Introduction to Supersymmetry in Elementary Particle Physics

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Lecture notes at www.desy.de/~simon/susy.html

Abstract

These lectures aim towards supersymmetry relevant for near-future high energy experiments, but some technical footing in supersymmetry and in symmetries in general is given first. We discuss various motivations for and consequences of a fermion-boson symmetry. The two most physically relevant types of supermultiplet are discussed, followed by a redetermination of their content and properties from the simpler superfield formalism in superspace, in which supersymmetry is naturally manifest. The construction of supersymmetric Lagrangians is determined, from which the minimally supersymmetric extension of the Standard Model and its consequences for grand unification are derived. The physically required soft supersymmetry breaking is applied to the MSSM to obtain constraints on the mass eigenstates and spectrum. We will begin with a self contained development of continuous internal and external symmetries of particles in general, followed by a determination of the external symmetry properties of fermions and bosons permitted in a relativistic quantum (field) theory, and we highlight the importance of the Lagrangian formalism in the implementation of symmetries and its applicability to the Standard Model and the MSSM.

Contents

1 Quantum mechanics of particles

1.1 Basic principles

Physical states represented by directions of vectors (rays) $|i\rangle$ in Hilbert space of universe.

Write conjugate transpose $|i\rangle^\dagger$ as $\langle i|$, scalar product $|j\rangle^\dagger \cdot |i\rangle$ as $\langle j|i\rangle$.

Physical observable represented by Hermitian operator $A = A^\dagger$ such that $\langle A\rangle_i = \langle i|A|i\rangle$.

Functions of observables represented by same functions of their operators, $f(A)$.

Errors $\langle i|A^2|i\rangle - \langle A\rangle_i^2$ etc. vanish when $|i\rangle = |a\rangle$,

where $A|a\rangle = a|a\rangle$, i.e. $|a\rangle$ is A eigenstate, real eigenvalue a. $|a\rangle$ form complete basis.

If 2 observables A, B do not commute, $[A, B] \neq 0$, eigenstates of A do not coincide with those of B.

If basis $|X(a, b)\rangle$ are A, B eigenstates, any $|i\rangle = \sum_X C_{iX}|X\rangle$ obeys $[A, B]|i\rangle = 0 \implies [A, B] = 0$.

If A, B commute, their eigenstates coincide.

$AB|a\rangle = BA|a\rangle = aB|a\rangle$, so $B|a\rangle \propto |a\rangle$.

Completeness relation: $\sum_a |a\rangle\langle a| = \mathbf{1}$.

Expand $|i\rangle = \sum_a W_{ia}|a\rangle$ then act from left with $\langle a'| \longrightarrow W_{ia'} = \langle a'|i\rangle$, so $|i\rangle = \sum_a |a\rangle\langle a|i\rangle$

Probability to observe system in eigenstate $|a\rangle$ of A to be in eigenstate $|b\rangle$ of B: $P_{a\to b} = |\langle b|a\rangle|^2$.

$P_{a\to b}$ are the only physically meaningful quantities, thus $|i\rangle$ and $e^{I\alpha}|i\rangle$ for any α represent same state.

$\sum_a P_{b\to a} = 1$ for some state $|b\rangle$ as expected. $|b\rangle = \sum_a \langle a|b\rangle|a\rangle$. Act from left with $\langle b|$ gives $1 = \sum_a \langle a|b\rangle\langle b|a\rangle$.

Principle of reversibility: $P_{a\to b} = P_{b\to a}$. $\langle b|a\rangle = \langle a|b\rangle^*$

Time dependence: Time evolution of states: $|i,t\rangle = e^{-IHt}|i\rangle$, H is *Hamiltonian* with energy eigenstates.

Probability system in state $|i\rangle$ observed in state $|j\rangle$ time t later $= |M_{i\to j}|^2$, transition amplitude $M_{i\to j} = \langle j|e^{-IHt}|i\rangle$.

Average value of observable Q evolves in time as $\langle Q\rangle_i(t) = \langle i|e^{IHt}Qe^{-IHt}|i\rangle$.

Q is conserved $\iff [Q,H] = 0$ ($\langle Q\rangle(t)$ independent of t).

1.2 Fermionic and bosonic particles

Particle's eigenvalues $= \sigma$. Particle states $|\sigma, \sigma', ...\rangle$ completely span Hilbert space. Vacuum is $|0\rangle = |\rangle$.

$|\sigma, \sigma', ...\rangle = \pm|\sigma', \sigma, ...\rangle$ for bosons/fermions. *Pauli exclusion principle*: $|\sigma, \sigma, \sigma'...\rangle = 0$ if σ fermionic.

$|\sigma, \sigma', ...\rangle$ and $|\sigma', \sigma, ...\rangle$ are same state, $|\sigma, \sigma', ...\rangle = e^{I\alpha}|\sigma', \sigma, ...\rangle = e^{2I\alpha}|\sigma, \sigma', ...\rangle \implies e^{I\alpha} = \pm 1$.

Creation/annihilation operators: $a^\dagger_\sigma|\sigma', \sigma'', ...\rangle = |\sigma, \sigma', \sigma'', ...\rangle$, so $|\sigma, \sigma', ...\rangle = a^\dagger_\sigma a^\dagger_{\sigma'} \dots |0\rangle$.

$[a^\dagger_\sigma, a^\dagger_{\sigma'}]_\mp = [a_\sigma, a_{\sigma'}]_\mp = 0$ (bosons/fermions).

$|\sigma, \sigma', ...\rangle = a^\dagger_\sigma a^\dagger_{\sigma'}|\dots\rangle = \pm|\sigma', \sigma, ...\rangle = \pm a^\dagger_{\sigma'} a^\dagger_\sigma|\dots\rangle$

a_σ removes σ particle $\implies$ $a_\sigma|0\rangle = 0$.

E.g. $(a_\sigma|\sigma', \sigma''\rangle)^\dagger \cdot |\sigma'''\rangle = \langle\sigma', \sigma''|(a^\dagger_\sigma|\sigma'''\rangle) = 0$ unless $\sigma = \sigma'$, $\sigma''' = \sigma''$ or $\sigma = \sigma''$, $\sigma''' = \sigma'$. i.e. $a_\sigma|\sigma', \sigma''\rangle = \delta_{\sigma\sigma''}|\sigma'\rangle \pm \delta_{\sigma\sigma'}|\sigma''\rangle$.

$[a_\sigma, a^\dagger_{\sigma'}]_\mp = \delta_{\sigma\sigma'}$.

e.g. 2 fermions $a^\dagger_{\sigma''} a^\dagger_{\sigma'''}|0\rangle$: Operator $a^\dagger_\sigma a_{\sigma'}$ replaces any σ' with σ, must still vanish when $\sigma'' = \sigma'''$.

Check: $(a^\dagger_\sigma a_{\sigma'}) a^\dagger_{\sigma''} a^\dagger_{\sigma'''}|0\rangle = -a^\dagger_\sigma a^\dagger_{\sigma''} a_{\sigma'} a^\dagger_{\sigma'''}|0\rangle + \delta_{\sigma'\sigma''} a^\dagger_\sigma a^\dagger_{\sigma'''}|0\rangle = -\delta_{\sigma'\sigma'''} a^\dagger_\sigma a^\dagger_{\sigma''}|0\rangle + \delta_{\sigma'\sigma''} a^\dagger_\sigma a^\dagger_{\sigma'''}|0\rangle$.

Expansion of observables: $Q = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} C_{NM;\sigma'_1 \dots \sigma'_N;\sigma_M \dots \sigma_1} \, a^\dagger_{\sigma'_1} \dots a^\dagger_{\sigma'_N} a_{\sigma_M} \dots a_{\sigma_1}$.

Can always tune the C_{NM} to give any values for $\langle 0 | a_{\sigma'_1} \dots a_{\sigma'_L} Q a^\dagger_{\sigma_1} \dots a^\dagger_{\sigma_K} | 0 \rangle$.

Commutations with additive observables: $[Q, a^\dagger_\sigma] = q(\sigma) a^\dagger_\sigma$ (no sum),

where Q is an observable such that for $|\sigma, \sigma', \dots\rangle$, total $Q = q(\sigma) + q(\sigma') + \dots$ and $Q|0\rangle = 0$ (e.g. energy).

Check for each particle state: $Q a^\dagger_\sigma |0\rangle = q(\sigma) a^\dagger_\sigma |0\rangle + a^\dagger_\sigma Q |0\rangle = q(\sigma) a^\dagger_\sigma |0\rangle$,

$Q a^\dagger_\sigma a^\dagger_{\sigma'} |0\rangle = a^\dagger_\sigma Q a^\dagger_{\sigma'} |0\rangle + q(\sigma) a^\dagger_\sigma a^\dagger_{\sigma'} |0\rangle = a^\dagger_\sigma a^\dagger_{\sigma'} Q |0\rangle + q(\sigma') a^\dagger_\sigma a^\dagger_{\sigma'} |0\rangle + q(\sigma) a^\dagger_\sigma a^\dagger_{\sigma'} |0\rangle = (q(\sigma) + q(\sigma')) a^\dagger_\sigma a^\dagger_{\sigma'} |0\rangle$ etc.

Note: Conjugate transpose is $[Q, a_\sigma] = -q(\sigma) a_\sigma$.

Number operator for particles with eigenvalues σ is $a^\dagger_\sigma a_\sigma$ (no sum). E.g. $(a^\dagger_\sigma a_\sigma) a^\dagger_\sigma a^\dagger_\sigma a^\dagger_{\sigma'} |0\rangle = 2 a^\dagger_\sigma a^\dagger_\sigma a^\dagger_{\sigma'} |0\rangle$.

Additive observable: $Q = \sum_\sigma Q_\sigma a^\dagger_\sigma a_\sigma$. E.g. $Q = H$, (free) Hamiltonian, $Q_\sigma = E_\sigma$, energy eigenvalues.

2 Symmetries in QM

2.1 Unitary operators

Symmetry is powerful tool: e.g. relates different processes.

Symmetry transformation is change in our point of view (e.g. spatial rotation / translation),

does not change experimental results. i.e. all $|i\rangle \longrightarrow U|i\rangle$ does not change any $|\langle j|i\rangle|^2$.

Continuous symmetry groups G require U unitary:

$\langle j|i\rangle \rightarrow \langle j|U^\dagger U|i\rangle = \langle j|i\rangle$, so $U^\dagger U = UU^\dagger = \mathbf{1}$, includes $U = \mathbf{1}$.

Wigner + Weinberg: General physical symmetry groups require U unitary,

or antiunitary: $\langle j|i\rangle \rightarrow \langle j|U^\dagger U|i\rangle = \langle j|i\rangle^* = \langle i|j\rangle$, e.g. (discrete) time reversal.

$\langle A\rangle$ unaffected (and $\langle f(A)\rangle$ in general), so must have $A \rightarrow UAU^\dagger$.

Transition amplitude $M_{i\rightarrow j}$ unaffected by symmetry transformation, i.e. $\langle j|U^\dagger e^{-IHt}U|i\rangle = \langle j|e^{-IHt}|i\rangle$,

which requires $[U, H] = 0$ if time translation and symmetry transformation commute.

Parameterize unitary operators as $U = U(\alpha)$, α_i real, $i = 1, ..., d(\mathrm{G})$.

$d(\mathrm{G})$ is dimension of G, minimum no. of paramenters required to distinguish elements.

Choose group identity at $\alpha = 0$, i.e. $U(0) = \mathbf{1}$.

So for small α_i, can write $U(\alpha) \approx \mathbf{1} + I t_i \alpha_i$.

t_i are the linearly independent *generators* of G. Since $U^\dagger U = \mathbf{1}$, $t_i = t_i^\dagger$, i.e. t_i are Hermitian.

$[U, H] = 0 \implies [t_i, H] = 0$, so conserved observables are generators.

Can replace all $t_i \to t_i' = M_{ij} t_j$ if M invertible and real, then $\alpha_i' = \alpha_j M^{-1}_{ji}$ real.

In general, $U(\alpha)U(\beta) = U(\gamma(\alpha, \beta))$ (up to possible phase $e^{I\rho}$, removable by enlarging group).

$U(\alpha) = \exp[It_i\alpha_i]$ if Abelian limit is obeyed: for $\beta_i \propto \alpha_i$, $U(\alpha)U(\beta) = U(\alpha+\beta)$

(usually true for physical symmetries, e.g. rotation about same line / translation in same direction).

Can write $U(\alpha) = [U(\alpha/N)]^N$, then for $N \to \infty$ is $[\mathbf{1} + It_i\alpha_i/N + O(1/N^2)]^N = \exp[It_i\alpha_i] + O\left(\frac{1}{N}\right)$.

Lie algebra: $[t_i, t_j] = IC_{ijk}t_k$, where C_{ijk} are the *structure constants*

appearing in $U(\alpha)U(\beta) = U(\alpha + \beta + \frac{1}{2}IC\alpha\beta + \text{cubic and higher})$

(where $(C\alpha\beta)_k = C_{ijk}\alpha_i\beta_j$, and no $O(\alpha^2)$ ensures $U(\alpha)U(0) = U(\alpha)$, likewise no $O(\beta^2)$).

LHS: $e^{I\alpha t}e^{I\beta t} \approx \left[\mathbf{1} + I\alpha t - \frac{1}{2}(\alpha t)^2\right] \times \left[\mathbf{1} + I\beta t - \frac{1}{2}(\beta t)^2\right]$

$$\approx \mathbf{1} + I(\alpha t + \beta t) - \tfrac{1}{2}\left[(\alpha t)^2 + (\beta t)^2 + \underline{2(\alpha t)(\beta t)}\right]$$

RHS: $e^{I(\alpha+\beta+\frac{1}{2}IC\alpha\beta)t} \approx \mathbf{1} + I(\alpha + \beta + \frac{1}{2}IC\alpha\beta)t - \frac{1}{2}\left[(\alpha + \beta + \frac{1}{2}IC\alpha\beta)t\right]^2$

$$\approx \mathbf{1} + I(\alpha t + \beta t + \underline{\tfrac{1}{2}IC\alpha\beta t}) - \tfrac{1}{2}\left[(\alpha t)^2 + (\beta t)^2 + \underline{(\alpha t)(\beta t) + (\beta t)(\alpha t)}\right],$$

i.e. $\frac{1}{2}IC_{ijk}\alpha_i\beta_j t_k = \frac{1}{2}[\alpha_i t_i, \beta_j t_j]$.

Now take all α_i, β_j zero except e.g. $\alpha_1 = \beta_2 = \epsilon \to IC_{12k}t_k = [t_1, t_2]$ etc., gives Lie algebra.

In fact, Lie algebra completely specifies group in *non-small* neighbourhood of identity.

This means that for $U(\alpha)U(\beta) = U(\gamma)$, $\gamma = \gamma(\alpha, \beta)$ can be found from Lie algebra.

We have shown this above in small neighbourhood of identity, i.e. to 2nd order in α, β, only.

Check to 3rd order: Write $X = \alpha_i t_i$, $Y = \beta_i t_i$, can verify that

$$\exp[IX]\exp[IY] = \exp[\underbrace{I(X+Y) - \frac{1}{2}[X,Y] + \frac{I}{12}([X,[Y,X]] + [Y,[X,Y]]) + \text{quadratic and higher}}_{\text{has the form } I\gamma_i t_i,\ \gamma_i \text{ real}}].$$

Lie algebra implies: 1. $C_{ijk} = -C_{jik}$ (antisymmetric in i, j). C_{ijk} can be chosen antisymmetric in i, j, k (see later).

2. C_{ijk} real.

Conjugate of Lie Algebra is $\left[t_j^\dagger, t_i^\dagger\right] = -IC_{ijk}^* t_k^\dagger$, which is negative of Lie Algebra because $t_i^\dagger = t_i$.

Thus $C_{ijk}^* t_k = C_{ijk} t_k$, but t_k linearly independent so $C_{ijk}^* = C_{ijk}$.

3. $t^2 = t_j t_j$ is invariant (commutes with all t_i, so tranforming give $e^{I\alpha_i t_i} t^2 e^{-I\alpha_k t_k} = t^2$).

$[t^2, t_i] = t_j[t_j, t_i] + [t_j, t_i]t_j = IC_{jik}\{t_j, t_k\} = 0$ by (anti)symmetry in (j, i) j, k.

Examples: Rest mass, total angular momentum. $t_j t_j$ doesn't have to include all j, only subgroup.

2.2 (Matrix) Representations

Physically, matrix representation of any symmetry group G of nature formed by particles:

General transformation of $a_\sigma^{(\dagger)}$: $U(\alpha)a_\sigma^\dagger U^\dagger(\alpha) = D_{\sigma\sigma'}(\alpha)a_{\sigma'}^\dagger$, with **invariant vacuum**: $U(\alpha)|0\rangle = |0\rangle$.

$U(\alpha)a_\sigma^\dagger|0\rangle$ is 1 particle state, so must be linear combination of $a_{\sigma'}^\dagger|0\rangle$, i.e. $U(\alpha)a_\sigma^\dagger|0\rangle = D_{\sigma\sigma'}(\alpha)a_{\sigma'}^\dagger|0\rangle$.

Thus $U(\alpha)a_\sigma^\dagger a_{\sigma'}^\dagger|0\rangle = D_{\sigma\sigma''}(\alpha)D_{\sigma'\sigma'''}(\alpha)a_{\sigma''}^\dagger a_{\sigma'''}^\dagger|0\rangle$ etc.

$D(\alpha)$ matrices furnish a representation of G, matrix generators $(t^i)_{\sigma'\sigma}$ with same Lie algebra $[t_i, t_j]_{\sigma\sigma'} = IC_{ijk}(t_k)_{\sigma\sigma'}$.

Since $U(\alpha)U(\beta) = U(\gamma)$, must have $D_{\sigma''\sigma'}(\alpha)D_{\sigma'\sigma}(\beta) = D_{\sigma''\sigma}(\gamma)$.

If Abelian limit of page 7 is obeyed, $D_{\sigma'\sigma}(\alpha) = \left(e^{I\alpha_i t_i}\right)_{\sigma'\sigma}$.

Similarity transformation $(t_i)_{\sigma\sigma'''} \longrightarrow (t_i')_{\sigma\sigma'''} = V_{\sigma\sigma'}(t_i)_{\sigma'\sigma''}(V^{-1})_{\sigma''\sigma'''}$ also a representation.

Particle states require V unitary.

$a_\sigma^\dagger|0\rangle \to a_\sigma'^\dagger|0\rangle = V_{\sigma\sigma'}a_{\sigma'}^\dagger|0\rangle$, then othogonality $\langle 0|a_{\sigma'}'a_\sigma'^\dagger|0\rangle = \delta_{\sigma'\sigma}$ requires $V^\dagger V = \mathbf{1}$.

But in general, representations don't have to be unitary.

In "reducible" cases, can similarity transform $t_i \to V t_i V^{-1}$ such that t_i is block-diagonal matrix,

each block furnishes a representation, e.g.:

$$(t_i)_{\sigma'\sigma} = \begin{pmatrix} (t_i)_{jk} & \mathbf{0} \\ \mathbf{0} & (t_i)_{\alpha\beta} \end{pmatrix}_{\sigma'\sigma} .$$

Particles of one block don't mix with those of other — can be treated as 2 separate "species".

Each block can have different t^2, corresponds to different particle species.

Irreducible representation: Matrices $(t_i)_{\sigma'\sigma}$ not block-diagonalizable by similarity transformation.

In this sense, these particles are elementary.

Size of matrix written as $m(r) \times m(r)$, where r labels representation.

Corresponds to single species, single value of t^2: $t^2 = C_2(r)\mathbf{1}$ (consistent with $[t^2, t_i] = 0$).

$C_2(r)$ is *quadratic Casimir operator* of representation r.

Fundamental representation of G: Generators written $(t_i)_{\alpha\beta}$.

Matrices representing elements used to define group G (also called defining representation).

Adjoint representation of G ($r = A$): Generators $(t_j)_{ik} = IC_{ijk}$, satisfy Lie algebra.

