)\overline{\theta}\gamma_5\not{\partial}\psi - \frac{1}{8}[\overline{\theta}\gamma_5\theta]^2\partial^2 A$.

$\Phi_\pm$ is obtained from $\Phi_\pm = A' - \overline{\theta}\psi' + \frac{1}{2}\overline{\theta}\theta F' - \frac{I}{2}\overline{\theta}\gamma_5\theta G' + \frac{I}{2}\overline{\theta}\gamma_5\gamma_\mu\theta\partial^\mu B' + \frac{1}{2}(\overline{\theta}\gamma_5\theta)\overline{\theta}\gamma_5\not{\partial}\psi' - \frac{1}{8}[\overline{\theta}\gamma_5\theta]^2\partial^2 A'$ with $F' = \pm G'$,

$B' = \mp IA'$, $\psi' = \sqrt{2}\psi_{L/R}$, i.e. by taking $N = \pm IM$, $B = \mp IC$, $\omega = \pm$ left/right-handed spinor in general form of superfield

on page 103, which is preserved under component field transformations on page 104.

Left / right transformation: $\delta\psi_{L/R} = \sqrt{2}(\partial_\mu\phi_\pm\gamma^\mu P_{R/L} + \mathscr{F}_\pm P_{L/R})\alpha$, $\delta\mathscr{F}_\pm = \sqrt{2}\,\overline{\alpha}\partial\!\!\!/\psi_{L/R}$, $\delta\phi_\pm = \sqrt{2}\,\overline{\alpha}\psi_{L/R}$.

So left-chiral supermultiplet here = that on page 95 ($\zeta_a = \psi_L$, $\phi = \phi_+$, $\mathscr{F} = \mathscr{F}_+$), right-chiral by conjugation.

Compact form: $\Phi_\pm(x,\theta) = \phi_\pm(x_\pm) \mp \sqrt{2}\,\theta^T_{L/R}E\;\psi_{L/R}(x_\pm) \pm \theta^T_{L/R}E\theta_{L/R}\;\mathscr{F}_\pm(x_\pm)$,

where $\theta_{L/R} = P_{L/R}\theta$, $x^\mu_\pm = x^\mu \pm \theta^T_R E\gamma^\mu\theta_L$.

Recall $\psi^T = \frac{1}{\sqrt{2}}\left(\zeta_a, \zeta^{\dagger\dot{b}}\right)$ from page 97. This is left / right-chiral superfield on page 106, by Taylor expansion in $\theta^T_R E\gamma^\mu\theta_L$:

Make use of $\theta^T = -\overline{\theta}\gamma_5 E$ (Majorana conjugation on page 96) and $E^2 = (\gamma_5 E)^2 = -\mathbf{1}$. Also $\underline{x^\mu_\pm = x^\mu \pm \frac{1}{2}\overline{\theta}\gamma_5\gamma^\mu\theta}$

(see alternative form for $x_\pm$ on page 108). Firstly, rewrite as $\underline{\Phi_\pm(x,\theta) = \phi_\pm(x_\pm) - \sqrt{2}\,\overline{\theta}P_{L/R}\psi(x_\pm) + \overline{\theta}P_{L/R}\theta\mathscr{F}_\pm(x_\pm)}$,

because $\theta^T_{L/R}E\psi_{L/R} = -\overline{\theta}\gamma_5 E\frac{1}{2}(1\pm\gamma_5)E\frac{1}{2}(1\pm\gamma_5)\psi = -\overline{\theta}\frac{1}{2}(-\gamma_5 \mp 1)\frac{1}{2}(1\pm\gamma_5)\psi = \pm\overline{\theta}P_{L/R}\psi$.

1st term in underlined equation above: $\phi_\pm(x_\pm) = \phi_\pm(x) \pm \frac{1}{2}(\overline{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\phi_\pm(x) - \frac{1}{2}(\frac{1}{2}\overline{\theta}\gamma_5\gamma^\mu\theta)(\frac{1}{2}\overline{\theta}\gamma_5\gamma^\nu\theta)\partial_\mu\partial_\nu\phi_\pm(x)$

$= \phi_\pm(x) \pm \frac{1}{2}(\overline{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\phi_\pm(x) - \frac{1}{2}(\frac{1}{2}\overline{\theta}\gamma_5\theta)^2\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu\phi_\pm(x)$, then use $\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} = g^{\mu\nu}$.

2nd term: $-\sqrt{2}\,\overline{\theta}P_{L/R}\psi(x_\pm) = -\sqrt{2}\,\overline{\theta}P_{L/R}\psi(x) \mp \frac{1}{\sqrt{2}}(\overline{\theta}\gamma_5\gamma^\mu\theta)\overline{\theta}\partial_\mu P_{L/R}\psi(x) = -\sqrt{2}\,\overline{\theta}P_{L/R}\psi(x) \mp \frac{1}{\sqrt{2}}(\overline{\theta}\gamma_5\theta)\overline{\theta}\gamma^\mu\partial_\mu P_{L/R}\psi(x)$.

3rd term: $\overline{\theta}P_{L/R}\theta\mathscr{F}_\pm(x_\pm) = \overline{\theta}P_{L/R}\theta\mathscr{F}_\pm(x) \pm \frac{1}{2}\overline{\theta}\gamma_5\gamma^\mu\theta\overline{\theta}P_{L/R}\theta\partial_\mu\mathscr{F}_\pm(x) = \overline{\theta}P_{L/R}\theta\mathscr{F}_\pm(x) + [(\theta^T_R E\gamma^\mu\theta_L)(\theta^T_{L/R}E\theta_{L/R})\partial_\mu\mathscr{F}_\pm(x)]$,

term in square brackets vanishes due to 3 occurences of "2-component" $\theta_{L/R}$.

Alternative form for $x_\pm$: $x_\pm^\mu = x^\mu \pm \frac{1}{2}\overline{\theta}\gamma_5\gamma^\mu\theta$.

Recall $x_\pm^\mu = x^\mu \pm \theta_R^T E\gamma^\mu\theta_L$. Now $\frac{1}{2}\overline{\theta}\gamma_5\gamma^\mu\theta = \frac{1}{2}\theta^T\gamma_5 E\gamma_5\gamma^\mu\theta = -\frac{1}{2}\theta^T\gamma_5 E\gamma^\mu\gamma_5\theta$ using $\{\gamma^\mu,\gamma_5\} = 0$. But $\gamma_5 = P_L - P_R$,

so $\frac{1}{2}\overline{\theta}\gamma_5\gamma^\mu\theta = \frac{1}{2}\theta^T(P_L - P_R)E\gamma^\mu(P_R - P_L)\theta = \frac{1}{2}\theta^T P_L E\gamma^\mu P_R\theta + \frac{1}{2}\theta^T P_R E\gamma^\mu P_L\theta - \frac{1}{2}\theta^T P_L E\gamma^\mu P_L\theta - \frac{1}{2}\theta^T P_R E\gamma^\mu P_R\theta$.

3rd and 4th terms zero because $P_{L/R}E\gamma^\mu P_{L/R} = EP_{L/R}\gamma^\mu P_{L/R} = E\gamma^\mu P_{R/L}P_{L/R} = 0$.

2nd term is $\frac{1}{2}\theta^T P_R E\gamma^\mu P_L\theta = -(\frac{1}{2}\theta^T P_R E\gamma^\mu P_L\theta)^T = -\frac{1}{2}\theta^T P_L(E\gamma^\mu)^T P_R\theta$ ($-$ sign because components of θ anticommute).

But $(E\gamma^\mu)^T = -E\gamma^\mu$, so 2nd term = 1st term, i.e. $\frac{1}{2}\overline{\theta}\gamma_5\gamma^\mu\theta = \theta_R^T E\gamma^\mu\theta_L$.

Relations between superspace directions: $x_+^* = x_-$.

I.e. show $\theta_R E\gamma^\mu\theta_L = \overline{\theta}\gamma_5\gamma^\mu\theta$ is "imaginary".

$x_+^* = x^\mu + \underline{\theta_R^\dagger E\gamma^{\mu *}\theta_L^*}$. But $\theta_R^T = (0, \theta^{\dagger\dot{b}})$, so $\theta_R^\dagger = (0, \theta^b) = \theta_L E\beta$, and $\theta_L = (\theta_a, 0)^T$ so $\theta_L^* = (\theta_{\dot{a}}^\dagger, 0)^T = E\beta\theta_R$,

and $\gamma^{\mu *} = -\beta\gamma^{\mu T}\beta$. These results together with $\beta^2 = -E^2 = 1$, $\beta E = E\beta$ and $E^T = -E$ give $\underline{\theta_R^\dagger E\gamma^{\mu *}\theta_L^*} = -\theta_R E\gamma^\mu\theta_L$.

Alternatively, use $x_\pm^\mu = x^\mu \pm \frac{1}{2}\overline{\theta}\gamma_5\gamma^\mu\theta$ and show $(\overline{\theta}\gamma_5\gamma^\mu\theta)^\dagger = -\overline{\theta}\gamma_5\gamma^\mu\theta$:

$(\overline{\theta}\gamma_5\gamma^\mu\theta)^\dagger = (\theta^\dagger\beta\gamma_5\gamma^\mu\theta)^\dagger = -\theta^\dagger\gamma_\mu^\dagger\gamma_5\beta\theta$ ($-$ sign because components of θ anticommute).

Then inserting $\beta^2 = 1$ in places, $(\overline{\theta}\gamma_5\gamma^\mu\theta)^\dagger = -(\theta^\dagger\beta)(\beta\gamma_\mu^\dagger\beta)(\beta\gamma_5\beta)\theta = \overline{\theta}\gamma^\mu\gamma_5\theta = -\overline{\theta}\gamma_5\gamma^\mu\theta$.

Compact form on page 107 is most general function of $x_\pm$ and $\theta_{L/R}$.

So any function of left-chiral superfields is left-chiral superfield (likewise right).

Conjugate of right-chiral superfield is left-chiral superfield. Chiral superfield decomposition, page 106: Use $(\overline{\theta}\gamma_5)^* = -\overline{\theta}\gamma_5$.

Superspace "direction" of chiral superfield: $\mathscr{D}_\mp \Phi_\pm = 0$, where *superderivative* $\mathscr{D}_\mp = P_{R/L}\mathscr{D}$.

Can be used to define left / right-chiral superfield.

Follows from $\mathscr{D}_\mp x_\pm^\mu = 0$, using $\mathscr{D}_{\mp\alpha} = \mp E_{\alpha\beta}\frac{\partial}{\partial\theta_{R/L\ \beta}} - (\gamma^\mu\theta_{L/R})_\alpha\frac{\partial}{\partial x^\mu}$. $D_- f(x,\theta) = 0$ implies $f(x,\theta) = f'(x_+,\theta_L)$, i.e. f is left-chiral:

Any function $f(x,\theta) = g(x_+,\theta_R,\theta_L) = \sum_a g^{(a)}(x_+)h^{(a)}(\theta_R,\theta_L)$, where each $h^{(a)}$ is all possible products

of between 1 and 4 components of θ. Using first $D_{-,\alpha}g^{(a)}(x_+) = 0$ then $D_{-,\alpha}h^{(a)}(\theta_R,\theta_L) = -E_{\alpha\beta}\frac{\partial}{\partial\theta_{R\ \beta}}h^{(a)}(\theta_R,\theta_L)$,

we require $E_{\alpha\beta}D_{-,\beta}f = \sum_a g^{(a)}(x_+)\frac{\partial}{\partial\theta_{R\ \alpha}}h^{(a)}(\theta_R,\theta_L) = 0$. Therefore if $h^{(a)}$ contains $\theta_{R\ \alpha}$, $g^{(a)}(x_+) = 0$.

Chiral superfield from superfield: For general superfield S, $\mathscr{D}_{\pm\alpha}\mathscr{D}_{\pm\beta}S$ (i.e. $\mathscr{D}_\pm^T E\mathscr{D}_\pm S$) is left / right-chiral.

$\mathscr{D}_{\pm\gamma}(\mathscr{D}_{\pm\alpha}\mathscr{D}_{\pm\beta}S) = 0$ because the $\mathscr{D}_{\pm\alpha}$ anticommute and there are only 2 of them.

$\mathscr{F}$-component of chiral superfield (coefficient of $\overline{\theta}P_L\theta$) is candidate for SUSY Lagrangian.

If $\mathscr{L} \propto [\Phi_\pm]_\mathscr{F} = \mathscr{F}_\pm$, then $\delta\mathscr{L} \propto \overline{\alpha}\partial\!\!\!/\psi_{L/R}$, i.e. a derivative, so action $\mathscr{A} = \int d^4x\mathscr{L}$ obeys $\delta\mathscr{A} = 0$.

Any left/right-chiral superfield $\Phi_\pm$ can be written $\Phi_\pm = D^2_{R/L} S$ respectively,

where S is a general superfield and $D^2_{R/L} = E_{\alpha\beta} D_{\mp,\alpha} D_{\mp,\beta}$.

Expand $S = f_1(x_+, \theta_L) + f_2(x_+, \theta_L)\theta_{R1} + f_3(x_+, \theta_L)\theta_{R2} + f_4(x_+, \theta_L)\theta_{R1}\theta_{R2}$.

But $\mathscr{D}_{-,\alpha} f(x_+, \theta_L) = 0$ because $\mathscr{D}_{-\alpha} = -E_{\alpha\beta}\frac{\partial}{\partial\theta_{R\ \beta}} - (\gamma^\mu\theta_L)_\alpha \frac{\partial}{\partial x^\mu}$ and $\mathscr{D}_{-\alpha} x_+ = 0$.

Thus $\mathscr{D}_{-\alpha} S = -E_{\alpha\beta}\frac{\partial}{\partial\theta_{R\ \beta}} S$, giving $D^2_{R/L} S = -2 f_4(x_+, \theta_L)$,

where $f_4(x_+, \theta_L)$ is any function of (x_+, θ_L), i.e. a general lef-chiral superfield.

3.5.2 Supersymmetric Actions

From now on, only work with left-chiral superfields, write $\Phi_+ = \Phi$, $\phi_+ = \phi$, $\mathscr{F}_+ = \mathscr{F}$.

Supersymmetric action from chiral superfields: $\mathscr{A} = \int d^4x[f]_{\mathscr{F}} + \int d^4x[f]^*_{\mathscr{F}} + \int d^4x\frac{1}{2}[K]_D$,

where f is left-chiral, K is real superfield.

Superpotential f is polynomial in left-chiral superfields.

Recall from page 109 that function of left-chiral superfields (but not complex conjugates thereof) is left-chiral superfield.

Superderivatives (and therefore derivatives) of left-chiral superfield is not left-chiral. Let S be a general superfield.

If (a term in) f contains a left-chiral superfield of form $\mathscr{D}_{-\alpha}\mathscr{D}_{-\beta}S$ (see chiral superfield from superfield on page 109),

can write $f = \mathscr{D}_R^2 h$ where h is product of S with left-chiral superfields,

since $\mathscr{D}_-$ annihilates all left-chiral superfields after differentiating with product rule.

But $\mathscr{D}_R^2 h \propto$ coeff. of $\theta_R^T E\theta_R$ in h (neglecting spacetime derivatives which don't contribute to $\mathscr{A}$ from now on).

Then $[\mathscr{D}_R^2 h]_{\mathscr{F}} \propto$ coeff. of $(\theta_L^T E\theta_L)(\theta_R^T E\theta_R) = (\overline{\theta}\gamma_5\theta)^2$ in h, i.e. $[h]_D$, so $\int d^4x[\mathscr{D}_R^2 h]_{\mathscr{F}} \propto \int d^4x[h]_D$,

i.e. $\mathscr{D}_R^2 S$ terms not necessary in f, can be included in $[K]_D$.

Kahler potential K depends on left / right-chiral superfields but is more general.

R quantum number assignments for potentials f and K: $\mathscr{R}_f = 2$, $\mathscr{R}_K = 0$.

First case: Want $\int d^4x[f]_{\mathscr{F}}$ to have $\mathscr{R}_{\mathscr{F}} = 0$. From chiral superfield decomposition on page 106, term in left-chiral superfield Φ containing $\mathscr{F}$ component is $\overline{\theta} P_L \theta \mathscr{F} = \pm\theta_L^T E \theta_L \mathscr{F}$. Since $\mathscr{R}_{\theta_L^T E \theta_L} = 2$ and want $\mathscr{R}_{\mathscr{F}} = 0$, must have $\mathscr{R}_f = 2$.

Second case: Want $\int d^4x[K]_D$ to have $\mathscr{R}_D = 0$. From general form of superfield on page 103, term containing D component is $\propto [\overline{\theta}\gamma_5\theta]^2(D + \frac{1}{2}\partial^2 C)$. But $\mathscr{R}_{[\overline{\theta}\gamma_5\theta]^2} = 0$, so for $\mathscr{R}_D = 0$ must have $\mathscr{R}_K = 0$.

For renormalizable theory of chiral superfields:

Superpotential f is at most cubic in left-chiral superfields Φ_n.

Operators O in $\mathscr{L}$ must have mass dimension $d_M(O) \leq 4$ for renormalizability. From definition of $\mathscr{D}$, $d_M(\theta) = -\frac{1}{2}$.

So $d_M([S]_{\mathscr{F}}) = d_M(S) + 1$, $d_M([S]_D) = d_M(S) + 2$, so operators (represented by S here) in f (K) have $d_M(S) \leq 3(2)$.

Since $d_M(\Phi_\pm) = d_M(\phi_\pm) = 1$, f can only be cubic polynomial in Φ_n.

General form of Kahler potential: $K(\Phi, \Phi^*) = \sum_{mn} g_{mn} \Phi_m^* \Phi_n$, g_{mn} Hermitian and positive-definite.

As noted above, operators in K have mass dimensionality ≤ 2. $\Phi_m\Phi_n$ terms (or conjugates thereof) don't appear, since they are left-chiral so have no D term. g_{mn} positive-definite to get correct commutation relations for fields.

Kahler potential part of $\mathscr{L}$: $\frac{1}{2}[K(\Phi,\Phi^*)]_D = -\partial_\mu\phi_n^*\partial^\mu\phi_n + \mathscr{F}_n^*\mathscr{F}_n - \frac{1}{2}\overline{\psi}_{nL}\gamma^\mu\partial_\mu\psi_{nL} + \frac{1}{2}(\partial_\mu\overline{\psi}_{nR})\gamma^\mu\psi_{nR}$.

Direct calculation gives $\frac{1}{2}[K(\Phi,\Phi^*)]_D = g_{mn}\left[-\partial_\mu\phi_m^*\partial^\mu\phi_n + \mathscr{F}_m^*\mathscr{F}_n - \frac{1}{2}\overline{\psi}_{mL}\gamma^\mu\partial_\mu\psi_{nL} + \frac{1}{2}(\partial_\mu\overline{\psi}_{mR})\gamma^\mu\psi_{nR}\right]$.

Take $\Phi'_m = N_{mn}\Phi_n$, $g'_{mn} = (N^\dagger g N)_{mn}$. Since g_{mn} Hermitian, choose unitary N to diagonalize it.

Diagonal terms positive to get positive coefficient for $-\partial_\mu\phi_n'^*\partial^\mu\phi_n'$, absorb them into ϕ_n' then drop primes.

Superpotential part of $\mathscr{L}$: $[f(\Phi)]_{\mathscr{F}} = -\frac{1}{2}\frac{\partial^2 f(\phi)}{\partial\phi_n\partial\phi_m}\overline{\psi}_{nR}\psi_{mL} + \mathscr{F}_n\frac{\partial f(\phi)}{\partial\phi_n}$. Similar for f^*.

Use compact form on page 107 to write $f(\Phi) = f(\phi_+(x_+) - \sqrt{2}\,\theta_L^T E\ \psi_L(x_+) + \theta_L^T E\theta_L\ \mathscr{F}_+(x_+))$.

Taylor expand this in $-\sqrt{2}\,\theta_L^T E\ \psi_L(x_+) + \theta_L^T E\theta_L\ \mathscr{F}_+(x_+)$. Note e.g. $\theta_{L\alpha}\theta_{L\beta} = E_{\alpha\beta}\theta_{L1}\theta_{L2}$ for $\alpha,\beta \le 1,2$, but $=0$ otherwise.

Replace x_+ with x because $\theta_R^T E\gamma^\mu\theta_L$ gives zero when acting on expression with 2 θ_L factors.

$\mathscr{L}$ from chiral superfields: $\mathscr{L} = -\partial_\mu\phi_n^*\partial^\mu\phi_n - \frac{1}{2}\overline{\psi}_{nL}\gamma^\mu\partial_\mu\psi_{nL} + \frac{1}{2}(\partial_\mu\overline{\psi}_{nR})\gamma^\mu\psi_{nR}$

$$-\frac{1}{2}\frac{\partial^2 f(\phi)}{\partial\phi_n\partial\phi_m}\overline{\psi}_{nR}\psi_{mL} - \frac{1}{2}\left(\frac{\partial^2 f(\phi)}{\partial\phi_n\partial\phi_m}\right)^*(\overline{\psi}_{nR}\psi_{mL})^* - V(\phi),$$

where $\mathscr{F}_n = -\left(\frac{\partial f(\phi)}{\partial\phi_n}\right)^*$ and *scalar field potential* $V(\phi) = \sum_n \left|\frac{\partial f(\phi)}{\partial\phi_n}\right|^2$.

Sum Kahler potential and superpotential parts (and conjugate) of $\mathscr{L}$ above, use field equations to get $\mathscr{F}_n = -\left(\frac{\partial f(\phi)}{\partial\phi_n}\right)^*$.

Zeroth order $\phi_{n0} = \langle 0|\phi_n|0\rangle$ at minimum of $V(\phi)$. If $\frac{\partial f(\phi)}{\partial \phi_n}\Big|_{\phi=\phi_0} = 0$ possible, $V(\phi_0) = 0$ and SUSY unbroken.

Tree-level expansion of V: $V(\phi) = (\mathscr{M}^\dagger \mathscr{M})_{mn} \Delta\phi_m^\dagger \Delta\phi_n$,

where $\Delta\phi_n = \phi_n - \phi_{n0}$, bosonic mass matrix $\mathscr{M}_{mn} = \frac{\partial^2 f}{\partial\phi_m \partial\phi_n}\Big|_{\phi=\phi_0}$.

$$V = \sum_n \left|\frac{\partial f}{\partial \phi_n}\right|^2 \simeq \sum_n \left|\frac{\partial^2 f}{\partial\phi_m\partial\phi_n}\Big|_{\phi=\phi_0} \Delta\phi_m\right|^2 = \Delta\phi_m^\dagger (\mathscr{M}^\dagger \mathscr{M})_{mn} \Delta\phi_n.$$

Matrix $\mathscr{M}$ is complex symmetric, so can be diagonalised by unitary matrix such that diagonal terms are real and positive.

From $\mathscr{L}$ from chiral superfields on page 113, fermionic mass term is $-\frac{1}{2}\mathscr{M}_{mn}\overline{\psi}_{nR}\psi_{mL}$,

so fermions and bosons have same mass, as required in SUSY.

Free Lagrangian density: $\mathscr{L}_0 = \sum_n \Big[-\partial_\mu \Delta\phi_n^* \partial^\mu \Delta\phi_n - m_n^2 \Delta\phi_n^* \Delta\phi_n$

$-\frac{1}{2}\overline{\psi}_{nL}\gamma^\mu\partial_\mu\psi_{nL} + \frac{1}{2}(\partial_\mu\overline{\psi}_{nR})\gamma^\mu\psi_{nR} - \frac{1}{2}m_n\overline{\psi}_{nR}\psi_{nL} - \frac{1}{2}m_n(\overline{\psi}_{nR}\psi_{nL})^* \Big]$.

$$\mathscr{L}_0 = -\partial_\mu\Delta\phi_n^*\partial^\mu\Delta\phi_n - (\mathscr{M}^\dagger\mathscr{M})_{mn}\Delta\phi_m^*\Delta\phi_n - \tfrac{1}{2}\overline{\psi}_{nL}\gamma^\mu\partial_\mu\psi_{nL} + \tfrac{1}{2}(\partial_\mu\overline{\psi}_{nR})\gamma^\mu\psi_{nR} - \tfrac{1}{2}\mathscr{M}_{mn}\overline{\psi}_{nR}\psi_{mL} - \tfrac{1}{2}\mathscr{M}_{mn}^*(\overline{\psi}_{nR}\psi_{mL})^*$$

from $\mathscr{L}$ from chiral superfields on page 113 and tree-level expansion of V above, then diagonalize $\mathscr{M}$.

Problems 3

1: By working in the chiral representation, show that $E^T = -E$, $(\gamma_5 E)^2 = -\mathbf{1}$, $(\gamma_5 E)^T = -\gamma_5 E$, $\gamma^{\mu *} = -\beta\gamma^{\mu T}\beta$, $\beta^2 = \mathbf{1}$, $\{\gamma_5 E, \beta\} = 0$, $[E, \beta] = 0$ and $(E\gamma^\mu)^T = -E\gamma^\mu$.

2: Show that $\overline{(\overline{\psi}^T)} = -\overline{\psi}\gamma_5 E$ and $\overline{\gamma^\mu\psi} = -\overline{\psi}\gamma^\mu$, where ψ is a Majorana spinor.

3: Using $\mathscr{Q}_\alpha = (\gamma_5 E)_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} + \gamma^\mu_{\alpha\beta}\theta_\beta\partial_\mu$ and $\overline{\mathscr{Q}}_\delta = \left(\frac{\partial}{\partial\theta}\right)_\delta - \theta_\alpha(\gamma_5 E\gamma^\mu)_{\alpha\delta}\partial_\mu$, show that $\{\mathscr{Q}_\alpha, \overline{\mathscr{Q}}_\beta\} = 2\gamma^\mu_{\alpha\beta}\partial_\mu$.

4: Show that $\{\mathscr{D}_\beta, \mathscr{Q}_\gamma\} = 0$, where $\mathscr{D}_\alpha = (\gamma_5 E)_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} - \gamma^\mu_{\alpha\beta}\theta_\beta\partial_\mu$.

5: Show that the result obtained by setting $\lambda = D = 0$ and $V_\mu = \partial_\mu B$, where B is scalar field, in a superfield is another superfield, called the chiral superfield. (Hint: show that these constraints are preserved under SUSY transformations.)

6: Show that the chiral superfield is the sum of two chiral superfields, called left- and right-handed chiral superfields respectively, for which $N = \pm IM$, $B = \mp IC$, $\omega = \pm$ left/right-handed spinor. (Hint: show that these constraints are preserved under SUSY transformations. Be careful not to confuse the component fields N, M, ... of the three different superfields.)

7: Show that $\mathscr{D}_\mp\Phi_\pm = 0$, where $\mathscr{D}_{\mp\alpha} = \mp E_{\alpha\beta}\frac{\partial}{\partial\theta_{R/L\ \beta}} - (\gamma^\mu\theta_{L/R})_\alpha\frac{\partial}{\partial x^\mu}$ and $x^\mu_\pm = x^\mu \pm \theta_R^T E\gamma^\mu\theta_L$.

8: Show that $(\theta_R E\gamma^\mu\theta_L)^* = \theta_R E\gamma^\mu\theta_L$.

9: For a general superfield S, calculate $\mathscr{D}_-^T E\mathscr{D}_- S$ and show that the result is the coefficient of $\theta_R^T E\theta_R$ in S up to spacetime derivatives.

10: Show that the D term of $\Phi\Phi'$ vanishes, where Φ and Φ' are left-chiral superfields.

3.6 Superspace integration

Define integration over Grassman variables θ_α as (no sum over $\alpha \neq \beta \neq \gamma$) $\int d\theta_\alpha \theta_\alpha = 1$, $\int d\theta_\alpha = 0$,

more generally e.g. $\int d\theta_\beta d\theta_\alpha \theta_\alpha \theta_\beta \theta_\gamma = \theta_\gamma$.

Because $(\overline{\theta}\gamma_5\theta)^2 = 8\theta_1\theta_2\theta_3\theta_4$, then $\int d^4x[S]_D = -\frac{1}{2}\int d^4x \int d^4\theta S(x,\theta)$.

Because $\theta_L^T E\theta_L = 2\theta_1\theta_2$ in compact form on page 107, then $\int d^4x[\Phi]_{\mathscr{F}} = \frac{1}{2}\int d^4x \int d^2\theta_L \Phi(x,\theta)$.

Define $\delta^{(4)}(\theta' - \theta)$ via $\int d^4\theta' \delta^{(4)}(\theta' - \theta) f(\theta') = f(\theta)$.

Solution is $\delta^{(4)}(\theta' - \theta) = (\theta'_1 - \theta_1)(\theta'_2 - \theta_2)(\theta'_3 - \theta_3)(\theta'_4 - \theta_4) = \frac{1}{4}\left[(\theta'_L - \theta_L)^T E\,(\theta'_L - \theta_L)\right]\left[(\theta'_R - \theta_R)^T E\,(\theta'_R - \theta_R)\right]$.

Note e.g. $\int d\theta_1 \frac{\partial}{\partial\theta_1} f = 0$, Try $f = 1$ amd $f = \theta_1$.

and $\int d\theta_1 f \frac{\partial}{\partial\theta_1} g = \int d\theta_1 \left(\frac{\partial}{\partial\theta_1} f\right) g$. Both sides zero for $(f,g) = (1,1)$ and $(1,\theta_1)$, both sides 1 for (θ_1,θ_1).

3.7 Supergraphs

(Super)graphs allow for preservation of Lorentz group symmetry (and SUSY) throughout calculations.

In e.g. theory of left-chiral superfields $\Phi_n^*(x,\theta)$ and their complex conjugates,

VEV of time-ordered product of component fields from

VEV of time-ordered product of superfields:

$\langle 0|T\Big\{\Phi_{n_1}(x_1,\theta_1),\Phi_{n_2}(x_2,\theta_2)\ldots\Big\}|0\rangle = \int\Big[\prod_{n,x,\theta} d\Phi_n(x,\theta)\Big]\, e^{I\mathscr{A}[\Phi]}\Phi_{n_1}(x_1,\theta_1)\Phi_{n_2}(x_2,\theta_2)\ldots$, where

Action from spacetime integration: $\mathscr{A}[\Phi] = \frac{1}{2}\int d^4x\,[\Phi_n^*(x,\theta)\Phi_n(x,\theta)]_D + 2\mathrm{Re}\int d^4x\,[f(\Phi)]_{\mathscr{F}}$.

To get supergraph Feynman rules, first write $\mathscr{A}[\Phi]$ in form $\mathscr{A} = \int d^4x \underline{\int d^4\theta} \times$ something.

Introduce superfields S_n such that $\Phi_n = \mathscr{D}_R^2 S_n$. No loss of generality - see page 110.

Action from superspace integration: $\mathscr{A}[\mathscr{D}_R^2 S] = \int d^4x \int d^4\theta \left(-\frac{1}{4} S_n^* \mathscr{D}_L^2 \mathscr{D}_R^2 S_n - \mathrm{Re}[\tilde{f}(\mathscr{D}_R^2 S, S)]\right)$,

where $\tilde{f}$ defined by $f(\mathscr{D}_R^2 S) = \mathscr{D}_R^2 \tilde{f}(\mathscr{D}_R^2 S, S)$. In $\tilde{f}$, all superfields acted on by $\mathscr{D}_R^2$ except one.

Recall $\mathscr{D}_R^2 S_n$ is left-chiral because $\mathscr{D}_{R\alpha}\mathscr{D}_{R\beta}\mathscr{D}_{R\gamma} = 0$.

So $\Phi_n^*(x,\theta)\Phi_n(x,\theta) = (\mathscr{D}_L^2 S_n^*)\mathscr{D}_R^2 S_n = S_n^* \mathscr{D}_L^2 \mathscr{D}_R^2 S_n + E_{\alpha\beta}\mathscr{D}_{L\alpha}((\mathscr{D}_{L\beta} S_n^*)\mathscr{D}_R^2 S_n + S_n^* \mathscr{D}_{L\beta}\mathscr{D}_R^2 S_n)$.

Now for any superfield T, $[\mathscr{D}_{L/R} T]_D = 0$, so $[\Phi_n^*(x,\theta)\Phi_n(x,\theta)]_D = [S_n^* \mathscr{D}_L^2 \mathscr{D}_R^2 S_n]_D$.

Because e.g. $(\mathscr{D}_R^2 T)^n = \mathscr{D}_R^2(T(\mathscr{D}_R^2 T)^{n-1})$ (follows from $(\mathscr{D}_R^2)^2 = 0$) then $f(\mathscr{D}_R^2 S) = \mathscr{D}_R^2 \tilde{f}(\mathscr{D}_R^2 S, S)$.

From page 111, $\int d^4x [\mathscr{D}_-^T E \mathscr{D}_- h]_{\mathscr{F}} \propto \int d^4x [h]_D$, i.e. $\int d^4x [\mathscr{D}_R^2 \tilde{f}(\mathscr{D}_R^2 S, S)]_{\mathscr{F}} = \int d^4x [\tilde{f}(\mathscr{D}_R^2 S, S)]_D$.

So $\mathscr{A} = \int d^4x \left(\frac{1}{2}\left[S_n^* \mathscr{D}_L^2 \mathscr{D}_R^2 S_n\right]_D + 2\mathrm{Re}\left[\tilde{f}(\mathscr{D}_R^2 S, S)\right]_D\right)$. Then use $\int d^4x [S]_D = -\frac{1}{2}\int d^4x \int d^4\theta S(x,\theta)$ from page 116.

Also we can replace $\prod_{n,x,\theta} d\Phi_n(x,\theta) \to \prod_{n,x,\theta} dS_n(x,\theta)$ in path-integral.

Consider lattice of N points, fields $\phi_i \longrightarrow \psi_i = \phi_{i+1} - \phi_i = C_{ij}\phi_j$. Then $d^N\psi = \det(C)d^N\phi = d^N\phi$ because $\det(C) = 1$.

Thus VEV of time-ordered product of superfields on page 117 becomes

$$\langle 0|T\{\Phi_{n_1}(x_1,\theta_1), \Phi_{n_2}(x_2,\theta_2)\dots\}|0\rangle = \mathscr{D}^2_{1R}\mathscr{D}^2_{2R}\dots\int\left[\prod_{n,x,\theta} dS_n(x,\theta)\right] e^{I\mathscr{A}[\mathscr{D}^2_R S]} S_{n_1}(x_1,\theta_1)S_{n_2}(x_2,\theta_2)\dots$$

$$= \mathscr{D}^2_{1R}\mathscr{D}^2_{2R}\dots\langle 0|T\{S_{n_1}(x_1,\theta_1), S_{n_2}(x_2,\theta_2)\dots\}|0\rangle.$$

Action invariant under $S_n \longrightarrow S_n + \mathscr{D}_R X_n$, where X_n is any superfield.

In $\mathscr{A}[\mathscr{D}^2_R S]$, $\mathscr{D}^2_R S \to \mathscr{D}^2_R S + \mathscr{D}^2_R \mathscr{D}_R X_n$ but $\mathscr{D}^2_R\mathscr{D}_R = 0$.

Propagator for left-chiral superfields: $\boxed{\Delta^\Phi_{nm} = \frac{1}{4}\mathscr{D}^2_R\mathscr{D}'^2_L\Delta_F(x-y)\delta^{(4)}(\theta-\theta')\delta_{nm}}$.

Propagator for S is $\Delta^S_{nm} = \frac{1}{4}\Delta_F(x-y)\delta^{(4)}(\theta-\theta')\delta_{nm} + \mathscr{D}_R$-terms (see page 120).

Δ^S_{nm} is superpropagator for line created by superfield $S^*_m(x',\theta')$ and destroyed by $S_n(x,\theta)$,

and $\Phi^*_m(x',\theta') = \mathscr{D}'^2_L S^*_m(x',\theta')$ and $\Phi_n(x,\theta) = \mathscr{D}^2_R S_n(x,\theta)$, so $\Delta^\Phi_{nm} = \mathscr{D}^2_R\mathscr{D}'^2_L\Delta^S_{nm}$.

Superpropagator: $\Delta^S_{nm} = \frac{1}{4}\Delta_F(x-y)\delta^{(4)}(\theta-\theta')\delta_{nm} + \mathscr{D}_R\text{-terms}$.

Propagator usually comes from quadratic terms in action. If i labels spacetime position (on lattice) as well as spin etc., so $\mathscr{A}_{\text{quad}}[\phi] = -D_{ij}\phi_i^*\phi_j$ where D is Hermitian, then $\Delta = D^{-1}$. But D not invertible if $\mathscr{A}_{\text{quad}}[\phi]$ invariant under $\phi_i \to \phi_i + \xi_i$, i.e. $D_{ij}\xi_j = 0$. Let u_ν be remaining eigenvectors of D, i.e. (no sum over ν unless explicitly indicated) $D_{ij}u_{\nu j} = d_\nu u_{\nu i}$, $u^*_{\nu i}u_{\nu' i} = \delta_{\nu\nu'}$, $u^*_{\nu i}\xi_i = 0$. In path integral $P_{a\ldots b\ldots} = \int\prod_i d\phi_i d\phi_i^* e^{\mathscr{A}_{\text{quad}}[\phi]}\phi_a\ldots\phi_b^*\ldots$, transform from ϕ_i to $\phi'_{(\nu)}$ with Jacobian $\mathscr{J}$, where $\phi_i = \phi'\xi_i + \sum_\nu \phi'_\nu u_{\nu i}$: With $\mathscr{C} = \mathscr{J}\int d\phi' d\phi'^*$ and, up to terms with 1+ factors of ξ,

$P_{a\ldots b\ldots} = \mathscr{C}\int\prod_\nu d\phi'_\nu d\phi'^*_\nu e^{-I\sum_\nu d_\nu|\phi'_\nu|^2}\left[\sum_\nu \phi'_\nu u_{\nu a}\right]\ldots\left[\sum_\nu \phi'_\nu u_{\nu b}\right]^*\ldots + \xi\text{−terms} = \sum_{\text{pairings}}[-I\Delta_{ab}] + \xi\text{−terms}$,

where propagator $\Delta_{ab} = \sum_\nu \frac{u_{\nu a}u^*_{\nu b}}{d_\nu}$. Replacing $\mathscr{A}_{\text{quad}}$ with full *gauge-invariant* $\mathscr{A}$, ξ-terms vanish because ϕ_i contracted with objects J_i obeying $J_i\xi_i = 0$. So, alternatively, solve $D_{ac}\Delta_{cb} = \Pi_{ab}$ to get Δ_{ab}, where projection operator $\Pi_{ab} = \sum_\nu u_{\nu a}u^*_{\nu b}$ obeys $\Pi^2 = \Pi$ and $\Pi\xi = 0$, because solution for Δ_{ab} unique up to these irrelevant ξ-terms. From action from superspace integration on page 118,

$\frac{1}{4}\mathscr{D}_L^2\mathscr{D}_R^2\Delta^S_{nm}(x,\theta;x',\theta') = \mathscr{P}\delta^{(4)}(x-x')\delta^{(4)}(\theta-\theta')\delta_{nm}$, where $\mathscr{P}$ obeys $\mathscr{P}^2 = \mathscr{P}$ and $\mathscr{P}\mathscr{D}_R X_n = 0$.

Solution for $\mathscr{P}$ is $\mathscr{P} = -\frac{1}{16}\Box^{-1}\mathscr{D}_L^2\mathscr{D}_R^2$, because $\mathscr{D}_R^2\mathscr{D}_L^2\mathscr{D}_R^2 = -16\Box\mathscr{D}_R^2$ and $\mathscr{D}_R^2\mathscr{D}_R = 0$,

so solution for Δ^S_{nm} is $\Delta^S_{nm} = \frac{1}{4}\Delta_F(x-y)\delta^{(4)}(\theta-\theta')\delta_{nm} + \mathscr{D}_R$-terms.

Generating functional: $Z[J,\mathscr{A}] = \int \prod_{x,l} d\phi_l(x) e^{I\mathscr{A}[\phi]+I\int d^4x \phi_l(x) J_l(x)}$. Contains all S-matrix elements.

From page 61, $\frac{\langle 0,\ \text{out}|T\left\{\phi_{l_A}(x_A),\phi_{l_B}(x_B),\ldots\right\}|0,\ \text{in}\rangle}{\langle 0,\ \text{out}|0,\ \text{in}\rangle} = \frac{1}{Z[J,\mathscr{A}]}\left(-I\frac{\delta}{\delta J_{l_A}(x_A)}\right)\left(-I\frac{\delta}{\delta J_{l_B}(x_A)}\right)\ldots Z[J,\mathscr{A}]\Big|_{J=0}$.

$Z[J,\mathscr{A}]$ = tree-level part of $Z[J,\Gamma]$, where

Quantum effective action: $\Gamma[\phi] = -\int d^4x \phi_l(x) J_{\phi l}(x) - I\ln Z[J_\phi,\mathscr{A}]$, J_ϕ is solution of $\phi_l = \frac{\delta(-I\ln Z[J,\mathscr{A}])}{\delta J_l(x)}\Big|_{J=J_\phi}$.

Define $Z[J,\Gamma,g] = \int \prod_{x,l} d\phi_l(x) e^{g^{-1}\left[I\Gamma[\phi]+I\int d^4x\phi_l(x)J_l(x)\right]}$, i.e. $Z[J,\Gamma,1] = Z[J,\Gamma]$. Propagator for $Z[J,\Gamma,g]$ is $\propto g$,

because it is inverse of coefficient of term in $g^{-1}\Gamma[\phi]$ which is quadratic in ϕ_l. Each vertex is $\propto g^{-1}$.

So for L = number of loops, $Z[J,\Gamma,g] = \sum_{L=0}^{\infty} g^{L-1} Z^{(L)}[J,\Gamma]$. Now take $g\to 0$, so integration in $Z[J,\Gamma,g]$ is dominated

by ϕ-configurations that give stationary phase, i.e. $\underline{Z[J,\Gamma,g] \propto e^{g^{-1}\left[I\Gamma[\phi_J]+I\int d^4x\phi_{Jl}(x)J_l(x)\right]}}$ where ϕ_J such that

$\frac{\delta}{\delta\phi_l(y)}\left[I\Gamma[\phi]+I\int d^4x\phi_l(x)J_l(x)\right]_{\phi=\phi_J} = 0$, or $\frac{\delta}{\delta\phi_l(x)}\Gamma[\phi]\Big|_{\phi=\phi_J} = -J_l(x)$. Thus $J_{\phi_J} = J$ because, from definition of $\Gamma[\phi]$ and J_ϕ,

$\frac{\delta}{\delta\phi_l(x)}\Gamma[\phi] = -\int d^4y\phi_k(y)\frac{\delta J_{\phi k}(y)}{\delta\phi_l(x)} - J_{\phi l}(x) + \int d^4y \frac{\delta(-I\ln Z[J])}{\delta J_k(y)}\Big|_{J=J_\phi}\frac{\delta J_{\phi k}(y)}{\delta\phi_l(x)} = J_{\phi l}(x)$. Proportionality constant in underlined equation

begins at $O(g^0)$ so, taking logs, coefficient of g^{-1} is $-I\ln Z^{(0)}[J,\Gamma] = \Gamma[\phi_J] + \int d^4x\phi_{Jl}(x)J_l(x) = -I\ln Z[J_{\phi_J},\mathscr{A}] = -I\ln Z[J,\mathscr{A}]$.

Coefficients of products of fields in $\Gamma[\phi]$ from corresponding connected 1 particle irreducible (1PI) graphs.

$\Gamma[S, S^*]$ is local in fermionic coordinates.

Each incoming (outgoing) external line is attached to vertex labelled (x, θ), gives a factor $S_n(x, \theta)$ $(S_n^*(x, \theta))$.

Each vertex gives factor equal to coefficient of corresponding product of fields in $\tilde{f}(\mathscr{D}_R^2 S, S)$.

Each propagator from vertex (x, θ) to vertex (x', θ') gives $-\frac{I}{4}\delta^{(4)}(\theta - \theta')\Delta_F(x - x')$.

Recall from page 118 that in $\tilde{f}$ all superfields acted on by $\mathscr{D}_R^2$ except one.

So a $\mathscr{D}_R^2$ $(\mathscr{D}_L^2)$ acts on all lines but one that come into (go out of) a vertex.

But $\Gamma[S, S^*]$ obtained by restricting to 1PI graphs and integrating over all xs and θs.

So use integration by parts to move all $\mathscr{D}_R^2$ $(\mathscr{D}_L^2)$ on propagators to external line factors $S_n(x, \theta)$ $(S_n^*(x, \theta))$.

Integrate out all $\delta^{(4)}(\theta - \theta')$ so that $\Gamma[S, S^*]$ is integral over single 4-D θ.

Graphs in $\Gamma[S, S^*]$ with loops are D-terms, the rest are $\mathscr{F}$-terms.

For V_n (V_n^*) vertices with n incoming (outgoing) lines, number of $\mathscr{D}_R^2$ ($\mathscr{D}_L^2$) is $N_R = \sum_n V_n(n-1)$ ($N_L = \sum_n V_n^*(n-1)$).

Let there be E (E^*) incoming (outgoing) external line factors $S_n(x,\theta)$ ($S_n^*(x,\theta)$), and I internal lines.

Because $I + E(E^*) = \sum_n nV_n(V_n^*)$ and number of loops is $L = I - \sum_n V_n - \sum_n V_n^* + 1$,

$N_R = L + \sum_n V_n^* + E - 1$ ($N_L = L + \sum_n V_n + E^* - 1$). Thus a graph with any loops has $N_R \geq E$ and $N_L \geq E$,

i.e. enough $\mathscr{D}_R^2$ ($\mathscr{D}_L^2$) operators to act on all external line factors $S_n(x,\theta)$ ($S_n^*(x,\theta)$) at least once

i.e. such graphs are integral over single 4-D θ of a functional of $\mathscr{D}_R^2 S_n = \Phi_n$ and $\mathscr{D}_L^2 S_n^* = \Phi_n^*$, which is a D-term.

Graphs with $N_R = E - 1$ $\mathscr{D}_R^2$ operators ($N_L = E - 1$ $\mathscr{D}_L^2$ operators) have both $L = 0$ and $V^* = 0$ ($V = 0$),

so there can only be one vertex and no loops, and there is a single $S_n(x,\theta)$ ($S_n^*(x,\theta)$) not acted on by $\mathscr{D}_R^2$ ($\mathscr{D}_L^2$).

But recall from page 118 that $\int d^4x \int d^4\theta \tilde{g}(\mathscr{D}_R^2 S, S) = \int d^4x \left[g(\mathscr{D}_R^2 S)\right]_{\mathscr{F}}$, i.e. such graphs are $\mathscr{F}$-terms.

Superpotential $f(S)$ not renormalized at any order in perturbation theory.

$f(S)$ contributes in form of $\mathscr{F}$-term, but $\mathscr{F}$-terms in $\Gamma[S, S^*]$ are free of loops (no infinite renormalization).

Also there is just one vertex, which must be the integrated $[f(S)]_{\mathscr{F}}$ (no finite renormalization).

3.8 SUSY current

SUSY current: $S^\mu = J^\mu + K^\mu$ (4-vector of Majorana spinors), where K^μ appears in $\delta\mathscr{L} = \overline{\alpha}\partial_\mu K^\mu$,

J is the Noether current in explicit result for Noether current when $\mathscr{L}$ is invariant on page 59: $-\overline{\alpha}J^\mu = \frac{\partial_R\mathscr{L}}{\partial(\partial_\mu\chi^l)}\delta\chi^l$.

But $\mathscr{L}$ is bot invariant under SUSY, so add K^μ to definition of SUSY current. Then

Conservation of SUSY current: $\partial_\mu S^\mu = 0$.

(Following argument is general.) From definition of J^μ above, $\overline{\alpha}\partial_\mu J^\mu = -\left(\partial_\mu \frac{\partial_R\mathscr{L}}{\partial(\partial_\mu\chi^l)}\right)\delta\chi^l - \frac{\partial_R\mathscr{L}}{\partial(\partial_\mu\chi^l)}\delta\partial_\mu\chi^l$.

Using Euler-Lagrange equations on page 58, $\overline{\alpha}\partial_\mu J^\mu = -\frac{\partial_R\mathscr{L}}{\partial\chi^l}\delta\chi^l - \frac{\partial_R\mathscr{L}}{\partial(\partial_\mu\chi^l)}\delta\partial_\mu\chi^l = -\delta\mathscr{L}$,

cancels $\delta\mathscr{L} = \overline{\alpha}\partial_\mu K^\mu$.

Generator of SUSY transformations: $\left[\int d^3x\overline{\alpha}S^0, \chi^l\right] = I\delta\chi^l$.

(Following argument is general.) Let $L = L(q, \dot{q})$, $\delta q = \delta q(q, \dot{q})$, canonical coordinates q are all fields χ^l at all spacetime points.

Recall $L = \int d^3x\mathscr{L}$, so $\delta L = \int d^3x\overline{\alpha}\partial_\mu K^\mu = \int d^3x\overline{\alpha}\frac{dK^0}{dt}$ or $\underline{\frac{\partial L}{\partial q^n}\delta q^n + \frac{\partial L}{\partial \dot{q}^n}\delta\dot{q}^n = \frac{dF}{dt}}$, where $F = \int d^3x\overline{\alpha}K^0$.

But $\frac{\partial L}{\partial q^n} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^n}$, so underlined equation is $\dot{Q} = 0$ where conserved charge $Q = -\frac{\partial L}{\partial \dot{q}^n}\delta q^n + F$.

So $Q = \int d^3x\overline{\alpha}S^0$, because $-\frac{\partial L}{\partial \dot{q}^n}\delta q^n = \int d^3x\overline{\alpha}J^0$ from the definition of the Noether current. Must show $[Q, \chi^l] = I\delta\chi^l$:

Now $[Q, q^m] = -\left[\frac{\partial L}{\partial \dot{q}^n}, q^m\right]\delta q^n - \frac{\partial L}{\partial \dot{q}^n}[\delta q^n, q^m] + [F, q^m]$. In 1st term, use QM result $\left[\frac{\partial L}{\partial \dot{q}^n}, q^m\right] = -I\delta^m_n$ (assuming no constraints).

For 2nd and 3rd terms consider general $[f(q, \dot{q}), q^m]$. We may write $f = \sum_{k=0}^\infty A_{n_1\ldots n_k}(q)\dot{q}_{n_1}\ldots\dot{q}_{n_k}$. Using QM result $[q^n, q^m] = 0$,

we find $[f(q, \dot{q}), q^m] = \sum_{k=0}^\infty A_{n_1\ldots n_k}(q)\sum_{r=1}^k \dot{q}_{n_1}\ldots\dot{q}_{n_{r-1}}[\dot{q}_{n_r}, q_m]\dot{q}_{n_{r+1}}\ldots\dot{q}_{n_k} = \frac{\partial f}{\partial \dot{q}_n}[\dot{q}_n, q_m]$

(in last expression, insert $[\dot{q}_n, q_m]$ in correct position among $\dot{q}$ products). Then $[Q, q^m] = I\delta q^m + \left(\frac{\partial F}{\partial \dot{q}^n} - \frac{\partial L}{\partial \dot{q}^l}\frac{\partial \delta q^l}{\partial \dot{q}^n}\right)[\dot{q}^n, q^m]$.

But $\frac{\partial F}{\partial \dot{q}^n} = \frac{\partial L}{\partial \dot{q}^l}\frac{\partial \delta q^l}{\partial \dot{q}^n}$: In underlined equation above, replace $\dot{X}$, where $\dot{X} = \delta\dot{q}^n$ on LHS and $\dot{X} = \frac{dF}{dt}$ on RHS,

with chain rule $\dot{X} = \dot{q}^l\frac{\partial X}{\partial q^l} + \ddot{q}^l\frac{\partial X}{\partial \dot{q}^l}$. Then $\frac{\partial L}{\partial q^n}\delta q^n + \dot{q}^l\frac{\partial L}{\partial \dot{q}^n}\frac{\partial \delta q^n}{\partial q^l} + \ddot{q}^l\frac{\partial L}{\partial \dot{q}^n}\frac{\partial \delta q^n}{\partial \dot{q}^l} = \dot{q}^n\frac{\partial F}{\partial q^n} + \ddot{q}^n\frac{\partial F}{\partial \dot{q}^n}$.

Match coefficients of $\ddot{q}^n$, which does not appear implicitly in any term.

Thus $[Q, q^m] = I\delta q^m$. Taking the time derivative also gives $[Q, \dot{q}^m] = I\delta\dot{q}^m$. Can be extended to cases with constraints.