Use *Jacobi identity* $[t_i, [t_j, t_k]] + [t_j, [t_k, t_i]] + [t_k, [t_i, t_j]] = 0.$

From Lie algebra, $[t_i, [t_j, t_k]] = I[t_i, C_{jkl}t_l] = -C_{jkl}C_{ilm}t_m$,

so Jacobi identity is $C_{jkl}C_{ilm} + C_{kil}C_{jlm} + C_{ijl}C_{klm} = 0$ (after removing contraction with linearly independent t_m),

or, from $C_{ijk} = -C_{jik}$, $-IC_{kjl}IC_{lim} + IC_{kil}IC_{ljm} - IC_{ijl}IC_{klm} = 0$, which from $(t_j)_{ik} = IC_{ijk}$ reads $[t_i, t_j]_{km} = IC_{ijl}(t_l)_{km}$.

Conjugate representation has generators $t_i^* = -t_i^T$ (obey the same Lie algebra as t_i).

If $-t_i^* = Ut_iU^\dagger$ (U unitary), then $e^{-I\alpha_i t_i^*} = Ue^{I\alpha_i t_i}U^\dagger$,

i.e. conjugate representation $\equiv$ original representation, $\longrightarrow$ representation is *real.*

For invariant matrix $\mathfrak{g}$ ("metric"), i.e. $e^{I\alpha_i t_i^T}\mathfrak{g}e^{I\alpha_i t_i} = \mathfrak{g}$, G transformation leaves $\phi^T\mathfrak{g}\psi$ invariant.

$$\phi^T\mathfrak{g}\psi \quad \to \quad \phi^T \underbrace{e^{I\alpha_i t_i^T}}_{=Ue^{-I\alpha_i t_i}U^\dagger} \mathfrak{g}e^{I\alpha_i t_i}\psi \quad = \quad \phi^T Ue^{-I\alpha_i t_i}U^\dagger \underbrace{e^{-I\alpha_i t_i^T}}_{=Ue^{I\alpha_i t_i}U^\dagger} \underbrace{e^{I\alpha_i t_i^T}\mathfrak{g}e^{I\alpha_i t_i}}_{=\mathfrak{g}}\psi$$

$$= \quad \phi^T Ue^{-I\alpha_i t_i}\underbrace{U^\dagger U}_{=\mathbf{1}}e^{I\alpha_i t_i}U^\dagger\mathfrak{g}\psi \quad = \quad \phi^T \underbrace{\left(e^{I\alpha_i t_i}U^\dagger\right)^\dagger e^{I\alpha_i t_i}U^\dagger}_{=\mathbf{1}}\mathfrak{g}\psi \quad = \quad \phi^T\mathfrak{g}\psi.$$

Semi-simple Lie algebra: no t_i that commutes with all other generators (no U(1) subgroup).

Semi-simple group's matrix generators must obey $\mathrm{tr}[t_i] = 0$.

Make all $\mathrm{tr}[t_i] = 0$ except one, $\mathrm{tr}[t_K]$, via $t'_i = M_{ij} t_i$ (this is just rotation of vector with components $\mathrm{tr}[t_i]$).

Determinant of $U(\alpha)U(\beta) = U(\gamma)$ is $e^{I\alpha_i \mathrm{tr}[t_i]} e^{I\beta_i \mathrm{tr}[t_i]} = e^{I\gamma_i \mathrm{tr}[t_i]}$, i.e. $(\alpha_i + \beta_i - \gamma_i)\mathrm{tr}[t_i] = 0$.

But only $\mathrm{tr}[t_K] \neq 0$, so $\alpha_K + \beta_K - \gamma_K = 0$.

Must have $C_{ijK}\alpha_i\beta_j = 0$ (recall $\gamma_K \simeq \alpha_K + \beta_K + C_{ijK}\alpha_i\beta_j$),

or $C_{ijK} = C_{Kij} = 0$ for all i, j, so from Lie group we have $[t_K, t_i] = 0$ for all i.

Normalization of generators chosen as $\text{tr}[t_i t_j] = C(r)\delta_{ij}$.

N_{ij} =tr$[t_i t_j]$ becomes $M_{ik}N_{kl}(M^T)_{lj}$ after basis transformation $t_i \to M_{ij}t_j$.

N_{ij} components of real symmetric matrix N, diagonalizable via MNM^T when M real, orthogonal.

Also N is positive definite matrix $\alpha^T N\alpha = \alpha_i \text{tr}[t_i t_j]\alpha_j$ =tr$[\alpha_i t_i \alpha_j t_j]$ =tr$[(\alpha_i t_i)^\dagger \alpha_j t_j] \geq 0$

(because for any matrix A, $\text{tr}[A^\dagger A] = A^\dagger_{\beta\alpha}A_{\alpha\beta} = A^*_{\alpha\beta}A_{\alpha\beta} = \sum_{\alpha\beta}|A_{\alpha\beta}|^2 \geq 0$).

After diagonalization, $N_{ij} = 0$ for $i \neq j$ and above implies $N_{ii} > 0$ (no sum).

Then multiply each t_i by real number c_i, changes $N_{ii} \to c_i^2 N_{ii} > 0$.

Choose c_i such that each N_{ii} (no sum over i) all equal to positive $C(r)$.

Representation dependence of quadratic Casimir operator: $C_2(r)m(r) = C(r)d(G)$.

Definition of quadratic Casimir operator gives $\text{tr}[t^2] = C_2(r)m(r)$.

Normalization of generators $\text{tr}[t_i t_j] = C(r)\delta_{ij} \implies \text{tr}[t^2] = C(r)d(G)$.

Example: 2 and 3 component repesentations of rotation group have different spins $\frac{1}{2}$ and 1.

Antisymmetric structure constants: $C_{ijk} = -\frac{I}{C(r)}\mathrm{tr}[[t_i, t_j]t_k]$.

From Lie algebra, $\mathrm{tr}[[t_i, t_j]t_l] = IC_{ijk}\mathrm{tr}[t_k t_l] = IC_{ijl}C(r)$.

Structure constants obey $C_{jki}C_{lki} = C(A)\delta_{jl}$.

In adjoint representation, quadratic Casimir operator on page 10 is $(t^2)_{jl} = C_2(A)\delta_{jl} = -C_{jik}C_{kil}$.

But $C_2(A) = C(A)$:

$C_2(A)m(A) = C(A)d(G)$ from representation dependence of quadratic Casimir operator on page 13,

and $m(A) = d(G)$.

2.3 External symmetries

2.3.1 Rotation group representations

(Spatial) rotation of vector $v \to Rv$ preserves $v^T v$, so R orthogonal ($R^T R = \mathbf{1}$).

Rotation $|\boldsymbol{\theta}|$ about $\boldsymbol{\theta}$: $U(\boldsymbol{\theta}) = e^{-I\boldsymbol{J}\cdot\boldsymbol{\theta}}$.

Lie algebra is $[J_i, J_j] = I\epsilon_{ijk}J_k$ (see later).

Or use J_3 and raising/lowering operators $J_\pm = (J_1 \pm IJ_2)$.

Irreducible spin j representations:

$(J_3^{(j)})_{m'm} = m\delta_{m'm}$ and $(J_\pm^{(j)})_{m'm} = [(j \mp m)(j \pm m + 1)]\delta_{m',m\pm1}$, where $m = -j, -j+1, ..., j$,

spin $j = 0, \frac{1}{2}, 1, ...$, number of components n.o.c.$= 2j+1$ and $\boldsymbol{J}^2 = j(j+1)$.

Let $|m,j\rangle$ be $J_3 = m$ and $\boldsymbol{J}^2 = F(j)$ orthonormal eigenstates.

$J_\pm$ changes m by ±1 because $[J_\pm, J_3] = \mp J_\pm$, so $J_\pm|m,j\rangle = C_\pm(m,j)|m\pm1,j\rangle$.

$C_\mp(m,j) = \sqrt{F(j) - m^2 \pm m}$ (states absorb complex phase):

$|C_\mp(m,j)|^2 = \langle m,j|J_\pm J_\mp|m,j\rangle$ ($J_\pm^\dagger = J_\mp$) and $J_\pm J_\mp = \boldsymbol{J}^2 - J_3^2 \pm J_3$.

Let j be largest m value for given $F(j)$

(m bounded because $m^2 = \langle m,j|J_3^2|m,j\rangle = F(j) - \langle m,j|J_1^2 + J_2^2|m,j\rangle < F(j)$).

$F(j) = j(j+1)$ because $J_+|j,j\rangle = 0$, so $J_-J_+|j,j\rangle = (F(j) - j^2 - j)|j,j\rangle = 0$.

Let $-j'$ be smallest m: $J_-|-j',j\rangle = 0 \implies F(j) = j'(j'+1)$, so $j' = j$ (other possibility $-j' = j+1 > j$).

So $m = -j, -j+1, ..., j$, i.e. n.o.c.$= 2j+1$. Since n.o.c. is integer, $j = 0, \frac{1}{2}, 1, ...$

Spin decomposition of tensors: e.g. 2nd rank tensor C_{ij} (n.o.c.=9), representations are $j = 0, 1, 2$:

scalar (n.o.c.=**1**) + antisymmetric rank 2 tensor (n.o.c.=**3**) + symmetric traceless rank 2 (n.o.c.=**5**) components.

$$C_{ij} = \tfrac{1}{3}\delta_{ij}C_{kk} + \tfrac{1}{2}(C_{ij} - C_{ji}) + \tfrac{1}{2}(C_{ij} + C_{ji} - \tfrac{2}{3}\delta_{ij}C_{kk}).$$

Component irreducible representations signified by **1** + **3** + **5**.

Counting n.o.c. shows they are equivalent respectively to

the $j = 0$ $(2j + 1 = \mathbf{1})$, $j = 1$ $(2j + 1 = \mathbf{3})$ and $j = 2$ $(2j + 1 = \mathbf{5})$ representations:

$\frac{1}{3}\delta_{ij}C_{kk} \to \frac{1}{3}\delta_{ij}C_{kk}$ is like scalar $\equiv$ spin 0,

$(\frac{1}{2}(C_{jk} - C_{kj}))|_{i\neq k,j} = \frac{1}{2}\epsilon_{ijk}C_{jk} \to R_{il}\frac{1}{2}\epsilon_{ljk}C_{jk}$ (because $\epsilon R^2 = R\epsilon$) is like vector $\equiv$ spin 1,

$\frac{1}{2}(C_{ij} + C_{ji} - \frac{2}{3}\delta_{ij}C_{ii}) \to R_{il}R_{jm}\frac{1}{2}(C_{lm} + C_{ml} - \frac{2}{3}\delta_{lm}C_{kk})$ is like rank 2 tensor $\equiv$ spin 2.

Direct product: 2 particles, spins j_1, j_2, also in a representation:

$e^{I\boldsymbol{J}\cdot\boldsymbol{\theta}}|m_1, j_1; m_2, j_2\rangle = (e^{I\boldsymbol{J}^{(j_1,j_2)}\cdot\boldsymbol{\theta}})_{m_1'm_2'm_1m_2}|m_1', j_1; m_2', j_2\rangle$ where $\boldsymbol{J}^{(j_1,j_2)}_{m_1'm_2'm_1m_2} = \boldsymbol{J}^{(j_1)}_{m_1'm_1}\delta_{m_2'm_2} + \delta_{m_1'm_1}\boldsymbol{J}^{(j_2)}_{m_2'm_2}$.

From $e^{I\boldsymbol{J}\cdot\boldsymbol{\theta}}|m_1, j_1; m_2, j_2\rangle = (e^{I\boldsymbol{J}^{(j_1)}\cdot\boldsymbol{\theta}})_{m_1'm_1}(e^{I\boldsymbol{J}^{(j_2)}\cdot\boldsymbol{\theta}})_{m_2'm_2}|m_1', j_1; m_2', j_2\rangle$, i.e. total rotation is rotation of each particle in turn.

Thus $J_3|m_1, j_1; m_2, j_2\rangle = (m_1 + m_2)|m_1, j_1; m_2, j_2\rangle$,

so $|m_1, j_1; m_2, j_2\rangle$ is combination of $J_3^{(j_1,j_2)}$, $\boldsymbol{J}^{(j_1,j_2)2}$ eigenstates $|2; m_1 + m_2, j\rangle$,

with $j = m_1 + m_2, m_1 + m_2 + 1, \ldots, j_1 + j_2$.

Triangle inequality: representation for (j_1, j_2) contains $j = |j_1 - j_2|, |j_1 - j_2| + 1, j_1 + j_2$.

$$\text{Number of orthogonal eigenstates } |m_1, j_1; m_2, j_2\rangle = \text{Number of orthogonal eigenstates } |2; m, j\rangle,$$

$$\text{i.e. } (2j_1 + 1)(2j_2 + 1) = \textstyle\sum_{j=j_{\min}}^{j_1+j_2}(2j + 1), \qquad \text{so } j_{\min} = |j_1 - j_2|.$$

Example: Representation for 2 spin 1 particles $(j_1, j_2) = (1, 1)$:

From triangle inequality, this is $\equiv$ sum of irreducible representations $j = 0, 1, 2$.

Also from tensor representation on page 17: product of 2 vectors $u_i v_j$ (2nd rank tensor) is $\mathbf{3} \times \mathbf{3} = \mathbf{1} + \mathbf{3} + \mathbf{5}$.

2.3.2 Poincaré and Lorentz groups

Poincaré (inhomogeneous Lorentz) group formed by coordinate transformations $x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$,

preserving spacetime separation: $g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} dx'^\mu dx'^\nu$. Implies:

Transformation of metric tensor: $g_{\rho\sigma} = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma$ or $(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu$.

Identity: $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$, $a^\mu = 0$.

Poincaré group defined by $U(\overline{\Lambda}, \overline{a}) U(\Lambda, a) = U(\overline{\Lambda}\Lambda, \overline{\Lambda} a + \overline{a})$.

This is the double transformation $x'' = \overline{\Lambda} x' + \overline{a} = \overline{\Lambda}(\Lambda x + a) + \overline{a}$.

Poincaré group generators: $J^{\mu\nu}$ and P^μ, appearing in $U(1+\omega, \epsilon) \simeq 1 + \frac{1}{2} I \omega_{\mu\nu} J^{\mu\nu} - I \epsilon_\mu P^\mu$.

Obtained by going close to identity, $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ and $a_\mu = \epsilon_\mu$.

Choose $J^{\mu\nu} = -J^{\nu\mu}$.

Allowed because $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$: Transformation of metric tensor reads $g_{\rho\sigma} = g_{\mu\nu}(\delta^\mu{}_\rho + \omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma) \simeq g_{\sigma\rho} + \omega_{\rho\sigma} + \omega_{\sigma\rho}$.

Transformation properties of P^μ, $J^{\mu\nu}$: $U(\Lambda,a)P^\mu U^\dagger(\Lambda,a) = \Lambda_\rho{}^\mu P^\rho$

and $U(\Lambda,a)J^{\mu\nu}U^\dagger(\Lambda,a) = \Lambda_\rho{}^\mu \Lambda_\sigma{}^\nu \left(J^{\rho\sigma} - a^\rho P^\sigma + a^\sigma P^\rho\right)$.

Apply Poincaré group to get $\underbrace{U(\Lambda,a)U(1+\omega,\epsilon)}_{=U(\Lambda(1+\omega),\Lambda\epsilon+a)}\ \underbrace{U^\dagger(\Lambda,a)}_{=U(\Lambda^{-1},-\Lambda^{-1}a)} = U(1+\Lambda\omega\Lambda^{-1},\Lambda(\epsilon-\omega\Lambda^{-1}a))$.

Expand both sides in ω,ϵ: $U(\Lambda,a)(1+\frac{1}{2}I\omega J - I\epsilon P)U^\dagger(\Lambda,a) = 1+\frac{1}{2}I\Lambda\omega\Lambda^{-1}J - I\Lambda(\epsilon-\omega\Lambda^{-1}a)P$, equate coefficients of ω,ϵ.

So P^μ transforms like 4-vector, J_{ij} like angular momentum.

Poincaré algebra: $I[J^{\rho\sigma},J^{\mu\nu}] = -g^{\sigma\nu}J^{\rho\mu} - g^{\rho\mu}J^{\sigma\nu} + g^{\sigma\mu}J^{\rho\nu} + g^{\rho\nu}J^{\sigma\mu}$ $\longrightarrow$ (homogeneous) Lorentz group,

$I[P^\mu,J^{\rho\sigma}] = g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho$, and $[P^\mu,P^\nu]=0$.

Obtained by taking $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$, $a_\mu = \epsilon_\mu$ in transformation properties of P^μ, $J^{\mu\nu}$, to first order in ω,ϵ gives

$$P^\mu - \tfrac{1}{2}I\omega_{\rho\sigma}[P^\mu,J^{\rho\sigma}] + I\epsilon_\nu[P^\mu,P^\nu] = P^\mu + \tfrac{1}{2}\omega_{\rho\sigma}g^{\mu\sigma}P^\rho - \tfrac{1}{2}\omega_{\rho\sigma}g^{\mu\rho}P^\sigma, \text{ and}$$

$$J^{\mu\nu} + \tfrac{1}{2}I\omega_{\rho\sigma}[J^{\rho\sigma},J^{\mu\nu}] - I\epsilon_\rho[P^\rho,J^{\mu\nu}] = J^{\mu\nu} - g^{\rho\mu}\epsilon_\rho P^\nu + g^{\rho\nu}\epsilon_\rho P^\mu + \tfrac{1}{2}\omega_{\rho\sigma}(g^{\rho\nu}J^{\sigma\mu} - g^{\sigma\nu}J^{\rho\mu}) + \tfrac{1}{2}\omega_{\rho\sigma}(g^{\sigma\mu}J^{\rho\nu} - g^{\rho\mu}J^{\sigma\nu}).$$

2.3.3 Relativistic quantum mechanical particles

Identify $H = P^0$, spatial momentum P^i, angular momentum $J_i = \frac{1}{2}\epsilon_{ijk}J^{jk}$ (i.e. $(J_1, J_2, J_3) = (J^{23}, J^{31}, J^{12})$).

$[H, \boldsymbol{P}] = [H, \boldsymbol{J}] = 0 \rightarrow \boldsymbol{P}, \boldsymbol{J}$ conserved.

Rotation group $[J_i, J_j] = I\epsilon_{ijk}J_k$ is subgroup of Poincaré group.

Explicit form of Poincaré elements: $U(\Lambda, a) = \underbrace{e^{-IP^\mu a_\mu}}_{\text{translate } a_\mu} \times \underbrace{e^{-I\boldsymbol{K}\cdot\hat{\boldsymbol{e}}\beta}}_{\text{boost along } \hat{\boldsymbol{e}} \text{ by } V=\sinh\beta} \times \underbrace{e^{I\boldsymbol{J}\cdot\boldsymbol{\theta}}}_{\text{rotate } |\boldsymbol{\theta}| \text{ about } \boldsymbol{\theta}}$

(V: magnitude of 4-velocity's spatial part),

where *boost generator* $\boldsymbol{K} = (J^{10}, J^{20}, J^{30})$, obeys $[J_i, K_j] = I\epsilon_{ijk}K_k$ and $[K_i, K_j] = -I\epsilon_{ijk}J_k$.

$\boldsymbol{K}$ not conserved: $[K_i, H] = IP_i$, because boost and time translation don't commute.

Lorentz transformation of 4-vectors: $(K_i)^\mu{}_\nu = I(\delta_{0\mu}\delta_{i\nu} + \delta_{i\mu}\delta_{0\nu})$ and $(J_i)^\mu{}_\nu = -I\epsilon_{0i\mu\nu}$, so

$$\left[e^{-I\boldsymbol{K}\cdot\hat{\boldsymbol{e}}\beta}\right]^\mu{}_\nu = \left[1 - IK_i\hat{e}_i\sinh\beta - (K_i\hat{e}_i)^2(\cosh\beta - 1)\right]^\mu{}_\nu \text{ and } \left[e^{I\boldsymbol{J}\cdot\hat{\boldsymbol{\theta}}\theta}\right]^\mu{}_\nu = \left[1 + IJ_i\hat{\theta}_i\sin\theta - \left(J_i\hat{\theta}_i\right)^2(1 - \cos\theta)\right]^\mu{}_\nu.$$

Follows directly from $(K_i\hat{e}_i)^3 = K_i\hat{e}_i$ and $(J_i\hat{\theta}_i)^3 = J_i\hat{\theta}_i$.