SUSY current for chiral supermultiplet: $\boxed{S^\mu = \sqrt{2}\left[(\partial\!\!\!/\phi_n)\gamma^\mu\psi_{nR} + \frac{\partial f}{\partial\phi_n}\gamma^\mu\psi_{nL} + (\phi_n \to \phi_n^*, \psi_{nR} \leftrightarrow \psi_{nL})\right]}$.

Noether current: $J^\mu = \frac{1}{\sqrt{2}}[2(\partial^\mu\phi_n^*)\psi_{nL} + 2(\partial^\mu\phi_n)\psi_{nR} + (\partial\!\!\!/\phi_n)\gamma^\mu\psi_{nR} + (\partial\!\!\!/\phi_n^*)\gamma^\mu\psi_{nL} - \mathscr{F}_n\gamma^\mu\psi_{nR} - \mathscr{F}_n^*\gamma^\mu\psi_{nL}]$:

Use explicit result for Noether current when $\mathscr{L}$ is invariant on page 59, where $\mathscr{L}$ from chiral superfields on page 113 is

$\mathscr{L} = -\partial_\mu\phi_n^*\partial^\mu\phi_n - \frac{1}{2}\overline{\psi}_{nL}\gamma^\mu\partial_\mu\psi_{nL} + \frac{1}{2}(\partial_\mu\overline{\psi}_{nR})\gamma^\mu\psi_{nR}$ + non-spacetime-derivative terms,

change in fields from left / right transformation on page 107 (excluding $\delta\mathscr{F}_\pm$).

Next, $K^\mu = \frac{1}{\sqrt{2}}\gamma^\mu\left[-(\partial\!\!\!/\phi_n)\psi_{nR} - (\partial\!\!\!/\phi_n^*)\psi_{nL} + \mathscr{F}_n^*\psi_{nL} + \mathscr{F}_n\psi_{nR} + 2\frac{\partial f}{\partial\phi_n}\psi_{nL} + 2\left(\frac{\partial f}{\partial\phi_n}\right)^*\psi_{nR}\right]$:

Recall $\delta\mathscr{L} = \overline{\alpha}\partial_\mu K^\mu$ from page 125. In supersymmetric action from chiral superfields on page 111,

use (from left / right transformation on page 107) $\delta[f]_{\mathscr{F}} = \sqrt{2}\,\overline{\alpha}\partial\!\!\!/[f]_{\psi_L}$,

and (from component field transformations on page 104) $\delta[K]_D = I\overline{\alpha}\gamma_5\partial\!\!\!/[K]_\lambda$.

Alternative SUSY current: $S^\mu = K^\mu + J^\mu + \frac{\sqrt{2}}{3}[\gamma^\mu, \gamma^\nu]\partial_\nu \left(\phi_n \psi_{nR} + \phi_n^* \psi_{nL}\right)$.

Can always add $\partial_\nu A^{\mu\nu}$ to S^μ ($A^{\mu\nu}$ is some antisymmetric tensor of spinors), because 1. it is conserved ($\partial_\mu \partial_\nu A^{\mu\nu} = 0$),

2. generator of SUSY transformations on page 125 unchanged because $\int d^3x \overline{\alpha} \partial_\nu A^{0\nu} = \int d^3x \overline{\alpha} \partial_i A^{0i} = 0$.

This gives

Measure of scale invariance violation: $\gamma_\mu S^\mu = -2\sqrt{2} \left(\phi_m \frac{\partial^2 f}{\partial \phi_n \partial \phi_m} - 2\frac{\partial f}{\partial \phi_n}\right) \psi_{nL} + (L \to R, \phi \to \phi^*)$.

From $\mathscr{L}$ from chiral superfields on page 113, Dirac equations are $\partial\!\!\!/ \psi_{mL} = -\left(\frac{\partial^2 f}{\partial \phi_m \partial \phi_n}\right)^* \psi_{nR}$

and same with $L \leftrightarrow R$ and $\phi \to \phi^*$. Use SUSY current for chiral supermultiplet on page 126 ($= K^\mu + J^\mu$)

and alternative SUSY current above.

because $\gamma_\mu S^\mu$ vanishes when f is 3rd order homogeneous polynomial ($f = C_{mnp}\phi_m \phi_n \phi_p$),

in which case coupling (C_{mnp}) is dimensionless.

3.9 Spontaneous supersymmetry breaking

Broken SUSY vacuum condition: $\langle 0|\mathscr{F}|0\rangle \neq 0$ or $\left.\frac{\partial f(\phi)}{\partial \phi_n}\right|_{\phi=\phi_0} \neq 0$ or $V(\phi_0) > 0$.

In left / right transformation on page 107, want to make one of $\langle 0|\delta\Psi_l|0\rangle \neq 0$, where $\Psi_l = \psi_{nL}$, ϕ_n or $\mathscr{F}_n$.

Cannot make $\langle 0|\psi_{nL}|0\rangle \neq 0$ since vacuum Lorentz invariant, nor $\langle 0|\partial_\mu \phi_n|0\rangle \neq 0$ since $\langle 0|\phi_n|0\rangle$ constant.

Only possibility is $\langle 0|\mathscr{F}_n|0\rangle \neq 0$ (whence $\langle 0|\delta\psi_{nL}|0\rangle = \sqrt{2}\,\overline{\alpha}\langle 0|\mathscr{F}_n|0\rangle$),

which from $\mathscr{L}$ from chiral superfields on page 113 is equivalent to last 2 statements.

Spontaneous SUSY breaking requires f such that $\left.\frac{\partial f(\phi)}{\partial \phi_n}\right|_{\phi=\phi_0} = 0$ has no solution.

Spontaneous SUSY breaking gives rise to massless spin $\frac{1}{2}$ *goldstino*.

$\left.\frac{\partial V}{\partial \phi_n}\right|_{\phi=\phi_0} = 0$. But $V = \sum_n \left|\frac{\partial f}{\partial \phi_n}\right|^2$, so $2\sum_n \frac{\partial^2 f}{\partial \phi_m \partial \phi_n}\left.\left(\frac{\partial f}{\partial \phi_n}\right)^*\right|_{\phi=\phi_0} = 0$, i.e. $\sum_n \mathscr{M}_{mn}\left.\left(\frac{\partial f}{\partial \phi_n}\right)^*\right|_{\phi=\phi_0} = 0$. But one+ of $\left.\frac{\partial f(\phi)}{\partial \phi_n}\right|_{\phi=\phi_0} \neq 0$,

so $\mathscr{M}$ has at least one zero eigenvalue, so from free Lagrangian density on page 114, since $\mathscr{M}$ eigenvalues are the m_n,

there is at least one linear combination of ψ_{nL} with zero mass.

In local SUSY, goldstino absorbed into longitudinal component of gravitino, gives it a mass.

3.9.1 O'Raifeartaigh Models

Theories in which left-chiral fields X_n, Y_i have $\mathscr{R} = 0, 2$. Most general superpotential is $f(X,Y) = \sum_i Y_i f_i(X)$.

From R quantum number assignments for potentials f and K on page 112, i.e. $\mathscr{R}_f = 2$ (X^*, Y^* not allowed since f left-chiral).

SUSY broken when no. fields $X <$ no. fields Y.

Write scalar components of X, Y as x, y. Condition $\frac{\partial f(x,y)}{\partial y_i} = 0$ implies $f_i(x) = 0$, i.e. more conditions than fields X_i, only possible to satisfy by careful choice of the $f_i(x)$.

Scalar field potential: $V = \sum_i |f_i(x)|^2 + \sum_n \left| \sum_i y_i \frac{\partial f_i(x)}{\partial x_n} \right|^2$.

From general superpotential above and definition of scalar field potential in $\mathscr{L}$ from chiral superfields on page 113.

Simplest (renormalizable) model: Fields X, Y_1, Y_2.

Then choice $f_1(X) = X - a$, $f_2(X) = X^2$ is general, for which $V = |x|^4 + |x-a|^2 + |y_1 + 2xy_2|^2$.

Renormalizability allows f_i to be quadratic only. Then take linear combinations of Y_i and shift and rescale X.

$\frac{\partial f}{\partial X} = \frac{\partial f}{\partial Y_1} = \frac{\partial f}{\partial Y_2} = 0$ only possible if we can make $f_1 = f_2 = 0$, only possible if $a = 0$.

3.10 Supersymmetric gauge theories

Gauge transformation of left-chiral supermultiplet: $(\phi, \psi_L, \mathscr{F})_n(x) \to \big(\exp[It_A\Lambda^A(x)]\big)_{nm} (\phi, \psi_L, \mathscr{F})_m(x)$.

Fields in same supermultiplet have same transformation properties under internal symmetry transformation U:

Since $[Q, a^{(\dagger)}_{B/F}]_{\mp} \sim a^{(\dagger)}_{F/B}$, where $a^{(\dagger)}_{B/F}$ are annihilation (creation) operators for bosonic and fermionic superpartners,

general transformation of $a^{(\dagger)}_{\sigma}$ on page 10 is same for $a^{(\dagger)}_{B}$, $a^{(\dagger)}_{F}$ because $[U, Q] = 0$ from page 89.

So from complete field on page 27, transformation same for all fields in supermultiplet.

Gauge transformation of left-chiral superfields: $\Phi(x, \theta) \to \exp[It_A\Lambda^A(x_+)]\Phi(x, \theta)$ (column vector Φ).

Implied by gauge transformation of left-chiral supermultiplet above and compact form on page 107.

Gauge transformation of right-chiral superfields: $\Phi^\dagger(x, \theta) \to \Phi^\dagger(x, \theta) \exp[-It_A\Lambda^A(x_-)]$.

From conjugate of gauge transformation of left-chiral superfields above. Note $\Lambda^A(x)$ is real function of x and $x_+^* = x_-$.

<u>Global gauge invariance $\implies$ local for $\int d^4x[f(\Phi)]_{\mathscr{F}}$</u> but not for $\int d^4x[K(\Phi, \Phi^\dagger)]_D$.

No derivatives / conjugates of Φ in f, so argument y of $\Lambda^A(y)$ irrelevant. But K also contains $\Phi^\dagger$,

so e.g. $\Phi_n^\dagger\Phi_n$ not invariant under gauge transformation of left and right-chiral superfield above because $\Lambda^A(x_+) \neq \Lambda^A(x_-)$.

Define *gauge connection* $\Gamma(x,\theta)$, with transformation property

Gauge transformation of gauge connection: $\Gamma(x,\theta) \rightarrow \exp[It_A\Lambda^A(x_-)]\Gamma(x,\theta)\exp[-It_A\Lambda^A(x_+)]$,

then "new" right-chiral superfield $\Phi^\dagger(x,\theta)\Gamma(x,\theta)$ has transformation property

Gauge transformation of new right-chiral superfield: $\Phi^\dagger(x,\theta)\Gamma(x,\theta) \rightarrow \Phi^\dagger(x,\theta)\Gamma(x,\theta)\exp[-It_A\Lambda^A(x_+)]$.

Global gauge invariance $\implies$ local for $\int d^4x[K(\Phi,\Phi^\dagger\Gamma)]_D$. Also implies invariance under

Extended gauge transformations: $\Phi(x,\theta) \rightarrow \exp[It_A\Omega^A(x,\theta)]\Phi(x,\theta)$,

$\Gamma(x,\theta) \rightarrow \exp[It_A\Omega^{A\dagger}(x,\theta)]\Gamma(x,\theta)\exp[-It_A\Omega^A(x,\theta)]$ with any left-chiral superfields $\Omega^A(x,\theta)$.

We are extending $\Lambda^A(x_+)$ to $\Omega^A(x,\theta) = \Lambda^A(x_+) - \sqrt{2}\,\theta_L^T E\,\psi_{\Lambda L}(x_+) + \theta_L^T E\theta_L\,\mathscr{F}_{\Lambda+}(x_+)$ in compact form on page 107.

$\Phi(x,\theta)$ is left-chiral (depends on x_+, θ_L only), so cannot introduce x_- or θ_R in $\Omega^A(x,\theta)$.

Hermitian connection: $\Gamma(x,\theta) = \Gamma^\dagger(x,\theta)$. Take $\Gamma \rightarrow \frac{1}{2}(\Gamma+\Gamma^\dagger)$ or $\frac{1}{2I}(\Gamma-\Gamma^\dagger)$, note Γ, $\Gamma^\dagger$ have same transformation.

Connection from gauge superfields: $\Gamma(x,\theta) = \exp[-2t_AV^A(x,\theta)]$, with real *gauge superfields* V_A.

Form preserved by gauge transformation of gauge connection above, new V^A independent of t_A representation.

See Baker-Hausdorff formula on page 9.

Extended gauge transformation of gauge superfields: $V^A(x,\theta) \to V^A(x,\theta) + \frac{I}{2}[\Omega^A(x,\theta) - \Omega^{A\dagger}(x,\theta)] + \ldots$,

where "…" = commutators of generators, which vanish for zero coupling and in U(1) (Abelian).

From extended gauge transformations and definition of gauge superfields above.

Gauge superfields in terms of components: $V^A(x,\theta) = C^A(x) - I\overline{\theta}\gamma_5\omega^A(x) - \frac{I}{2}\overline{\theta}\gamma_5\theta M^A(x) - \frac{1}{2}\overline{\theta}\theta N^A(x)$

$$+\frac{I}{2}\overline{\theta}\gamma_5\gamma^\mu\theta V_\mu^A(x) - I\overline{\theta}\gamma_5\theta\overline{\theta}[\lambda^A(x) + \frac{1}{2}\not{\partial}\omega^A(x)] - \frac{1}{4}(\overline{\theta}\gamma_5\theta)^2(D^A(x) + \frac{1}{2}\partial^2 C^A(x)).$$

From general form of superfield on page 103.

Transformation superfields in terms of components: $\Omega^A(x,\theta) = W^A(x) - \sqrt{2}\overline{\theta}P_L w^A(x) + \mathscr{W}^A(x)\overline{\theta}P_L\theta$

$$+\frac{1}{2}\overline{\theta}\gamma_5\gamma_\mu\theta\partial^\mu W^A(x) - \frac{1}{\sqrt{2}}\overline{\theta}\gamma_5\theta\overline{\theta}\not{\partial}P_L w^A(x) - \frac{1}{8}(\overline{\theta}\gamma_5\theta)^2\partial^2 W^A(x).$$

From chiral superfield decomposition on page 106, taking left-chiral part (Φ^+ there).

Transformation of gauge supermultiplet fields: $C^A(x) \to C^A(x) - \mathrm{Im}W^A(x) + \ldots$,

$\omega^A(x) \to \omega^A(x) + \frac{1}{\sqrt{2}}w^A(x) + \ldots$, $V_\mu^A(x) \to V_\mu^A(x) + \partial_\mu \mathrm{Re}W^A(x) + \ldots$, $M^A(x) \to M^A(x) - \mathrm{Re}\mathscr{W}^A(x) + \ldots$,

$N^A(x) \to N^A(x) + \mathrm{Im}\mathscr{W}^A(x) + \ldots$, $\lambda^A(x) \to \lambda^A(x) + \ldots$, $D^A(x) \to D^A(x) + \ldots$.

From all 3 above results. Again, "…" = commutators of generators, which vanish for zero coupling and in U(1) (Abelian).

Wess-Zumino gauge: $C^A, \omega^A, M^A, N^A = 0$ $\rightarrow$ $V^A(x,\theta) = \frac{I}{2}\overline{\theta}\gamma_5\gamma^\mu\theta V^A_\mu(x) - I\overline{\theta}\gamma_5\theta\overline{\theta}\lambda^A(x) - \frac{1}{4}(\overline{\theta}\gamma_5\theta)^2 D^A(x)$.

Transformation of supermultiplet fields on page 132: take $\mathrm{Im}W^A(x) = C^A(x)$, $w^A(x) = -\sqrt{2}\omega^A(x)$, $\mathscr{W}^A(x) = M^A(x) - IN^A(x)$.

Sufficient for Abelian case because "…" $= 0$. For non-Abelian case, add nth order (in gauge couplings) terms

to $\mathrm{Im}W^A(x)$, $w^A(x)$, $\mathscr{W}^A(x)$ to cancel nth order terms from commutators of $m \leq n-1$th order terms.

Wess-Zumino gauge is not supersymmetric.

Ensuring $\delta C^A, \delta\omega^A, \delta M^A, \delta N^A = 0$ in component field transformations on page 104, requires also having $V^A_\mu, \lambda^A = 0$,

which requires $\delta V^A_\mu, \delta\lambda^A = 0$. $\delta V^A_\mu = 0$ satisfied, but $\delta\lambda^A = 0$ requires $D^A = 0$, i.e. $V^A = 0$.

Gauge invariant $\mathscr{L}$ for chiral supermultiplet: $\frac{1}{2}[\Phi^\dagger\Gamma\Phi]_D = -\frac{1}{2}(D_\mu\phi)^\dagger D^\mu\phi - \frac{1}{2}\overline{\psi}_L\gamma^\mu D_\mu\psi_L + \frac{1}{2}\mathscr{F}^\dagger\mathscr{F}$

$+I\sqrt{2}\overline{\psi}_L t_A\lambda^A\phi - \frac{1}{2}D^A\phi^\dagger t_A\phi + \text{h.c.}$, where covariant derivative $D_\mu = \partial_\mu - It_A V^A_\mu$

(gauge transformation for gauge superfield derived next on page 134).

In Wess-Zumino gauge, $\Gamma(x,\theta) = 1 - I\overline{\theta}\gamma_5\gamma_\mu\theta t_A V^A_\mu(x) - \frac{1}{2}\overline{\theta}\gamma_5\gamma^\mu\theta\overline{\theta}\gamma_5\gamma^\nu\theta t_A t_B V^A_\mu(x)V^B_\nu(x) + 2I\overline{\theta}\gamma_5\theta t_A\overline{\theta}\lambda^A(x) + \frac{1}{2}(\overline{\theta}\gamma_5\theta)^2 t_A D^A(x)$.

Then use left-chiral superfield in chiral superfield decomposition on page 106.

Note D term is coefficient of $-\frac{1}{4}(\overline{\theta}\gamma_5\theta)^2$ minus $\frac{1}{2}\partial^2\times$ first "ϕ" term.

Gauge transformation of gauge supermultiplet fields: $\delta_{\text{gauge}} V_\mu^A(x) = C_{ABC} V_\mu^B(x)\Lambda^C(x) + \partial_\mu \Lambda^A(x)$

(which is infinitesimal version of transformation of gauge fields on page 66), and

$\delta_{\text{gauge}}\lambda^A(x) = C_{ABC}\lambda^B(x)\Lambda^C(x)$, $\delta_{\text{gauge}} D^A(x) = C_{ABC} D^B(x)\Lambda^C(x)$ (i.e. λ^A, D^A in adjoint representation).

Firstly $\Lambda^A(x_+) = \Lambda^A(x) + \frac{1}{2}(\overline{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\Lambda^A(x) - \frac{1}{8}(\overline{\theta}\gamma_5\theta)^2\partial^2\Lambda^A(x)$. Then gauge transformation of gauge connection on page 131 (using connection from gauge superfields) reads $\exp[-2t_A V^{A\prime}(x,\theta)] = \exp[I t_A \Lambda^{A\dagger}(x_+)]\exp[-2t_A V^A(x,\theta)]\exp[-I t_A\Lambda^A(x_+)]$.

Write as $\underline{\exp[-2t_A V^{A\prime}(x,\theta)] = e^a e^X e^b}$ where $\boxed{X = -2t_A V^A(x,\theta) = -2t_A[\frac{I}{2}\overline{\theta}\gamma_5\gamma_\mu\theta V_\mu^A(x) - I\overline{\theta}\gamma_5\theta\overline{\theta}\lambda^A(x) - \frac{1}{4}(\overline{\theta}\gamma_5\theta)^2 D^A(x)]}$, and

small quantities $\boxed{b + a = 2t_A \mathrm{Im}\Lambda^A(x_+) = -I\overline{\theta}\gamma_5\gamma_\mu\theta t_A\partial^\mu\Lambda^A(x)}$, $\boxed{b - a = -2It_A \mathrm{Re}\Lambda^A(x_+) = -2It_A[\Lambda^A(x) - \frac{1}{8}(\overline{\theta}\gamma_5\theta)^2\partial^2\Lambda^A(x)]}$.

To first order in a, b, $\underline{-2t_A V^{A\prime}(x,\theta) = X + \frac{1}{2}[X, b-a] + b + a + \ldots}$ and "$\ldots$" means higher order in a, b,

as well as terms of $O(X^n)O(b-a)$ with integer $n \geq 2$ which vanish:

since $(\mathbf{1}, \gamma_5) = P_L \pm P_R$, $X \sim \theta_L\theta_R(\times$ further θ factors), while $b + a \sim \theta_L\theta_R$, so $O(X^2)O(b-a) \sim \theta_L^3\theta_R^3(\times\ldots) = 0$.

So $\underline{V^{A\prime}(x,\theta) = V^A(x,\theta) + C_{ABC}V^B(x,\theta)\Lambda^C(x) + \frac{I}{2}\overline{\theta}\gamma_5\gamma_\mu\theta\partial^\mu\Lambda^A(x)}$,

compare superfield expansion in Wess-Zumino gauge on page 133.

Wess-Zumino gauge is gauge invariant under ordinary gauge transformations.

Last underlined equation above shows that $\delta C^A = \delta\omega^A = \delta M^A = \delta N^A = 0$.

3.10.1 Gauge-invariant Lagrangians

Construct gauge-invariant Lagrangian from

Gauge-covariant spinor superfield: $2t_A W^A_{L\alpha}(x,\theta) = \mathscr{D}_-^T E \mathscr{D}_- \exp[2t_A V^A(x,\theta)] \mathscr{D}_{+\alpha} \exp[-2t_A V^A(x,\theta)]$

which is left-chiral because $\mathscr{D}_- W_L = 0$ ($\mathscr{D}_{\alpha-}\mathscr{D}_{\beta-}\mathscr{D}_{\gamma-} = 0$), i.e.

Extended gauge transformation of W^A_L: $2t_A W^A_{L\alpha}(x,\theta) \to \exp[It_A\Omega^A(x,\theta)] 2t_A W^A_{L\alpha}(x,\theta) \exp[-It_A\Omega^A(x,\theta)]$.

From extended gauge transformations and connection from gauge superfields on page 131,

$2t_A W^A_{L\alpha} \to \mathscr{D}_-^T E \mathscr{D}_- \exp[It_A\Omega^A] \exp[2t_A V^A] \exp[-It_A\Omega^{A\dagger}]\ \mathscr{D}_{+\alpha}\ \exp[It_A\Omega^{A\dagger}] \exp[-2t_A V^A] \exp[-It_A\Omega^A]$.

Use product rule for the $\mathscr{D}_\pm$ here. Since Ω^A is left-chiral, $\mathscr{D}_-\Omega^A = \mathscr{D}_+\Omega^{A\dagger} = 0$.

Then $2t_A W^A_{L\alpha} \to \exp[It_A\Omega^A] \mathscr{D}_-^T E \mathscr{D}_- \exp[2t_A V^A] \exp[-It_A\Omega^{A\dagger}] \exp[It_A\Omega^{A\dagger}] \mathscr{D}_{+\alpha} \exp[-2t_A V^A] \exp[-It_A\Omega^A]$

$= \exp[It_A\Omega^A] \mathscr{D}_-^T E \mathscr{D}_- \exp[2t_A V^A] \underline{\mathscr{D}_{+\alpha}} \exp[-2t_A V^A] \underline{\exp[-It_A\Omega^A]}$. Not finished, because $\underline{\mathscr{D}_{+\alpha} \exp[-It_A\Omega^A]} \neq 0$

(however, $\mathscr{D}_-^T E \mathscr{D}_- \exp[-It_A\Omega^A]$ does vanish). But $\mathscr{D}_-^T E \mathscr{D}_- \mathscr{D}_{+\alpha} = \mathscr{D}_{+\alpha} \mathscr{D}_-^T E \mathscr{D}_- - 4\left[P_L \not{\partial} \mathscr{D}_-\right]_\alpha$, so $\mathscr{D}_-^T E \mathscr{D}_- \mathscr{D}_{+\alpha} \exp[-It_A\Omega^A] = 0$.

So $2t_A W^A_{L\alpha} \to \exp[It_A\Omega^A] \left\{ \mathscr{D}_-^T E \mathscr{D}_- \exp[2t_A V^A] \mathscr{D}_{+\alpha} \exp[-2t_A V^A] \right\} \exp[-It_A\Omega^A]$,

where all derivatives in quantity between $\Big\{$ and $\Big\}$ evaluated before multiplying on left / right with $\exp[\pm It_A\Omega^A]$.

Form of W_L^A: $W_L^A(x,\theta) = \lambda_L^A(x_+) + \frac{1}{2}\gamma^\mu\gamma^\nu\theta_L F_{\mu\nu}^A(x_+) + \theta_L^T E\theta_L \not{D}\lambda_R^A(x_+) - I\theta_L D^A(x_+)$.

Note $\exp[-2t_A V^A(x,\theta)] = 1 - I\overline{\theta}\gamma_5\gamma_\mu\theta t_A V_\mu^A(x) - \frac{1}{2}\overline{\theta}\gamma_5\gamma^\mu\theta\overline{\theta}\gamma_5\gamma^\nu\theta t_A t_B V_\mu^A(x) V_\nu^B(x) + 2I\overline{\theta}\gamma_5\theta t_A\overline{\theta}\lambda^A(x) + \frac{1}{2}(\overline{\theta}\gamma_5\theta)^2 t_A D^A(x)$.

After performing all $\mathscr{D}_\pm$ in gauge-covariant spinor superfield on page 135, choose gauge where $V_\mu^A(X) = 0$ at given $x = X$, gives $W_L^A(X,\theta) = \lambda_L^A(X_+) + \frac{1}{2}\gamma^\mu\gamma^\nu\theta_L(\partial_\mu V_\nu^A(X_+) - \partial_\nu V_\mu^A(X_+)) + \theta_L^T E\theta_L \not{\partial}\lambda_R^A(X_+) - I\theta_L D^A(X_+)$. Then convert to gauge covariant form consistent with this, i.e. $\partial_\mu V_\nu^A - \partial_\nu V_\mu^A \to F_{\mu\nu}^A$ (the non-Abelian field strength of page 67) and $\not{\partial} \to \not{D}$.

Alternative form of W_L^A: $\mathscr{D}_+ E W_L^A = \mathscr{D}_- E W_R^A$ with $\not{D} \to \not{\partial}$, $\epsilon^{\mu\nu\rho\sigma}\partial_\rho f_{\mu\nu} = 0$.

Note W_R^A from W_L^A via $x_+ \to x_-$, $R \leftrightarrow L$, $E \to -E$. Implies form of W_L^A above, by direct calculation.

Gauge supermultiplet Lagrangian: $\mathscr{L}_{\text{gauge}} = -\frac{1}{2}\text{Re}([W_L^{AT} E W_L^A]_{\mathscr{F}}) = -\frac{1}{4}F_{\mu\nu}^A F^{A\mu\nu} - \frac{1}{2}\overline{\lambda}^A(\not{D}\lambda)^A + \frac{1}{2}D^A D^A$.

From form of W_L^A above.

Additional non perturbative part: $\mathscr{L}_\theta = -\frac{g^2\theta}{16\pi^2}\text{Im}\left([W_L^{AT} E W_L^A]_{\mathscr{F}}\right) = -\frac{g^2\theta}{16\pi^2}\left(I\overline{\lambda}^A\not{D}\gamma_5\lambda^A - \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}F^{A\mu\nu}F^{A\rho\sigma}\right)$,

where g is coupling appearing in t_A.

Fayet-Iliopolis term: $\mathscr{L}_{\text{FI}} = \xi D$, where ξ is arbitrary constant, D is for Abelian supermultiplet..

Corresponding action is supersymmetric, because $\delta D = I\overline{\alpha}\gamma_5\not{\partial}\lambda$ is derivative.

Explicit check that action from gauge supermultiplet Lagrangian $\mathscr{L}_{\text{gauge}}$ on page 136 is supersymmetric.

In $V^A_\mu(X) = 0$ gauge, component field transformations on page 104 at $x = X$: $\delta V^A_\mu = \overline{\alpha}\gamma_\mu\lambda^A$, $\delta D^A = I\overline{\alpha}\gamma_5\partial\!\!\!/\lambda^A$, and

$\delta\lambda^A = (\frac{1}{4}F^A_{\mu\nu}[\gamma^\nu, \gamma^\mu] + I\gamma_5 D^A)\alpha$ where $F^A_{\mu\nu} = \partial_\mu V^A_\nu - \partial_\nu V^A_\mu + C_{ABC}V^B_\mu V^C_\nu$ is non-Abelian field strength of page 67.

Then using $\delta(F^A_{\mu\nu}F^{A\mu\nu}) = 2F^A_{\mu\nu}\delta F^{A\mu\nu}$ etc., $\underline{\delta(F^A_{\mu\nu}F^{A\mu\nu}) = 2F^{A\mu\nu}\overline{\alpha}(\gamma_\nu\partial_\mu - \gamma_\mu\partial_\nu)\lambda^A}$, $\underline{\delta(D^A D^A) = 2ID^A\overline{\alpha}\gamma_5\partial\!\!\!/\lambda^A}$

and $\underline{\delta(\overline{\lambda}^A D\!\!\!\!/\,_{AB}\lambda^B) = (\delta\overline{\lambda}^A)D\!\!\!\!/\,_{AB}\lambda^B + \overline{\lambda}^A D\!\!\!\!/\,_{AB}\delta\lambda^B + \overline{\lambda}^A(\delta D\!\!\!\!/\,_{AB})\lambda^B = 2\overline{\alpha}[\frac{1}{4}F^A_{\mu\nu}[\gamma^\mu, \gamma^\nu] + I\gamma_5 D]\partial\!\!\!/\lambda^A + C_{ABC}\overline{\lambda}^A\delta(V\!\!\!/^B)\lambda^C}$

(we will show $C_{ABC}\overline{\lambda}^A\delta(V\!\!\!/^B)\lambda^C = 0$ later). For this last expression, use $[\gamma^\mu, \gamma^\nu]\gamma^\rho = -2g^{\mu\rho}\gamma^\nu + 2g^{\nu\rho}\gamma^\mu - 2I\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5$

(expand in 16 matrices on page 45, Lorentz and space inversion invariance limits this to these three terms.

1st coefficient by taking $\mu\nu\rho = 121$: $[\gamma^1, \gamma^2]\gamma^1 = -2g^{11}\gamma^2$, correct because from anticommutation relations for γ^μ on page 44,

$\gamma^1\gamma^2 = -\gamma^2\gamma^1$ and $\gamma^{1\ 2} = 2$. Similarly for $\mu\nu\rho = 211$ to get 2nd coefficient. 3rd coefficient from $\mu\nu\rho = 123$:

$[\gamma^1, \gamma^2]\gamma^3 = -2I\epsilon^{1230}\gamma_0\gamma_5$, LHS is $2\gamma^1\gamma^2\gamma^3 = -2\gamma_0\gamma^0\gamma^1\gamma^2\gamma^3 = -2I\gamma_0\gamma_5$).

So $\delta(\overline{\lambda}^A\partial\!\!\!/\lambda^A) = -F^{A\mu\nu}\overline{\alpha}(\gamma_\nu\partial_\mu - \gamma_\mu\partial_\nu)\lambda^A - I\epsilon^{\mu\nu\rho\sigma}F^A_{\mu\nu}\overline{\alpha}\gamma_\sigma\gamma_5\partial_\rho\lambda^A + 2ID^A\overline{\alpha}\gamma_5\partial\!\!\!/\lambda^A$. 2nd term replaceable, after integration by parts,

by $I\epsilon^{\mu\nu\rho\sigma}(\partial_\rho F^A_{\mu\nu})\overline{\alpha}\gamma_\sigma\gamma_5\lambda^A$, which vanishes since e.g. $\epsilon^{\mu\nu\rho\sigma}\partial_\rho\partial_\mu V^A_\nu = 0$, 1st and 3rd terms cancel with $\delta(F^A_{\mu\nu}F^{A\mu\nu})$ and $\delta(D^A D^A)$.

Thus we are left with $C_{ABC}\overline{\lambda}^A\delta(V\!\!\!/^B)\lambda^C = C_{ABC}\overline{\lambda}^A\gamma_\mu\lambda^C\overline{\alpha}\gamma^\mu\lambda^B = 0$,

where antisymmetry of C_{ABC} used in 1st step. Last step requires explicit calculation.

Lagrangian for chiral and gauge supermultiplet fields:

$$\mathscr{L} = -(D_\mu\phi)^\dagger_n(D^\mu\phi)_n - \tfrac{1}{2}\overline{\psi}_{nL}\gamma^\mu(D_\mu\psi_L)_n + \tfrac{1}{2}D_\mu\overline{\psi}_{nL}\gamma^\mu\psi_{nL} - \frac{1}{2}\frac{\partial^2 f}{\partial\phi_n\partial\phi_m}\psi^T_{nL}E\psi_{mL} - \frac{1}{2}\left(\frac{\partial^2 f}{\partial\phi_n\partial\phi_m}\right)^*\left(\psi^T_{nL}E\psi_{mL}\right)^*$$

$$-V(\phi) + I\sqrt{2}\overline{\psi}_{nL}(t_A)_{nm}\lambda^A\phi_m - I\sqrt{2}\phi^\dagger_n\overline{\lambda}^A(t_A)_{nm}\psi_{mL} - \tfrac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} - \tfrac{1}{2}\overline{\lambda}^A(\not{D}\lambda)^A + \frac{g^2\theta}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{A\mu\nu}F^{A\rho\sigma},$$

where potential $V(\phi) = \frac{\partial f(\phi)}{\partial\phi_n}\left(\frac{\partial f(\phi)}{\partial\phi_n}\right)^* + \frac{1}{2}\left(\xi^A + \phi^*_n(t_A)_{nm}\phi_m\right)\left(\xi^A + \phi^*_k(t_A)_{kl}\phi_l\right)$.

Sum of gauge invariant $\mathscr{L}$ for chiral supermultiplet on page 133, superpotential part of $\mathscr{L}$ on page 113 and 3 Lagrangians on page 136 gives $\mathscr{L} = \frac{1}{2}\left[\Phi^\dagger\exp\left(-2t_AV^A\right)\Phi\right]_D + 2\mathrm{Re}[f(\Phi)]_{\mathscr{F}} - \frac{1}{2}\mathrm{Re}\left(W^{AT}_LEW^A_L\right)_{\mathscr{F}} - \xi^AD^A - \frac{g^2\theta}{16\pi^2}\mathrm{Im}(W^{AT}_LEW^A_L)_{\mathscr{F}}$,

or explicitly (using Majorana conjugation on page 96, e.g. $\overline{\psi}_R = \psi^\dagger P_R\beta = \overline{\psi}P_L = \psi^TE\gamma_5P_L = \psi^TP_LE\gamma_5 = \psi^T_LE\gamma_5$)

$$\mathscr{L} = -(D_\mu\phi)^\dagger_n(D^\mu\phi)_n - \tfrac{1}{2}\overline{\psi}_n\gamma^\mu(D_\mu\psi)_n + \mathscr{F}^\dagger_n\mathscr{F}_n - \mathrm{Re}\frac{\partial^2 f}{\partial\phi_n\partial\phi_m}\psi^T_nE\psi_m + 2\mathrm{Re}\frac{\partial f(\phi)}{\partial\phi_n}\mathscr{F}_n - 2\sqrt{2}\mathrm{Im}(t_A)_{nm}\overline{\psi}_{nL}\lambda^A\phi_m$$

$$+2\sqrt{2}\mathrm{Im}(t_A)_{mn}\overline{\psi}_{nR}\lambda^A\phi^\dagger_m - \phi^\dagger_n(t_A)_{nm}\phi_mD^A - \xi^AD^A + \tfrac{1}{2}D^AD^A - \tfrac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} - \tfrac{1}{2}\overline{\lambda}^A(\not{D}\lambda)^A + \frac{g^2\theta}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{A\mu\nu}F^{A\rho\sigma}.$$

Then use field equations for auxiliary fields: $\mathscr{F}_n = -\left(\frac{\partial f(\phi)}{\partial\phi_n}\right)^\dagger$ and $D^A = \xi^A + \phi^*_n(t_A)_{nm}\phi_m$.

3.10.2 Spontaneous supersymmetry breaking in gauge theories

Unbroken SUSY vacuum: $\mathscr{F}_{n0} = -\left[\frac{\partial f(\phi)}{\partial \phi_n}\right]_{\phi=\phi_0} = 0$, $D_0^A = \xi^A + \phi_{n0}^*(t_A)_{nm}\phi_{m0} = 0 \Longleftrightarrow V(\phi) = 0$.

Can write $V = \mathscr{F}_n^*\mathscr{F}_n + D^A D^A > 0$, so $V = 0$ (if allowed) is a minimum. In this case, $\mathscr{F}_{n0} = D_0^A = 0$.

From left / right transformation on page 107 and component field transformations on page 104,

$\langle 0|\delta\psi_L = \delta\mathscr{F} = \delta\phi = \delta\lambda|0\rangle = 0$. Argument holds in reverse.

Note: no overconstraining on M ϕ components: $\mathscr{F}_{n0} = D_0^A = 0$ is M conditions,

not $M + D$, where D is dimensionality of group, i.e. $A = 1, \ldots, D$,

because only $M - D$ conditions needed to satisfy $\mathscr{F}_{n0} = \left.\frac{\partial f(\phi)}{\partial \phi}\right|_{\phi=\phi_0} = 0$: $f(\Phi)$ invariant under extended gauge transformations

on page 131, $\frac{\partial f(\Phi)}{\partial \Omega^A} = 0 = \frac{\partial f(\Phi)}{\partial \Phi_n}\frac{\partial([e^{It_B\Omega^B}]_{nm}\Phi_m)}{\partial \Omega^A}\Big|_{\Omega^C=0} = \frac{\partial f(\Phi)}{\partial \Phi_n}(It_A\Phi)_n$, i.e. $\frac{\partial f(\phi)}{\partial \phi_n}(t_A\phi)_n = 0$, which is already D conditions.

Existence of any supersymmetric field configuration $\Longrightarrow$ unbroken SUSY vacuum.

From above, SUSY field configuration has $V = 0$, which is absolute minimum so lower than V for non SUSY field configuration.

(Now let $\xi^A = 0$.) To check vacuum is unbroken SUSY, enough to check that $\frac{\partial f(\phi)}{\partial \phi_n} = 0$ can be satisfied.

$f(\phi)$ has no $\phi^\dagger$, so invariant under $\phi \to e^{I\Lambda^A t_A}\phi$ with Λ^A complex. If $\frac{\partial f(\phi)}{\partial \phi_n} = 0$ true for $\widetilde{\phi}$, true for $\phi^\Lambda = e^{I\Lambda^A t_A}\widetilde{\phi}$.

Choose Λ^A such that $\phi^{\Lambda\dagger}\phi^\Lambda$ minimum (which exists because it is real and positive), i.e. $\frac{\partial}{\partial \Lambda^A}(\phi^{\Lambda\dagger}\phi^\Lambda) \propto \phi_n^{\Lambda\dagger}(t_A)_{nm}\phi_m^\Lambda = 0$,

i.e. there exists a field configuration such that $D^A = 0$, so unbroken SUSY vacuum condition obeyed.

So break SUSY by

1. Making $\frac{\partial f}{\partial \phi} = 0$ impossible (already considered on page 128)

or

2. Fayet-Iliopoulos term ξD. Simple example:

2 left-chiral $\Phi^{\pm}$ with U(1) quantum numbers $\pm e$,

so spinor components are left-handed parts of electron / positron.

Then only possibility is $f(\Phi) = m\Phi^+\Phi^-$,

so from V defined on page 138, $V(\phi^+, \phi^-) = m^2|\phi^+|^2 + m^2|\phi^-|^2 + (\xi + e^2|\phi^+|^2 - e^2|\phi^-|^2)^2$,

which cannot vanish for $\xi \neq 0$, so SUSY broken.

Note U(1) symmetry intact for $|\xi| < \frac{m^2}{2e^2}$, since minimum at $\phi^+ = \phi^- = 0$.

Tree-level mass sum rule: $\mathscr{C}=0$, where $\mathscr{C}=\sum_{\text{fermions}} m^2 - \sum_{\text{bosons}} m^2$. For unbroken/broken SUSY.

Take new scalar fields $\Delta\phi_n = \phi_n - \phi_{n0}$. Then quadratic part of V defined on page 138 in unitarity gauge $\phi^\dagger(t_A\phi_0)=0$ is

$$V_{\text{quad}} = \tfrac{1}{2}\begin{pmatrix}\Delta\phi\\ \Delta\phi^*\end{pmatrix}^\dagger M_0^2 \begin{pmatrix}\Delta\phi\\ \Delta\phi^*\end{pmatrix}, \text{ where } \underline{M_0^2 = \begin{pmatrix}\mathscr{M}^*\mathscr{M} + (t_A\phi_0)(t_A\phi_0)^\dagger + D_0^A t_A & \dots \\ \dots & \mathscr{M}\mathscr{M}^* + ((t_A\phi_0)(t_A\phi_0)^\dagger)^* + D_0^A t_A^T\end{pmatrix}}$$

($\mathscr{M}$ defined in tree-level expansion of V on page 114). From Lagrangian for chiral and gauge supermultiplet fields

on page 138, terms bilinear in fermion fields are $\mathscr{L}_{1/2} = -\frac{1}{2}\begin{pmatrix}\psi_L\\ \lambda\end{pmatrix}^T M \begin{pmatrix}\psi_L\\ \lambda\end{pmatrix}$, where $M = \begin{pmatrix}\mathscr{M} & I\sqrt{2}(t_A\phi_0)^* \\ I\sqrt{2}(t_A\phi_0)^* & 0\end{pmatrix}$,

so $\underline{M^\dagger M = \begin{pmatrix}\mathscr{M}^\dagger\mathscr{M} + 2(t_A\phi_0)(t_A\phi_0)^\dagger & \dots \\ \dots & 2\phi_0^\dagger t_B t_A \phi_0\end{pmatrix}}$. For gauge fields, $\mathscr{L}_V = -V_\mu^A (M_V^2)_{AB} V^{B\mu}$ where $\underline{(M_V^2)_{AB} = \phi_0^\dagger\{t_B, t_A\}\phi_0}$.

Underlined results above give $\text{Tr}M_0^2 = 2\text{Tr}(\mathscr{M}^*\mathscr{M}) + \text{Tr}M_V^2 + 2D_0^A \text{Tr}t_A$ and $\text{Tr}(M^\dagger M) = \text{Tr}(\mathscr{M}^*\mathscr{M}) + 2\text{Tr}M_V^2$.

Eliminate $\text{Tr}(\mathscr{M}^*\mathscr{M})$ to get $\text{Tr}M_0^2 - 2\text{Tr}(M^\dagger M) + 3\text{Tr}M_V^2 = 2D_0^A\text{Tr}t_A$.

But trace $=\sum$ eigenvalues, so $\sum_{\text{spin }0}\text{mass}^2 - 2\sum_{\text{spin }\frac{1}{2}}\text{mass}^2 + 3\sum_{\text{spin }1}\text{mass}^2 = 2D_0^A\text{Tr}t_A$.

1 spin-3 component for scalar boson, 2 for spin-$\frac{1}{2}$ fermion, 3 for spin-1 gauge boson,

i.e. $\sum_{\text{fermions}} m^2 - \sum_{\text{bosons}} m^2 = 2D_0^A\text{Tr}t_A$

In non-Abelian theories, $\text{Tr}t_A = 0$. Can have $\text{Tr}t_A \neq 0$ for U(1) but gives graviton-graviton-U(1) anomaly.

$\mathscr{C}$ is coefficient of quadratic divergence in vacuum energy, i.e. breaking SUSY does not affect UV structure.

4 The Minimally Supersymmetric Standard Model

4.1 Left-chiral superfields

Assign left-handed SM fermions to left-chiral superfields according to table 4.1.1.

Table 4.1.1: MSSM equivalent of table 2.9.1 on page 68, for the left-chiral superfields. Supermultiplets for e.g. U are written u_L for the left-handed up quark and $\tilde{u}_L$ for its scalar superpartner, the up squark. Likewise for $\overline{U}$ we write $u_R^\dagger$ and $\tilde{u}_R^\dagger$.

Names	Label	Representation under SU(3)$_C\times$SU(2)$_L\times$U(1)$_Y$
(S)quarks	$Q = (U, D)$	$(\mathbf{3}, \mathbf{2}, \frac{1}{6})$
	$\overline{U}$	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})$
	$\overline{D}$	$(\overline{\mathbf{3}}, \mathbf{1}, \frac{1}{3})$
(S)leptons	$L = (N, E)$	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$
	$\overline{E}$	$(\mathbf{1}, \mathbf{1}, 1)$

Baryon / lepton number violating terms:

$$[Q_a^K(\epsilon L^M)_a\overline{D}^N]_{\mathscr{F}} = [(D^K N^M - U^K E^M)\overline{D}^N]_{\mathscr{F}},\ [(E^K N^M - N^K E^M)\overline{E}^N]_{\mathscr{F}},\ [\overline{D}^K\overline{D}^M\overline{U}^N]_{\mathscr{F}},$$

SU(3)$\times$SU(2)$\times$U(1) invariant terms not in SM,

allows proton decay $p \longrightarrow \pi^0 + e^+$ in $\sim$minutes, while experiment gives $> 10^{32}$ years.

Relevant processes are $u_R d_R \to (\tilde{s}_R^* \text{ or } \tilde{b}_R^*) \to \overline{e}_L\overline{u}_L$ and $u_L \to u_L$. (No $u_R d_R \to \tilde{d}_R^*$: coupling is $\lambda_{KMN}\overline{D}^K\overline{D}^M\overline{U}^N$ with λ_{KMN} antisymmetric in KM since $\overline{D}^K\overline{D}^M$ antisymmetric in colour indices to get colour singlet coupling.)

Rule out some / all baryon lepton number violating terms on page 143 by symmetry, e.g.:

1.) Rule out all by baryon number (B) and lepton number (L) assignments

$B_{U,\overline{U}} = B_{D,\overline{D}} = \pm\frac{1}{3}$, $L_{U,\overline{U}} = L_{D,\overline{D}} = 0$, $B_{N,\overline{N}} = B_{E,\overline{E}} = 0$, $L_{N,\overline{N}} = L_{E,\overline{E}} = \pm 1$, $B_{\theta_L} = B_{\theta_R} = L_{\theta_{L/R}} = 0$.

Or rule out some by requiring only conservation of linear combination, e.g. L, B, $B - L$ etc.

2.) Rule out $[\overline{D}^K \overline{D}^M \overline{U}^N]_{\mathscr{F}}$ by $L_N = L_E = 0$, $L_U = L_D = L_{\overline{U}} = L_{\overline{D}} = -1$, $L_{\overline{E}} = -2$.

"Conventional" L for quarks, leptons by taking $L_{\theta_{L/R}} = \pm 1$. L for squarks, sleptons is then unconventional.

But continuous global symmetries are dubious in string theories.

3.) Rule out all by R parity, a *discrete* global symmetry, $= 1$ for SM particles and $= -1$ for their superpartners,

i.e. $\Pi_R = (-1)^{2s}(-1)^{3(B-L)}$. Lightest SUSY particle (LSP) (lightest particle with $\Pi_R = -1$) completely stable.

Cannot decay into other $\Pi_R = -1$ particles (heavier), or into $\Pi_R = 1$ particles only (violates R-parity conservation).

Colliders: sparticles produced in pairs.

Non-zero vacuum expectation values for scalar components of $N^K \to$ charged lepton, charge $-\frac{e}{3}$ quark masses

via 1st, 2nd terms in baryon / lepton number violating terms on page 143, but charge $\frac{2e}{3}$ quarks remain massless.

Spontaneous breakdown of SU(2)×U(1) for massive quarks, leptons, $W^\pm$, Z by Higgs superfields in table 4.1.2:

Table 4.1.2: Required Higgs superfields in the MSSM.

Names	Label	Representation under $\mathrm{SU}(3)_C\times\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$
Higgs(ino)	$H_1 = (H_1^0, H_1^-)$	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$
	$H_2 = (H_2^+, H_2^0)$	$(\mathbf{1}, \mathbf{2}, \frac{1}{2})$

Higgs-chiral superfield couplings in superpotential $[f(\Phi)]_{\mathscr{F}}$: $(K, M = \text{generation})$

$$h^D_{KM}[(D^K H_1^0 - U^K H_1^-)\overline{D}^M]_{\mathscr{F}},\ h^E_{KM}[(E^K H_1^0 - N^K H_1^-)\overline{E}^M]_{\mathscr{F}},\ h^U_{KM}[(D^K H_2^+ - U^K H_2^0)\overline{U}^M]_{\mathscr{F}}.$$

E.g. last term is just $Q_a(\epsilon H_2)_a\overline{U}$, leads to $-G_u^{KM}\overline{\mathscr{Q}}^K_{L\ a}(\epsilon\phi_H)^\dagger_a u_R^M$ term in $\mathscr{L}_{\text{Higgs}-\text{fermion}}$ on page 70.

Note second Higgs H_2 needed for last term, because we need a $(\mathbf{1}, \mathbf{2}, \frac{1}{2})$ left-chiral superfield. ($H_1^\dagger$ is $(\mathbf{1}, \mathbf{2}, \frac{1}{2})$, but is right-chiral.)

$\langle 0|\phi_{H_1^0}|0\rangle \neq 0$ gives mass to d-type quarks and charged leptons, $\langle 0|\phi_{H_2^0}|0\rangle \neq 0$ gives mass to u-type quarks.

In superpotential part of $\mathscr{L}$ on page 113, fermions get mass from first term $-\frac{1}{2}\frac{\partial^2 f(\phi)}{\partial\phi_n\partial\phi_m}\overline{\psi}_{nR}\psi_{mL}$.

So from e.g. first term of Higgs-chiral superfield couplings in superpotential $[f(\Phi)]_{\mathscr{F}}$ above,

mass term for d^K from $\frac{\partial^2 f(\phi)}{\partial\tilde{d}_L^K\partial\tilde{d}_R^{K*}}\overline{d}_R^K d_L^K = \phi_{H_1^0}\overline{d}_R^K d_L^K$.

For more Higgs superfields, number of H_1 and H_2 type superfields must be equal.

Higgsinos produce SU(2)-SU(2)-U(1) anomalies: For H_1, anomaly $\propto \sum t_3^2 y = \left(\frac{1}{2}g\right)^2\left(\frac{1}{2}g'\right) + \left(-\frac{1}{2}g\right)^2\left(\frac{1}{2}g'\right) = \frac{1}{2}g^2 g'$,

for H_2, anomaly $\propto \sum t_3^2 y = \left(\frac{1}{2}g\right)^2\left(-\frac{1}{2}g'\right) + \left(-\frac{1}{2}g\right)^2\left(-\frac{1}{2}g'\right) = -\frac{1}{2}g^2 g'$. No anomalies from gauginos, in adjoint representation.

Most general renormalizable Lagrangian for a gauge theory with R parity or $B-L$ conserved consists of

1. sum of $[\Phi^* \exp(-V)\Phi]_D$ terms for quark, lepton and Higgs chiral superfields,

2. sum of $[\epsilon_{\alpha\beta} W_\alpha W_\beta]_{\mathscr{F}}$ for gauge superfields,

3. sum of Higgs-chiral superfield couplings in superpotential $[f(\Phi)]_{\mathscr{F}}$ on page 145, and

4. μ **term**: $\mathscr{L}_\mu = \mu\left[H_1^T \epsilon H_2\right]_{\mathscr{F}} = \mu\left[H_2^+ H_1^- - H_2^0 H_1^0\right]_{\mathscr{F}}$.

   μ has no radiative corrections due to perturbative non-renormalization theorem for $\mathscr{F}$ term on page 123.

Gauge hierarchy problem on page 80 explicitly solved as follows:

Unbroken SUSY: 1-loop correction to Higgs mass from any particle cancelled by that particle's superpartner.

For broken SUSY, replace Λ_{UV}^2 with $\sim \Delta\mathrm{mass}^2$ of supermultiplet,

so 1-loop correction to Higgs mass from top is $\frac{|\kappa_t|^2}{8\pi^2}\Delta m_s^2$, where Δm_s^2 is mass splitting between top and stop.

No fine-tuning if this is $\lesssim 1$ TeV, so since $|\kappa_t|^2 \sim 1$, stop mass is $< \sqrt{8\pi^2} \sim 10$ TeV.