Particles: Use P^μ, $P^2 = m^2$ eigenstates. Distinguish momentum $\boldsymbol{p}$ from list σ, i.e. $a^\dagger_\sigma \to a^\dagger_\sigma(\boldsymbol{p})$.

Commutation relations for $a^{(\dagger)}_\sigma(\boldsymbol{p})$: $[a^\dagger_\sigma(\boldsymbol{p}), a^\dagger_{\sigma'}(\boldsymbol{p}')]_\mp = [a_\sigma(\boldsymbol{p}), a_{\sigma'}(\boldsymbol{p}')]_\mp = 0$

and $[a_\sigma(\boldsymbol{p}), a^\dagger_{\sigma'}(\boldsymbol{p}')]_\mp = \delta_{\sigma\sigma'}(2p^0)\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')$.

As on page 3, but with different normalization (Lorentz invariant).

Application of general transformation of $a^{(\dagger)}_\sigma$ on page 9 to Lorentz transformation

complicated by mixing between σ and $\boldsymbol{p}$.

Simplify by finding 2 transformations, one that mixes σ and one that mixes $\boldsymbol{p}$, separately.

Lorentz transformation for $a_\sigma^{(\dagger)}(\boldsymbol{p})$: $U(\Lambda)a_\sigma^\dagger(\boldsymbol{p})U^\dagger(\Lambda) = D_{\sigma'\sigma}(W(\Lambda,p))a_{\sigma'}^\dagger(\boldsymbol{\Lambda p})$.

Same as general transformation on page 9, but because $\sigma \to \{\sigma, \boldsymbol{p}\}$, mixing of σ with $\boldsymbol{p}$ must be allowed.

Construction of W: Choose reference momentum k and transformation L to mix k but not σ (defines σ):

$L^\mu{}_\nu(p)k^\nu = p^\mu$ and $U(L(p))a_\sigma^\dagger(\boldsymbol{k})|0\rangle = a_\sigma^\dagger(\boldsymbol{p})|0\rangle$.

Then $W(\Lambda,p) = L^{-1}(\Lambda p)\Lambda L(p)$. W mixes σ but not k, i.e. belongs to *little group* of k: $W^\mu{}_\nu k^\nu = k^\mu$.

General transformation is $U(\Lambda)\underline{a_\sigma^\dagger(\boldsymbol{p})|0\rangle} = U(\Lambda)\underline{U(L(p))a_\sigma^\dagger(\boldsymbol{k})|0\rangle}$, using definition of L.

Multiplying by $1 = U(L(\Lambda p))U(L^{-1}(\Lambda p))$ gives

$U(\Lambda)a_\sigma^\dagger(\boldsymbol{p})|0\rangle = U(L(\Lambda p))\ \underline{U(L^{-1}(\Lambda p))\ U(\Lambda)U(L(p))}a_\sigma^\dagger(\boldsymbol{k})|0\rangle = U(L(\Lambda p))\underline{U(W(\Lambda,p))}a_\sigma^\dagger(\boldsymbol{k})|0\rangle$.

But W doesn't change k, so $U(\Lambda)a_\sigma^\dagger(\boldsymbol{p})|0\rangle = U(L(\Lambda p))D_{\sigma'\sigma}(W(\Lambda,p))a_{\sigma'}^\dagger(\boldsymbol{k})|0\rangle$.

Then $L(\Lambda p)$ changes k to Λp but doesn't change σ'.

Poincaré transformation for $a_\sigma^{(\dagger)}(\boldsymbol{p})$: $U(\Lambda,a)a_\sigma^\dagger(\boldsymbol{p})U^\dagger(\Lambda,a) = e^{-I\Lambda p\cdot a}D_{\sigma'\sigma}(W(\Lambda,p))a_{\sigma'}^\dagger(\boldsymbol{\Lambda p})$.

In Lorentz transformation for $a_\sigma^{(\dagger)}(\boldsymbol{p})$, use $U(\Lambda,a) = e^{-IP^\mu a_\mu}U(\Lambda)$ (from explit form of Poincaré elements on page 21).

2.3.4 Quantum field theory

Lorentz invariant QM: $H = \int d^3x \mathscr{H}(x)$, scalar field $\mathscr{H}(x)$ (i.e. $U(\Lambda, a)\mathscr{H}(x)U^\dagger(\Lambda, a) = \mathscr{H}(\Lambda x + a)$),

obeys *cluster decomposition principle* (two processes with large spatial separation evolve independently).

Causality: $[\mathscr{H}(x), \mathscr{H}(y)] = 0$ when $(x-y)^2 \geq 0$. Required for Lorentz invariance of S-matrix.

Intuitive reason: signal can't propagate between 2 spacelike separated events.

In QM (general): H built from $a_\sigma^{(\dagger)}$. In QFT: H built from $\mathscr{H}(x)$ built from products of

Fields $\psi_l^{-c}(x) = \int D^3p\; v_{l\sigma}(x; \boldsymbol{p}) a_\sigma^{c\dagger}(\boldsymbol{p})$ and $\psi_l^{+}(x) = \int D^3p\; u_{l\sigma}(x; \boldsymbol{p}) a_\sigma(\boldsymbol{p})$, obeying

Lorentz invariant momentum space volume $D^3p = \frac{d^3p}{2p^0} = d^4p\delta(p^2 + m^2)\theta(p^0)$ obeys $D^3\Lambda p = D^3p$.

Poincaré transformation for fields: $U(\Lambda, a)\psi_l^{(c)\pm}(x)U^\dagger(\Lambda, a) = D_{ll'}(\Lambda^{-1})\psi_{l'}^{(c)\pm}(\Lambda x + a)$.

Take single particle species in representation labelled j, allow $a_\sigma^{c\dagger}(\boldsymbol{p}) \neq a_\sigma^\dagger(\boldsymbol{p})$.

x **dependence of** u, v: $u_{l\ \sigma}(x;\boldsymbol{p}) = e^{Ip\cdot x}u_{l\ \sigma}(\boldsymbol{p})$, $v_{l\ \sigma}(x;\boldsymbol{p}) = e^{-Ip\cdot x}v_{l\ \sigma}(\boldsymbol{p})$.

Take $\Lambda = 1$ in Poincaré transformation for fields and for $a_\sigma^{(\dagger)}(\boldsymbol{p})$ on page 24 and 23,

e.g. $U(1,a)\psi_l^+(x)U^\dagger(1,a) = \int D^3p\ u_{l\sigma}(x;\boldsymbol{p})U(1,a)a_\sigma(\boldsymbol{p})U^\dagger(1,a) = \int D^3p\ \underline{e^{Ip\cdot a}u_{l\sigma}(x;\boldsymbol{p})}a_\sigma(\boldsymbol{p})$

$= \psi_l^+(x+a) = \int D^3p\ \underline{u_{l\sigma}(x+a;\boldsymbol{p})}a_\sigma(\boldsymbol{p})$, equate coefficients of $a_\sigma(\boldsymbol{p})$ (underlined), gives $u_{l\ \sigma}(x;\boldsymbol{p})e^{Ip\cdot a} = u_{l\sigma}(x+a;\boldsymbol{p})$.

Klein-Gordon equation: $(\partial^2 - m^2)\psi_l^{\pm(c)}(x) = 0$.

Act on e.g. $\psi_l^{-c}(x) = \int D^3p\ e^{-Ip\cdot x}v_{l\ \sigma}(\boldsymbol{p})a_\sigma^{c\dagger}(\boldsymbol{p})$ with $(\partial^2 - m^2)$, use $p^2 = -m^2$.

Transformation of u, v: $D_{ll'}(\Lambda)u_{l'\sigma}(\boldsymbol{p}) = D^{(j)}_{\sigma'\sigma}(W(\Lambda,p))u_{l\sigma'}(\boldsymbol{\Lambda p})$, $D_{ll'}(\Lambda)v_{l'\sigma}(\boldsymbol{p}) = D^{(j)}_{\sigma\sigma'}(W^{-1}(\Lambda,p))v_{l\sigma'}(\boldsymbol{\Lambda p})$.

E.g. consider v, use Poincaré transformation for fields and for $a_\sigma^{(\dagger)}(\boldsymbol{p})$ on page 24 and 23.

LHS: $U(\Lambda,a)\psi_l^{-c}(x)U^\dagger(\Lambda,a) = \int D^3p\ v_{l\sigma}(x;\boldsymbol{p})U(\Lambda,a)a_\sigma^{c\dagger}(\boldsymbol{p})U^\dagger(\Lambda,a) = \int D^3p\ \underline{e^{-I\Lambda p\cdot a}v_{l\sigma}(x;\boldsymbol{p})D^{(j)}_{\sigma'\sigma}(W(\Lambda,p))}a_{\sigma'}^{c\dagger}(\boldsymbol{\Lambda p})$, and

RHS: $D_{ll'}(\Lambda^{-1})\psi_{l'}^{-c}(\Lambda x+a) = \int D^3p\ D_{ll'}(\Lambda^{-1})v_{l\sigma}(x+\Lambda a;\boldsymbol{p})a_\sigma^{c\dagger}(\boldsymbol{p}) = \int D^3\Lambda p\ \underline{D_{ll'}(\Lambda^{-1})v_{l\sigma}(x+\Lambda a;\boldsymbol{\Lambda p})}a_\sigma^{c\dagger}(\boldsymbol{\Lambda p})$ $(p \to \Lambda p)$.

Use $D^3\Lambda p = D^3p$, equate coefficient of $a_\sigma^{c\dagger}(\boldsymbol{\Lambda p})$ (underlined), multiply by $D_{l''l}(\Lambda)D^{(j)}_{\sigma''\sigma}(W^{-1}(\Lambda,p))$.

$\boldsymbol{p}$ **dependence of** u, v: $u_{l\sigma}(\boldsymbol{p}) = D_{ll'}(L(p))u_{l'\sigma}(\boldsymbol{k})$. ($\boldsymbol{p}$ dependence of v is the same.)

In transformation of u, v, take $p = k$ so $L(p) = 1$, $\Lambda = L(q)$ so $\Lambda p = L(q)k = q$, then $W(\Lambda,p) = L^{-1}(\Lambda p)L(q) = 1$.

2.3.5 Causal field theory

Since $[\psi_l^+(x), \psi_{l'}^{-c}]_\mp \neq 0$, causality on page 24 only gauranteed by taking $\mathscr{H}(x)$ to be functional of

complete field $\psi_l(x) = \kappa\psi_l^+(x) + \lambda\psi_l^{-c}(x)$ (so representations $D_{ll'}(\Lambda^{-1})$ for $\psi_l^+(x)$ and $\psi_l^{-c}(x)$ the same), with

Causality: $[\psi_l(x), \psi_{l'}(y)]_\mp = [\psi_l(x), \psi_{l'}^\dagger(y)]_\mp = 0$ when $(x-y)^2 > 0$ by suitable choice of κ, λ.

Now $\langle 0|H|0\rangle = \infty$, i.e. consistency with gravity not gauranteed by QFT.

Complete field: $\psi_l(x) = \int D^3p \left[\kappa e^{Ip\cdot x} u_{l\sigma}(\boldsymbol{p}) a_\sigma(\boldsymbol{p}) + \lambda e^{-Ip\cdot x} v_{l\sigma}(\boldsymbol{p}) a_\sigma^{c\dagger}(\boldsymbol{p})\right]$.

2.3.6 Antiparticles

$\mathscr{H}(x)$ **commutes with conserved additive** Q: $[Q, \mathscr{H}(x)] = 0$.

Imposed in order to satisfy $[Q, H] = 0$.

This is achieved as follows:

Commutation of fields with conserved additive Q: $[Q, \psi_l(x)] = -q_l \psi_l(x)$, and

Field construction of $\mathscr{H}(x)$: $\mathscr{H} = \sum \psi_{l_1}^{L_1} \psi_{l_2}^{L_2} \ldots \psi_{m_1}^{M_1 \dagger} \psi_{m_2}^{M_2 \dagger} \ldots$ with $q_{l_1}^{L_1} + q_{l_2}^{L_2} + \ldots - q_{m_1}^{M_1} - q_{m_2}^{M_2} - \ldots = 0$

(M_i, L_j label particle species).

Antiparticles: For every particle species there is another species with opposite conserved quantum numbers.

Commutation of fields with conserved additive Q implies $[Q, \psi_l^{-c}(x)] = -q_l \psi_l^{-c}(x)$ and $[Q, \psi_l^{+}(x)] = -q_l \psi_l^{+}(x)$,

but since $[Q, a_\sigma^{c\dagger}] = q_\sigma^c a_\sigma^{c\dagger}$ and $[Q, a_\sigma] = -q_\sigma a_\sigma$ (no sum) from page 4, $q_l = q_\sigma^c$ and $q_l = -q_\sigma$, i.e. $q_\sigma^c = -q_\sigma$.

2.3.7 Spin in relativistic quantum mechanics

Lorentz group algebra simplified by choosing generators $A_i = \frac{1}{2}(J_i - IK_i)$ and $B_i = \frac{1}{2}(J_i + IK_i)$,

behaves like 2 independent rotations: $[A_i, A_j] = I\epsilon_{ijk}A_k$, $[B_i, B_j] = I\epsilon_{ijk}B_k$ and $[A_i, B_j] = 0$,

i.e. relativistic particle of type (A, B) (i.e. in eigenstate of $\boldsymbol{A}^2 = A(A+1)$, $\boldsymbol{B}^2 = B(B+1)$)

$\equiv$ 2 particles at rest, ordinary spins A, B (in representation sense).

In terms of degrees of freedom, $(A, B) = (\mathbf{2}\boldsymbol{A}\mathbf{+1}) \times (\mathbf{2}\boldsymbol{B}\mathbf{+1})$.

Triangle inequality: ordinary spin $\boldsymbol{J} = \boldsymbol{A} + \boldsymbol{B}$, so $j = |A - B|, |A - B| + 1, ..., A + B$.

Derived on page 18.

Can have eigenstates of $\boldsymbol{A}^2 = A(A+1)$, $\boldsymbol{B}^2 = B(B+1)$ and $\boldsymbol{J}^2 = j(j+1)$ *simultaneously*,

because $[\boldsymbol{J}^2, A_i] = [\boldsymbol{J}^2, B_i] = 0$. Use $\boldsymbol{J}^2 = (\boldsymbol{A} + \boldsymbol{B})^2$ and $\boldsymbol{A}$, $\boldsymbol{B}$ commutation relations.

Simultaneous eigenstate also with any $\boldsymbol{K} = I(\boldsymbol{A} - \boldsymbol{B})$ or $\boldsymbol{K}^2$ not possible.

Example: $(A,B)=(\frac{1}{2},\frac{1}{2})$ is representation of 4-vector: $(A_3)^\mu{}_\nu$, $(B_3)^\mu{}_\nu$ can have eigenvalues $\pm\frac{1}{2}$ only.

From triangle inequality on page 28, $j=0,1$.

Also follows from tensor representation on page 17: $\mathbf{2}\times\mathbf{2}=\mathbf{1}+\mathbf{3}$.

More generally, rank N tensor is $(\frac{1}{2},\frac{1}{2})^N=\sum_{A=0}^{\frac{N}{2}}\sum_{B=0}^{\frac{N}{2}}(A,B)$,

i.e. $(\frac{1}{2},\frac{1}{2})^N=\left(\frac{N}{2},\frac{N}{2}\right)$ +lower spins, where $\left(\frac{N}{2},\frac{N}{2}\right)\equiv$ traceless symmetric rank N tensor, $j=0,...,N$.

$(N,0)$ and $(0,N)$ are purely spin $j=N$.

2.3.8 Irreducible representation for fields

If particles created by $a_\sigma^{(c)\dagger}(\boldsymbol{p})$ have spin j,

must take field with same spin j but any (A, B) consistent with triangle inequality on page 28:

$\psi_{ab}(x) = \int D^3p \left[\kappa\, e^{Ip\cdot x} u_{ab\ \sigma}(\boldsymbol{p}) a_\sigma(\boldsymbol{p}) + \lambda\, e^{-Ip\cdot x} v_{ab\ \sigma}(\boldsymbol{p}) a_\sigma^{c\dagger}(\boldsymbol{p})\right]$ $(l = ab,\ l' = a'b')$,

Lorentz transformation uses generators $\boldsymbol{A}_{a'b'ab} = \boldsymbol{J}^{(A)}_{a'a}\delta_{b'b}$, $\boldsymbol{B}_{a'b'ab} = \delta_{a'a}\boldsymbol{J}^{(B)}_{b'b}$

where $a = -A, -A+1, ..., A$ and $b = -B, -B+1, ..., B$.

2.3.9 Massive particles

Choose $k = (0,0,0,m)$ (momentum of particle at rest). Then W is an element of the spatial rotation group,

i.e. rotation group apparatus of subsubsection 2.3.1 applies to relativity too.

$L(p)$ **in terms of** $\boldsymbol{K}$: $L(p) = \exp[-I\hat{\boldsymbol{p}}\cdot\boldsymbol{K}\beta]$. In this case, $L(p)$ includes boost in $\boldsymbol{p}$ direction by 4-velocity $\frac{|\boldsymbol{p}|}{m} = \sinh\beta$.

Conditions on u, v: $\boldsymbol{J}^{(j)}_{\sigma'\sigma} u_{ab\ \sigma'}(0) = \boldsymbol{J}^{(A)}_{aa'} u_{a'b\ \sigma}(0) + \boldsymbol{J}^{(B)}_{bb'} u_{ab'\ \sigma}(0)$, $-\boldsymbol{J}^{(j)*}_{\sigma'\sigma} v_{ab\ \sigma'}(0) = \boldsymbol{J}^{(A)}_{aa'} v_{a'b\ \sigma}(0) + \boldsymbol{J}^{(B)}_{bb'} v_{ab'\ \sigma}(0)$.

In transformation of u, v on page 25, take $\Lambda = R$ and $\boldsymbol{p} = 0$

(so $\boldsymbol{p} = \boldsymbol{k}$, $Rp = p$, $L(p) = 1$, $W(R,p) = L^{-1}(Rp)RL(p) = L^{-1}(p)R = R$),

so e.g. $D_{ll'}(R)v_{l'\sigma}(0) = D^{(j)}_{\sigma\sigma'}(R^{-1})v_{l\sigma'}(0)$, and use $D^{(j)}_{\sigma\sigma'}(R^{-1}) = D^{(j)*}_{\sigma'\sigma}(R)$ because irreducible representations of R are unitary.

Take $l = ab$ etc., $D_{aba'b'}(R)v_{a'b'\sigma}(0) = D^{(j)*}_{\sigma'\sigma}(R)v_{ab\sigma'}(0)$. Generators of $D^{(j)*}(R)$ and $D(R)$ are $-\boldsymbol{J}^{(j)*}$ and $A+B$.

u, v **relation**: $v_{ab\ \sigma}(0) = (-1)^{j+\sigma} u_{ab\ -\sigma}(0)$ up to normalization.

Conditions on u, v with $-\boldsymbol{J}^{(j)*}_{\sigma\sigma'} = (-1)^{\sigma-\sigma'}\boldsymbol{J}^{(j)}_{-\sigma,-\sigma'}$ from page 16

gives $v_{ab\ \sigma}(0) \propto (-1)^{\sigma} u_{ab\ -\sigma}(0)$, absorb proportionality constant into u, v.

The $\boldsymbol{p}$ dependence of u, v are the same, so $v_{ab\ \sigma}(\boldsymbol{p}) = (-1)^{j+\sigma} u_{ab\ -\sigma}(\boldsymbol{p})$.

2.3.10 Massless particles

Take reference vector $k = (0, 0, 1, 1)$.

Little group transformation: $W(\theta, \mu, \nu) \simeq \mathbf{1} + I\theta J_3 + I\mu M + I\nu N$ with $M = J_2 + K_1$, $N = -J_1 + K_2$.

W has 3 degrees of freedom: For $(t_i)^\mu{}_\nu k^\mu = 0$, take $t_i = (J_3, M, N)$, check with Lorentz transformation of 4-vectors on page 21.