Flavour changing processes suppressed to below experimental bounds if squark masses $\sim$ equal,

so if gauge hierarchy problem solved by SUSY, all squark masses < 10 TeV.

4.2 Supersymmetry and strong-electroweak unification

Mentioned in grand unification subsubsection, page 76. Assume SUSY unbroken in most of range $< M_X$.

1-loop modifications to SM $\beta_i(g_i(\mu_r)) = \mu_r \frac{d}{d\mu_r} g_i(\mu_r)$ from new SUSY particles, with n_s Higgs chiral superfields:

$$\beta_1 = \frac{5n_g g'^3}{36\pi^2} \to \frac{g'^3}{4\pi^2}\left(\frac{5n_g}{6} + \frac{n_s}{8}\right) \implies \frac{1}{g'^2(\mu_r)} = \frac{1}{g'^2(M_X)} + \frac{1}{2\pi^2}\left(\frac{5n_g}{6} + \frac{n_s}{8}\right)\ln\left(\frac{M_X}{\mu_r}\right)$$

$$\beta_2 = \frac{g^3}{4\pi^2}\left(-\frac{11}{6} + \frac{n_g}{3}\right) \to \frac{g^3}{4\pi^2}\left(-\frac{9}{6} + \frac{n_g}{2} + \frac{n_s}{8}\right) \implies \frac{1}{g^2(\mu_r)} = \frac{1}{g^2(M_X)} + \frac{1}{2\pi^2}\left(-\frac{3}{2} + \frac{n_g}{2} + \frac{n_s}{8}\right)\ln\left(\frac{M_X}{\mu_r}\right)$$

$$\beta_3 = \frac{g_s^3}{4\pi^2}\left(-\frac{11}{4} + \frac{n_g}{3}\right) \to \frac{g_s^3}{4\pi^2}\left(-\frac{9}{4} + \frac{n_g}{2}\right) \implies \frac{1}{g_s^2(\mu_r)} = \frac{1}{g_s^2(M_X)} + \frac{1}{2\pi^2}\left(-\frac{9}{4} + \frac{n_g}{2}\right)\ln\left(\frac{M_X}{\mu_r}\right).$$

Take $\mu_r = m_Z$, set $\sqrt{\frac{5}{3}} g'(M_X) = g(M_X) = g_s(M_X)$, solution (where e defined on page 72, $\sin^2\theta_W$ on page 71)

1. $\sin^2\theta_W(m_Z) = \frac{18 + 3n_s + \frac{e^2(m_Z)}{g_s^2(m_Z)}(60 - 2n_s)}{108 + 6n_s}$

2. $\ln\left(\frac{M_X}{m_Z}\right) = \frac{8\pi^2}{e^2(m_Z)} \frac{1 - \frac{8e^2(m_Z)}{3g_s^2(m_Z)}}{18 + n_s}.$

For measured values $\sin^2\theta_W = 0.231$, $\frac{e^2(m_Z)}{4\pi} = (128)^{-1}$, $\frac{g_s^2(m_Z)}{4\pi} = 0.118$, $m_Z = 91.2\text{GeV}$,

equation 1. above gives $n_s = 2$, then equation 2. above gives $M_X = 2 \times 10^{16}\text{GeV}$.

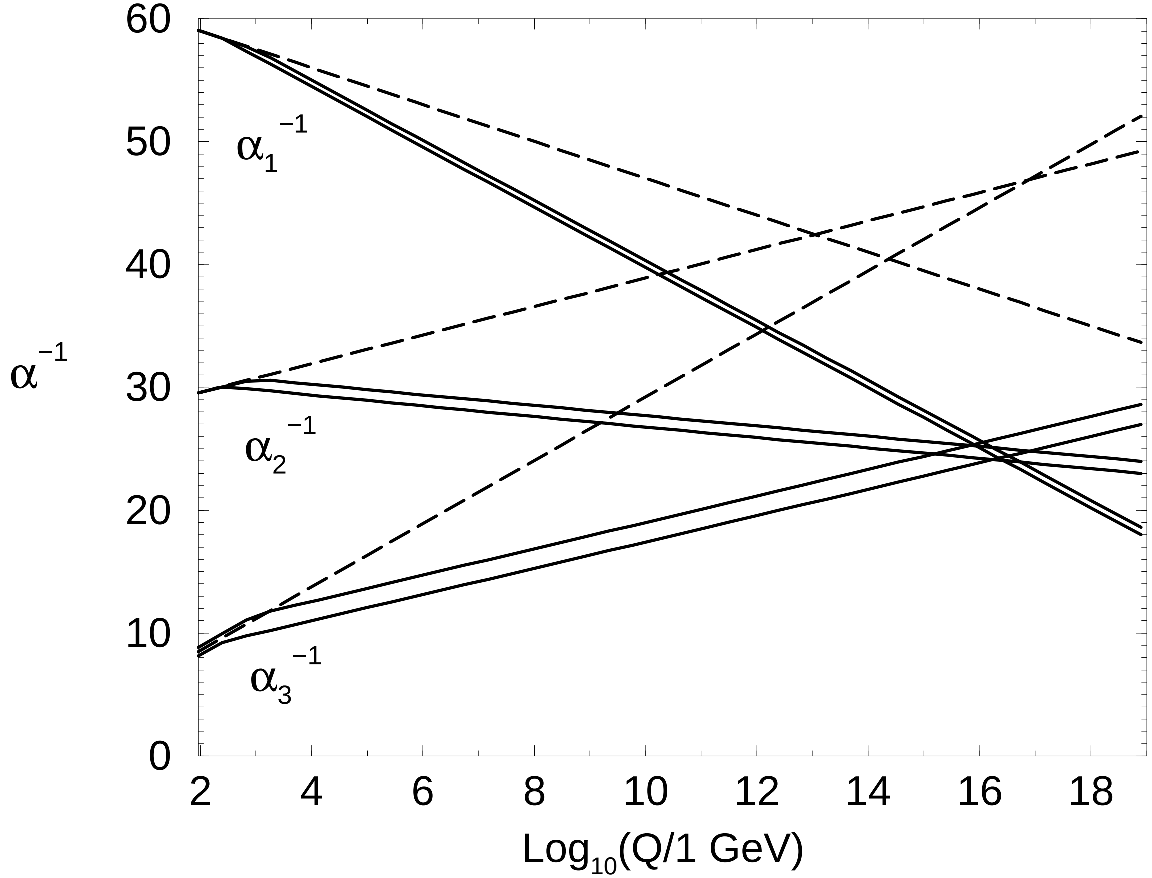


Figure 4.1: 2-loop RG evolution of inverse of $\alpha_i = \frac{g_i^2}{4\pi}$ in the SM (dashed) and MSSM (solid). $\alpha_3(m_Z)$ is varied between 0.113 and 0.123, sparticle mass thresholds between 250 GeV and 1 TeV.

4.3 Supersymmetry breaking in the MSSM

In effective MSSM Lagrangian, SUSY must be broken.

SUSY breaking terms must not reintroduce hierarchy problem.

Terms with coupling's mass dimension $\leq 0 \to |\kappa_\psi|^2 - \lambda_\phi = O(\ln \Lambda_{UV})$ (see page 80), so $\delta m_H^2 \sim \Lambda_{UV}^2 \ln \Lambda_{UV}$.

Superrenormalizable terms (coupling's mass dimension > 0, $\sim$ power of some M) $\to \delta m_H^2 \sim M^2 \ln \Lambda_{UV}$,

OK provided $M \sim m_{\mathrm{SUSY}} \lesssim 10$ TeV. So SUSY breaking terms must be superrenormalizable, called *soft terms*.

Soft SUSY breaking R parity / $B-L$ conserving SM invariant terms:

(Sum over SU(2), SU(3) indices and generations K, M)

1. $\mathscr{L}_{\rm SR} \supset \sum_S -M^{S2}_{KM}\phi^{K\dagger}_S\phi^M_S$, where $S = Q, \overline{U}, \overline{D}, L, \overline{E}$ superfields and ϕ_S their component scalars,

2. $\mathscr{L}_{\rm SR} \supset \sum_X \overline{\lambda}_X m_X \lambda_X$, where $X =$ gluino, wino, bino,

3. Trilinear terms: $\mathscr{L}_{\rm SR} \supset -A^D_{KM}h^D_{KM}(\phi^K_Q)^T\epsilon\phi_{H_1}\phi^M_{\overline{D}} - C^D_{KM}h^D_{KM}(\phi^K_Q)^T\phi^*_{H_2}\phi^M_{\overline{D}}$

$$-A^E_{KM}h^E_{KM}(\phi^K_L)^T\epsilon\phi_{H_1}\phi^M_{\overline{E}} - C^E_{KM}h^E_{KM}(\phi^K_L)^T\phi^*_{H_2}\phi^M_{\overline{E}}$$

$$-A^U_{KM}h^U_{KM}(\phi^K_Q)^T\epsilon\phi_{H_2}\phi^M_{\overline{U}} - C^U_{KM}h^U_{KM}(\phi^K_Q)^T\phi^*_{H_1}\phi^M_{\overline{U}}$$

where $h^{D,E,U}_{KM}$ defined in Higgs-chiral superfield couplings in superpotential $[f(\Phi)]_{\mathscr{F}}$ on page 145,

4. $\mathscr{L}_{\rm SR} \supset -\frac{1}{2}B\mu\phi^T_{H_2}\epsilon\phi_{H_1} - \frac{1}{2}m^2_{H_1}\phi^\dagger_{H_1}\phi_{H_1} - \frac{1}{2}m^2_{H_2}\phi^\dagger_{H_2}\phi_{H_2}$ where μ defined in μ term on page 146.

Recall Hermitian conjugate is added to $\mathscr{L}$. So $\mathscr{L}_{\rm SR} \supset -{\rm Re}\big\{B\mu\phi^T_{H_2}\epsilon\phi_{H_1}\big\} - m^2_{H_1}\phi^\dagger_{H_1}\phi_{H_1} - m^2_{H_2}\phi^\dagger_{H_2}\phi_{H_2}$.

Choose H_1, H_2 superfields' phases so $B\mu$ real, positive: $\mathscr{L}_{\rm SR} \supset -B\mu{\rm Re}\big\{\phi^T_{H_2}\epsilon\phi_{H_1}\big\} - m^2_{H_1}\phi^\dagger_{H_1}\phi_{H_1} - m^2_{H_2}\phi^\dagger_{H_2}\phi_{H_2}$

To respect approximate symmetries, choose A^S_{KM}, $B \sim 1$:

A^S_{KM} for chiral symmetry: Reflected by small Yukawa coupling of light quarks.

B for Peccei-Quinn symmetry: Reflected by small μ term on page 146.

C^S_{KM} terms involve scalar components of left- and right-chiral superfields,

$\rightarrow$ quadratic divergences $\Longrightarrow$ fine-tuning and hierarchy problems.

In fact these divergences from scalar tadpole graphs which disappear into vacuum,

cannot occur since no SM invariant scalars.

Note SM superpartners acquire mass even if no electroweak symmetry breaking (i.e. if SM particles massless).

A^S_{KM}, B, C^S_{KM} are arbitrary and complex $\rightarrow > 100$ parameters even without C^S_{KM} terms.

But expect soft terms to arise from some underlying principle.

SUSY breaking at tree-level for 3 generations ruled out.

Tree-level mass sum rule on page 142 holds for each set of colour and charge values.

Gives e.g. $2(m_d^2 + m_s^2 + m_b^2) \simeq 2(5\ \text{GeV})^2 = \sum$ of all masses for bosonic degrees of freedom with charge $-e/3$.

So each squark mass is $< \sqrt{2}\ 5\ \text{GeV} \simeq 7\ \text{GeV}$.

Predicts squark mass(es) too small, would have effect in accurately measured $e^+e^- \longrightarrow$ hadrons.

This is good, otherwise get fine-tuning: tree-level SUSY breaking would require mass parameter M in Lagrangian.

Would affect all SM masses, so M must coincidentally be $\sim$ electroweak symmetry breaking scale $v = 246$ GeV.

No SUSY breaking at tree-level implies no SUSY breaking at all orders.

SUSY breaking at tree-level for $\geq$3 generations ruled out.

General argument against tree-level SUSY breaking: Consider first underlined equation on page 142.

To conserve colour and charge, only allowed non-zero D_0^A terms are for y of U(1) (D_1) and t_3 of SU(2) (D_2).

Squarks are colour triplets, so have zero vacuum expectation values. So for $(\tilde{u}_L, \tilde{u}_R)$,

$M_{0U}^2 = \begin{pmatrix} \mathscr{M}_U^* \mathscr{M}_U - g'\frac{1}{6}D_1 + g\frac{1}{2}D_2 & \cdots \\ \cdots & \mathscr{M}_U \mathscr{M}_U^* + \frac{2}{3}g'D_1 \end{pmatrix}$. For $(\tilde{d}_L, \tilde{d}_R)$, $M_{0D}^2 = \begin{pmatrix} \mathscr{M}_D^* \mathscr{M}_D - g'\frac{1}{6}D_1 - g\frac{1}{2}D_2 & \cdots \\ \cdots & \mathscr{M}_D \mathscr{M}_D^* - \frac{1}{3}g'D_1 \end{pmatrix}$.

Second underlined equation on page 142 gives mass-squared matrix $\mathscr{M}_U^* \mathscr{M}_U$ for charge $\frac{2e}{3}$ quarks

and $\mathscr{M}_D^* \mathscr{M}_D$ for charge $-\frac{e}{3}$ quarks. Let v_x, $x = u$, d be unit eigenvector for quark of lowest mass, i.e. $\mathscr{M}_X^* \mathscr{M}_X v_x = m_x^2 v_x$.

Then mass^2 of lightest charge $\frac{2e}{3}$ squark $< \begin{pmatrix} 0 \\ v_u* \end{pmatrix}^\dagger M_{0U}^2 \begin{pmatrix} 0 \\ v_u* \end{pmatrix} = m_u^2 + \frac{2}{3}g'D_1$,

mass^2 of lightest charge $-\frac{e}{3}$ squark $< \begin{pmatrix} 0 \\ v_d* \end{pmatrix}^\dagger M_{0D}^2 \begin{pmatrix} 0 \\ v_d* \end{pmatrix} = m_d^2 - \frac{1}{3}g'D_1$,

so regardless of value / sign of D_1 there is at least one squark lighter than u or d quark.

In fact, since $D_1 \sim m_{\mathrm{SUSY}}^2$, get squark with negative mass which breaks colour and charge conservation.

Possible solution is new U(1) gauge superfield, so mass^2 of lightest charge $(\frac{2e}{3}, -\frac{e}{3})$ squark $< (m_u^2 + \frac{2}{3}g'D_1, m_d^2 - \frac{1}{3}g'D_1) + \tilde{g}\tilde{D}$.

Alternative breaking: Introduce new strong-force-like gauge field, asymptotically free coupling $\frac{\mathscr{G}^2(\mu_r)}{8\pi^2} \simeq \frac{b}{\ln \mu_r / m_{\text{SUSY}}}$,

so $\mu_r \gg m_{\text{SUSY}}$ gives $\mathscr{G}(\mu_r) \ll 1$ (SUSY unbroken), $\mu_r \sim m_{\text{SUSY}}$ gives $\mathscr{G}(\mu_r) \sim 1$ (SUSY broken).

Allows for $m_{\text{SUSY}} \ll M_X$ without introducing M_X into theory as new mass scale.

SUSY broken either non-perturbatively or by scalar field potential with non-zero vacuum expectation value.

To get $m_{\text{SUSY}} \sim 10\text{TeV} \ll M_X$, $\mathscr{G}(M_X)$ does not have to be very small (no fine-tuning).

New force clearly does not interact with SM particles,

so SUSY breaking occurs via $\langle 0|\mathscr{F}_{\text{hidden}}|0\rangle \neq 0$ in *hidden sector* of particles that interact via new force,

communicated to observed particles by interactions felt by both hidden and observed particles,

namely gravity (gravity-mediated SUSY breaking) or SM interactions (gauge-mediated SUSY breaking):

1. Gravity-mediated SUSY breaking: $\boxed{m_{\text{SUSY}} \sim \frac{\langle 0|\mathscr{F}_{\text{hidden}}|0\rangle}{M_P}}$ $\implies$ $\sqrt{\langle 0|\mathscr{F}_{\text{hidden}}|0\rangle} \sim 10^{11}$ GeV. (No C^S_{KM} terms.)

From dimensional analysis, subject to "no SUSY breaking conditions" $m_{\text{SUSY}} \to 0$ as $\langle 0|\mathscr{F}_{\text{hidden}}|0\rangle \to 0$ or $M_P \to \infty$.

2. Gauge-mediated SUSY breaking: Messenger particle of mass $M_{\text{messenger}}$ couples to $\langle 0|\mathscr{F}_{\text{hidden}}|0\rangle$

and to MSSM particles via SM interactions (loops). Then $\boxed{m_{\text{SUSY}} \sim \frac{\alpha_i}{4\pi}\frac{\langle 0|\mathscr{F}_{\text{hidden}}|0\rangle}{M_{\text{messenger}}}}$.

For $\sqrt{\langle 0|\mathscr{F}_{\text{hidden}}|0\rangle} \sim M_{\text{messenger}}$, $\implies$ $\sqrt{\langle 0|\mathscr{F}_{\text{hidden}}|0\rangle} \sim 10^5$ GeV. (Very small C^S_{KM} terms.)

Flavour blindness of above SUSY breaking mechanisms simplifies soft terms in Lagrangian at $\mu_r = M_X$.

E.g. in minimal supergravity (mSUGRA), K on page 112, which appears in supergravity Lagrangian,

is diagonal for hidden and observed sectors, i.e. $K = \sum_{i=\text{observed, hidden}} |\Phi_i|^2$.

Gives *organising principle* (for mSUGRA) (recall $S = Q, \overline{D}, \overline{U}, L, \overline{E}$)

$M^{S2}_{KM} = m_0^2 \delta_{KM}, \quad m^2_{H_1} = m^2_{H_2} = m_0^2, \quad m_{\text{gluino}} = m_{\text{wino}} = m_{\text{bino}} = m_{1/2}, \quad A^S_{KM} = A_0 \delta_{KM}, \quad C^S_{KM} = 0.$

This or similar principle expected: small experimental upper bounds on flavour changing, CP violating processes.

Flavour changing processes allowed because squark, slepton mass matrices not necessarily diagonalised

in same basis as quark, lepton mass matrices.

Most stringent limits for quarks are on $K^0 - \overline{K}^0$ transitions involving $d_L \rightarrow$ gluino + $\tilde{d}_L^K$, $\tilde{d}_L^K \rightarrow$ gluino + s_L,

most stringent limits for leptons are on $\mu \rightarrow e\gamma$ decays.

CP violating processes allowed due to many new phases in MSSM, can have large effect at low SM energies.

Upper bound on electric dipole moments of neutron and atoms and molecules

requires CP violating phases $\lesssim 10^{-2}$ or some superpartner masses $\gtrsim 1$ TeV.

4.4 Electroweak symmetry breaking in the MSSM

Consider Higgs H_1 and H_2 superfields' scalar components $\phi_{H_1} = (\phi_{H_1^0}, \phi_{H_1^-})^T$ and $\phi_{H_2} = (\phi_{H_2^+}, \phi_{H_2^0})^T$.

Scalar Higgs potential: $V = \frac{g^2}{2}\left|\phi^\dagger_{H_1}\phi_{H_2}\right|^2 + \frac{g^2+g'^2}{8}\left(\phi^\dagger_{H_1}\phi_{H_1} - \phi^\dagger_{H_2}\phi_{H_2}\right)^2$

$$+(m^2_{H_1} + |\mu|^2)\phi^\dagger_{H_1}\phi_{H_1} + (m^2_{H_2} + |\mu|^2)\phi^\dagger_{H_2}\phi_{H_2} + B\mu\mathrm{Re}(\phi^T_{H_1}\epsilon\phi_{H_2}).$$

From potential in Lagrangian for chiral and gauge supermultiplet fields page 138,

with $\xi^A = 0$ and $f(H_1, H_2) = \mu H_1^T \epsilon H_2$ (the μ term on page 146),

together with part 4. of $\mathscr{L}_{\mathrm{SR}}$ in soft SUSY breaking terms on page 150.

V **bounded from below**: $2|\mu|^2 + m^2_{H_1} + m^2_{H_2} \geq B\mu$.

Terms in V which are quartic in ϕ_{H_1} and ϕ_{H_2} are positive or zero. When positive, V has a minimum.

However, quartic terms vanish when $\phi_{H_1} = (\phi, 0)^T$ and $\phi_{H_2} = (0, \phi)^T$, and then $V = (2|\mu|^2 + m^2_{H_1} + m^2_{H_2})|\phi|^2 - B\mu\mathrm{Re}(\phi^2)$,

which must not go to $-\infty$ as $|\phi| \to \infty$.

Definition of vacuum: $\langle 0|\phi_{H_2^+}|0\rangle = \langle 0|\phi_{H_1^-}|0\rangle = 0$, $\langle 0|\phi_{H_i^0}|0\rangle = v_i$ real,

so $(m_{H_1}^2 + |\mu|^2)v_1 + \frac{g^2+g'^2}{4}(v_1^2 - v_2^2)v_1 - \frac{1}{2}B\mu v_2 = 0$ and $1 \leftrightarrow 2$. Around vacuum, $\phi_{H_i^0} = v_i + \phi_i$.

Define $V^{\text{neutral}} = V$ when charged $\phi_{H_1^-} = \phi_{H_2^+} = 0$.

Choose minimum $\phi_{H_2^+} = 0$ by SU(2) rotation. $\frac{\partial V}{\partial \phi_{H_2^+}} = 0 \implies \phi_{H_1^-} = 0$ (or $B\mu = -\frac{g^2}{2}\phi^\dagger_{H_1^0}\phi^\dagger_{H_2^0}$, but then $v_i = 0$).

Then $V = V^{\text{neutral}}$, where $V^{\text{neutral}} = \frac{g^2+g'^2}{8}\left(|\phi_{H_1^0}|^2 - |\phi_{H_2^0}|^2\right)^2 + (m_{H_1}^2 + |\mu|^2)|\phi_{H_1^0}|^2 + (m_{H_2}^2 + |\mu|^2)|\phi_{H_2^0}|^2 - B\mu\text{Re}(\phi_{H_1^0}\phi_{H_2^0})$.

Let stationary point of V^{neutral} be at $\phi_{H_i^0} = v_i$, i.e. for $\phi_{H_i^0} = v_i + \phi_i$, must have $\left.\frac{\partial V^{\text{neutral}}}{\partial \phi_i}\right|_{\phi_i=0} = 0$,

i.e. coefficient of term $\propto \phi_i$ vanishes. Gives $(m_{H_1}^2 + |\mu|^2)v_1^* + \frac{g^2+g'^2}{4}(v_1^2 - v_2^2)v_1^* - \frac{1}{2}B\mu v_2 = 0$ and $1 \leftrightarrow 2$.

Last result implies v_2 real if v_1 real. Adjust relative phase between $\phi_{H_1^0}$ and $\phi_{H_2^0}$ so that v_1 is real.

Parameter relations with v_1 and v_2: $B\mu = m_A^2 \sin 2\beta$ where $\tan\beta = \frac{v_2}{v_1}$, $m_A^2 = 2|\mu|^2 + m_{H_1}^2 + m_{H_2}^2$, and

$m_{H_1}^2 + |\mu|^2 = \frac{1}{2}m_A^2 - \frac{1}{2}(m_A^2 + m_Z^2)\cos 2\beta$, $m_{H_2}^2 + |\mu|^2 = \frac{1}{2}m_A^2 + \frac{1}{2}(m_A^2 + m_Z^2)\cos 2\beta$ with $m_Z^2 = \frac{1}{2}(g^2 + g'^2)(v_1^2 + v_2^2)$.

From definition of vacuum above. Compare m_Z here with m_Z on page 71.

Limit on β: $0 \le \beta \le \frac{\pi}{2}$. $B\mu = m_A^2 \sin 2\beta$ above, but V bounded from below on page 157 is $m_A^2 \ge B\mu$, so $0 \le \sin 2\beta \le 1$.

Quadratic (mass) terms in V^{neutral}: $V_{\text{quad}}^{\text{neutral}} = \frac{1}{2}m_Z^2 \cos 2\beta \left(|\phi_1|^2 - |\phi_2|^2\right) + m_Z^2 \left(\text{Re}(\cos\beta\phi_1 - \sin\beta\phi_2)\right)^2$

$+\frac{1}{2}m_A^2(|\phi_1|^2 + |\phi_2|^2) - \frac{1}{2}(m_A^2 + m_Z^2)\cos 2\beta(|\phi_1|^2 - |\phi_2|^2) - m_A^2 \sin 2\beta \text{Re}(\phi_1\phi_2)$.

Neutral Higgs particle masses: m_A^2, 0, $m_H^2 = \frac{1}{2}\left(m_A^2 + m_Z^2 + \sqrt{(m_A^2 - m_Z^2)^2 + 4m_A^2 m_Z^2 \sin^2 2\beta}\right)$,

$m_h^2 = \frac{1}{2}\left(m_A^2 + m_Z^2 - \sqrt{(m_A^2 - m_Z^2)^2 + 4m_A^2 m_Z^2 \sin^2 2\beta}\right)$ (zero mass particle is Goldstone boson).

Real, imaginary parts of ϕ_i decouple in $V_{\text{quad}}^{\text{neutral}}$ (no terms like e.g. $\text{Re}(\phi_1)\text{Im}(\phi_2)$).

First two masses associated with $\text{Im}(\phi) = (\text{Im}(\phi_1), \text{Im}(\phi_2))$: Write $\text{Im}(\phi)$ dependence of $V_{\text{quad}}^{\text{neutral}}$ as $\text{Im}(\phi)^T M^2_{\text{Im}(\phi)} \text{Im}(\phi)$,

then $M^2_{\text{Im}(\phi)} = \begin{pmatrix} \frac{1}{2}m_A^2(1-\cos 2\beta) & \frac{1}{2}m_A^2 \sin 2\beta \\ \frac{1}{2}m_A^2 \sin 2\beta & \frac{1}{2}m_A^2(1+\cos 2\beta) \end{pmatrix}$. Eigenvalues are m_A^2 and 0, corresponding eigenstates C odd.