Choice of states: $(J_3, M, N)|k, \sigma\rangle = (\sigma, 0, 0)|k, \sigma\rangle$.

Since $[M, N] = 0$, try eigenstates for which $M|k, m, n\rangle = m|k, m, n\rangle$, $N|k, m, n\rangle = n|k, m, n\rangle$.

Then m, n continuous degrees of freedom, unobserved: $[J_3, M] = IN$, so $M(1 - I\theta J_3)|k, m, n\rangle = (m - n\theta)(1 - I\theta J_3)|k, m, n\rangle$,

i.e. $(1 - I\theta J_3)|k, m, n\rangle$ is eigenvector of M, eigenvalue $m - n\theta$.

Similarly, $[J_3, N] = -IM$, so $(1 - I\theta J_3)|k, m, n\rangle$ is eigenvector of N, eigenvalue $n + m\theta$.

Avoid this problem by taking $m = n = 0$, so left with states $J_3|k, \sigma\rangle = \sigma|k, \sigma\rangle$.

Since $J_3 = \boldsymbol{J} \cdot \hat{\boldsymbol{k}}$, σ is *helicity*, component of spin in direction of motion.

Representation for massless particles: $D_{\sigma'\sigma}(W) = e^{I\theta\sigma}\delta_{\sigma'\sigma}$.

$U(W)|k, \sigma\rangle = (1 + I\theta J_3 + I\mu M + I\nu N)|k, \sigma\rangle = (1 + I\theta\sigma)|k, \sigma\rangle$. For finite θ, $U(W)|k, \sigma\rangle = e^{I\theta\sigma}|k, \sigma\rangle$.

p **dependence of** u: $u_{l\,\sigma}(\boldsymbol{p}) = D_{ll'}(L(p))u_{l\,\sigma}(\boldsymbol{k})$. ($\boldsymbol{p}$ dependence of v is the same.) As on page 25.

Little group transformation of u, v: $u_{l\,\sigma}(\boldsymbol{k})e^{I\theta(W,k)\sigma} = D_{ll'}(W)u_{l\,\sigma}(\boldsymbol{k})$, $v_{l\,\sigma}(\boldsymbol{k})e^{-I\theta(W,k)\sigma} = D_{ll'}(W)v_{l\,\sigma}(\boldsymbol{k})$.

Transformation of u, v on page 25 reads $u_{l\,\sigma}(\boldsymbol{\Lambda p})e^{I\theta(\Lambda,p)\sigma} = D_{ll'}(\Lambda)u_{l\,\sigma}(\boldsymbol{p})$. Take $\Lambda = W$, $p = k$.

Rotation of u, v: $(J_3)_{ll'}u_{l'\,\sigma}(\boldsymbol{k}) = \sigma u_{l\,\sigma}(\boldsymbol{k})$, $(J_3)_{ll'}v_{l'\,\sigma}(\boldsymbol{k}) = -\sigma v_{l\,\sigma}(\boldsymbol{k})$.

Take W to be rotation about 3-axis in little group transformation of u, v.

M, N **transformation of** u, v: $M_{ll'}u_{l\,\sigma}(\boldsymbol{k}) = N_{ll'}u_{l\,\sigma}(\boldsymbol{k}) = 0$. Same for v.

Take $W = (1 + I\mu M + I\nu N)$ in little group transformation of u, v.

u, v **relation**: $v_{l\,\sigma}(\boldsymbol{p}) = u^*_{l\,\sigma}(\boldsymbol{p})$.

Implied (up to proportionality constant) by rotation (note $(J_3)_{ll'}$ is imaginary) and M, N transformation of u, v.

Allowed helicities for fields in given (A, B) **representation**: $\sigma = \pm(B - A)$ for particle/antiparticle.

$\boldsymbol{J} = \boldsymbol{A} + \boldsymbol{B}$, so rotation of u is $\sigma u_{ab\,\sigma}(0) = (a + b)u_{ab\,\sigma}(0)$. M, N transformation of u is $M_{aba'b''}u_{a'b'\,\sigma} = N_{aba'b'}u_{a'b'\,\sigma} = 0$.

Using $M = IA_- - IB_+$ and $N = -A_- - B_+$, where $A_\pm = A_1 \pm IA_2$, $B_\pm = B_1 \pm IB_2$ are usual raising/lowering operators,

$(A_-)_{aba'b'}u_{a'b'\,\sigma} = (B_+)_{aba'b'}u_{a'b'\,\sigma} = 0$, i.e. must have $a = -A$, $b = B$ or $u_{ab\,\sigma} = 0$. So $\sigma = B - A$.

Similar for v, gives $\sigma = A - B$.

2.3.11 Spin-statistics connection

Determine which of $\mp$ for given j is possible for causality on page 26 to hold. Demand more general condition

$[\psi_{ab}(x), \tilde{\psi}^{\dagger}_{\tilde{a}\tilde{b}}(y)]_{\mp} = \int D^3p \; \pi_{ab,\tilde{a}\tilde{b}}(\boldsymbol{p}) \left[\kappa\tilde{\kappa}^* e^{Ip\cdot(x-y)} \mp \lambda\tilde{\lambda}^* e^{-Ip\cdot(x-y)}\right] = 0$ for $(x-y)^2 > 0$,

where $\psi_{ab}, \tilde{\psi}_{\tilde{a}\tilde{b}}$ for same particle species and $\pi_{ab,\tilde{a}\tilde{b}}(\boldsymbol{p}) = u_{ab\ \sigma}(\boldsymbol{p})\tilde{u}^*_{\tilde{a}\tilde{b}\ \sigma}(\boldsymbol{p}) = v_{ab\ \sigma}(\boldsymbol{p})\tilde{v}^*_{\tilde{a}\tilde{b}\ \sigma}(\boldsymbol{p})$.

In field on page 30, use commutation relations for $a^{(\dagger)}_{\sigma}(\boldsymbol{p})$ on page 22.

$u_{ab\ \sigma}(\boldsymbol{p})\tilde{u}^*_{\tilde{a}\tilde{b}\ \sigma}(\boldsymbol{p}) = v_{ab\ \sigma}(\boldsymbol{p})\tilde{v}^*_{\tilde{a}\tilde{b}\ \sigma}(\boldsymbol{p})$ holds for massive particles from u, v relation on page 31.

($u_{ab\ \sigma}(\boldsymbol{p})\tilde{u}^*_{\tilde{a}\tilde{b}\ \sigma}(\boldsymbol{p}) = [v_{ab\ \sigma}(\boldsymbol{p})\tilde{v}^*_{\tilde{a}\tilde{b}\ \sigma}(\boldsymbol{p})]^*$ in massless case from u, v relation on page 33).

Relation between κ, λ of different massive fields: $\kappa\tilde{\kappa}^* = \pm(-1)^{2\tilde{A}+2B}\lambda\tilde{\lambda}^*$.

Explicit calculation shows $\pi_{ab,\tilde{a}\tilde{b}}(\boldsymbol{p}) = P_{ab,\tilde{a}\tilde{b}}(\boldsymbol{p}) + 2\sqrt{\boldsymbol{p}^2+m^2}\,Q_{ab,\tilde{a}\tilde{b}}(\boldsymbol{p})$,

where $(P,Q)(\boldsymbol{p})$ are polynomial in $\boldsymbol{p}$, obey $(P,Q)(-\boldsymbol{p}) = (-1)^{2\tilde{A}+2B}(P,-Q)(\boldsymbol{p})$.

Take $(x-y)^2 > 0$, use frame $x^0 = y^0$, write $\Delta_+(x) = \int D^3p e^{Ip\cdot x}$:

$[\psi_{ab}, \tilde{\psi}_{\tilde{a}\tilde{b}}]_{\mp} = \left[\kappa\tilde{\kappa}^* \mp (-1)^{2\tilde{A}+2B}\lambda\tilde{\lambda}^*\right] P_{ab,\tilde{a}\tilde{b}}(-I\boldsymbol{\nabla})\Delta_+(\boldsymbol{x}-\boldsymbol{y},0) + \left[\kappa\tilde{\kappa}^* \pm (-1)^{2\tilde{A}+2B}\lambda\tilde{\lambda}^*\right] Q_{ab,\tilde{a}\tilde{b}}(-I\boldsymbol{\nabla})\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$.

Commutator must vanish for $\boldsymbol{x} \neq \boldsymbol{y}$, so coefficient of P zero.

Relations between κ, λ of single field: $|\kappa|^2 = |\lambda|^2$ and $\pm(-1)^{2A+2B} = 1$.

For $A = \tilde{A}$, $B = \tilde{B}$, relation between κ, λ of different fields reads $|\kappa|^2 = \pm(-1)^{2A+2B}|\lambda|^2$.

Spin-statistics: Bosons (fermions) have even (odd) $2j$ and vice versa.

From triangle inequality on page 28,

$j-(A+B)$ is integer, so $\pm(-1)^{2j} = 1$, i.e. in $[\psi_{ab}, \tilde{\psi}_{\tilde{a}\tilde{b}}]_{\mp}$, must have $-$ $(+)$ for even (odd) $2j$.

Relation between κ, λ of single massive field: $\lambda = (-1)^{2A}e^{Ic}\kappa$, c is the same for all fields.

Divide relation between κ, λ of different massive fields on page 34 by $|\tilde{\kappa}|^2 = |\tilde{\lambda}|^2$: $\frac{\kappa}{\tilde{\kappa}} = \pm(-1)^{2\tilde{A}+2B}\frac{\lambda}{\tilde{\lambda}} = (-1)^{2A+2\tilde{A}}\frac{\lambda}{\tilde{\lambda}}$.

Absorb κ into field, e^{Ic} into $a_\sigma^{c\dagger}(\boldsymbol{p})$ (does not affect commutation relations on page 22).

$(-1)^{2A}$ can't be absorbed into $a_\sigma^{c\dagger}(\boldsymbol{p})$ since this is independent of A,

nor absorbed into v since this is already chosen. So

Massive irreducible field: $\psi_{ab}(x) = \int D^3p \;\left[e^{Ip\cdot x}u_{ab\;\sigma}(\boldsymbol{p})a_\sigma(\boldsymbol{p}) + (-1)^{2A}e^{-Ip\cdot x}v_{ab\;\sigma}(\boldsymbol{p})a_\sigma^{c\dagger}(\boldsymbol{p})\right]$,

or more fully as $\psi_{ab}(x) = \int D^3p\; D^{(j)}_{aba'b'}(L(p))\left[e^{Ip\cdot x}u_{a'b'\;\sigma}(0)a_\sigma(\boldsymbol{p}) + (-1)^{2A+j+\sigma}e^{-Ip\cdot x}u_{a'b'\;-\sigma}(0)a_\sigma^{c\dagger}(\boldsymbol{p})\right]$.

Use u, v relation on page 31.

2.4 External symmetries: fermions

2.4.1 Spin $\frac{1}{2}$ fields

$2j$ is odd $\longrightarrow$ particles are fermions. $j = \frac{1}{2}$ representations include $(A, B) = \left(\frac{1}{2}, 0\right)$ and $(A, B) = \left(0, \frac{1}{2}\right)$,

In each case, group element acts on 2 component *spinor* $X_{ab}^{(A,B)}$,

i.e. $X^{\left(\frac{1}{2},0\right)} \equiv X_L = \left(X_{\frac{1}{2},0}^{\left(\frac{1}{2},0\right)}, X_{-\frac{1}{2},0}^{\left(\frac{1}{2},0\right)}\right)$ ("left-handed")

and $X^{\left(0,\frac{1}{2}\right)} \equiv X_R = \left(X_{0,\frac{1}{2}}^{\left(0,\frac{1}{2}\right)}, X_{0,-\frac{1}{2}}^{\left(0,\frac{1}{2}\right)}\right)$ ("right-handed").

"Handedness"/chirality refers to eigenstates of helicity for *massless* particles (see later).

X is a field operator, but argument x suppressed.

Lorentz transformation of spinors: $U(\Lambda)X_{L/R}U^\dagger(\Lambda) = h_{L/R}(\Lambda)X_{L/R}$ (D acts on the 2 a, b components).

From explicit form of Lorentz group elements on page 21, $h_{L/R}(\Lambda) = e^{I\boldsymbol{J}^{\left(\frac{1}{2},0\right)/\left(0,\frac{1}{2}\right)}\cdot\boldsymbol{\theta}}e^{-I\boldsymbol{K}^{\left(\frac{1}{2},0\right)/\left(0,\frac{1}{2}\right)}\cdot\hat{\boldsymbol{e}}\beta}$, where

Lorentz group generators for spinors: $J_i^{\left(\frac{1}{2},0\right)} = \frac{1}{2}\sigma_i,\ K_i^{\left(\frac{1}{2},0\right)} = I\frac{1}{2}\sigma_i$, $J_i^{\left(0,\frac{1}{2}\right)} = \frac{1}{2}\sigma_i,\ K_i^{\left(0,\frac{1}{2}\right)} = -I\frac{1}{2}\sigma_i$,

where σ_i are the *Pauli* σ *matrices* $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

which obey rotation group algebra $\left[\frac{1}{2}\sigma_i, \frac{1}{2}\sigma_j\right] = I\epsilon_{ijk}\frac{1}{2}\sigma_k$.

Follows from $J_i = J_i^{(A)} + J_i^{(B)}$ and $K_i = I(J_i^{(A)} - J_i^{(B)})$ on page 28,

and $J_i^{(0)} = 0$ and $J_i^{\left(\frac{1}{2}\right)} = \frac{1}{2}\sigma_i$ from irreducible representation for spin j on page 16.

Explicit form of $h_{L/R}$: $h_{L/R} = e^{I\frac{1}{2}\sigma_i\theta_i}e^{\mp\frac{1}{2}\sigma_i\hat{e}_i\beta}$. Note σ matrices are Hermitian.

Product of σ **matrices**: $\sigma_i\sigma_j = \delta_{ij} + I\epsilon_{ijk}\sigma_k$. Follows by explicit calculation.

Direct calculation of $h_{L/R}$: $h_{L/R} = (\cos\frac{\theta}{2} + I\sigma_i\hat{\theta}_i\sin\frac{\theta}{2})(\cosh\frac{\beta}{2} \mp \sigma_i\hat{e}_i\sinh\frac{\beta}{2})$.

From product of σ matrices, $T^2 = \mathbf{1}$ where $T = \sigma_i\hat{e}_i$ ($T = \sigma_i\hat{\theta}_i$). Then $e^{xT} = \cosh x + T\sinh x$ ($e^{IxT} = \cos x + IT\sin x$).

Write $h_L = h$ and $X_a = (X_1, X_2) = X_L = X^{\left(\frac{1}{2},0\right)}$, which transforms as $\boxed{1.\ X'_a = h_a{}^b X_b}$.

Can also defined spinor transforming with h^*: Use dotted indices for h^*, so $\boxed{2.\ X'^{\dagger}_{\dot{a}} = h^*_{\dot{a}}{}^{\dot{b}} X^{\dagger}_{\dot{b}}}$.

Then $h_R = h^{*-1T}$. From explicit form of $h_{L/R}$ on page 37.

Use upper indices for h^{-1}, so $X^{\dagger\dot{a}} = (X^{\dagger\dot{1}}, X^{\dagger\dot{2}}) = X_R = X^{\left(0,\frac{1}{2}\right)}$, Dotted indices because h^* is used.

transformation is $\boxed{4.\ X'^{\dagger\dot{a}} = \left(h^{*-1T}\right)^{\dot{a}}{}_{\dot{b}} X^{\dagger\dot{b}}}$, where we define $(h^T)^a{}_b = h_b{}^a$.

Conjugate of this turns dotted indices into undotted indices, so $\boxed{3.\ X'^a = \left(h^{-1T}\right)^a{}_b X^b}$.

Conjugate of spinors: $(X^a)^\dagger = X^{\dagger\dot{a}}$, $(X_a)^\dagger = X^\dagger_{\dot{a}}$. Definition of $X^{\dagger\dot{a}}$ in terms of X^a, $X^\dagger_{\dot{a}}$ in terms of X_a.

Check that if X_a transforms as transformation 1., $X^\dagger_{\dot{a}}$ transforms as transformation 2.: $X'^{\dagger}_{\dot{a}} = (X'_a)^\dagger = (h_a{}^b X_b)^\dagger = h^{*\dot{b}}_{\dot{a}} X^\dagger_{\dot{b}}$.

Spinor metric: $\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\epsilon_{ab} = -\epsilon^{ab}$. Note $\epsilon_{ab}\epsilon^{bc} = \delta_a{}^c$. ϵ is unitary matrix.

Pseudo reality: $\epsilon^{ac}\sigma_c{}^d\epsilon_{db} = -(\sigma^T)^a{}_b$.

Follows by explicit calculation. Since $\sigma_i^* = \sigma_i^T$, shows rotation group representation by σ matrices is real (see page 11).

$X^\dagger_{\dot{a}}$ is $\left(0, \frac{1}{2}\right)$, i.e. right-handed, like $X^{\dagger\dot{a}}$.

Pseudo reality implies h^*, h_R same up to unitary similarity transformation: $\epsilon^{\dot{a}\dot{c}}h^{*\dot{d}}_{\dot{c}}\epsilon_{\dot{d}\dot{b}} = (h^{*-1T})^{\dot{a}}{}_{\dot{b}}$.

For unitary representations, follows because $A = B^\dagger$ from definition of page 28, so conjugation makes $\left(\frac{1}{2}, 0\right) \to \left(0, \frac{1}{2}\right)$.

This means dotted indices are for right-handed ($\left(0, \frac{1}{2}\right)$) fields.

(Similarly, undotted indices are for left-handed ($\left(\frac{1}{2}, 0\right)$) fields.)

Raising and lowering of spinor indices: $X^a = \epsilon^{ab} X_b$. This is definition of X^a in terms of X_a.

Follows that $X_a = \epsilon_{ab} X^b$. Same definition / behaviour for dotted indices.

Check that if X_a transforms as transformation 1. on page 38, X^a transforms as transformation 3.: $X'^a = \epsilon^{ab} X'_b = \epsilon^{ab} h_b{}^c X_c$.

From pseudo reality, $\epsilon^{ab} h_b{}^c \epsilon_{cd} = \left(h^{-1T}\right)^a{}_d$, or $\epsilon^{ab} h_b{}^c = \left(h^{-1T}\right)^a{}_b \epsilon^{bc}$, so $X'^a = \left(h^{-1T}\right)^a{}_b \epsilon^{bc} X_c = \left(h^{-1T}\right)^a{}_b X^b$.

Right-handed from left-handed fields: $X^{\dagger \dot{a}} = \left(\epsilon^{ab} X_b\right)^\dagger$.

So all fields can be expressed in terms of left-handed fields.

Scalar from 2 spinors: $XY = X^a Y_a = -Y_a X^a = Y^a X_a = YX$.

$X^{a\prime} Y'_a = \left(h^{-1T}\right)^a{}_c X^c h_a{}^b Y_b = \left(h^{-1}\right)_c{}^a h_a{}^b X^c Y_b$. X, Y anticommute (spinor operators).

Hermitian conjugate of scalar: $(XY)^\dagger = (X^a Y_a)^\dagger = (Y_a)^\dagger (X^a)^\dagger = Y^\dagger_{\dot{a}} X^{\dagger \dot{a}} = Y^\dagger X^\dagger$.

4-vector σ matrices $\sigma^{\mu}_{a\dot{b}}$: $\sigma^{i}_{a\dot{b}} = (\sigma_i)_{a\dot{b}}$, $\sigma^{0}_{a\dot{b}} = (\sigma_0)_{a\dot{b}}$ and $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

4-vector σ matrices with raised indices: $\overline{\sigma}^{\mu\ \dot{a}b} = \epsilon^{bc}\sigma^{\mu}_{c\dot{d}}\epsilon^{\dot{a}\dot{d}}$, so $\overline{\sigma}^{i\ \dot{a}b} = -(\sigma_i)^{\dot{a}b}$, $\overline{\sigma}^{0\ \dot{a}b} = (\sigma_0)^{\dot{a}b}$.

Second result follows from definition by explicit calculation.

Inner product of 4-vector σ matrices: $g_{\rho\omega}\sigma^{\omega}_{e\dot{f}}\overline{\sigma}^{\rho\ \dot{b}a} = -2\delta_{e}^{\ a}\delta_{\dot{f}}^{\ \dot{b}}$.