So m_A^2 must be positive to ensure $\phi_i = 0$ is local minimum, i.e. that eigenvalues of $\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}$ positive. Similarly,

last two masses are eigenvalues of $M^2_{\text{Re}(\phi)} = \begin{pmatrix} \frac{1}{2}m_A^2(1-\cos 2\beta) + \frac{1}{2}m_Z^2(1+\cos 2\beta) & -\frac{1}{2}(m_A^2 + m_Z^2)\sin 2\beta \\ -\frac{1}{2}(m_A^2 + m_Z^2)\sin 2\beta & \frac{1}{2}m_A^2(1+\cos 2\beta) + \frac{1}{2}m_Z^2(1-\cos 2\beta) \end{pmatrix}$.

Heaviest neutral scalar: $m_H^2 > m_A^2, m_Z^2$. **Lightest neutral scalar:** $m_h^2 < m_A^2, m_Z^2$.

Large top to bottom quark mass ratio suggests large $\tan\beta = \frac{v_2}{v_1} \to m_H^2 \simeq \text{Max}(m_A^2, m_Z^2)$, $m_h^2 \simeq \text{Min}(m_A^2, m_Z^2)$.

Define $V^{\text{charged}} = V$ when $\phi_1 = \phi_2 = 0$.

Quadratic (mass) terms in V^{charged}:

$V^{\text{charged}}_{\text{quad}} = \frac{1}{2}(m_W^2 + m_A^2)\left(|\phi_{H_1^-}|^2(1-\cos 2\beta) + |\phi_{H_2^+}|^2(1+\cos 2\beta) + 2\sin 2\beta \text{Re}(\phi_{H_1^-}\phi_{H_2^+})\right)$, with $m_W^2 = \frac{1}{2}g^2(v_1^2+v_2^2)$.

Compare m_W here with m_W on page 71.

Charged Higgs particle masses: 0, $m_C^2 = m_W^2 + m_A^2$.

$V^{\text{charged}}_{\text{quad}} = (\phi_{H_1^-}, \phi_{H_2^+}) M_C^2 (\phi^*_{H_1^-}, \phi^*_{H_2^+})^T$, where $M_C^2 = \frac{1}{2}(m_W^2 + m_A^2)\begin{pmatrix} 1-\cos 2\beta & \sin 2\beta \\ \sin 2\beta & 1+\cos 2\beta \end{pmatrix}$.

Above masses are eigenvalues of M_C^2.

Results for neutral and charged scalar fields modified most significantly by radiative corrections from top quark, which has largest Yukawa couplings to Higgs. Most significant modification to above results are the *increases*

$$m_H^2 = \frac{1}{2}\left(m_A^2 + m_Z^2 + \Delta_t + \sqrt{((m_A^2 - m_Z^2)\cos 2\beta + \Delta_t)^2 + (m_A^2 + m_Z^2)^2 \sin^2 2\beta}\right)$$

$m_h^2 = \frac{1}{2}\left(m_A^2 + m_Z^2 + \Delta_t - \sqrt{((m_A^2 - m_Z^2)\cos 2\beta + \Delta_t)^2 + (m_A^2 + m_Z^2)^2 \sin^2 2\beta}\right)$, where $\Delta_t = \frac{3\sqrt{2}m_t^4 G_F}{2\pi^2 \sin^2\beta} \ln \frac{M_{\text{st}}^2}{m_t^2}$.

By taking stop mass $M_{\text{st}} > 300$ GeV, $\tan\beta > 10$, get $m_h > m_Z$.

Condition which leads to electroweak symmetry breaking: $4(m^2_{H_1}+|\mu|^2)(m^2_{H_2}+|\mu|^2) \leq (B\mu)^2$.

From parameter relations with v_1 and v_2 on page 158, $4(m^2_{H_1}+|\mu|^2)(m^2_{H_2}+|\mu|^2) = m_A^4 \sin^2 2\beta - m_Z^2(m_Z^2+2m_A^2)\cos^2 2\beta$.

But $B\mu = m_A^2 \sin 2\beta$, so $4(m^2_{H_1}+|\mu|^2)(m^2_{H_2}+|\mu|^2) = (B\mu)^2 - m_Z^2(m_Z^2+2m_A^2)\cos^2 2\beta$, so inequality follows.

These inequalities mean that second derivative matrix of V has negative eigenvalue at SU(2) respecting point $\phi_{H_1} = \phi_{H_2} = 0$,

i.e. V unstable (not minimum) there: Quadratic part of scalar Higgs potential on page 157 can be written as four terms

$\tilde{\phi}^T M_\phi^2 \tilde{\phi}$, where $\tilde{\phi}$ is separately the imainary and real parts of $(\phi_{H_1^-}, \phi_{H_2^+})$ and $(\phi_{H_1^0}, \phi_{H_2^0})$, and $M_\phi^2 = \begin{pmatrix} m^2_{H_1}+|\mu|^2 & \pm\frac{1}{2}B\mu \\ \pm\frac{1}{2}B\mu & m^2_{H_2}+|\mu|^2 \end{pmatrix}$.

So mass eigenvalues m^2 obey $(m^2_{H_1}+|\mu|^2-m^2)(m^2_{H_2}+|\mu|^2-m^2) - \frac{1}{4}(B\mu)^2 = 0$,

solutions are $2m^2 = 2|\mu|^2 + m^2_{H_1} + m^2_{H_2} \pm \sqrt{(2|\mu|^2+m^2_{H_1}+m^2_{H_2})^2 + (B\mu)^2 - 4(m^2_{H_1}+|\mu|^2)(m^2_{H_2}+|\mu|^2)}$.

Inequality above implies one of these solutions is negative.

Example: For $\tan\beta = \infty$ $(\beta = \frac{\pi}{2})$, parameter relations with v_1 and v_2 on page 158

imply $m^2_{H_1}+|\mu|^2 > 0$ and $m^2_{H_2}+|\mu|^2 < 0$, so electroweak symmetry broken.

Alternatively, radiative corrections give $\frac{d}{d\ln\mu_r} m^2_{H_2} = x\ h^U_{33} + \ldots$ (h^U_{33} is top quark Yukawa coupling), $x > 0$.

(Similarly for stop masses, smaller x.) So although $m^2_{H_2} > 0$ at $\mu_r = M_X$, may have $m^2_{H_2} < (\ll) 0$ at $\mu_r = v$.

4.5 Sparticle mass eigenstates

Consider gauginos and higgsinos. Particles with different SU(2)×U(1)

but same $\mathrm{U}(1)_{\mathrm{e.m.}}$ transformation properties can mix after electroweak symmetry breaking.

Neutralinos: 4 neutral fermionic mass eigenstates $\widetilde{\chi}_i^0$, $i = 1$(lightest), ..., 4,

mixtures of bino, neutral wino, neutral higgsinos.

Bilinear terms in these fields appearing in Lagrangian can be written $-\frac{1}{2}(\lambda^0)^T M_{\widetilde{\chi}^0}\lambda^0$, where $\lambda^0 = (\lambda_{\mathrm{bino}}, \lambda_{\mathrm{neutralwino}}, \lambda_{H_1^0}, \lambda_{H_2^0})$,

$$M_{\widetilde{\chi}^0} = \begin{pmatrix} m_{\mathrm{bino}} & 0 & -c_\beta s_W m_Z & s_\beta s_W m_Z \\ 0 & m_{\mathrm{wino}} & c_\beta c_W m_Z & -s_\beta s_W m_Z \\ -c_\beta s_W m_Z & c_\beta c_W m_Z & 0 & -\mu \\ s_\beta s_W m_Z & -s_\beta c_W m_Z & -\mu & 0 \end{pmatrix}$$ ($s_\beta = \sin\beta$ etc.). μ dependent part from μ term on page 146.

m_Z dependent part from 1. for $\Phi = H_i$ and $V = \mathrm{SU(2)\times U(1)}$ fields after electroweak symmetry breaking on page 146.

m_{bino}, m_{wino} dependent terms from SUSY breaking terms for gauginos from 2. on page 150.

$M_{\widetilde{\chi}^0}$ symmetric, diagonalized by unitary matrix.

$\widetilde{\chi}_1^0$ can be LSP, and therefore candidate for cold dark matter.

Charginos: 4 charged fermionic mass eigenstates $\widetilde{\chi}_i^{\pm}$, $i = 1, 2$, mixtures of charged winos and charged higgsinos.

Bilinear terms in these fields appearing in Lagrangian can be written $-\frac{1}{2}(\lambda^+)^T M_{\widetilde{\chi}^c}\lambda^-$,

where $\lambda^+ = (\lambda^+_{\text{charged wino}}, \lambda_{H_2^+})$, $\lambda^- = (\lambda^-_{\text{charged wino}}, \lambda_{H_1^-})$, $M_{\widetilde{\chi}^c} = \begin{pmatrix} m_{\text{wino}} & I\sqrt{2}m_W s_\beta \\ I\sqrt{2}m_W c_\beta & \mu \end{pmatrix}$.

Define $\lambda^c = (\lambda^+, \lambda^-)^T$, $\mathscr{M}_{\widetilde{\chi}^c} = \begin{pmatrix} 0 & M^\dagger_{\widetilde{\chi}^c} \\ M_{\widetilde{\chi}^c} & 0 \end{pmatrix}$, so contribution to Lagrangian including hermitian conjugate

is $-\frac{1}{2}(\lambda^c)^T \mathscr{M}_{\widetilde{\chi}^c}\lambda^c$. Squared mass eigenvalues can be obtained from diagonalization of $\mathscr{M}^2_{\widetilde{\chi}^c} = \begin{pmatrix} M^\dagger_{\widetilde{\chi}^c} M_{\widetilde{\chi}^c} & 0 \\ 0 & M_{\widetilde{\chi}^c} M^\dagger_{\widetilde{\chi}^c} \end{pmatrix}$,

gives $m^2_{\widetilde{\chi}^+_{1,2}} = m^2_{\widetilde{\chi}^-_{1,2}} = \frac{1}{2}\left(m^2_{\text{wino}} + 2m_W^2 + |\mu|^2 \pm \sqrt{(m^2_{\text{wino}} - |\mu|^2)^2 + 4m_W^4\cos^2 2\beta + 4m_W^2(m^2_{\text{wino}} + |\mu|^2 - 2m_{\text{wino}}\mathrm{Re}(\mu\sin 2\beta))}\right)$.

Slepton, squark mass matrix: $m^2_{\text{squark}} = \begin{pmatrix} m^2_{L\text{squark}} & m^{T2}_{LR\text{squark}} \\ m^2_{LR\text{squark}} & m^2_{R\text{squark}} \end{pmatrix}$,

where $m^2_{L/R\text{squark},KM} = M^{Q/\overline{U}\ 2}_{KM} + (m^2_{\text{quark},K} + m_Z^2(T_3 - Q\sin^2\theta_W)\cos 2\beta)\delta_{KM}$ (no K sum),

M^{S2}_{KM} from 1. on page 150, m_{quark} from unbroken SUSY condition $m_{\text{squark},KK} = m_{\text{quark},K}$ (no K sum),

and term proportional to m_Z^2 from Higgs-chiral superfield couplings in superpotential $[f(\Phi)]_{\mathscr{F}}$ on page 145.

$m^2_{LR\text{squark},KM} = (A^U_{KM} - \delta_{KM} m_{\text{quark}}\mu\tan\beta)m_{\text{quark}}$. First term from 3. on page 150, second term from $\left|\frac{\partial f}{\partial\phi_{H_1^0}}\right|^2$.

Further reading

Stephen P. Martin, hep-ph/9709356 Very topical (not much basics)

— helps you understand what people are talking about

Ian J. R. Aitchison, hep-ph/0505105 Complete derivations up to and including the MSSM

Michael E. Peskin, 0801.1928 [hep-ph] Recent, readable

Steven Weinberg, The Quantum Theory Of Fields III Very complete derivations, quite formal but clear

5 Supergravity

5.1 Spinors in curved spacetime

Gravity formulated in terms of integer rank matter tensors

and the metric tensor $g_{\mu\nu}$ for coordinates x_μ over whole of curved spacetime.

Supergravity also involves matter spinors, not generalizable to arbitrary coordinate systems.

So, describe spinor at point X using locally inertial coordinates $y^a_X(x)$, $a = 0, 1, 2, 3$,

which transform as $y^a \to y'^a = \Lambda^a{}_b(x) y^b$ (Lorentz transformation)

and have "flat" metric tensor $\eta_{ab} = \mathrm{diag}(1, 1, 1, -1)$.

Principle of equivalence $\Longrightarrow$ action can be expressed in terms of

1. matter spinors, 2. integer rank matter tensors in $y^a_X(x)$,

and 3. transformation between locally inertial and general coordinates, described by *vierbein*:

Vierbein: $e^a{}_\mu(X) = \left.\frac{\partial y^a_X(x)}{\partial x^\mu}\right|_{x=X}$.

Vierbein-metric tensor relation: $g_{\mu\nu}(x) = \eta_{ab} e^a{}_\mu(x) e^b{}_\nu(x)$.

Curved space transformation of vierbein: $e'^a{}_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} e^a{}_\mu(x)$.

Lorentz transformation of vierbein: $e^a{}_\mu(x) \to \Lambda^a{}_b(x) e^b{}_\mu(x)$.

A vector may be described by quantities $V^a(x)$

transforming as a vector under local Lorentz transformations ($V^a(x) \to \Lambda^a{}_b(x) V^b(x)$)

and as a scalar under general coordinate transformations,

or quantities v^μ which transform vice versa.

Two types of quantities related by $V^a = e^a{}_\mu v^\mu$.

Generalize to integer rank tensors.

A spinor may be described by quantities $\psi_\alpha(x)$

transforming as a spinor under local Lorentz transformations ($\psi_\alpha(x) \to D_{\alpha\beta}(\Lambda(x))\psi_\beta(x)$)

but *necessarily* as a scalar under general coordinate transformations.

Define covariant derivative of object ψ with a spinor index, $\mathscr{D}_\mu\psi$, to obey

Transformation of covariant derivative: $\mathscr{D}_\mu\psi \to D(\Lambda)\mathscr{D}_\mu\psi$ under local Lorentz transformations,

$\mathscr{D}_\mu\psi \to \frac{\partial x^\nu}{\partial x'\mu}\mathscr{D}_\nu\psi$ under general coordinate transformations.

Achieve this by writing

Covariant derivative of object ψ with a spinor index: $\mathscr{D}_\mu\psi = \psi_{;\mu} + \Omega_\mu\psi$,

where $[\Omega_\mu]_{\alpha\beta}(x) = \frac{1}{2}I[\mathscr{J}_{ab}]_{\alpha\beta}\omega_\mu^{ab}(x)$,

with $\mathscr{J}_{ab}$ the matrix generators of the Lorentz group and $\omega_\mu^{ab}(x)$ the *spin connection*.

and using

Choice of spin connection: $\omega_\mu^{ab} = g^{\nu\lambda}e^a{}_\nu e^b{}_{\lambda;\mu}$. Results in the transformation of covariant derivative above.

5.2 Weak field supergravity

Work in weak field limit, i.e. $e^a{}_\mu(x) \simeq \delta^a_\mu$, then $a \to \mu$ etc.

Weak field representation: $e_{\mu\nu}(x) = \eta_{\mu\nu} + 2\kappa\phi_{\mu\nu}$, i.e. $g_{\mu\nu}(x) = \eta_{\mu\nu} + 2\kappa(\phi_{\mu\nu} + \phi_{\nu\mu})$, where $\kappa = \sqrt{8\pi G}$.

Infinitesimal curved space transformation: $x^\mu \longrightarrow x^\mu + \xi^\mu(x)$.

Infinitesimal Lorentz transformation: $\Lambda^a{}_b(x) = \delta^a{}_b + \omega^a{}_b(x)$.

Particle content of supergravity: spin-2 *graviton* $h_{\mu\nu} = \phi_{\mu\nu} + \phi_{\nu\mu}$ and spin-$\frac{3}{2}$ *gravitino* ψ_μ.

Graviton transformation: $\phi_{\mu\nu}(x) \longrightarrow \phi_{\mu\nu}(x) + \frac{1}{2\kappa}\left[-\frac{\partial\xi_\mu(x)}{\partial x_\nu} + \omega_{\mu\nu}(x)\right]$ so $h_{\mu\nu}(x) \longrightarrow h_{\mu\nu}(x) + \frac{1}{\kappa}\left[-\frac{\partial\xi_\mu(x)}{\partial x_\nu}\right]$.

Follows from performing infinitessimal curved space and Lorentz transformations above.

Gravitino transformation: $\psi_\mu(x) \longrightarrow \psi_\mu(x) + \partial_\mu\psi(x)$.

Required for low energy interactions, similar to requirement of invariance under $A^\mu(x) \longrightarrow A^\mu(x) + \partial^\mu\alpha(x)$ found on page 55.

Goal: Put $\phi_{\mu\nu}$, ψ_μ into one superfield, ξ_μ, ψ into another.

Graviton, gravitino as functions of supermultiplet components of

Metric superfield: $H_\mu(x,\theta) = C^H_\mu(x) - I[\overline{\theta}\gamma_5]\omega^H_\mu(x) - \frac{I}{2}\left[\overline{\theta}\gamma_5\theta\right] M^H_\mu(x) - \frac{1}{2}[\overline{\theta}\theta]N^H_\mu(x) + \frac{I}{2}[\overline{\theta}\gamma_5\gamma_\nu\theta]V^{H\ \nu}_\mu(x)$

$$-I[(\overline{\theta}\gamma_5\theta)\overline{\theta}]\left(\lambda^H_\mu(x) + \frac{1}{2}\not{\partial}\omega^H_\mu(x)\right) - \frac{1}{4}[\overline{\theta}\gamma_5\theta]^2\left(D^H_\mu(x) + \frac{1}{2}\partial^2 C^H_\mu(x)\right),$$

This is general form of superfield on page 103, but with extra spacetime index μ.

Specifically,

Graviton, gravitino components of metric superfield H_μ: $\phi_{\mu\nu}(x) = V^H_{\mu\nu}(x) - \frac{1}{3}\eta_{\mu\nu}V^{H\lambda}_{\ \ \lambda}(x)$,

$$\frac{1}{2}\psi_\mu(x) = \lambda^H_\mu - \frac{1}{3}\gamma_\mu\gamma^\rho\lambda^H_\rho(x) - \frac{1}{3}\gamma_\mu\partial^\rho\omega^H_\rho(x).$$

Graviton, gravitino transformations on page 168 equivalent to

Transformation of metric superfield: $H_\mu(x,\theta) \to H_\mu(x,\theta) + \Delta_\mu(x,\theta)$, where

Transformation superfield:

$$\Delta_\mu(x,\theta) = C^\Delta_\mu(x) - I[\overline{\theta}\gamma_5]\omega^\Delta_\mu(x) - \frac{I}{2}\left[\overline{\theta}\gamma_5\theta\right] M^\Delta_\mu(x) - \frac{1}{2}[\overline{\theta}\theta]N^\Delta_\mu(x) + \frac{I}{2}[\overline{\theta}\gamma_5\gamma_\nu\theta]V^{\Delta\ \nu}_\mu(x)$$

$$-I[(\overline{\theta}\gamma_5\theta)\overline{\theta}]\left(\lambda^\Delta_\mu(x) + \frac{1}{2}\not{\partial}\omega^\Delta_\mu(x)\right) - \frac{1}{4}[\overline{\theta}\gamma_5\theta]^2\left(D^\Delta_\mu(x) + \frac{1}{2}\partial^2 C^\Delta_\mu(x)\right).$$

See later for dependence of these fields on ψ, ξ_μ of graviton, gravitino transformations on page 168.

Gravity-matter coupling: $\mathscr{A}_{\text{int}} = 2\kappa \int d^4x \left[H_\mu \Theta^\mu\right]_D$, where

Supercurrent from left-chiral superfields: $\Theta_\mu = \frac{I}{12}\left[4\Phi_n^\dagger \partial_\mu \Phi_n - 4\Phi_n \partial_\mu \Phi_n^\dagger + (\overline{\mathscr{D}}\Phi_n^\dagger)\gamma_\mu(\mathscr{D}\Phi_n)\right]$.

Supercurrent is a superfield, contains conserved current and energy momentum tensor:

Supercurrent conservation law: $\gamma^\mu \mathscr{D}\Theta_\mu = \mathscr{D}\frac{2}{3}\text{Im}\left[M\frac{\partial f(\Phi)}{\partial M}\right]$. Use the field equations $\left(\overline{\mathscr{D}}_L \mathscr{D}_L\right)\Phi_n = -4\left(\frac{\partial f(\Phi)}{\partial \Phi_n}\right)^*$.

M here defined as follows: Each coupling constant λ_i in f is written $\lambda_i = M^{d_M(\lambda_i)}\tilde{\lambda}_i$, where $\tilde{\lambda}_i$ dimensionless.

SUSY current from supercurrent components: $S^\mu = -2\omega^{\Theta\mu} + 2\gamma^\mu\gamma^\nu\omega_\nu^\Theta$.

Note S^μ is alternative SUSY current on page 127.

Energy-momentum tensor from supercurrent components: $T_{\mu\nu} = -\frac{1}{2}V_{\mu\nu}^\Theta - \frac{1}{2}V_{\nu\mu}^\Theta + \eta_{\mu\nu}V^{\Theta\lambda}_{\ \ \lambda}$, obeys

Energy-momentum tensor conservation law: $\partial_\mu T^{\mu\nu} = 0$. Follows from supercurrent conservation law above.

Relation between energy-momentum tensor and momentum: $\int d^3x T^{0\nu} = P^\nu$.

Follows from SUSY transformation of ω_μ^Θ, take time component and integrate over x: $I\left\{\int d^3x S^0, \overline{Q}\right\} = 2\gamma_\nu \int d^3x T^{0\nu}$.

But $\int d^3x S^0 = Q$ (as in bosonic generator case), then use relation with momentum for any N on page 87.

Constraint on transformation superfield Δ_μ: $\Delta_\mu = \overline{\mathscr{D}}\gamma_\mu\Xi$, where Ξ obeys $(\overline{\mathscr{D}}\mathscr{D})(\overline{\mathscr{D}}\Xi) = 0$.

Ensures $\mathscr{A}_{\text{int}}$ in gravity-matter coupling on page 170 is invariant under transformation of metric superfield on page 169,

follows from supercurrent conservation law on page 170: Let $X = \frac{2}{3}\mathrm{Im}\left[M\frac{\partial f(\Phi)}{\partial M}\right]$.

Then $\delta\mathscr{A}_{\text{int}} = \int d^4x \left[\Theta^\mu \left(\overline{\mathscr{D}}\gamma_\mu\Xi\right)\right]_D = -\int d^4x \left[(\overline{\mathscr{D}}\Theta^\mu)\gamma_\mu\Xi\right]_D = -\int \left[(\overline{\mathscr{D}}X)\Xi\right]_D$.

But X is a chiral superfield, can be written $X = (\overline{\mathscr{D}}\mathscr{D})\Omega$ where Ω is a general superfield.

So $\delta\mathscr{A}_{\text{int}} = \int d^4x \left[\Omega(\overline{\mathscr{D}}\mathscr{D})(\overline{\mathscr{D}}\Xi)\right]_D = 0$.

Recall transformation parameters ξ_μ, ψ in graviton, gravitino transformations on page 168.

Transformation parameter components of transformation superfield:

$V^\Delta_{\mu\nu} + V^\Delta_{\nu\mu} = -\frac{1}{2\kappa}\left[\frac{\partial\xi_\mu}{\partial x_\nu} + \frac{\partial\xi_\nu}{\partial x_\mu} - 2\eta_{\mu\nu}\frac{\partial\xi^\lambda}{\partial x^\lambda}\right]$, $\lambda^\Delta_\mu - \frac{1}{3}\gamma_\mu\gamma^\rho\lambda^\Delta_\rho - \frac{1}{3}\gamma_\mu\partial^\rho\omega^\Delta_\rho = \frac{1}{2}\partial_\mu\psi$.

From constraint on transformation superfield Δ_μ on page 171, and graviton, gravitino transformations on page 168.

Further constraints on components of transformation superfield: $-\frac{1}{2}\epsilon^{\nu\mu\kappa\sigma}\partial_\kappa V^\Delta_{\nu\mu} = D^{\Delta\sigma} + \partial^\sigma\partial^\rho C^\Delta_\rho$, $\partial^\mu M^\Delta_\mu = \partial^\mu N^\Delta_\mu = 0$.

Again, from constraint on transformation superfield Δ_μ on page 171.

Auxiliary fields: $b^\sigma = D^{H\sigma} + \frac{1}{2}\epsilon^{\nu\mu\kappa\sigma}\partial_\kappa V^H_{\nu\mu} + \partial^\sigma\partial^\rho C^H_\rho$, $s = \partial^\mu M^H_\mu$, $p = \partial^\mu N^H_\mu$, invariant.

From further constraints on components of transformation superfield above.

Choose $C^H_\mu = V^H_{\mu\nu} - V^H_{\nu\mu} = \phi_{\mu\nu} - \phi_{\nu\mu} = \omega^H_\nu = 0$. Then $h_{\mu\nu} = 2\phi_{\mu\nu}$.

Can be done by suitable choice of C^Δ_μ, $V^\Delta_{\mu\nu} - V^\Delta_{\nu\mu}$, ω^Δ_ν in transformation of metric superfield on page 169.

To summarize, components of superfields are:

$\Theta_\mu \ni T^{\kappa\sigma}, S^\sigma, \mathscr{R}_\sigma, \mathscr{M}, \mathscr{N}, \qquad H_\mu \ni h_{\kappa\sigma}, \psi_\sigma, b^\sigma, s, p, \qquad$ where $\mathscr{R}^\mu = 2C^{\Theta\mu}$, $M^\Theta_\mu = \partial_\mu\mathscr{M}$, $N^\Theta_\mu = \partial_\mu\mathscr{N}$.

Gravity-matter coupling in terms of components: $\mathscr{A}_{\text{int}} = \kappa\int d^4x[T^{\kappa\sigma}h_{\kappa\sigma} + \frac{1}{2}\overline{S}^\sigma\psi_\sigma + \mathscr{R}_\sigma b^\sigma - 2\mathscr{M}s - 2\mathscr{N}p]$.