Outer product of 4-vector σ matrices: $\sigma^{\nu}_{a\dot{b}}\overline{\sigma}^{\rho\ \dot{b}a} = -2g^{\nu\rho}$.

$X^a\sigma^{\mu}_{a\dot{b}}Y^{\dagger\dot{b}}$ is vector.

Need to show $X'^a\sigma^{\mu}_{a\dot{b}}Y'^{\dagger\dot{b}} = \Lambda^{\mu}_{\ \nu}X^a\sigma^{\nu}_{a\dot{b}}Y^{\dagger\dot{b}}$. Since $X'^a\sigma^{\mu}_{a\dot{b}}Y'^{\dagger\dot{b}} = \left(h^{-1T}\right)^{a}_{\ c}X^c\sigma^{\mu}_{a\dot{b}}\left(h^{*-1T}\right)^{\dot{b}}_{\ \dot{d}}Y^{\dagger\dot{d}}$,

need to show $\left(h^{-1}\right)_{a}^{\ c}\sigma^{\mu}_{c\dot{d}}\left(h^{*-1T}\right)^{\dot{d}}_{\ \dot{b}} = \Lambda^{\mu}_{\ \nu}\sigma^{\nu}_{a\dot{b}}$. Contracting with $\overline{\sigma}^{\rho\ \dot{b}a}$ and using outer product of 4-vector σ matrices

gives $\Lambda^{\mu\rho} = -\frac{1}{2}\overline{\sigma}^{\rho\ \dot{b}a}\left(h^{-1}\right)_{a}^{\ c}\sigma^{\mu}_{c\dot{d}}\left(h^{*-1T}\right)^{\dot{d}}_{\ \dot{b}} = -\frac{1}{2}\mathrm{tr}\left[\overline{\sigma}^{\rho}h^{-1}\sigma^{\mu}h^{*-1T}\right]$, which is equivalent because σ matrices linearly independent

(or multiply this by $g_{\rho\omega}\sigma^{\omega}_{e\dot{f}}$ and use inner product of 4-vector σ matrices).

Verify last result using direct calculation of $h_{L/R}$ on page 37 and Lorentz transformation of 4-vectors on page 21.

Convenient to put left- and right-handed fields together as 4 component spinor:

$$\psi^T = \left(C_1 X_{\frac{1}{2},0}^{\left(\frac{1}{2},0\right)}, C_1 X_{-\frac{1}{2},0}^{\left(\frac{1}{2},0\right)}, C_2 Y_{0,\frac{1}{2}}^{\left(0,\frac{1}{2}\right)}, C_2 Y_{0,-\frac{1}{2}}^{\left(0,\frac{1}{2}\right)} \right)$$

, where C_1, C_2 scalar constants.

Lorentz group generators: $J_i = \begin{pmatrix} \frac{1}{2}\sigma_i & 0 \\ 0 & \frac{1}{2}\sigma_i \end{pmatrix}$ and $K_i = \begin{pmatrix} I\frac{1}{2}\sigma_i & 0 \\ 0 & -I\frac{1}{2}\sigma_i \end{pmatrix}$.

Follows from Lorentz group generators for spinors on page 37.

This is the *chiral* (*Weyl*) representation,

others representations from similarity transformation: $J_i' = VJ_iV^{-1}$, $K_i' = VK_iV^{-1}$, $X' = VX$ etc.

In our index notation, $\psi = \begin{pmatrix} X_a \\ Y^{\dagger\dot{b}} \end{pmatrix}$. Preferable to express in terms of left-handed fields only: $\psi = \begin{pmatrix} X_a \\ (\epsilon^{bc}Y_c)^\dagger \end{pmatrix}$.

2.4.2 Spin $\frac{1}{2}$ in general representations

Any 4×4 matrix can be constructed from sums/products of gamma matrices (next page):

Gamma matrices (chiral representation): $\gamma^0 = -I \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$ and $\gamma^i = -I \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$, or $\gamma^\mu = -I \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}$.

Gamma matrix construction of 4×4 matrix same in representations related by similarity transformation $\gamma'^\mu = V\gamma^\mu V^{-1}$.

Anticommutation relations for γ^μ: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. This is representation independent.

Check by explicit calculation in chiral representation.

Define $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -I\gamma^0\gamma^1\gamma^2\gamma^3$. Check last equality explicitly in chiral representation.

Then $P_L = \frac{1}{2}(1 + \gamma_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ projects out left-handed spinor: $P_L \begin{pmatrix} X_a \\ (\epsilon^{bc} Y_c)^\dagger \end{pmatrix} = \begin{pmatrix} X_a \\ 0 \end{pmatrix}$,

similarly $P_R = \frac{1}{2}(1 - \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ projects out right-handed spinor: $P_R \begin{pmatrix} X_a \\ (\epsilon^{bc} Y_c)^\dagger \end{pmatrix} = \begin{pmatrix} 0 \\ (\epsilon^{bc} Y_c)^\dagger \end{pmatrix}$.

Note $P_{L/R}$ are projection operators: $P_{L/R}^2 = P_{L/R}$, $P_{L/R} P_{R/L} = 0$.

Any 4×4 matrix is linear combination of $\mathbf{1}$, γ^μ, $[\gamma^\mu, \gamma^\nu]$, $\gamma^\mu\gamma_5$, γ_5.

Because these are 16 non-zero linearly independent 4×4 matrices.

Non-zero because their squares, calculated from anticommutation relations on page 43, are non-zero.

Linearly independent because they are orthogonal if we define scalar product of any two to be trace of their matrix product:

$\mathrm{tr}[\mathbf{1}\gamma^\mu] = 0$, because $\mathrm{tr}[\gamma^\mu] = 0$ in chiral representation, therefore in any other representation.

$\mathrm{tr}[\mathbf{1}[\gamma^\mu, \gamma^\nu]] = 0$ by (anti)symmetry. From anticommutation relations, $\mathrm{tr}[\mathbf{1}\gamma^\mu\gamma_5] = -\mathrm{tr}[\gamma_5\gamma^\mu] = -\mathrm{tr}[\gamma^\mu\gamma_5] = 0$

and also $\mathrm{tr}[\mathbf{1}\gamma_5] = \mathrm{tr}[\gamma_5] = 0$ by commuting γ^0 from left to right.

Next, $\mathrm{tr}[\gamma^\rho[\gamma^\mu, \gamma^\nu]] = 0$: From anticommutation relations, $\gamma_5^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^{0\,2}\gamma^{1\,2}\gamma^{2\,2}\gamma^{3\,2} = \mathbf{1}$.

So $\mathrm{tr}[\gamma^\rho[\gamma^\mu, \gamma^\nu]] = \mathrm{tr}[\gamma_5^2\gamma^\rho[\gamma^\mu, \gamma^\nu]] = -\ \mathrm{tr}[\gamma_5\gamma^\rho[\gamma^\mu, \gamma^\nu]\gamma_5] = \mathrm{tr}[\gamma_5\gamma^\rho[\gamma^\mu, \gamma^\nu]\gamma_5]$.

To show $\mathrm{tr}[\gamma^\rho\gamma^\mu\gamma_5] = 0$, first consider case $\rho = \mu$. Then $\gamma^\rho\gamma^\mu = g^{\rho\rho}$, and result is $\propto \mathrm{tr}[\gamma_5] = 0$.

If e.g. $\rho = 1$, $\mu = 2$, $\mathrm{tr}[\gamma^\rho\gamma^\mu\gamma_5] = I\mathrm{tr}[\gamma^0\gamma^3] = 0$. $\mathrm{tr}[\gamma^\mu\gamma_5] = 0$ already shown.

$\mathrm{tr}[[\gamma^\mu, \gamma^\nu]\gamma^\rho\gamma_5] = 0$ from anticommutation relations. $\mathrm{tr}[[\gamma^\mu, \gamma^\nu]\gamma_5] = 0$ because $\mathrm{tr}[\gamma^\rho\gamma^\mu\gamma_5] = 0$. Finally $\mathrm{tr}[\gamma^\mu\gamma_5\gamma_5] = \mathrm{tr}[\gamma^\mu] = 0$.

So all spinorial observables expressible as representation independent sums/products of gamma matrices.

Lorentz group generators from γ^μ: $J^{\mu\nu} = -\frac{I}{4}[\gamma^\mu, \gamma^\nu]$.

Check by explicit calculation in chiral representation using $J_i = \frac{1}{2}\epsilon_{ijk}J_{jk}$ and $\boldsymbol{K} = (J^{10}, J^{20}, J^{30})$.

Similarity transformation on page 42 equivalent to similarity transformation on page 43.

Infinitesimal Lorentz transformation of γ^μ: $I[J^{\mu\nu}, \gamma^\rho] = g^{\nu\rho}\gamma^\mu - g^{\mu\rho}\gamma^\nu$,

Follows from anticommutation relations for γ^μ on page 43 and Lorentz group generators from γ^μ.

Lorentz transformation of γ^μ: $D(\Lambda)\gamma^\mu D^{-1}(\Lambda) = \Lambda_\nu{}^\mu\gamma^\nu$, i.e. γ^μ transforms like a vector.

Agrees with infinitesimal Lorentz transformation of γ^μ, because for infinitesimal case,

LHS is $(1 + \frac{1}{2}I\omega_{\rho\sigma}J^{\rho\sigma})\gamma^\mu(1 - \frac{1}{2}I\omega_{\omega\eta}J^{\omega\eta}) = \gamma^\mu - \frac{1}{2}I\omega_{\rho\sigma}[\gamma^\mu, J^{\rho\sigma}]$, and RHS is $(\delta_\nu{}^\mu + \omega_{\rho\sigma}g^{\mu\sigma})\gamma^\rho = \gamma^\mu - \frac{1}{2}\omega_{\rho\sigma}(g^{\mu\rho}\gamma^\sigma - g^{\mu\sigma}\gamma^\rho)$.

General boost: $\gamma^\mu(\Lambda p)_\mu = D(\Lambda)\ \gamma^\mu p_\mu\ D^{-1}(\Lambda)$.

From Lorentz transformation of γ^μ.

Reference boost: $\frac{\gamma^\mu p_\mu}{m} = -D(L(p))\gamma^0 D^{-1}(L(p))$. Equivalent of $p = L(p)k$ on page 23, $-\gamma^0 m = \gamma^\mu k_\mu$.

In general boost, take $p = k$, $\Lambda = L(q)$. Then $\gamma^\mu q_\mu = D(L(q))\ \gamma^\mu k_\mu\ D^{-1}(L(q)) = -D(L(q))\ \gamma^0 m\ D^{-1}(L(q))$.

Parity transformation matrix: $\beta = I\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$. Note $\beta^2 = \mathbf{1}$.

Pseudo-unitarity of Lorentz transformation: $J^{\mu\nu\dagger} = \beta J^{\mu\nu}\beta$, $D^\dagger = \beta D^{-1}\beta$.

$\beta\gamma^0\beta = \gamma^0 = -\gamma^{0\dagger}$ and $\beta\gamma^i\beta = -\gamma^i = -\gamma^{i\dagger}$, or $\beta\gamma^\mu\beta = -\gamma^{\mu\dagger}$. Then use Lorentz group generators from γ^μ on page 45.

Infinitesimal $D^\dagger$ is $1 - \frac{1}{2}I\omega_{mu\nu}J^{\mu\nu\dagger} = \beta(1 - \frac{1}{2}I\omega_{mu\nu}J^{\mu\nu})\beta$.

Adjoint spinor: $\overline{X} = X^\dagger\beta$. Allows construction of scalars, vectors etc. from spinors:

Covariant products: $\overline{X}Y$ is scalar, $\overline{X}\gamma^\mu Y$ is vector.

First case: $\overline{X'}Y' = X'^\dagger\beta Y' = X^\dagger D^\dagger\beta DY$.

From pseudo-unitarity of Lorentz transformation, $D^\dagger\beta = \beta D^{-1}\beta^2 = \beta D^{-1}$, so $\overline{X'}Y' = X^\dagger\beta D^{-1}DY = X^\dagger\beta Y$.

Second case: $\overline{X'}\gamma^\mu Y' = X^\dagger D^\dagger\beta\gamma^\mu DY = X^\dagger\beta D^{-1}\gamma^\mu DY$ as before.

Lorentz transformation of γ^μ on page 45 can be rewritten $D^{-1}\gamma^\mu D = \left(\Lambda^{-1}\right)_\nu^{\ \mu}\gamma^\nu = \Lambda^\mu{}_\nu\gamma^\nu$, so $\overline{X'}\gamma^\mu Y' = \Lambda^\mu{}_\nu X^\dagger\beta\gamma^\nu Y$.

Vanishing products: $\overline{\psi}_{R/L}\psi_{L/R}$ and $\overline{\psi}_{L/R}\gamma^\mu\psi_{L/R}$, where $\psi_{L/R} = P_{L/R}\psi$ (see page 43 for $P_{L/R}$).

Using $\gamma_5^\dagger = \gamma_5$ in chiral representation and $\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu$, $\overline{\psi}_R\psi_L = \psi^\dagger P_L\beta P_L\psi = \psi^\dagger\beta P_R P_L\psi = 0$.

Similarly, $\overline{\psi}_L\gamma^\mu\psi_L = \psi^\dagger P_R\beta\gamma^\mu P_L\psi = \psi^\dagger\beta\gamma^\mu P_R P_L\psi = 0$.

2.4.3 The Dirac field

Group 4 possibilities for $u_{ab\ \sigma}^{(A,B)}$ together as 4 component spinor: $u_\sigma^T = \left(u_{\frac{1}{2},0,\sigma}^{(\frac{1}{2},0)}, u_{-\frac{1}{2},0,\sigma}^{(\frac{1}{2},0)}, u_{0,\frac{1}{2},\sigma}^{(0,\frac{1}{2})}, u_{0,-\frac{1}{2},\sigma}^{(0,\frac{1}{2})} \right)$.

Likewise, $v_\sigma^T = \left(-v_{\frac{1}{2},0,\sigma}^{(\frac{1}{2},0)}, -v_{-\frac{1}{2},0,\sigma}^{(\frac{1}{2},0)}, v_{0,\frac{1}{2},\sigma}^{(0,\frac{1}{2})}, v_{0,-\frac{1}{2},\sigma}^{(0,\frac{1}{2})} \right)$.

First 2 v components multiplied by $(-1)^{2A} = -1$ to remove it from massive irreducible field on page 35.

Dirac field: $\psi(x) = \begin{pmatrix} X_a \\ (\epsilon^{bc} Y_c)^\dagger \end{pmatrix} = \int \frac{d^3p}{(2\pi)^3} \left[e^{Ip\cdot x} u_\sigma(\boldsymbol{p}) a_\sigma(\boldsymbol{p}) + e^{-Ip\cdot x} v_\sigma(\boldsymbol{p}) a_\sigma^{c\dagger}(\boldsymbol{p}) \right]$.

(Note $D^3p \to d^3p$ for convention.)

Anticommutation relations for spin $\frac{1}{2}$: $[a_\sigma(\boldsymbol{p}), a_{\sigma'}^\dagger(\boldsymbol{p}')]_+ = (2\pi)^3 \delta_{\sigma\sigma'} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')$ (i.e. no $2p^0$ factor).

***p* dependence of spin $\frac{1}{2}$** u, v: $u_\sigma(\boldsymbol{p}) = \sqrt{\frac{m}{p^0}} D(L(p)) u_\sigma(0)$ and $v_\sigma(\boldsymbol{p}) = \sqrt{\frac{m}{p^0}} D(L(p)) v_\sigma(0)$.

This is just the $\boldsymbol{p}$ dependence of u, v on page 25.

Condition on spin $\frac{1}{2}$ u, v: $-\frac{1}{2}\sigma^*_{i\ \sigma'\sigma} v^{(0,\frac{1}{2})}_{0b\ \sigma'}(0) = \frac{1}{2}\sigma_{i\ bb'} v^{(0,\frac{1}{2})}_{0b'\ \sigma}(0)$, $\frac{1}{2}\sigma_{i\ \sigma'\sigma} u^{(0,\frac{1}{2})}_{0b\ \sigma'}(0) = \frac{1}{2}\sigma_{i\ bb'} u^{(0,\frac{1}{2})}_{0b'\ \sigma}(0)$.

From conditions on u, v on page 31.

Spin $\frac{1}{2}$ u, v relation: $v_{1\text{ or }2\ \sigma}(0) = -(-1)^{\frac{1}{2}+\sigma} u_{1\text{ or }2\ -\sigma}(0)$ and $v_{3\text{ or }4\ \sigma}(0) = (-1)^{\frac{1}{2}+\sigma} u_{3\text{ or }4\ -\sigma}(0)$.

From u, v relation on page 31, $v^{(\frac{1}{2},0)}_{a0\ \sigma}(0) = (-1)^\sigma u^{(\frac{1}{2},0)}_{a0\ -\sigma}(0)$ and $v^{(0,\frac{1}{2})}_{0b\ \sigma}(0) = (-1)^\sigma u^{(0,\frac{1}{2})}_{0b\ -\sigma}(0)$.

Then multiply $v^{(\frac{1}{2},0)}_{a0\ \sigma}(0)$ by $(-1)^{2A} = -1$ as discussed on page 47.

Form of u, v: $u^T_{\sigma=\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(1\ 0\ 1\ 0)$, $u^T_{\sigma=-\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(0\ 1\ 0\ 1)$, $v^T_{\sigma=\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(0\ 1\ 0\ -1)$, $v^T_{\sigma=-\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(-1\ 0\ 1\ 0)$.

Solution to condition on spin $\frac{1}{2}$ v is $v^{(0,\frac{1}{2})}_{0,-\frac{1}{2},\frac{1}{2}} = -v^{(0,\frac{1}{2})}_{0,\frac{1}{2},-\frac{1}{2}}$ and $v^{(0,\frac{1}{2})}_{0,\frac{1}{2},\frac{1}{2}} = v^{(0,\frac{1}{2})}_{0,-\frac{1}{2},-\frac{1}{2}} = 0$. Components for u constrained similarly.

Use spin $\frac{1}{2}$ u, v relation, and adjust normalizations of $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$ parts individually.

Massless u, v: $u^T_{\sigma=\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(0\ 0\ 1\ 0)$, $u^T_{\sigma=-\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(0\ 1\ 0\ 0)$, $v^T_{\sigma=\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(0\ 1\ 0\ 0)$, $v^T_{\sigma=-\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}(0\ 0\ 1\ 0)$.

From allowed helicities for fields in given (A,B) representation on page 33, e.g. for $\left(\frac{1}{2},0\right)$ field such as neutrino, only allowed $\sigma = 0 - \frac{1}{2}$ for particle and $\sigma = \frac{1}{2} - 0$ for antiparticle, i.e. $\psi_L(x) = \int \frac{d^3p}{(2\pi)^3}\left[e^{Ip\cdot x} u^{(\frac{1}{2},0)}_{-\frac{1}{2}}(\boldsymbol{p}) a_{-\frac{1}{2}}(\boldsymbol{p}) + e^{-Ip\cdot x} v^{(\frac{1}{2},0)}_{\frac{1}{2}}(\boldsymbol{p}) a^{c\dagger}_{\frac{1}{2}}(\boldsymbol{p})\right]$.

Majorana particle = antiparticle: $a^{c\dagger}_\sigma(\boldsymbol{p}) = a^\dagger_\sigma(\boldsymbol{p})$, so $\psi^T_M = \left(X_a, X^{\dagger\dot{b}}\right)$ (i.e. $Y_a = X_a$ on page 47).

2.4.4 The Dirac equation

Representation independent definition of spin $\frac{1}{2}$ u, v: $(I\gamma^\mu p_\mu + m)u_\sigma(\boldsymbol{p}) = 0$ and $(-I\gamma^\mu p_\mu + m)v_\sigma(\boldsymbol{p}) = 0$.

For u and v, reference boost gives $-I\frac{\gamma^\mu p_\mu}{m}u_\sigma(\boldsymbol{p}) = D(L(p))\beta D^{-1}(L(p))u_\sigma(\boldsymbol{p}) = \sqrt{\frac{m}{p^0}}D(L(p))\beta u_\sigma(0)$,

last step from $\boldsymbol{p}$ dependence of spin $\frac{1}{2}$ u, v on page 47.