For dynamic part of gravitational action, use:

Einstein superfield E_μ: $C^E_\mu = b_\mu$, $\omega^E_\mu = \frac{3}{2}L_\mu - \frac{1}{2}\gamma_\mu\gamma^\nu L_\nu$, $M^E_\mu = \partial_\mu s$, $N^E_\mu = \partial_\mu p$,

$V^E_{\mu\nu} = -\frac{3}{2}E_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}E^\rho{}_\rho + \frac{1}{2}\epsilon_{\nu\mu\sigma\rho}\partial^\sigma b^\rho$, $\lambda^E_\mu = \partial_\mu\gamma^\nu\omega^E_\nu - \not{\partial}\omega^E_\mu$, $D^E_\mu = \partial_\mu\partial^\nu b_\nu - \partial^2 b_\mu$, where $L^\nu = I\epsilon^{\nu\mu\kappa\rho}\gamma_5\gamma_\mu\partial_\kappa\psi_\rho$,

and *linearized Einstein tensor* $E_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\nu h^\lambda{}_\lambda + \partial^2 h_{\mu\nu} - \partial_\mu\partial^\lambda h_{\lambda\nu} - \partial_\nu\partial^\lambda h_{\lambda\mu} - \eta_{\mu\nu}\partial^2 h^\lambda{}_\lambda + \eta_{\mu\nu}\partial^\lambda\partial^\rho h_{\lambda\rho}\right)$

$$= \frac{1}{2\kappa}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right).$$

Dynamic part of gravitational Lagrangian: $\mathscr{L}_E = \frac{4}{3}[E_\mu H^\mu]_D = E_{\mu\nu}h^{\mu\nu} - \frac{1}{2}\overline{\psi}_\mu L^\mu - \frac{4}{3}(s^2 + p^2 - b_\mu b^\mu)$.

Invariant under transformation of metric superfield on page 169.

Now put together matter Lagrangian $\mathscr{L}_M$, and $\mathscr{L}_E$ and integrand of $\mathscr{A}_{\text{int}}$ above and eliminate auxiliary fields:

Lagrangian of nature: $\mathscr{L} = \mathscr{L}_M + E_{\mu\nu}h^{\mu\nu} - \frac{1}{2}\overline{\psi}_\mu L^\mu + \kappa[T^{\kappa\sigma}h_{\kappa\sigma} + \frac{1}{2}\overline{S}^\sigma\psi_\sigma]$ $+\frac{3}{4}\kappa^2\left(\mathscr{M}^2 + \mathscr{N}^2 - \frac{1}{4}\mathscr{R}_\mu\mathscr{R}^\mu\right)$.

Field equations give $s = -6\kappa\mathscr{M}/8$, $p = -6\kappa\mathscr{N}/8$, $b_\mu = -6\kappa\mathscr{R}_\mu/16$.

Everything except $\mathscr{L}_M$ is of order κ^2.

Vacuum energy density: $\rho_{\mathrm{VAC}} = -\mathscr{L}_{\mathrm{VAC}} = -\mathscr{L}_{\mathrm{M\ VAC}} - \frac{3}{4}\kappa^2(\mathscr{M}^2 + \mathscr{N}^2)$.

Only s and p can acquire vacuum expectation values.

Solution to Einstein field equations (ρ_{VAC} uniform) for $\rho_{\mathrm{VAC}} \gtrless 0$ is de Sitter / anti de Sitter space:

spacetime embedded in 5-D space with $x_5^2 \pm \eta_{\mu\nu}x^\mu x^\nu = R^2$ and $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \pm dx_5^2$ for $\rho_{\mathrm{VAC}} \gtrless 0$.

$\rho_{\mathrm{VAC}} < 0$ corresponds to O(3,2), which includes $N = 1$ SUSY,

but $\rho_{\mathrm{VAC}} > 0$ corresponds to O(4,1), which excludes unbroken $N = 1$ SUSY.

In unbroken SUSY, $\mathscr{L}_{\mathrm{M\ VAC}} = 0$ so from vacuum energy density above, $\rho_{\mathrm{VAC}} < 0$.

$\rho_{\mathrm{VAC}} < 0$ is unstable. However, anti de Sitter space cannot form

since positive energy $S_1 > |\rho_{\mathrm{VAC}}|$ is needed for its surface tension.

Local supersymmetry: α in definition of superfield on page 101 becomes dependent on x.

Only change is to matter action: $\delta \int d^4x \mathscr{L}_M = -\int d^4x \overline{S}^\mu(x) \partial_\mu \alpha(x)$ (usual definition of current).

Cancelled by term $\kappa \int d^4x \frac{1}{2} \overline{S}^\sigma \psi_\sigma$ in gravity-matter coupling in terms of components on page 172

if SUSY transformation of gravitino modified to: $\delta\psi_\mu \to \delta\psi_\mu + \frac{2}{\kappa}\partial_\mu \alpha$.

But $\psi_\mu \to \psi_\mu + \frac{2}{\kappa}\partial_\mu\alpha$ is of same form as gravitino transformation on page 168,

i.e. leaves term $-\frac{1}{2}\overline{\psi}_\mu L^\mu$ in dynamic part of gravitational Lagrangian $\mathscr{L}_E$ on page 173 unchanged.

So supersymmetric gravity is locally supersymmetric.

5.3 Supergravity to all orders

Lagrangian of nature (page 173) not supersymmetric at $O(\kappa)$ (from matter - gravity supermultiplet interaction).

Solution: keep adding terms of higher order in κ to SUSY transformations

until SUSY, local Lorentz and general coordinate transformations form closed algebra,

then keep adding terms to action till it is invariant under all these transformations.

Define $\mathscr{D}_\mu = \partial_\mu + \frac{1}{8}[\gamma_a, \gamma_b]\omega_\mu^{ab}$, $\omega_\mu^{ab} = e^a{}_\lambda e^b{}_{\nu;\mu} g^{\lambda\nu} + \frac{\kappa^2}{4}\left[e^b{}_\nu \left(\overline{\psi}_\mu \gamma^a \psi^\nu\right) + e^a{}_\nu e^b{}_\rho \left(\overline{\psi}_\nu \gamma_\mu \psi^\rho\right) - e^a{}_\nu \left(\overline{\psi}_\mu \gamma^b \psi^\nu\right)\right]$,

This is the covariant derivative on page 167, with a gravitino-dependent term added to the spin connection,

and we have used $[\mathscr{J}_{ab}]_{\alpha\beta} = -\frac{I}{4}[\gamma_a, \gamma_b]$. in the Dirac representation.

$\gamma_\mu = e^a{}_\mu \gamma_a$, $e = \sqrt{\mathrm{Det} g}$, $L^\mu = I\gamma_5 \gamma_\nu D_\rho \psi_\sigma \epsilon^{\mu\nu\rho\sigma}$.

Then for the metric superfield

Local SUSY transformation of gravity: $\delta\psi_\mu = \frac{2}{\kappa}\mathscr{D}_\mu\alpha + 2I\gamma_5(b_\mu - \frac{1}{3}\gamma_\mu\gamma_\rho b^\rho)\alpha + \frac{2}{3}\gamma_\mu(s - I\gamma_5 p)\alpha$,

$\delta s = \frac{1}{4e}\left(\overline{\alpha}\gamma_\mu L^\mu\right) + \frac{\kappa}{2}\left(\overline{\alpha}[I\gamma_5 b^\nu - s\gamma^\nu - Ip\gamma_5\gamma^\nu]\psi_\nu\right)$, $\delta p = -\frac{I}{4e}\left(\overline{\alpha}\gamma_5\gamma_\mu L^\mu\right) + \frac{\kappa}{2}\left(\overline{\alpha}[b^\nu + Is\gamma_5\gamma^\nu - p\gamma^\nu]\psi_\nu\right)$,

$\delta b_\mu = \frac{3I}{4e}\left(\overline{\alpha}\gamma_5(L_\mu - \frac{1}{3}\gamma_\mu\gamma_\rho L^\rho)\right) + \frac{\kappa}{2}b_\nu\left(\overline{\alpha}\gamma^\nu\psi_\mu\right) + \frac{I\kappa}{2}\left(\overline{\psi}_\mu\gamma_5(s - I\gamma_5 p)\alpha\right) - \frac{I\kappa}{4}\epsilon_{\mu\nu\kappa\sigma}b^\nu\left(\overline{\alpha}\gamma_5\gamma^\kappa\psi^\sigma\right)$, $\delta e^a{}_\mu = \kappa\left(\overline{\alpha}\gamma^a\psi_\mu\right)$.

Action for pure supergravity: $I_{\mathrm{SUGRA}} = \int d^4x\left[-\frac{e}{2\kappa^2}R - \frac{1}{2}\overline{\psi}_\mu L^\mu - \frac{4e}{3}\left(s^2 + p^2 - b_\mu b^\mu\right)\right]$.

$s = p = b_\mu = 0$ gives action in absence of matter.

Vierbein curvature tensor from spin connection: $R^{ab}_{\mu\nu} = \omega^{ab}_{\mu,\nu} - \omega^{ab}_{\nu,\mu} + \omega^{ac}_\nu\omega^b_{\mu c} - \omega^{ac}_\mu\omega^b_{\nu c}$.

Vierbein and Riemann-Christoffel curvature tensors: $R^{ab}_{\mu\nu} = e^a{}_\kappa e^b{}_\lambda R_{\mu\nu}{}^{\kappa\lambda}$.

For matter superfields, first consider general scalar superfield:

First we define

Covariant derivatives: $\mathscr{D}_a C = e_a{}^\mu \left[\partial_\mu C - \frac{I\kappa}{2}\left(\overline{\psi}_\mu \gamma_5 \omega\right)\right]$,

$\mathscr{D}_a \omega = e_a{}^\mu \left[\partial_\mu \omega + \frac{1}{8}\omega_\mu^{cb}[\gamma_c, \gamma_b]\omega - I\kappa b_\mu \gamma_5 \omega - \frac{\kappa}{2}\left(\not{V} - I\gamma_5 \not{\partial} C - M + I\gamma_5 N\right)\psi_\mu\right]$,

$\mathscr{D}_a \lambda = e_a{}^\mu \left[\partial_\mu \lambda + \frac{1}{8}\omega_\mu^{cb}[\gamma_c, \gamma_b]\lambda + I\kappa b_\mu \gamma_5 \lambda + \frac{\kappa}{8}[\gamma_b, \gamma_c]\psi_\mu F_{bc} - \frac{I\kappa}{2}\gamma_5 D\psi_\mu\right]$,

$F_{ab} = e_a{}^\mu e_b{}^\nu \left[\partial_\mu V_\nu + \frac{\kappa}{2}\partial_\mu\left(\overline{\psi}_\nu \omega\right) - \frac{\kappa}{2}\left(\overline{\psi}_\mu \gamma_\nu \lambda\right)\right] - a \leftrightarrow b$,

then

Local SUSY transformation of scalar supermultiplet:

$\delta C = I\left(\overline{\alpha}\gamma_5 \omega\right)$, $\delta\omega = [-I\gamma_5 \not{\mathscr{D}} C - M + I\gamma_5 N + \not{V}]\alpha$,

$\delta M = -\left(\overline{\alpha}[\lambda + \not{\mathscr{D}}\omega]\right) + \frac{2\kappa}{3}\left(\overline{\alpha}[s - I\gamma_5 p + I\gamma_5 \not{b}]\omega\right)$, $\delta N = I\left(\overline{\alpha}\gamma_5[\lambda + \not{\mathscr{D}}\omega]\right) + \frac{2I\kappa}{3}\left(\overline{\alpha}[s - I\gamma_5 p + I\gamma_5 \not{b}]\gamma_5 \omega\right)$,

$\delta V_a = \left(\overline{\alpha}\gamma_a \lambda\right) + \left(\overline{\alpha}\mathscr{D}_a \omega\right) + \frac{\kappa}{3}\left(\overline{\alpha}[s - I\gamma_5 p + I\gamma_5 \not{b}]\gamma_a \omega\right)$, $\delta\lambda = -\frac{1}{4}[\gamma^a, \gamma^b]\alpha F_{ab} + ID\gamma_5 \alpha$, $\delta D = I\left(\overline{\alpha}\gamma_5 \not{\mathscr{D}}\lambda\right)$.

Chiral superfield: $\lambda = D = 0$, $V_\nu + \frac{1}{2}\kappa\left(\overline{\psi}_\nu\omega\right) = \partial_\nu Z$.

Compare with definition of chiral superfield on page 106.

From local SUSY transformation of scalar supermultiplet on page 178, $F_{ab} = 0$ so $\delta\lambda = 0$, and $\mathscr{D}_a\lambda = 0$ so $\delta D = 0$.

Finally $\delta\left[V_\nu + \frac{1}{2}\kappa\left(\overline{\psi}_\nu\omega\right)\right] = \partial_\nu\left(\overline{\alpha}\omega\right)$.

Otherwise, define left-chiral supermultiplet as on page 106.

Local SUSY transformation of left-chiral supermultiplet: $\delta\phi = \sqrt{2}\left(\overline{\alpha}\psi_L\right)$,

$\delta\psi_L = \sqrt{2}(\partial\!\!\!/\phi)\alpha_R - \kappa\gamma^\mu\left(\overline{\psi}_\mu\psi_L\right)\alpha_R + \sqrt{2}\mathscr{F}\alpha_L$, $\delta\mathscr{F} = \sqrt{2}\left(\overline{\alpha}\mathscr{D}\!\!\!\!/\,\psi_L\right) - \frac{2\kappa}{3}\left(\overline{\alpha}\left[s - Ip - I b\!\!\!/\right]\psi_L\right)$

From local SUSY transformation of scalar supermultiplet on page 178.

Recall $\omega = -I\gamma_5\psi$ so $\mathscr{D}\!\!\!\!/\,\psi_L$ from $\mathscr{D}\!\!\!\!/\,\omega$ from covariant derivatives on page 178.

D and $\mathscr{F}$ terms not locally supersymmetric.

Instead use

Modified D-term: $[S]_{\text{modified } D} = e\Big[D^S - \frac{I\kappa}{2}\left(\overline{\psi}^\mu\gamma_\mu\gamma_5\lambda^S\right) + \frac{4\kappa}{3}[-sN^S + pM^S - b^\mu V^S_\mu]$

$-\frac{I\kappa}{3}\left(\overline{\omega^S}\gamma_5\not{L}\right) - \frac{\kappa^2}{4}\epsilon^{\mu\rho\sigma\tau}V^S_\sigma\left(\overline{\psi}_\rho\gamma_\tau\psi_\mu\right)\frac{\kappa^2}{8}\epsilon^{\mu\rho\sigma\tau}\left(\overline{\omega^S}\psi_\sigma\right)\left(\overline{\psi}_\rho\gamma_\tau\psi_\mu\right)\Big] - \frac{2\kappa^2}{3}C^S\mathscr{L}_{\text{SUGRA}}$

and

Modified $\mathscr{F}$-term: $[X]_{\text{modified } \mathscr{F}} = e\left[\mathscr{F}^X + \frac{\kappa}{\sqrt{2}}\left(\overline{\psi}_{\mu R}\gamma^\mu\psi^X_L\right) + \frac{\kappa^2}{4}\left(\overline{\psi}_{\mu R}[\gamma^\mu,\gamma^\nu]\psi_{\nu R}\right)\phi^X + 2\kappa(s - Ip)\phi^X\right]$.

Then

Local SUSY Lagrangian with matter: $\mathscr{L} = \mathscr{L}_{\text{SUGRA}} + \frac{1}{2}[K(\Phi,\Phi^*)]_{\text{modified } D} + 2\text{Re}[f(\Phi)]_{\text{modified } \mathscr{F}}$.

Compare with supersymmetric Lagrangian from chiral superfields on page 111.

Consider other possibilities to add to the Lagrangian.

Supermultiplet **I** with $C = 1$, all other fields zero: $[\mathbf{I}]_{\text{modified } D} = -\frac{2\kappa^2}{3}\mathscr{L}_{\text{SUGRA}}$, i.e. nothing new.

Left-chiral supermultiplet **I** with $\phi = 1$, all other fields zero: $\text{Re}([\mathbf{I}]_{\text{modified } \mathscr{F}}) = e\left[\frac{\kappa^2}{4}\left(\overline{\psi}_\mu[\gamma^\mu, \gamma^\nu]\psi_\nu\right) + 4\kappa s\right]$.

Gives gravitino mass $m_g = c\kappa^2$ and

Vacuum energy density: $-3\kappa^2 c^2$.

Add $c\int d^4x\text{Re}([\mathbf{I}]_{\text{modified } \mathscr{F}})$ to action for pure supergravity on page 177, minimize with respect to s, p, b^μ.

Then $s = 3c\kappa/2$, $p = b^\mu = 0$, so $c\int d^4x\text{Re}([\mathbf{I}]_{\text{modified } \mathscr{F}}) = 3\kappa^2 c^2 \int d^4x e$.

Can be used to cancel positive vacuum energy density ρ_{VAC} due to SUSY breaking,

making vacuum Lorentz-invariant,

which fixes the relation between gravitino mass m_g and ρ_{VAC} to $m_g = \sqrt{8\pi G\rho_{\text{VAC}}/3}$.

Let $\tilde{g}_{\mu\nu} = (1 - \kappa^2 K/3) g_{\mu\nu}$ (called a *Weyl transformation*).

Let $\tilde{R}_{\mu\nu}$ be the curvature tensor calculated using metric $\tilde{g}_{\mu\nu}$, $\tilde{e} = \sqrt{\mathrm{Det}\tilde{g}}$.

We define the *modified Kahler potential* d via $1 - \frac{\kappa^2}{3} d = \exp\left(-\frac{\kappa^2 d}{3}\right)$, then define $g_{nm} = \frac{\partial^2 d}{\partial\phi_n \partial\phi_m^*}$, $L_m = \frac{\partial f}{\partial\phi_m} + \kappa^2 f \frac{\partial d}{\partial\phi_m}$

and the potential $V = \exp(\kappa^2 d)\left[(g^{-1})_{nm} L_m L_n^* - 3\kappa^2 |f|^2\right]$.

Bosonic part of local SUSY Lagrangian with matter: $\mathscr{L}_{\text{bosonic}}/\tilde{e} = -\frac{1}{2\kappa^2}\tilde{R}^{\mu}{}_{\mu} - g_{nm} D_\mu \phi_n D^\mu \phi_m^* - V$.

From local SUSY Lagrangian with matter on page 180, but with gauge fields included.

Modification with gauge fields: $V \to V + \frac{1}{2}\mathrm{Re}(f^{-1})_{AB} \frac{\partial d}{\partial\phi_n} t^A_{nm} \phi_m \left(\frac{\partial d}{\partial\phi_k} t^B_{kl} \phi_l\right)^*$

and $\mathscr{L}_{\text{bosonic}}/\tilde{e} \to \mathscr{L}_{\text{bosonic}}/\tilde{e} - \frac{1}{4}\mathrm{Re}(f_{AB} F^A_{\mu\nu} F^{B\mu\nu}) - \frac{1}{8}\mathrm{Im}(f_{AB} F^A_{\mu\nu} F^B_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma})$.

Terms quadratic in gaugino fields: $\mathscr{L}^{(2)}_{\text{gaugino}}/\tilde{e} = -\frac{1}{2}\mathrm{Re} f_{AB} \overline{\lambda}_A \not{D} \lambda_B + \frac{1}{2}\exp(\kappa^2 d)\mathrm{Re}(g^{-1})_{nm} L_m \left(\frac{\partial f_{AB}}{\partial\phi_n}\right)^* \overline{\lambda}_A \lambda_B$.

5.4 Anomaly-mediated SUSY breaking

$O(\kappa)$ effect only, results in gaugino masses and A_{ij} and B.

Generalize X defined on page 171 to include gauge superfields W, i.e. $X = \frac{2}{3}\mathrm{Im}\left[M\frac{\partial f(\Phi,W)}{\partial M}\right]$.

Applies also to dimensionless couplings with scale dependence coming from quantum mechanical anomaly.

In gravity-matter coupling in terms of components on page 172, give non-zero expectation values to s and p, so

1st order SUSY-breaking interaction: $\mathscr{L}_{\text{int}} = -\frac{4}{3}\kappa\mathrm{Re}[(\langle s\rangle + I\langle p\rangle)M\frac{\partial f(\phi,\lambda_L)}{\partial M}]$.

> A^X and B^X are A- and B- components of X defined in chiral superfield decomposition on page 106.
>
> Then use supercurrent conservation law on page 170 to get $\mathscr{M} = B^X$ and $\mathscr{N} = -A^X$.
>
> Then $\mathscr{L}_{\text{int}} = 2\kappa\mathrm{Im}[(\langle s\rangle + I\langle p\rangle)A^X + IB^X]$. Use $A^X + IB^X = \phi^X = X|_{\theta=0} = \frac{2M}{3I}\frac{\partial f(\phi,\lambda_L)}{\partial M}$.

Gaugino-mass term: $\mathscr{L}_{\text{gaugino mass}} = \mathrm{Re}\left[\tilde{m}_{\text{gaugino}}\lambda_L^A E\lambda_L^A\right]$, where $\tilde{m}_{\text{gaugino}} = -\frac{\beta(g)}{g}\frac{2\kappa}{3}(\langle s\rangle + I\langle p\rangle)$.

> Use gauge supermultiplet Lagrangian on page 136 and 1st order SUSY-breaking interaction above.

5.5 Gravity-mediated SUSY breaking

Includes squark and slepton squared masses, which are of $O(\kappa^2)$.

Assume a new strong force, group G, with scale $m_W \ll \Lambda \ll m_{\mathrm{Pl}}$,

weak at scale m_{Pl}, but strong at scale $<< m_{\mathrm{Pl}}$.

Φ_r: includes MSSM superfields, invariant under G,

Z_k: hidden sector superfields neutral under SM gauge group, not under G.

In any term in the superpotential, must be at least two of ϕ_r or Z_k for invariance under the groups,

so form of superpotential (cubic in superfields) must be $f(\Phi) + \tilde{f}(Z)$.

SUSY breaking 1: Scalar fields in Z_k of $O(\Lambda)$, makes $\phi_r = O(\kappa\Lambda^2)$.

Cancellation of large vacuum energy requires some fine-tuning in $\tilde{f}$ at minimum.

Superpotential contains soft SUSY breaking terms, with $B = O(1)$ and A,C small.

Leading non-renormalizable term of $O(\kappa\Phi^2 Z^2)$, gives μ-term on page 146 with $\mu = O(\kappa\Lambda^2)$.

For $\kappa\Lambda^2 = O(1)$TeV (MSSM scale),

$\Lambda = O(10^{11})$ GeV in range of scales for Peccei-Quinn symmetry breaking

allowed by astronomical observations to resolve the strong CP problem.

SUSY breaking 2: Hidden sector gauge couplings gives non-perturbative superpotential

for scalar fields y of *modular superfields* Y,

which could be e.g. low energy ($\ll O(\kappa^{-1})$) parameters describing compactified extra dimensions of radius $O(\kappa)$.

After imposing various symmetries $f(\Phi, Y, Z) = f_{rst}(\kappa Y)\Phi_r\Phi_s\Phi_t + f_{klm}(\kappa Y) Z_k Z_l Z_m +$ higher powers in Φ, Z.

Gives negligible $C^{D,E,U}_{KM}$ in trilinear terms on page 150.

For $\Lambda = O(10^{13})$ GeV (almost OK), gives desired $\mu = O(\kappa^2\Lambda^3)$.

Gaugino masses similar to slepton and squark masses.

However, *gauge-mediated* SUSY breaking yields generation-independent squark and slepton masses.

6 Higher dimensions

6.1 Spinors in higher dimensions

Lie algebra of $O(d-1,1)$: $I[J^{\rho\sigma}, J^{\mu\nu}] = -g^{\sigma\nu}J^{\rho\mu} - g^{\rho\mu}J^{\sigma\nu} + g^{\sigma\mu}J^{\rho\nu} + g^{\rho\nu}J^{\sigma\mu}$.

(Homogeneous) Lorentz group on page 21.

Fundamental spinor representation of $O(d-1,1)$: $J^{\mu\nu} = -\frac{I}{4}[\gamma_\mu, \gamma_\nu]$,

where γ_μ obey Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$.

This choice obeys Lie algebra of $O(d-1,1)$ above.

$\underline{d = 2n}$

Creation/annihilation matrices from γ_μ: $a_u = \frac{1}{2}(\gamma_{2u-1} + I\gamma_{2u})$, $u = 1, 2, \ldots, n$.

We are defining $\gamma_{2n} = -I\gamma_0$. Creation/annihilation properties $\{a_u, a_v^\dagger\} = \delta_{uv}$ and $\{a_u, a_v\} = \{a_u^\dagger, a_v^\dagger\} = 0$

follow from Clifford algebra of the γ_μ. a_u and $a_u^\dagger$ form alternative basis to γ_μ.

Dimensionality of Clifford algebra space: γ_μ are $2^n \times 2^n$ matrices.

Define vectors $|0\rangle$ such that $a_u^\dagger|0\rangle = 0$ and $|s_1, s_2, \ldots s_n\rangle = a_1^{s_1} a_2^{s_2} \ldots a_n^{s_n}|0\rangle$.

Because $a_u^2 = 0$, $s_i = 0$ and 1 only, so there are 2^n independent vectors.

$J^{\mu\nu}$ are reducible, sum of 2 irreducible representations $J_\pm^{\mu\nu} = J^{\mu\nu}\left(\frac{1 \pm \gamma_{2n+1}}{2}\right)$.

Define $\gamma_{2n+1} = I^n \gamma_1 \gamma_2 \ldots \gamma_{2n}$ (generalization of γ_5 in $d = 4$).

$\gamma_{2n+1}^2 = 1$ so use representation in which states have with eigenvalues ± 1 of γ_{2n+1}.

Using $\{\gamma_{2n+1}, \gamma_\mu\} = 0$ for $\mu = 1, 2, \ldots, 2n-1, 0$ (from Clifford algebra), $[\gamma_{2n+1}, J^{\mu\nu}] = 0$,

i.e. no mixing between $\gamma_{2n+1} = \pm 1$ states when acting with $J^{\mu\nu}$.

In this representation we have block diagonal $J^{\mu\nu} = J_+^{\mu\nu} \oplus J_-^{\mu\nu}$.

$d = 2n + 1$

Exactly as for $d = 2n$, and now Lorentz generators furnish irreducible representation of Lorentz group.

Everything is as before: $J^{\mu\nu}$, γ^μ are the same $2^n \times 2^n$ matrices as for $d = 2n$ with $\mu, \nu = 1, 2, \ldots, 2n-1, 0, 2n+1$

and again $\gamma_{2n+1} = I^n \gamma_1 \gamma_2 \ldots \gamma_{2n}$.

Now the "γ_5" matrix $\gamma_1 \gamma_2 \ldots \gamma_{2n} \gamma_{2n+1} = I^n$, i.e. no non-trivial matrix that commutes with $J^{\mu\nu}$.

6.2 Algebra

In d dimensions, work in Euclidean space (set $x^d = Ix^0$), group is $O(d)$ with unitary representations.

Then for $\mu(1)\nu(1) = d1$, $\mu(2)\nu(2) = 23$, $\mu(3)\nu(3) = 45$, ..., we have $[J_{\mu(\alpha)\nu(\alpha)}, J_{\mu(\beta)\nu(\beta)}] = 0$.

See Poincaré algebra on page 21.

Take any subset Q_n of the fermionic generators that form complete representation of Lorentz group.

Can combine Q_n so that $\underline{[J_{\mu(\alpha)\nu(\alpha)}, Q] = -\sigma^{\mu(\alpha)\nu(\alpha)}Q}$ (Q is each of the Q_n) with $\sigma^{\mu(\alpha)\nu(\alpha)}$ real numbers.

Q belongs to Q_n, where Q_n forms complete representation of Lorentz group,

so $[J_{d1}, Q_i] = -\sigma^{d1}_{ij}Q_j$, $[J_{23}, Q_i] = -\sigma^{23}_{ij}Q_j$, ..., where matrices σ^{AB} are Hermitian. Then $[J_{23}, [J_{d1}, Q_i]] = \sigma^{d1}_{ij}\sigma^{23}_{jk}Q_k$.

Because $[J_{d1}, J_{23}] = 0$, this is equal to $[J_{d1}, [J_{23}, Q_i]] = \sigma^{23}_{ij}\sigma^{d1}_{jk}Q_k$, i.e. $\sigma^{23}\sigma^{d1} = \sigma^{d1}\sigma^{23}$.

Diagonalize σ^{d1} with unitary matrix U, i.e. $U^\dagger\sigma^{d1}U = \Lambda$, where $\Lambda_{ij} = \lambda_i\delta_{ij}$ (no sum).

Then $\sigma^{23}U\Lambda U^\dagger = U\Lambda U^\dagger\sigma^{23}$ or $\tilde{\Lambda}\Lambda = \Lambda\tilde{\Lambda}$ where $\tilde{\Lambda} = U^\dagger\sigma^{23}U$, or $\lambda_i\tilde{\Lambda}_{ij} = \lambda_j\tilde{\Lambda}_{ij}$ (no sum).

If $\lambda_i \neq \lambda_j$, $\tilde{\Lambda}_{ij} = 0$, i.e. $\tilde{\Lambda}_{ij}$ is also diagonal.

Fermionic symmetry generator Q is in fundamental spinor representation of Lorentz group.

Let operator O be of weight w when $[J_{d1}, O] = -wO$. So from underlined equation on page 190, Q has $w = \sigma_{d1}$.

In Minkowski space ($J_{01} = IJ_{d1}$), $[J_{01}, O] = -IwO$. Because $J_{01}^\dagger = J_{01}$, $-[J_{01}, O^\dagger] = IwO^\dagger$, i.e. $O^\dagger$ also has weight w.

So $\{Q, Q^\dagger\}$ has weight $2\sigma_{d1}$. From CM theorem, $\{Q, Q^\dagger\}$ is linear combination of scalars, P_μ and $J_{\mu\nu}$.

But $w(\text{scalars}) = 0$, and from Poincaré algebra on page 21 (with i, j such that $2 \leq i, j \leq d-1$),

$w(P_0 \pm P_1) = w(J_{0i} \pm J_{1i}) = \pm 1$, $w(P_i) = w(J_{ij}) = w(J_{10}) = 0$,

so $2\sigma_{d1} = \pm 1$ (not zero, which is excluded by spin-statistics), i.e. $\sigma_{d1} = \pm 1/2$.

In Euclidean formalism, rotational invariance applied to underlined equation above implies all $\sigma_{\mu(\alpha)\nu(\alpha)} = \sigma_{d1}$.

So Q must belong to (direct sum of) fundamental spinor representation(s).

Commutation with momentum: $[Q, P^\mu] = 0$.

$[P_0 \pm P_1, [P_0 \pm P_1, Q]] = 0$ because $|w| = 3/2$ or $5/2$. So with $Q_\pm = [P_0 \pm P_1, Q]$, $\underline{[P_0 \pm P_1, [P_0 \pm P_1, \{Q, Q^\dagger\}]] = -2\{Q_\pm, Q_\pm^\dagger\}}$.

But $\{Q, Q^\dagger\}$ is linear combination of the scalars, P_μ and $J_{\mu\nu}$, and $P_0 \pm P_1$ commutes with the scalars and P_μ,

while $[P_0 \pm P_1, J_{\mu\nu}]$ forms linear combinations of P_μ which commutes with the other $P_0 \pm P_1$ in the underlined equation,

so $\{Q_\pm, Q_\pm^\dagger\} = 0$ so $Q_\pm = 0$, i.e. all Q commute with P_0, P_1 and by Lorentz invariance all P_μ.

General anticommutation relations: $\{Q_n, Q_m\} = \Gamma^\mu_{nm} P_\mu + Z_{nm}$.

$[P_\mu, \{Q_n, Q_m\}] = 0$. But $[J_{\mu\nu}, P_\rho] \neq 0$ so $\{Q_n, Q_m\}$ contains no linear combinations of $J_{\mu\nu}$.

Z_{nm} are the *central charges*, conserved scalar generators that commute with everything.

Z_{nm} are scalars, so commute with P_μ and $J_{\mu\nu}$. From general anticommutation relations (page 192), Poincaré algebra (page 21),

$[J_{d1}, \{Q_n, Q_m\}]$ is $-(w(Q_n) + w(Q_m))\{Q_n, Q_m\} = \Gamma_{0nm}P_1 - \Gamma_{1nm}P_0$, or $-(w(Q_n) + w(Q_m))[\Gamma^\mu_{nm}P_\mu + Z_{nm}] = \Gamma_{0nm}P_1 - \Gamma_{1nm}P_0$.

Terms in last expression of weight $w = 0$ give $-(w(Q_n) + w(Q_m))[Z_{nm} + \sum_{i=2}^{d-1} \Gamma^i_{nm}P_i] = 0$.

Spatial rotation of last expression gives $-(w(Q_n) + w(Q_m))Z_{nm} = 0$. So to allow $Z_{nm} \neq 0$, must have $w(Q_n) = -w(Q_m)$.

Now let $|w(Q_l)| \neq |w(Q_{n(m)})|$ (Q_l exists in each irreducible representation for $d \geq 4$).

From super-Jacobi identity $[Q_l, \{Q_m, Q_n\}] + [Q_m, \{Q_n, Q_l\}] + [Q_n, \{Q_l, Q_m\}] = 0$.

Last 2 terms vanish because $\{Q_{n(m)}, Q_l\}$ have some $\sigma \neq 0$, i.e. are linear combinations of P's but not Z's,

and P's commute with Q's. So $[Q_l, \{Q_m, Q_n\}] = [Q_l, Z_{mn}] = 0$.

But Z_{mn} is a Lorentz scalar, so must commute with all Q's in the irreducible representation.

So from general anticommutation relations the Z's commute with each other.

Commutation of central charges and internal symmetry generators: $[t_i, Z_{rs}] = 0$.

Same derivation as for commutation of central charges and internal symmetry generators on page 89.

6.3 Massless multiplets

$[Q, P^\mu] = 0$, so use eigenstates of P^μ. Define little group (pages 24, 33) to leave $k = (0, \ldots, 0, 1, 1)$ invariant.

"spin" j is maximum value of any $J_{\mu\nu}$. Impose condition $j \leq 2$ (as on page 56), and one $j = 2$ particle.

In Hilbert space of little group, only need SUSY generators of weight $\sigma_{d1} = 1/2$.

Any SUSY generator Q has weight $\pm 1/2$, so $\{Q, Q^\dagger\}$ has weight ± 1 ($Q^\dagger$ has same weight as Q from page 191),

i.e. $\{Q, Q^\dagger\} \propto P^0 \pm P^1$. But in this Hilbert space $P^0 - P^1 = 0$.

Further divide generators into two classes with $\sigma_{23} = \pm 1/2$.

Anticommutation of generators of the same σ_{23}: $\{Q_{\sigma_{23}=\pm 1/2}, Q'_{\sigma_{23}=\pm 1/2}\} = 0$.

Since all Q for the little group have $\sigma_{d1} = 1/2$ and $P^0 + P^1$ has $\sigma_{d1} = 1$, $\{Q_{\sigma_{23}=\pm 1/2}, Q'_{\sigma_{23}=\pm 1/2}\} \propto P^0 + P^1$.

But $P^0 + P^1$ has $\sigma_{23} = 0$.

Limit on number of fermionic generators: $\boxed{\mathscr{N} \leq 32}$.

Consider state $|\lambda\rangle$ of spin j and eigenvalue $\lambda > 0$ of J_{23}. Let state be annihilated by SUSY generators of $\sigma_{23} = -1/2$

(e.g., but not i.e., when $\lambda = j$). States with $J_{23} = \lambda - k/2$ formed by acting on $|\lambda\rangle$ with k SUSY generators of $\sigma_{23} = 1/2$.

(None of these states vanish because acting on them with the adjoints of the k SUSY generators gives back $|\lambda\rangle$.)

There are $\mathscr{N}/4$ SUSY generators with $\sigma_{23}(= \sigma_{d1}) = 1/2$ which all anticommute

(see anticommutation of generators of the same σ_{23} on page 194).

Minimum J_{23} is then $\lambda - \mathscr{N}/8$ (for $k = \mathscr{N}/4$). Then impose $j \leq 2$ and $j - \mathscr{N}/8 \geq -2$.

Limit on spacetime dimensionality: $\boxed{d \leq 11}$.

In $d = 2n$ or $2n + 1$ dimensions, a fundamental spinor representation like (a subset of) the SUSY generators has 2^n components.

For $2^n \leq 32$, need $n \leq 5$.

For $d = 11$, must have $N = 1$.

Construct massless multiplet by acting on state with J_{23} eigenvalue of 2, count number of degrees of freedom.

Gives one graviton and one gravitino. This may be the low energy limit of M-theory.

For $d = 10$, get massless particle spectrum of 3 kinds of superstring:

Type IIA: 16 generators of each chirality ($\mathscr{N} = 32$)

Type IIB: 32 generators of same chirality ($\mathscr{N} = 32$)

Heterotic: 16 generators of same chirality ($\mathscr{N} = 16$)

Rank 1 gauge vector fields couple to particles,

but in general rank $p+1$ *forms* (antisymmetric tensors) couple to extended objects of spatial dimensionality p.

These $p+1$ forms are conserved bosonic quantities other than those allowed by CM theorem

so they can appear on the RHS of the SUSY anticommutations.

But SUSY generators still belong to the fundamental spinor representations of the Lorentz group.

Let $A_{\dots}$ be a $p+1$ form. We now show that the weights of its components are 0 or ± 1.

Lorentz transform of $A_{\dots 1 \dots}$ ("1" can only appear once in $A_{\dots}$ because it is antisymmetric)

is $[J_{1d}, A_{\dots 1 \dots}] = (\dots + (J_{1d})^{\mu}{}_{1} + \dots)\, A_{\dots \mu \dots}$. But $(J_{1d})_{\mu\nu} = \pm 1$ when $\mu\nu = 1d$ or $d1$, otherwise $(J_{1d})_{\mu\nu} = 0$.

So $[J_{1d}, A_{\dots 1 \dots}] = A_{\dots d \dots}$. Likewise $[J_{1d}, A_{\dots d \dots}] = -A_{\dots 1 \dots}$. Thus $A_{\dots d \dots} \pm A_{\dots 1 \dots}$ has weight ± 1.

Also $[J_{1d}, A_{\dots 1 \dots d \dots}] = 0$ so $A_{\dots 1 \dots d \dots}$ has weight 0.

If "$\dots$" in $A_{\dots}$ does not contain "1" or "d", then it has weight 0 because $[J_{1d}, A_{\dots}] = 0$,

Then as before the SUSY generators have weight $\pm 1/2$.

7 Extended SUSY

$N = 2$ global SUSY, massless supermultiplets (or massive "short" supermultiplets discussed on page 94):

Contains supermultiplets (and their CPT-conjugates (helicities reversed)):

- 1 gauge boson of helicity +1, 2 fermions of +1/2 as doublet under SU(2) R-symmetry, 1 boson of 0

and hypermultiplets (and their CPT-conjugates):

- 2 fermions of each helicity $\pm 1/2$, 2 bosons of 0 as doublet under SU(2) R-symmetry

Supermultiplet mass limit for $N = 2$: $M \geq |Z_{12}|/2$.

Follows from lower mass bound in extended SUSY on page 94.

We will only consider "short" supermultiplets, so $N = 2$ **supermultiplet mass**: $M = |Z_{12}|/2$ *exactly*.

See discussion at end of page 94. Equality holds exactly because corrections cannot turn short multiplets into full multiplets.

Gauge Lagrangian for $N = 2$: Start with general $N = 1$ Lagrangian with fields of $N = 2$ gauge supermultiplet.

An $N = 2$ theory must have $N = 1$ SUSY, i.e. $N = 2$ is a special case of $N = 1$.

To make this $N = 1$ Lagrangian become $N = 2$, impose 2nd SUSY as discrete R-symmetry. We choose

R-symmetry for $N = 2$ generators: $Q_1 \to Q_2$, $Q_2 \to -Q_1$.

Leaves result for anticommuting generators on page 88 invariant.

$N = 2$ supermultiplet $\equiv$ an $N = 1$ chiral superfield $\Phi(\phi, \psi, \mathscr{F})$ + an $N = 1$ gauge superfield $V(V^\mu, \lambda, D)$.

R-symmetry for $N = 2$ generators above becomes

R-symmetry for $N = 2$ fields: $\psi \to \lambda$, $\lambda \to -\psi$.

No superpotential is allowed.

Superpotential would depend only on Φ, not V,

so would give mass and interaction terms for ψ but not λ, not allowed by R-symmetry.

Define $(D_\mu X)_A = \partial_\mu X_A + C_{ABC} V_{B\mu} X_C$ for $X = \psi, \lambda, \phi$, and $f_{A\mu\nu} = \partial_\mu V_{A\nu} - \partial_\nu V_{A\mu} + C_{ABC} V_{B\mu} V_{C\nu}$.

Gauge-invariant Lagrangian for $N = 2$: $\mathscr{L} = (D_\mu \phi)^*_A (D^\mu \phi)_A - \frac{1}{2}\overline{\psi_A}(\not{D}\psi)_A - \frac{1}{2}\overline{\lambda_A}(\not{D}\lambda)_A - \frac{1}{4} f_{A\mu\nu} f_A^{\mu\nu}$

$+\sqrt{2} C_{ABC} \overline{\psi_B} \frac{1}{2}(1 - \gamma_5) \lambda_A \phi_C - \sqrt{2} C_{ABC} \overline{\lambda_A} \frac{1}{2}(1 + \gamma_5) \psi_C \phi^*_B - V(\phi, \phi^*) + \frac{g^2 \theta}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} f_A^{\mu\nu} f_A^{\rho\sigma}$,

where $V(\phi, \phi^*) = 2C_{ABC} \mathrm{Re}\phi_B \mathrm{Im}\phi_C C_{ADE} \mathrm{Re}\phi_D \mathrm{Im}\phi_E$ is real and positive.

Use $N = 1$ Lagrangian for chiral and gauge supermultiplet fields on page 138,

combine each chiral and gauge supermultiplet into $N = 2$ supermultiplet with R-symmetry imposed (i.e. $f = 0$).

Then set auxiliary fields such that Lagrangian is stationary: $\mathscr{F}_A = 0$ and, with $\xi_A = 0$, $D_A = -I C_{ABC} \phi^*_B \phi_C$.

$N = 2$ SUSY has 2 $N = 1$ SUSYs:

Lagrangian can be seen as $N = 1$ SUSY with multiplets $(\phi, \psi, \mathscr{F})$ and (V^μ, λ, D),

or as $N = 1$ SUSY with multiplets $(\phi, \lambda, \mathscr{F})$ and $(V^\mu, -\psi, D)$.

Use R-symmetry for $N = 2$ fields on page 199.

Minimum at $V = 0$ when ϕ_A such that $C_{ABC} \mathrm{Re}\phi_B \mathrm{Im}\phi_C = 0$ (not only $\phi_A = 0$).

Such values of ϕ not equivalent (e.g. give different masses to gauge bosons on breaking gauge symmetries).

Example: SU(2) gauge theory, $C_{ABC} = \epsilon_{ABC}$. Then

Lagrangian: $\frac{1}{e^2}\left[-(D_\mu\phi)^*_A(D^\mu\phi)_A - \frac{1}{2}\overline{\psi_A}(\not{D}\psi)_A - 2\sqrt{2}\mathrm{Re}\epsilon_{ABC}\lambda^T_{AL}\epsilon\psi_{CL}\phi^*_B - \frac{1}{2}\overline{\lambda_A}(\not{D}\lambda)_A\right]$

$+\frac{\theta}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}f^{\mu\nu}_a f^{\rho\sigma}_A - V(\phi,\phi^*)$, where $V = 2\epsilon_{ABC}\mathrm{Re}\phi_B\mathrm{Im}\phi_C\epsilon_{ADE}\mathrm{Re}\phi_D\mathrm{Im}\phi_E$.

From gauge-invariant Lagrangian for $N = 2$ on page 200, with e absorbed into fields.

For $V = 0$, consider $\phi_1 = \phi_2 = 0$, $\phi_3 = a$. Particles with index 1 and 2 have mass $2|a|$, index 3 is massless.

Naively removing massive fields, and dropping the index "3" from now on,

Massless Lagrangian: $\mathscr{L} = \frac{1}{e^2}\left[-(\partial_\mu a)^*(\partial^\mu a) - \frac{1}{2}\overline{\psi}(\not{\partial}\psi) - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} - \frac{1}{2}\overline{\lambda}\not{\partial}\lambda\right] + \frac{\theta}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}f^{\mu\nu}f^{\rho\sigma}$.

Note this is a free field theory, because $N = 2$ does not allow a superpotential, and there is now just one supermultiplet.

Properly integrating out massive degrees of freedom gives non-renormalizable terms, dominant contributions

at tree-level from interactions of smallest number of fermions, auxilliary fields and derivatives in interaction:

Non-renormalizable low-energy Lagrangian: $\mathscr{L} = \frac{1}{2}\left[K(\Phi,\Phi^*)\right]_D - \frac{1}{2}\mathrm{Re}\left[T(\Phi)(W^T_L\epsilon W_L)\right]_{\mathscr{F}}$.

Gauge-covariant spinor superfield on page 135 relates W_L to V, and $\Phi = \Phi(a,\psi,\mathscr{F})$, $V = V(V_\mu,\lambda,D)$. Count

Feynman diagrams' energy powers with general $\mathscr{L} = \frac{1}{2}\left[K\left(\Phi,\Phi^\dagger e^{-2t_AV_A}\right)\right]_D + 2\mathrm{Re}[f(\Phi)]_{\mathscr{F}} - \frac{1}{2}\mathrm{Re}\left[H_{AB}(\Phi)(W^T_{AL}\epsilon W_{BL})\right]_{\mathscr{F}}$.

Low-energy Lagrangian: $\mathscr{L} = \frac{\partial^2 K(a,a^*)}{\partial a \partial a^*}\left[-\frac{1}{2}\overline{\psi}\partial\!\!\!/\psi + |\mathscr{F}|^2 - \partial_\mu a \partial^\mu a^*\right] - \mathrm{Re}\left(\frac{\partial^3 K(a,a^*)}{\partial^2 a \partial a^*}\overline{\psi}\psi_L \mathscr{F}^*\right)$

$-\frac{1}{2}\mathrm{Re}\left(\frac{\partial^3 K(a,a^*)}{\partial^2 a \partial a^*}\overline{\psi}\gamma^\mu\gamma_5\psi\partial_\mu a\right) + \frac{1}{4}\frac{\partial^4 K(a,a^*)}{\partial^2 a \partial^2 a^*}\overline{\psi}\psi_L\overline{\psi}\psi_R + \frac{1}{4}\mathrm{Re}\left(\overline{\lambda}\lambda_L\overline{\psi}\psi_L\frac{d^2T(a)}{da^2}\right) - \frac{1}{2}\mathrm{Re}\left(\overline{\lambda}\lambda_L\mathscr{F}\frac{dT(a)}{da}\right)$

$+\mathrm{Re}\left(T(a)\left[-\frac{1}{2}\overline{\lambda}\partial\!\!\!/(1-\gamma_5)\lambda - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{1}{8}I\epsilon_{\mu\nu\rho\sigma}f^{\mu\nu}f^{\rho\sigma} + \frac{1}{2}D^2\right]\right) + \frac{\sqrt{2}}{4}\mathrm{Re}\left(\frac{dT(a)}{da}\left[-\overline{\lambda}\gamma^\mu\gamma^\nu\psi_L f_{\mu\nu} + 2I\overline{\lambda}\psi_L D\right]\right).$

Component field expansion of non-renormalizable low-energy Lagrangian on page 201.

$N=2$ **conditions**: $(a,\psi,\mathscr{F}) \to (a,\lambda,\mathscr{F}^*)$, $(V_\mu,\lambda,D) \to (V_\mu,-\psi,-D)$ invariance, $T = \frac{1}{4\pi I}\frac{dh}{da}$, $K = \mathrm{Im}\left(\frac{a^* h}{4\pi}\right)$.

Begin with requirement of invariance under $(\psi,\lambda) \to (\lambda,-\psi)$. So for coefficients of $\overline{\psi}\partial\!\!\!/\psi$ and $\overline{\lambda}\partial\!\!\!/\lambda$ to be equal,

require $\frac{\partial^2 K(a,a^*)}{\partial a \partial a^*} = \mathrm{Re}\,(T(a))$, which is equivalent to expressions above for $T(a)$, $K(a,a^*)$ in terms of some function $h(a)$.

Also insures equality of coefficients of $\overline{\psi}\gamma^\mu\gamma_5\psi$ and $\overline{\lambda}\gamma^\mu\gamma_5\lambda$ (the latter appears after integration by parts of $\frac{1}{2}\mathrm{Re}(T(a)\overline{\lambda}\partial\!\!\!/\gamma_5\lambda)$).

Low-energy Lagrangian in h: $\mathscr{L} = \frac{1}{4\pi}\mathrm{Im}\Bigg\{\frac{dh}{da}\Big[-\frac{1}{2}\overline{\psi}\partial\!\!\!/(1-\gamma_5)\psi - \frac{1}{2}\overline{\lambda}\partial\!\!\!/(1-\gamma_5)\lambda - \partial_\mu a\partial^\mu a^* + |\mathscr{F}|^2$

$+\frac{1}{2}D^2 - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{1}{8}I\epsilon_{\mu\nu\rho\sigma}f^{\mu\nu}f^{\rho\sigma}\Big] + \frac{d^2h}{da^2}\left[-\frac{1}{2}\overline{\psi}\psi_L\mathscr{F}^* - \frac{1}{2}\overline{\lambda}\lambda_L\mathscr{F} + \frac{1}{\sqrt{2}}I\overline{\lambda}\psi_L D - \frac{\sqrt{2}}{4}\overline{\lambda}\gamma^\mu\gamma^\nu\psi_L f_{\mu\nu}\right] + \frac{1}{4}\frac{d^3h}{da^3}\overline{\lambda}\lambda_L\overline{\psi}\psi_L\Bigg\}.$

Tree-level h: $h = \left[\frac{4\pi I}{e^2} + \frac{\theta}{2\pi}\right]a$. Compare low-energy Lagrangian in h, page 202, with massless Lagrangian, page 201.

Low-energy action in h: $\boxed{I = \frac{1}{8\pi}\int d^4x \mathrm{Im}\left([\Phi^* h(\Phi)]_D - \left[h'(\Phi) W_L^T \epsilon W_L\right]_{\mathscr{F}}\right)}$.

From non-renormalizable low-energy Lagrangian, page 201, and results for T, K from h in $N=2$ conditions, page 202.

Term for SUSY Maxwell equations: $\boxed{\Delta I = \frac{1}{8\pi}\int d^4x \tilde{V} \mathrm{Re}\left(\left[\mathscr{D}_L^T E W_L\right]_D\right)}$, $\tilde{V}$ is real superfield.

Gives SUSY Maxwell equations $\mathrm{Re}\left(\mathscr{D}_L E W_L^A\right) = 0$, which is the alternative form of W_L^A on page 136.

Full effective action: $\boxed{\tilde{I} = \frac{1}{8\pi}\int d^4x \mathrm{Im}\left(\left[\frac{1}{h'(\Phi)}(\tilde{W}_L^T E \tilde{W}_L)\right]_{\mathscr{F}} + [\Phi^* h(\Phi)]_D\right)}$.

Integrate ΔI above by parts, gives $\Delta I = -\frac{1}{4\pi}\int d^4x \mathrm{Im}\left(\left[(\tilde{W}_L^T E W_L)\right]_{\mathscr{F}}\right)$, where $\tilde{W}_L = \frac{I}{4}(\mathscr{D}_R^T E \mathscr{D}_R)\mathscr{D}_L \tilde{V}$.

Add to low-energy action in h above, integrate over W_L which amounts to setting $W_L = -\frac{\tilde{W}_L}{h'(\Phi)}$.

Duality: $N=2$ effective field theory with scalar field value a and h function $h(a)$

is physically equivalent to one with scalar field value $a_D = h(a)$ and h function $\tilde{h}(a_D) = -a$.

Full effective action above can also be written $\tilde{I} = \frac{1}{8\pi}\int d^4x \mathrm{Im}\left(\left[\tilde{h}'(\tilde{\Phi})(\tilde{W}_L^T E \tilde{W}_L)\right]_{\mathscr{F}} + [\tilde{\Phi}^* \tilde{h}(\tilde{\Phi})]_D\right)$,

where $\tilde{\Phi} = h(\Phi)$, $\tilde{h}(\tilde{\Phi}) = -\Phi$.

Tree-level dual h: $\boxed{\tilde{h}(a_D) = -\frac{1}{\left[\frac{4\pi I}{e^2} + \frac{\theta}{2\pi}\right]} a_D}$. From tree-level h on page 202.