In the chiral representation, and therefore any other representation, $\beta u_\sigma(0) = u_\sigma(0)$ and $\beta v_\sigma(0) = -v_\sigma(0)$,

so $-I\frac{\gamma^\mu p_\mu}{m}u_\sigma(\boldsymbol{p}) = \sqrt{\frac{m}{p^0}}D(L(p))u_\sigma(0) = u_\sigma(\boldsymbol{p})$, last step from $\boldsymbol{p}$ dependence of spin $\frac{1}{2}$ u, v again, likewise $-I\frac{\gamma^\mu p_\mu}{m}v_\sigma(\boldsymbol{p}) = -v_\sigma(\boldsymbol{p})$.

Dirac equation: $(\gamma^\mu \partial_\mu + m)\,\psi_l^{\pm(c)}(x) = 0$.

Act on Dirac field on page 47 with $(\gamma^\mu\partial_\mu + m)$, then use representation independent definition of spin $\frac{1}{2}$ u, v.

Consistent with Klein-Gordon equation $\left(\partial^2 - m^2\right)\psi^{\pm(c)}(x) = 0$.

From page 25. To check, act on Dirac equation from left with $(\gamma^\nu\partial_\nu - m)$: $0 = (\gamma^\nu\partial_\nu - m)\left(\gamma^\mu\partial_\mu + m\right)\psi^{\pm(c)}$

$= \left(\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu - m^2\right)\psi^{\pm(c)} = \left(\frac{1}{2}\{\gamma^\nu,\gamma^\mu\}\partial_\nu\partial_\mu - m^2\right)\psi^{\pm(c)} = \left(g^{\mu\nu}\partial_\nu\partial_\mu - m^2\right)\psi^{\pm(c)}$

using anticommutation relations for γ^μ on page 43.

2.4.5 Dirac field equal time anticommutation relations

Projection operators from u, v: $u_{l\ \sigma}(\boldsymbol{p})\overline{u}_{l'\ \sigma}(\boldsymbol{p}) = \frac{1}{2p^0}\left(-I\gamma^\mu p_\mu + m\right)_{ll'}$, $v_{l\ \sigma}(\boldsymbol{p})\overline{v}_{l'\ \sigma}(\boldsymbol{p}) = \frac{1}{2p^0}\left(-I\gamma^\mu p_\mu - m\right)_{ll'}$.

Define $N_{ll'}(p) = u_{l\ \sigma}(\boldsymbol{p})\ \overline{u}_{l'\ \sigma}(\boldsymbol{p}) = \frac{m}{p^0}[D(L(p))u_\sigma(0)]_l\ [u^\dagger_\sigma(0)D^\dagger(L(p))\beta]_{l'}$ from $\boldsymbol{p}$ dependence of spin $\frac{1}{2}$ u, v on page 47.

so $N(p) = \frac{m}{p^0}D(L(p))N(0)D^{-1}(L(p))$.

Explicit calculation from the form of u, v on page 48 gives $N(0) = \frac{1}{2}(\beta + \mathbf{1})$ which is true in any representation.

So $N(p) = \frac{1}{2p^0}D(L(p))\left(I\gamma^0 m + m\right)D^{-1}(L(p))$, then use reference boost on page 45.

Equal time anticommutation relations: $[\psi_l(\boldsymbol{x},t), \psi^\dagger_{l'}(\boldsymbol{y},t)]_+ = \delta_{ll'}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$.

Define $R_{ll'} = [\psi_l(\boldsymbol{x},t), \psi^\dagger_{l'}(\boldsymbol{y},t)]_+ = \int \frac{d^3p}{(2\pi)^3}e^{I\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\left[u_{l\ \sigma}(\boldsymbol{p})[\overline{u}(\boldsymbol{p})\beta]_{l'\ \sigma} + v_{l\ \sigma}(-\boldsymbol{p})[\overline{v}(-\boldsymbol{p})\beta]_{l'\ \sigma}\right]$,

using Dirac field and anticommutation relations for spin $\frac{1}{2}$ on page 47. Then in "v" term, take $\boldsymbol{p} \to -\boldsymbol{p}$.

From projection operators from u, v, $R = \int \frac{d^3p}{(2\pi)^3 2p^0}e^{I\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\left[\left(I\gamma^0 p^0 - I\boldsymbol{\gamma}\cdot\boldsymbol{p} + m\right) + \left(I\gamma^0 p^0 + I\boldsymbol{\gamma}\cdot\boldsymbol{p} - m\right)\right]\beta = \mathbf{1}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$.

2.5 External symmetries: bosons

Scalar boson field: $\psi(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}(2p^0)^{\frac{1}{2}}} \left[e^{Ip\cdot x}a(\boldsymbol{p}) + e^{-Ip\cdot x}a^{c\dagger}(\boldsymbol{p})\right]$.

From general form of irreducible field on page 35, where $u(\boldsymbol{p}) = u(0)$ and $v(\boldsymbol{p}) = u(\boldsymbol{p})$, absorb overall $u(0)$ into field.

Vector boson field $(\frac{1}{2}, \frac{1}{2})$, **spin** 1: $\psi^\mu(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}(2p^0)^{\frac{1}{2}}} \left[e^{Ip\cdot x}u^\mu_\sigma(\boldsymbol{p})a_\sigma(\boldsymbol{p}) + e^{-Ip\cdot x}v^\mu_\sigma(\boldsymbol{p})a^{c\dagger}_\sigma(\boldsymbol{p})\right]$, $u^0_\sigma(0) = v^0_\sigma(0) = 0$.

Transformation of u, v on page 25 for $\Lambda = R$ gives e.g. $(J^{(j)}_k J^{(j)}_k)_{\sigma'\sigma} u^\mu_{\sigma'} = (J_k J_k)^\mu{}_\nu u^\nu_\sigma$.

But $(J^{(j)}_k J^{(j)}_k)_{\sigma'\sigma} = j(j+1)\delta_{\sigma'\sigma}$ and $(J_k J_k)^i{}_j = 2\delta^i{}_j$, $(J_k J_k)^0{}_\mu = 0$ from Lorentz transformation of 4-vectors on page 21,

so $j(j+1)u^i_\sigma(0) = 2u^i_\sigma(0)$, $j(j+1)u^0_\sigma(0) = 0$, i.e. $j = 0$ and $u^i_\sigma(0) = 0$ or $j = 1$ and $u^0_\sigma(0) = 0$.

Projection operator for vector boson $(\frac{1}{2}, \frac{1}{2})$, **spin** 1: $u^\mu_\sigma(\boldsymbol{p})u^{\nu *}_\sigma(\boldsymbol{p}) = v^\mu_\sigma(\boldsymbol{p})v^{\nu *}_\sigma(\boldsymbol{p}) = \frac{1}{2p^0}\left(g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\right)$.

Derivation similar to that for projection operators from u, v on page 50.

Projection operator shows there is problem for $m \to 0$. From allowed helicities for fields in given (A, B) representation on page 33, can't construct $(\frac{1}{2}, \frac{1}{2})$ 4-vector field, where helicity $\sigma = 0$, from massless helicity $\sigma = \pm 1$ particle. But can construct 4-component field:

Massless helicity ± 1 field: $A^\mu(x) = \int D^3p \left[e^{Ip\cdot x} u^\mu_\sigma(\boldsymbol{p}) a_\sigma(\boldsymbol{p}) + e^{-Ip\cdot x} v^\mu_\sigma(\boldsymbol{p}) a^{c\dagger}_\sigma(\boldsymbol{p})\right]$, where $\sigma = \pm 1$.

Lorentz transformation of massless helicity ± 1 polarization vector: $e^{-I\theta\sigma} u_\sigma(\boldsymbol{p}) = \Lambda^{-1} u_\sigma(\boldsymbol{\Lambda p}) + \omega(W, \boldsymbol{k}) p$.

Simplest approach: take p dependence and rotation of u on page 33 to be true, gives $u^\mu_\sigma(\boldsymbol{k}) \propto (1, \sigma, 0, 0)$.

Then M, N transformation on page 33 cannot be true, in fact $M u_\sigma \propto (0, 0, 1, 1) \propto k$ and likewise for N.

Then $D(W) u_\sigma(\boldsymbol{k}) = e^{I\theta\sigma}[u_\sigma(\boldsymbol{k}) + \omega(W, \boldsymbol{k}) k]$. Since $D(\Lambda) = \Lambda$, multiplying this from the left by $e^{-I\theta\sigma} \Lambda^{-1} D(\Lambda L(p))) D(W^{-1}) = e^{-I\theta\sigma} \Lambda^{-1} L(\Lambda p)$ gives result.

Lorentz transformation of massless helicity ± 1 field: $U(\Lambda) A^\mu(x) U^\dagger(\Lambda) = \Lambda_\nu{}^\mu A^\nu(\Lambda x) + \partial^\mu \alpha(x)$.

Use Lorentz transformation of $u_{\mu\ \sigma}$ above in $U(\Lambda) A^\mu(x) U^\dagger(\Lambda) = \int D^3p \left[e^{Ip\cdot x} u^\mu_\sigma(\boldsymbol{p}) e^{-I\theta(\Lambda,p)\sigma} a_\sigma(\boldsymbol{\Lambda p}) + e^{-Ip\cdot x} v^\mu_\sigma(\boldsymbol{p}) e^{I\theta(\Lambda,p)\sigma} a^{c\dagger}_\sigma(\boldsymbol{\Lambda p})\right]$.

As for Poincaré transformation for fields on page 24, up to gauge transformation.

This implies $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is Lorentz covariant, as expected:

it is antisymmetric, i.e. is $(1, 0)$ or $(0, 1)$ if $F^{\mu\nu} = \pm \frac{I}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$, so from $\sigma = \pm(A - B)$, can have $\sigma = \pm 1$.

Brehmstrahlung: Adding emission of massless helicity $\pm j$ boson with momentum $q \simeq 0$

to process with particles n with momenta p_n modifies amplitude by factor $\propto u_{\mu_1\mu_2\ldots\mu_j\ \sigma}(\mathbf{q}) \sum_n \eta_n \frac{g_n p_n^{\mu_1} p_n^{\mu_2} \ldots p_n^{\mu_j}}{p \cdot q}$,

where g_n is coupling of boson to fermion n and $\eta_n = \pm 1$ for outgoing / incoming particles.

Lorentz invariance condition: $q_{\mu_1} \sum_n \eta_n \frac{g_n p_n^{\mu_1} p_n^{\mu_2} \ldots p_n^{\mu_j}}{p_n \cdot q} = 0$.

Can show this for massless helicity ± 1 boson: Lorentz transformation of polarization vector on page 52

implies amplitude not Lorentz invariant unless this is true.

Einstein's principle of equivalence: Helicity ± 2 bosons have identical coupling to all fermions.

For soft emssion of graviton from process involving multiple fermions of momentum p_n,

Lorentz invariance condition reads $\sum_n \eta_n g_n p_n^{\mu_2} = 0$. But momentum conservation is $\sum_n \eta_n p_n^{\mu_2} = 0$, so g_n same for all particles.

Constraint on particle spins: Massless particles must have helicity ≤ 2 and ≥ -2.

Lorentz invariance condition can be written $\sum_n \eta_n g_n p_n^{\mu_2} \ldots p_n^{\mu_j} = 0$.

For $j > 2$ this overconstrains $2 \to 2$ processes, since momentum conservation alone $\implies$ it depends on scattering angle only.

2.6 The Lagrangian Formalism

2.6.1 Generic quantum mechanics

Lagrangian formalism is natural framework for QM implementation of symmetry principles.

Can be applied to *canonical* fields (e.g. Standard Model):

Fields $\psi_l(\boldsymbol{x}, t)$ behave as canonical coordinates, i.e. with conjugate momenta $p_l(\boldsymbol{x}, t)$ such that

$[\psi_l(\boldsymbol{x}, t), p_{l'}(\boldsymbol{y}, t)]_\mp = I\delta^3(\boldsymbol{x} - \boldsymbol{y})\delta_{ll'}$ and $[\psi_l(\boldsymbol{x}, t), \psi_{l'}(\boldsymbol{y}, t)]_\mp = [p_l(\boldsymbol{x}, t), p_{l'}(\boldsymbol{y}, t)]_\mp = 0$ as usual in QM.

In practice, find suitable $p_l(\boldsymbol{x}, t)$ by explicit calculation of $[\psi_l(x), \psi^\dagger_{l'}(y)]_\mp$.

Lagrangian formalism: Action $\mathscr{A}[\psi, \dot{\psi}] = \int_{-\infty}^{\infty} dt L[\psi(t), \dot{\psi}(t)]$ is stationary,

where Lagrangian $L[\psi(t), \dot{\psi}(t)] = \int d^3x\ p_l(\boldsymbol{x}, t)\dot{\psi}_l(\boldsymbol{x}, t) - H[\psi(t), p(t)]$.

Coordinates obey field equations $\dot{\psi}_l = \frac{\delta H}{\delta p_l}$ when $\mathscr{A}$ is stationary.

2.6.2 Relativistic quantum mechanics

If $L[\psi(t), \dot{\psi}(t)] = \int d^3x \mathscr{L}(\psi(x), \partial_\mu \psi(x))$,

Lagrangian density $\mathscr{L}(x)$ is scalar and $\mathscr{A} = \int d^4x \mathscr{L}(x)$ is Lorentz invariant.

In practice, determine $\mathscr{L}$ from classical field theory, e.g. electrodynamics,

then $\mathscr{L}(\psi(x), \partial_\mu \psi(x)) = p_l(x)\dot{\psi}_l(x) - \mathscr{H}(\psi(x), p(x))$ and p_l from $\mathscr{L}$: $p_l(x) = \frac{\partial \mathscr{L}(x)}{\partial \dot{\psi}_l(x)}$.

Stationary $\mathscr{A}$ requirement gives field equations $\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi_l)} = \frac{\partial \mathscr{L}}{\partial \psi_l}$ (*Euler-Lagrange equations*),

e.g. Klein-Gordon equation for free spin 0 field, Dirac equation for free spin $\frac{1}{2}$ field etc.

(some definition of derivative with respect to operator field ψ_l must be given here).

$\mathscr{A}$ must be real. Let $\mathscr{A}$ depend on N real fields. Stationary real and imaginary parts of $\mathscr{A} \to 2N$ field equations.

Noether's theorem: Symmetries imply conservation:

$\mathscr{A}$ invariant under $\psi_l(x) \to \psi_l(x) + I\alpha\mathscr{F}_l[\psi; x] \to$ conserved current $J^\mu(x)$, $\partial_\mu J^\mu = 0$, for stationary $\mathscr{A}$.

If α made dependent on x, $\mathscr{A}$ no longer invariant. But change must be $\delta\mathscr{A} = \int d^4x J^\mu \partial_\mu \alpha(x) = -\int d^4x \partial_\mu J^\mu \alpha(x)$

so that $\delta\mathscr{A} = 0$ when α constant. Now take $\mathscr{A}$ stationary: $\delta\mathscr{A} = 0$ even though α depends on x, so $\partial_\mu J^\mu = 0$.

Scalar boson $(0,0)$: $\mathscr{L}_{\text{scalar}} = -\frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}m^2\psi^2$.

Scalar boson field on page 51 implies $\left[\psi(\boldsymbol{x},t),\dot{\psi}(\boldsymbol{y},t)\right]_- = \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$, i.e. $p=\dot{\psi}$, consistent with p_l from $\mathscr{L}$ on page 55.

Field equations on page 55 give Klein-Gordon equation $\left(\partial^2 - m^2\right)\psi = 0$ as required. Note ψ is a single operator.

Dirac fermion $(\frac{1}{2},0)+(0,\frac{1}{2})$: $\mathscr{L}_{\text{Dirac}} = -\overline{\psi}\left(\gamma^\mu\partial_\mu + m\right)\psi$.

Recall ψ is a column of 4 operators, and covariant quantities on page 46.

Recall equal time anticommutation relations on page 50, $[\psi_l(\boldsymbol{x},t),\psi^\dagger_{l'}(\boldsymbol{x},t)]_+ = \delta_{ll'}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$, so $p=\psi^\dagger$,

consistent with p_l from $\mathscr{L}$ on page 55. Field equations give Dirac equation $\left(\gamma^\mu\partial_\mu + m\right)\psi = 0$ as required.

Vector boson $(\frac{1}{2},\frac{1}{2})$, **spin** 1: $\mathscr{L}_{\text{spin 1 vector}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2\psi_\mu\psi^\mu$, where $F_{\mu\nu} = \partial_\mu\psi_\nu - \partial_\nu\psi_\mu$.

From vector boson field and projection operator for vector boson $(\frac{1}{2},\frac{1}{2})$, spin 1, on page 51,

$\left[\psi_i(\boldsymbol{x},t),\dot{\psi}_j(\boldsymbol{y},t)+\partial_j\psi^0(\boldsymbol{y},t)\right]_- = \delta_{ij}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$, i.e. conjugate momentum to ψ_i is $p_i = \dot{\psi}_i + \partial_i\psi^0 = F^{i0}$,

consistent with p_l from $\mathscr{L}$ on page 55. ψ_0 is auxilliary field because $p_0 = 0$.

Also $\partial_\mu\psi^\mu = 0$ and Klein-Gordon equation $\left(\partial^2 - m^2\right)\psi^\mu = 0$, which is found from field equations on page 55.

2.7 Path-Integral Methods

Follows from Lagrangian formalism. Assume H is quadratic in the p_l.

Gives direct route from Lagrangian to calculations, all symmetries manifestly preserved along the way.

Can work in simpler classical limit then return to QM later.

Result is that bosons described by ordinary numbers, fermions by Grassmann variables.

LSZ reduction gives S-matrix from vacuum matrix elements of time ordered product of functions of fields,

given by path integral as $\frac{\langle 0,\ \mathrm{out}|T\left\{\psi_{l_A}(x_A),\psi_{l_B}(x_B),\ldots\right\}|0,\ \mathrm{in}\rangle}{\langle 0,\ \mathrm{out}|0,\ \mathrm{in}\rangle} = \frac{\int\prod_{x,l} d\psi_l(x)\psi_{l_A}(x_A)\psi_{l_B}(x_B)\ldots e^{I\mathscr{A}[\psi]}}{\int\prod_{x,l} d\psi_l(x)e^{I\mathscr{A}[\psi]}}$.

Contribution mostly from field configurations for which $\mathscr{A}$ is minimal, i.e. fluctuation around classical result.

Noether's theorem again (see page 55): $\mathscr{A}$ invariant under $\psi_l(x) \to \psi_l(x) = \psi_l(x) + I\alpha\mathscr{F}_l[\psi;x]$.

So if α dependent on x, $\int\prod_{x,l} d\psi_l(x)\exp[I\mathscr{A}] \to \int\prod_{x,l} d\psi_l(x)\exp\left[I\left(\mathscr{A} - \int d^4x\partial_\mu J^\mu(x)\alpha(x)\right)\right]$

assuming measure $\prod_{x,l} d\psi_l(x)$ invariant. This is just change of variables, so $\langle\partial_\mu J^\mu(x)\rangle = 0$.

2.8 Internal symmetries

Consider unitary group representations.

Unitary U(N): elements can be represented by $N \times N$ unitary matrices U ($U^\dagger U = \mathbf{1}$).

Dimension d(U(N))$= N^2$.

$2N^2$ degrees of freedom in complex $N \times N$ matrix, $U^\dagger U = \mathbf{1}$ is N^2 conditions

or N^2 Hermitian $N \times N$ matrices: N diagonal reals, $N^2 - N$ off-diagonal complexes but lower half conjugate to upper.

Special unitary SU(N): same as U(N) but U's have unit determinant ($\det(U) = 1$).

Thus $\mathrm{tr}[t_i] = 0$, i.e. group is semi-simple.

d(SU(N))$= N^2 - 1$.

Fundamental representation denoted $\boldsymbol{N}$.

Normalization of fundamental representation: $\mathrm{tr}[t_i t_j] = \frac{1}{2}\delta_{ij}$ (i.e. $C(\boldsymbol{N}) = \frac{1}{2}$).

U(1) (Abelian group): elements can be represented by phase $e^{Iq\alpha}$. One generator: the real number q.

SU(2): Fundamental representation denoted **2**, spin $\frac{1}{2}$ representation of rotation group.

SU(2) is actually the *universal covering group* of rotation group. 3 generators $t_i = \frac{\sigma_i}{2}$, $[t_i, t_j] = I\epsilon_{ijk}t_k$.

Adjoint representation denoted **3**. $C(\mathbf{3}) = 2$.

$$\epsilon_{jki}\epsilon_{lki} = (d(\mathrm{SU}(2)) - 1)\delta_{jl} = 2\delta_{jl}.$$

2 representation is real, $\mathbf{2} = \overline{\mathbf{2}}$ (i.e. $-\frac{\sigma_i^*}{2} = U\frac{\sigma_i}{2}U^\dagger$, pseudoreal), and $\mathfrak{g}_{\alpha\beta} = \epsilon_{\alpha\beta}$ and $\delta_{\alpha\beta}$.

SU(3): 8 generators $\frac{\lambda_i}{2}$, structure constants f_{ijk}.

Fundamental representation **3**: λ_i are 3×3 *Gell-Mann matrices*. Adjoint representation **8**. $C(\mathbf{8}) = 3$.

Group is complex, $\mathbf{3} \neq \overline{\mathbf{3}}$.

Example: $\mathbf{3} \times \overline{\mathbf{3}} = \mathbf{8} + \mathbf{1}$, i.e. quark and antiquark can be combined to behave like gluon or colour singlet.

2.8.1 Abelian gauge invariance

Global gauge invariance: Consider complex fermion / boson field $\psi_l(x)$, arbitrary spin.

Each operator in $\mathscr{L}_{\text{free}}$ is product of $(\partial_\mu)\psi_l$ with $(\partial_\mu)\psi_l^\dagger$,

invariant under U(1) transformation $\psi_l \to e^{Iq\alpha}\psi_l$ (whence $\partial_\mu\psi_l \to e^{Iq\alpha}\partial_\mu\psi_l$)

if α independent of spacetime coords. q are U(1) generators, or *charges*.

Local gauge invariance: Find $\mathscr{L}$ invariant when $\alpha = \alpha(x)$. Leads to renormalizable interacting theory.

In $\mathscr{L}_{\text{free}}$, $\partial_\mu\psi_l \to \partial_\mu e^{Iq\alpha}\psi_l = e^{iq\alpha}\left[\partial_\mu\psi_l + \underline{Iq(\partial_\mu\alpha)\psi_l}\right] \neq e^{Iq\alpha}\partial_\mu\psi_l$, $\therefore$ replace ∂_μ by 4-vector "derivative" D_μ,

such that D_μ gauge transformation $D_\mu\psi_l \to e^{Iq\alpha}D_\mu\psi_l$. Simplest choice: $D_\mu - \partial_\mu$ is 4-component field:

Covariant derivative: $D_\mu = \partial_\mu - IqA_\mu(x)$.

Gauge transformation: $A_\mu \to A_\mu + \partial_\mu\alpha$ whenever $\psi_l \to e^{Iq\alpha}\psi_l$.

Write transformation as $A_\mu \to A'_\mu(\alpha)$. Require $D_\mu\psi_l \to D'_\mu e^{iq\alpha}\psi_l = e^{iq\alpha}D_\mu\psi_l$,

i.e. $e^{Iq\alpha}\left[\partial_\mu\psi_l - Iq(\partial_\mu\alpha)\psi_l - IqA'_\mu\psi_l\right] = e^{Iq\alpha}\left[\partial_\mu\psi_l - IqA_\mu\psi_l\right]$, so $A'_\mu = A_\mu + \partial_\mu\alpha$.

Use D_ν to find invariant (free) Lagrangian for A_μ, quadratic in $(\partial_\nu)A_\mu$:

From D_μ gauge transformation, $D_\mu D_\nu \ldots \psi_l \to e^{Iq\alpha} D_\mu D_\nu \ldots \psi_l$. Products $D_\mu D_\nu \ldots$ contain spurious ∂_ρs, but

$F_{\mu\nu}$ **from** D_μ: $qF_{\mu\nu} = I\,[D_\mu, D_\nu]$, where electromagnetic field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

$$[D_\mu, D_\nu]\,\psi_l = \big(\underbrace{[\partial_\mu, \partial_\nu]}_{=0} + Iq([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu]) - q^2\underbrace{[A_\mu, A_\nu]}_{=0}\big)\psi_l.$$

$F_{\mu\nu}$ is gauge invariant.

$F_{\mu\nu}\psi_l \to F'_{\mu\nu} e^{Iq\alpha}\psi_l = e^{Iq\alpha} F_{\mu\nu}\psi_l$, i.e. $F'_{\mu\nu} = F_{\mu\nu}$, or $F_{\mu\nu} \to F_{\mu\nu}$. Also check explicitly from $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Conversely, choose A^μ to be massless helicity ± 1 field, whose Lorentz transformation on page 52 implies Lorentz invariant free Lagrangian for A^μ must be gauge invariant.

$F_{\mu\nu}$ in representation of U(1): Since $F_{\mu\nu} \to F_{\mu\nu}$, $F_{\mu\nu}$ transforms in adjoint representation of U(1).

Example: QED Lagrangian for fermions: $\mathscr{L}_{\text{Dirac, QED}} = \underbrace{-\overline{\psi}\,(\gamma^\mu D_\mu + m)\,\psi}_{\mathscr{L}_{\text{Dirac, free}} + Iq\overline{\psi}\gamma^\mu A_\mu\psi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

Interactions due to $Iq\overline{\psi}\gamma^\mu A_\mu\psi$. Most general Lagrangian locally gauge invariant under U(1) ($\psi \to e^{Iq\alpha}\psi$, $A_\mu \to A_\mu + \partial_\mu\alpha$), assuming P, T invariance and no mass dimension > 4 terms (Wilson: no contribution).

2.8.2 Non-Abelian gauge invariance

Global gauge invariance: Each term in $\mathscr{L}_{\text{free}}$ proportional to $(\partial^\mu)\psi_{l\ \gamma}(\partial_\mu)\psi^\dagger_{l'\ \gamma}$, $\gamma = 1, \ldots, N$.

Then $\mathscr{L}_{\text{free}}$ invariant under $\psi_{l\ \gamma} \to U_{\gamma\delta}\psi_{l\ \delta}$, where $U = \exp[I\alpha_i t_i]$, α_i spacetime independent.

So ψ_γ is in fundamental representation of group G =SU(N) formed by matrices $U_{\gamma\delta}$, $i = 1, \ldots, d(G)$.

Local gauge invariance: Spacetime derivatives in Lagrangian appear as

Covariant derivative: $D_\mu = \partial_\mu - IA_\mu(x)$ with $A^\mu = A^\mu_i t_i$,

t_i contain couplings, A^μ_i for $i = 1, \ldots, d(G)$ are (for) massless helicity ± 1 gauge fields.

To achieve $D_\mu\psi \to UD_\mu\psi$, require

Transformation of covariant derivative: $D_\mu \to UD_\mu U^\dagger$, which requires

Transformation of gauge fields: $A_\mu \to UA_\mu U^\dagger - I\,(\partial_\mu U)\,U^\dagger$.

Non-Abelian field strength: $T_i F^i_{\mu\nu} = F_{\mu\nu} = I[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - I[A_\mu, A_\nu]$.

$F_{\mu\nu}$ in adjoint representation: $F_{\mu\nu} \to U F_{\mu\nu} U^\dagger$.

In infinitesimal case, $F_i t_{i\ \alpha\beta} \to F^i \left[(1 + I\alpha_k t_k) t_i (1 - I\alpha_k t_k)\right]_{\alpha\beta} = F^i \left[t_i + I\alpha_k \left[t_k, t_i\right]\right]_{\alpha\beta}$

$= F^i \left[t_i + I\alpha_k (IC_{kij}) t_j\right]_{\alpha\beta} = F^i \left[t_{i\ \alpha\beta} + I\alpha_k (t^k_{ji}) t_{j\ \alpha\beta}\right] = F^i t_{j\ \alpha\beta} \left[\delta_{ji} + I\alpha_k (t^k_{ji})\right] = F^i t_{j\ \alpha\beta} U_{ji} = U_{ij} F_j t_{i\ \alpha\beta}$, i.e. $F^i \to U_{ij} F^j$.

Example: QCD Lagrangian for fermions: $\mathscr{L}_{\text{Dirac, QCD}} = -\overline{\psi}_\alpha \left(\gamma^\mu D_{\alpha\beta\ \mu} + m\right) \psi_\beta - \frac{1}{4} F^i_{\mu\nu} F^{i\ \mu\nu}$.

More general result is $-\frac{1}{4} g_{ij} F^i_{\mu\nu} F^{j\ \mu\nu}$, but can always diagonalize and rescale so $g_{ij} \to \delta_{ij}$.

2.9 The Standard Model

Symmetry of vacuum is G=SU(3)$_{\text{colour}}\times$U(1)$_{\text{e.m.}}$ gauge group.

SM: At today's collider energies, some "hidden" (broken) symmetries become apparent:

G=SU(3)$_{\text{colour}}\times$SU(2)$_{\text{weak isospin}}\times$U(1)$_{\text{weak hypercharge}}$.

Table 2.9.1: SM fermions and their SU(3)$_C\times$SU(2)$_L\times$U(1)$_Y$ representations, written as (SU(3)$_C$ rep.,SU(2)$_L$ rep.,U(1)$_Y$ hypercharge = generator / [coupling $\equiv Y$]). The SU(3)$_C$ charges (3 for quarks, none for leptons) are not shown but, since SU(2)$_L$ is broken, particles differing only in T_3 (component of weak isospin SU(2)$_L$) are shown explicitly, namely u_L / ν_e ($T_3 = 1/2$) and d_L / e_L ($T_3 = -1/2$). Recall $\psi_{L/R} = \frac{1}{2}(1 \pm \gamma_5)\psi$. Note e.g. u_L annihilates u_L^- and creates u_R^+, and ν_e is left-handed.

Names	Label	Representation under SU(3)$_C\times$SU(2)$_L\times$U(1)$_Y$
Quarks	$\mathscr{Q}_L = (u_L, d_L)$	$(\mathbf{3}, \mathbf{2}, \frac{1}{6})$
	$u_R^\dagger$	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})$
	$d_R^\dagger$	$(\overline{\mathbf{3}}, \mathbf{1}, \frac{1}{3})$
Leptons	$\mathscr{E}_L = (\nu_e, e_L)$	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$
	$e_R^\dagger$	$(\mathbf{1}, \mathbf{1}, 1)$

Gauge fields are written as

SU(3): $\widehat{g}_\mu = g_{i\mu}\, g_s \frac{\lambda_i}{2}$

SU(2): $\widehat{A}_\mu = A_{i\mu}\, g \frac{\sigma_i}{2}$

U(1): $\widehat{B}_\mu = B_\mu\, g' Y$.

We have only discussed the "1st generation" of fermions, in fact $(\mathscr{E}_L, e_R^\dagger, \mathscr{Q}_L, u_R^\dagger, d_R^\dagger)^K$, $K = 1, 2, 3$.

where $(e^1, e^2, e^3) = (e, \mu, \tau)$, $(\nu_e^1, \nu_e^2, \nu_e^3) = (\nu_e, \nu_\mu, \nu_\tau)$, $(u^1, u^2, u^3) = (u, c, t)$ and $(d^1, d^2, d^3) = (d, s, b)$.

Allow mixing between particles of different generations with same transformation properties.

From Table 2.9.1, can construct the full Lagrangian by including all renormalizable invariant $(\mathbf{1}, \mathbf{1}, 0)$ terms.

These are all possible terms of form $\overline{\psi}^K \gamma^\mu D_\mu \psi^K$:

$$\mathscr{L}_{\text{quark}} = \overline{\mathscr{Q}}_L^K \gamma^\mu [\partial_\mu - I(\widehat{g}_\mu + \widehat{A}_\mu + \widehat{B}_\mu)] \mathscr{Q}_L^K \quad + \quad \overline{u}_R^K \gamma^\mu [\partial_\mu - I(\widehat{g}_\mu + \widehat{B}_\mu)] u_R^K \quad + \quad \overline{d}_R^K \gamma^\mu [\partial_\mu - I(\widehat{g}_\mu + \widehat{B}_\mu)] d_R^K.$$

$$\mathscr{L}_{\text{lepton}} = \overline{\mathscr{E}}_L^K \gamma^\mu [\partial_\mu - I(\widehat{A}_\mu + \widehat{B}_\mu)] \mathscr{E}_L^K \quad + \quad \overline{e}_R^K \gamma^\mu [\partial_\mu - I \widehat{B}_\mu] e_R^K.$$

More general $\overline{\psi}^K \gamma^\mu D_\mu R^{KM} \psi^M$ for some constant matrix R not allowed, gives terms $\overline{\psi}^K \gamma^\mu \partial_\mu \psi^M$ for $K \neq M$.

$$\mathscr{L}_{\text{spin 1}} = -\tfrac{1}{4} F_{\mu\nu}^i(\widehat{g}) F^{i\ \mu\nu}(\widehat{g}) - \tfrac{1}{4} F_{\mu\nu}^i(\widehat{A}) F^{i\ \mu\nu}(\widehat{A}) - \tfrac{1}{4} F_{\mu\nu}(\widehat{B}) F^{\mu\nu}(\widehat{B}).$$

Recall only real part of $\mathscr{L}$ to be taken.

2.9.1 Higgs mechanism

Mass terms $m\overline{\psi}_{L/R}\psi_{R/L}$ are all $(\mathbf{1},\mathbf{2},\pm\frac{1}{2})$ $\rightarrow$ violate gauge symmetry and thus renormalizability.

Instead introduce Yukawa coupling $\lambda\phi_H\overline{\psi}_{L/R}\psi_{R/L}$ which is $(\mathbf{1},\mathbf{1},0)$, i.e. invariant (thus renormalizable),

so ϕ_H is $(\mathbf{1},\overline{\mathbf{2}}=\mathbf{2},\frac{1}{2})$ scalar field, then hide ("break") symmetry so that $\lambda\langle 0|\phi_H|0\rangle = m$.

$\mathscr{L}_{\text{Higgs}} = \mathscr{L}_{\text{pure Higgs}} + \mathscr{L}_{\text{Higgs}-\text{fermion}}$.

Writing $D_\mu = \partial_\mu - I(\widehat{A}_\mu + \widehat{B}_\mu)$ and $\text{SU}(2)_L$ components $\phi_H^T = (\phi_{H\ 1}, \phi_{H\ 2}) = (\phi_H^+, \phi_H^0)$, $(\epsilon\phi_H^\dagger)^T = (\phi_H^{0\dagger}, -\phi_H^{+\dagger})$,

$\mathscr{L}_{\text{pure Higgs}} = -\frac{1}{2}(D_\mu\phi_H)^\dagger D^\mu\phi_H - V(\phi_H)$, *Higgs potential* $V(\phi_H) = \frac{m_H^2}{2}\phi_H^\dagger\phi_H + \frac{\lambda}{4}(\phi_H^\dagger\phi_H)^2$

$\mathscr{L}_{\text{Higgs}-\text{fermion}} = -G_e^{KM}\overline{\mathscr{E}}^K_{L\ a}\phi_{H\ a}e_R^M - G_u^{KM}\overline{\mathscr{Q}}^K_{L\ a}(\epsilon\phi_H)^\dagger_a u_R^M - G_d^{KM}\overline{\mathscr{Q}}^K_{L\ a}\phi_{H\ a}d_R^M$.

All three terms are $(\mathbf{1},\mathbf{1},0)$, i.e. invariant. Consider e.g. second term: From table 2.9.1, $Y = -\frac{1}{6} - \frac{1}{2} + \frac{2}{3} = 0$.

Write $\text{SU}(2)_L$ transformation of ϕ_H as $\phi'_{H\ a} = U_{ab}\phi_{H\ b}$ ($U = e^{I\frac{1}{2}\sigma_i\alpha_i}$ from page 59),

so $\underline{(\epsilon\phi'_H)^\dagger_a = (\epsilon\phi_H)^\dagger_b U^{-1*}_{ba}}$ from $\epsilon_{ac}\sigma_{cd}\epsilon^T_{db} = -\sigma^*_{ab}$. (Same transformation as ϕ_H^T: $\phi'_{H\ a} = \phi_{H\ b}U^T_{ba} = \phi_{H\ b}U^{-1*}_{ba}$.)

Also $\mathscr{Q}'_{L\ a} = U_{ab}\mathscr{Q}_{L\ b}$, so $\underline{\mathscr{Q}'^\dagger_{L\ a} = U^*_{ac}\mathscr{Q}^\dagger_{L\ c}}$, so $\mathscr{Q}'^\dagger_{L\ a}(\epsilon\phi'_H)^\dagger_a = \mathscr{Q}^\dagger_{L\ a}(\epsilon\phi_H)^\dagger_a$. Note u_R is an $\text{SU}(2)_L$ singlet.

$m_H^2 > 0$: Stationary $\mathscr{L}$ (vacuum) occurs when all fields vanish.

Spontaneous symmetry breaking (SSB): $m_H^2 < 0$: tree level vacuum obeys $\frac{\partial V(\phi_H)}{\partial \phi}\Big|_{\phi_H=\phi_{H0}} = 0$

$\implies |\phi_{H0}^2| = v^2 = \frac{|m_H^2|}{\lambda}$. From Higgs potential on page 66.

Infinite number of choices for $\phi_{0H} = \langle 0|\phi_H|0\rangle$. Take general $\phi_H = e^{Iv\xi_i(x)\frac{\sigma_i}{2}} \begin{pmatrix} 0 \\ v+\eta(x) \end{pmatrix}$.

Vacuum taken as $\xi_i = \eta = 0$, no longer invariant under symmetry transformations.

$\boxed{\mathscr{L}_{\text{gauge mass}} = -m_W^2 W_\mu^\dagger W^\mu - \frac{1}{2} m_Z^2 Z_\mu Z^\mu}$, where $m_Z = \frac{v}{2}\sqrt{g^2+g'^2}$, $m_W = \frac{v}{2} g = m_Z \cos\theta_W$, $\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}$,

$W_\mu = \frac{1}{\sqrt{2}}(A_{1\ \mu} - IA_{2\ \mu})$, $Z_\mu = \cos\theta_W A_{3\ \mu} - \sin\theta_W B_\mu$.

From $\mathscr{L}_{\text{pure Higgs}}$. Start with $\mathscr{L}_{\text{gauge mass}} = -\frac{1}{2}\left|\left(gA_{i\ \mu}\frac{\sigma_i}{2} + g'B_\mu\frac{1}{2}\right)\begin{pmatrix} 0 \\ v \end{pmatrix}\right|^2 = -\frac{1}{2}\left|\begin{pmatrix} g(A_{1\ \mu} + IA_{2\ \mu}) \\ -gA_{3\ \mu} + g'B_\mu \end{pmatrix}\frac{v}{2}\right|^2$.

$\boxed{\mathscr{L}_{\text{gauge dynamic}} = -\frac{1}{2}|\partial^\mu W^\nu - \partial^\nu W^\mu|^2 - \frac{1}{4}|\partial^\mu Z^\nu - \partial^\nu Z^\mu|^2 - \frac{1}{4}|\partial^\mu A^\nu - \partial^\nu A^\mu|^2}$,

where $A_\mu = \sin\theta_W A_{3\ \mu} + \cos\theta_W B_\mu$ must be massless photon.

From $\mathscr{L}_{\text{spin 1}}$. Start with $\mathscr{L}_{\text{gauge dynamic}} = -\frac{1}{4}|\partial^\mu A_i^\nu - \partial^\nu A_i^\mu|^2 - \frac{1}{4}|\partial^\mu B^\nu - \partial^\nu B^\mu|^2$.

In $\mathscr{L}_{\text{quark}}$ and $\mathscr{L}_{\text{lepton}}$, between left-handed fermions:

$$(\widehat{A}_\mu + \widehat{B}_\mu)_L = \begin{pmatrix} e\left(\frac{1}{2}+Y\right)A_\mu + \left(\frac{g}{2}\cos\theta_W - g'Y\sin\theta_W\right)Z_\mu & \frac{g}{\sqrt{2}}W_\mu^\dagger \\ \frac{g}{\sqrt{2}}W_\mu & e\left(-\frac{1}{2}+Y\right)A_\mu + \left(-\frac{g}{2}\cos\theta_W - g'Y\sin\theta_W\right)Z_\mu \end{pmatrix},$$

between right-handed: $(\widehat{A}_\mu + \widehat{B}_\mu)_R = eYA_\mu + (-g'\sin\theta_W Y)Z_\mu$, where $e = g\sin\theta_W = g'\cos\theta_W$.

So charge/e is $Q = T_3 + Y$, i.e. charge of u_L, d_L, ν_e, e_L is $\frac{2}{3}$, $-\frac{1}{3}$, 0, -1 and of u_R, d_R, e_R is $\frac{2}{3}$, $-\frac{1}{3}$, -1.

$\boxed{\mathscr{L}_{\text{fermion mass}} = -\overline{e}_L^K m_e^{KM} e_R^M - \overline{u}_L^K m_u^{KM} u_R^M - \overline{d}_L^K m_d^{KM} d_R^M}$, where $m_\psi^{KM} = G_\psi^{KM} v$. From $\mathscr{L}_{\text{quark/lepton}}$.

Can always transform $u_R^{K\prime} = A_{u_R}^{KM} u_R^M$, likewise for u_L, d_L, d_R, ν_e, e_L, e_R.

A matrices must be unitary so that kinetic terms retain their previous forms, $\overline{u}_R^{K\prime} \gamma^\mu \partial_\mu u_R^{K\prime}$ etc.

Choose A matrices such that new mass matrices $m_u' = A_{u_L} m_u A_{u_R}^\dagger$ etc. diagonal, entries $m_u^{K\prime}$:

$\mathscr{L}_{\text{fermion mass}} = \sum_K -\overline{e}_L^{K\prime} m_e^K e_R^{K\prime} - \overline{u}_L^{K\prime} m_u^{K\prime} u_R^{K\prime} - \overline{d}_L^{K\prime} m_d^{K\prime} d_R^{K\prime}$.

Then $\mathscr{L}_{W\ -\text{fermion}} \propto \overline{d}_L^K \gamma^\mu W_\mu u_L^K + \overline{e}_L^K \gamma^\mu W_\mu \nu_e^K = \overline{d}_L^{K\prime} \gamma^\mu W_\mu (V^\dagger)^{KN} u_L^{N\prime} + \overline{e}_L^{K\prime} \gamma^\mu W_\mu \nu_e^{K\prime\prime}$

(proportionality constant is $-\frac{Ig}{\sqrt{2}}$), where *CKM matrix* $V = A_{u_L} A_{d_L}^{-1}$.

Analogous leptonic matrix absorbed into $\nu_e^{K\prime\prime} = \left(A_{e_L} A_{\nu_e}^{-1}\right)^{KN} \nu_e^{N\prime}$.

(In contrast to u_L^K, any combination of ν_e^K is mass eigenstate because mass matrix is zero.)

2.9.2 Some remaining features

Neutrino mass by adding to $\mathscr{L}_{\text{Higgs}-\text{fermion}}$ a term $-G_{\nu_e}\overline{\mathscr{E}}^K_a(\epsilon\phi_H)^\dagger_a\nu_{eR} \to -\overline{\nu}_e m_{\nu_e}\nu_{eR}$, where ν_{eR} is $(\mathbf{1},\mathbf{1},0)$.

Expect $m_{\nu_e} \sim v$ to be similar order of magnitude to quark and charged lepton masses.

Also allowed SM invariant term $-\frac{1}{2}\nu^T_{eR}M_R\nu_{eR}$, can only come from higher scale symmetry breaking,

so $M_R \gg v \sim m_{\nu_e}$, i.e. no right handed neutrinos at low energy.

Gives $\mathscr{L}_{\text{neutrino mass}} = -\frac{1}{2}(\overline{\nu}_e \ \ \nu^T_{eR})\begin{pmatrix} 0 & m_{\nu_e} \\ m_{\nu_e} & M_R \end{pmatrix}\begin{pmatrix} \overline{\nu}^T_e \\ \nu_{eR} \end{pmatrix} \simeq -\frac{1}{2}(\overline{\nu}'_e \ \ \nu'^T_{eR})\begin{pmatrix} -\frac{m^2_{\nu_e}}{M_R} & 0 \\ 0 & M_R \end{pmatrix}\begin{pmatrix} \overline{\nu}'^T_e \\ \nu'_{eR} \end{pmatrix}$,

i.e. *seesaw mechanism*: mass $\frac{m^2_{\nu_e}}{M_R}$ of (almost) left handed ν'_e (i.e. m_{ν_e} suppressed by $\frac{m_{\nu_e}}{M_R}$).

Invariance with respect to parity P, charge conjugation C and time reversal T transformations.

CPT conserved, but CP-violation due to phases in CKM matrix.

CP and P violating terms $\frac{\theta}{64\pi^2}\epsilon^{\kappa\lambda\rho\sigma}F^i_{\kappa\lambda}F^i_{\rho\sigma}$ allowed, but are total derivatives and therefore non-perturbative.

Current observation suggest θ consistent with zero (no CP violation in QCD).

Cancelled by (harmless) anomaly (subsubsection 2.9.4) of global symmetry $\psi_f \to e^{I\gamma_5\alpha_f}\psi_f$ when $\sum\alpha_f = -\frac{1}{2}\theta$,

but this introduces unobserved CP violating phase $e^{-I\theta}$ on quark masses.

Peccei-Quinn mechanism: where $(\epsilon\phi_H)^\dagger$ in $\mathscr{L}_{\text{Higgs}-\text{fermion}}$ on page 66 is replaced with second Higgs,

which transforms differently to first Higgs and can soak up this phase at least at some GUT scale.

2.9.3 Grand unification

Suppose SM unifies to single group G at scale M_X, then $t^{\mathrm{U}(1)_Y}$ (diagonal), $t_i^{\mathrm{SU}(2)_L}$, $t_j^{\mathrm{SU}(3)_C}$ are generators of G.

Tracelessness requires sum of Y values (=elements of $t^{\mathrm{U}(1)_Y}$) to vanish, which is the case from Table 2.9.1.

Normalization of generators as on page 13, so $\mathrm{tr}[t^{\mathrm{U}(1)_Y 2}] = \mathrm{tr}[t_i^{\mathrm{SU}(2)_L 2}] = \mathrm{tr}[t_j^{\mathrm{SU}(3)_c 2}]$, so

$g_s^2(M_X) = g^2(M_X) = \frac{5}{3}g'^2(M_X)$ (after dividing by 2×no. generations). Implies $\sin^2\theta_W(M_X) = \frac{3}{8}$ from page 67.

Within couplings' exp. errors, unification occurs (provided $N = 1$ SUSY is included) at $M_X = 2 \times 10^{16}$ GeV.

To find simplest unification with no new particles, note SM particles are *chiral*, require complex representations:

In general, define all particles f_L to be left-handed, then antiparticles $f_R = f_L^\dagger$ are right-handed.

Then if f_L in representation $\boldsymbol{R}$ of some group G, f_R is in representation $\overline{\boldsymbol{R}}$.

If f_L, f_R equivalent (have same transformation properties), then $\boldsymbol{R} = \overline{\boldsymbol{R}}$, i.e. pseudoreal representation.

SM particles $f_L = (\mathscr{E}_L, e_R^\dagger, \mathscr{Q}_L, u_R^\dagger, d_R^\dagger)$ require complex representation because $f_R = f_L^\dagger$ inequivalent, e.g. $\overline{\mathbf{3}} \neq \mathbf{3}$.

Pseudoreal representation possible if particle content enlarged to $f_L \to F_L$ so that F_L, $F_R = F_L^\dagger$ equivalent.

e.g. in SO(10), can fit 15 particles of each generation into real **16** representation, requires adding 1 ν_{eR}.

SU(5) is simplest unification.

Since SU(3)×SU(2)×U(1) ⊂ SU(5), all internal symmetries accounted for by fermions ψ_α with $\alpha = 1, ..., 5$.

$$\text{Choose } t_i^{\mathrm{SU}(3)} = g_s \begin{pmatrix} & \frac{\lambda_i}{2} & & 0 & 0 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = t_i^{\mathrm{SU}(3)_C},\ i = 1, ..., 8, \qquad t_i^{\mathrm{SU}(2)} = g \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \multicolumn{2}{c}{\frac{\sigma_i}{2}} \\ 0 & 0 & 0 & & \end{pmatrix} = t_i^{\mathrm{SU}(2)_L},\ i = 1, ..., 3.$$

U(1) generator must commute with generators above and be traceless. Tentatively take

$$t^{\mathrm{U}(1)} = 2g' \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} = 2t^{\mathrm{U}(1)_Y}.$$

Then fermions form $\overline{\mathbf{5}}$ (fundamental) and $\mathbf{10}$ representations of SU(5):

$$\begin{pmatrix} \overline{d}_R^1 \\ \overline{d}_R^2 \\ \overline{d}_R^3 \\ e_L \\ -\nu_{eL} \end{pmatrix}, \begin{pmatrix} 0 & \overline{u}_R^3 & -\overline{u}_R^2 & u_L^1 & d_L^1 \\ & 0 & \overline{u}_R^1 & u_L^2 & d_L^2 \\ & & 0 & u_L^3 & d_L^3 \\ & \text{—“} & & 0 & \overline{e}_R \\ & & & & 0 \end{pmatrix}.$$

(Note: all particles left-handed, 1,2,3 superscripts are colour indices, $\mathbf{10}$ matrix is antisymmetric).

In SO(10), 1 generation fits into $\mathbf{16} = \mathbf{1} + \overline{\mathbf{5}} + \mathbf{10}$, and $\mathbf{1}$ is identified with right-handed neutrino.

2.9.4 Anomalies

Gauge anomalies modify symmetry relations (Ward identities), spoils renormalizability and maybe unitarity.

Anomaly occurs because $\mathscr{A} = \int d^4x \mathscr{L}$ respects symmetry, but not measure $\int \prod_{x,l} d\psi_l(x) = d[\psi]d[\overline{\psi}]d[A]$.

Relevant example: Let $\mathscr{A}$ be invariant under $\psi_\alpha = U_{k\ \alpha\beta}\psi_\beta$,

where $U_k = \exp[I\gamma_5 \alpha t_k]$ is *chiral symmetry* (global) and each ψ_α is a Dirac field.

Problem: although $\mathscr{A}$ is invariant, measure is not: Resulting change in path-integral $\int d[\psi]d[\overline{\psi}]d[A] \exp[I\mathscr{A}]$

is "as if" $\mathscr{L}$ changes by $\alpha \mathscr{J}_k[A]$, where $\mathscr{J}_k = -\frac{1}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F_i^{\mu\nu}F_j^{\rho\sigma}\mathrm{tr}[\{t_i, t_j\}t_k]$.

Noether's theorem on page 57: $\int d[\psi]d[\overline{\psi}]d[A] \exp[I\mathscr{A}] \to \int d[\psi]d[\overline{\psi}]d[A] \exp[I\left(\mathscr{A} + \int d^4x \alpha(x)\left[\mathscr{J}_k - \partial_\mu J_k^\mu(x)\right]\right)]$,

i.e. conservation violation: $\langle \partial_\mu J_k^\mu(x)\rangle_A = -\frac{1}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F_i^{\mu\nu}F_j^{\rho\sigma}\mathrm{tr}[\{t_i, t_j\}t_k]$ ($\langle\rangle_A$ means no A integration).

Anomalous (non-classical) triangle diagrams between J_k^μ, $F_i^{\mu\nu}$ and $F_j^{\rho\sigma}$ modify Ward identities.

Physical theories must be *anomaly free* (i.e. $\mathrm{tr}[\{t_i, t_j\}t_k]$ cancel), e.g. real representations: $\mathrm{tr}[\{t_i, t_j\}t_k] = 0$.

$\mathrm{tr}[\{t_i^*, t_j^*\}t_k^*] = \mathrm{tr}[\{(-Ut_iU^\dagger), (-Ut_jU^\dagger)\}(-Ut_kU^\dagger)] = -\mathrm{tr}[\{t_i, t_j\}t_k]$. But $\mathrm{tr}[\{t_i^*, t_j^*\}t_k^*] = \mathrm{tr}[\{t_i^T, t_j^T\}t_k^T] = \mathrm{tr}[\{t_i, t_j\}t_k]$.

SM is anomaly free. SM in $\mathbf{10} + \overline{\mathbf{5}}$ of SU(5), in real representation $\mathbf{16}$ of SO(10) (thus $\mathrm{tr}[\{t_i, t_j\}t_k]_{\overline{\mathbf{5}}} = -\mathrm{tr}[\{t_i, t_j\}t_k]_{\mathbf{10}}$).

3 Supersymmetry: development

3.1 Why SUSY?

Attractive features of SUSY:

1. Eliminates fine tuning in Higgs mass.

2. Gauge coupling unification.

3. Radiative electroweak symmetry breaking: SUSY $\Longrightarrow$ Higgs potential on page 66 with $\mu^2 < 0$.

4. Excess of matter over anti-matter (large CP violation, not in SM) possible from SUSY breaking terms.

5. Cold dark matter may be stable neutral lightest SUSY particle (LSP) = gravitino / lightest neutralino.

6. Gravity may be described by local SUSY = *supergravity*.

SM is accurately verified but incomplete — e.g. does not + cannot contain gravity,

so must break down at / before energies around Planck scale $M_P = (8\pi G)^{-1/2} = 2.4 \times 10^{18}$ GeV.

In fact, SM cannot hold without modification much above 1 TeV, otherwise we have

Gauge hierarchy problem: Since $v = |\langle 0|\phi_H|0\rangle| = 246\text{GeV}$ and $\lambda = O(1)$, $|m_H| = |\sqrt{\lambda} v| = O(100)$ GeV.

If $\Lambda_{\text{UV}} > O(1)\text{TeV}$, fine tuning between

Δm_H^2 from quantum loop corrections (Fig. 3.1) and tree level (bare) m_H^2:

$$\Delta m_H^2 = \frac{\lambda_\phi}{8\pi^2}\Lambda_{\text{UV}}^2 - \underbrace{3}_{\text{colour``3''}} \frac{|\kappa_t|^2}{8\pi^2}\Lambda_{\text{UV}}^2 + \underbrace{\cdots}_{\text{smaller terms}} .$$

(Largest from $-\frac{\lambda_\phi}{4}|\phi_H|^4$ and top quark ($\kappa_t \simeq 1$ because $m_t \simeq v$).)

No similar problem for fermion and gauge boson masses, but these masses affected by m_H^2.

Avoid fine tuning by taking $\Lambda_{\text{UV}} \sim 1$ TeV, i.e. modify SM above this scale.

(a) Fermion field ψ, Lagrangian term $-\kappa_\psi \phi_H \overline{\psi}\psi$, giving 1-loop contribution to Higgs mass of $-\frac{|\kappa_\psi|^2}{8\pi^2}\Lambda_{\mathrm{UV}}^2$.

(b) Boson field ϕ, Lagrangian term $\lambda_\phi |\phi_H|^2 |\phi|^2$, giving 1-loop contribution to Higgs mass of $\frac{\lambda_\phi}{8\pi^2}\Lambda_{\mathrm{UV}}^2$.

Figure 3.1: Fermion and boson contributions to Higgs mass parameter m_H^2.

One solution: Higgs is composite of new fermions bound by new strong force at $\Lambda_{\mathrm{UV}} \simeq 1$ TeV $\to$ difficult.

Alternatively, forbid bare $m_H^2|\phi_H|^2$ term by some new symmetry $\delta\phi_H = \epsilon \times$ something.

Various choices for "something" bosonic (leads to "little Higgs" models, extra dimensions).

For a standard symmetry, "something" would be $I\,[Q_a, \phi_H]$, i.e. $\phi_H \to e^{I\epsilon Q_a}\phi_H e^{-I\epsilon Q_a}$.

Try "fermionic" generator Q_a, which must be a $\left(\frac{1}{2}, 0\right)$ spinor (so ϵ a spinor of Grassmann variables).

Relation with momentum: $\{Q_a, Q^\dagger_{\dot b}\} = 2\sigma^\mu_{a\dot b} P_\mu$.

$\{Q_a, Q^\dagger_{\dot b}\}$ is $\left(\frac{1}{2},0\right) \times \left(0,\frac{1}{2}\right) = \left(\frac{1}{2},\frac{1}{2}\right)$ from triangle inequality. Only candidate is P^μ (see Coleman-Mandula theorem later).

Lorentz invariance requires combination $\sigma^\mu_{a\dot b} P_\mu$, factor 2 comes from suitable normalization of Q_a.

Note $\{Q_a, Q^\dagger_{\dot b}\} \neq 0$ because for any state $|X\rangle$,

$\langle X|\{Q_a, (Q_a)^\dagger\}|X\rangle = \langle X|Q_a\,(Q_a)^\dagger\,|X\rangle + \langle X|\,(Q_a)^\dagger\,Q_a|X\rangle = \left|\,(Q_a)^\dagger\,|X\rangle\right|^2 + \left|Q_a|X\rangle\right|^2 \geq 0$. If equality holds for all $|X\rangle$, $Q_a = 0$.

At least one of P_μ non-zero on every state, so Q_a affects every state, not just Higgs,

i.e. <u>every</u> particle has a *superpartner* with opposite statistics and spin difference of 1/2,

together called a *supermultiplet*. This fermion-boson symmetry is *supersymmetry*.

For every fermion field (component) ψ_f with $-\kappa_f \phi_H \overline{\psi}_f \psi_f$, introduce boson field ϕ_f with $-\lambda_f |\phi_H|^2 |\phi_f|^2$.

From Fig. 3.1, contribution of this supermultiplet (component) to Δm_H^2 is $\Delta m_H^2 = \underbrace{\frac{\lambda_f}{8\pi^2}\Lambda_{\rm UV}^2}_{\rm boson} \underbrace{-\frac{|\kappa_f|^2}{8\pi^2}\Lambda_{\rm UV}^2}_{\rm fermion}$.

Just requiring fermion-boson symmetry guarantees $\lambda_f = |\kappa_f|^2$, and $\Delta m_H^2 = 0$ (+ finite terms) to all orders.

3.2 Haag-Lopuszanski-Sohnius theorem

Reconsider symmetries: So far assumed generators are bosonic. Now generalize to include fermionic ones.

Generalize additive observable on page 4 to $Q = Q_{\sigma\sigma'} a^\dagger_\sigma a_{\sigma'}$.

(If $Q_{\sigma\sigma'}$ are components of Hermitian matrix, unitary transformation of particle states gives back original result.)

Since Q is bosonic, $a^\dagger_\sigma$, $a^\dagger_{\sigma'}$ both bosons or both fermions, i.e. $Q_{\sigma\sigma'} = 0$ if $a^\dagger_\sigma$ bosonic, $a^\dagger_{\sigma'}$ fermionic, or vice versa.

SUSY: Allow for Q's containing fermionic parts to also be generators of symmetries that commute with S-matrix.

For convenience, distinguish between fermionic and bosonic parts of any Q.

Fermionic $Q = Q_{\sigma\rho} a^\dagger_\sigma a_\rho + R_{\sigma\rho} a^\dagger_\rho a_\sigma$, where σ sums over bosonic particles, ρ over fermionic particles.

Such a generator converts bosons into fermions and vice versa.

E.g. action of Q on 1 fermion + 1 boson state using (anti) commutation relations on page 3 gives

$Q a^\dagger_{\rho'} a^\dagger_{\sigma'} |0\rangle = Q_{\sigma\rho'} a^\dagger_\sigma a^\dagger_{\sigma'} |0\rangle + R_{\sigma'\rho} a^\dagger_\rho a^\dagger_{\rho'} |0\rangle$.

But as a symmetry implies there are fermions and bosons with similar properties.

Identify symmetry generators t_i also with fermionic Q.

Graded parameters α_i, β_j obey $\alpha_i\beta_j = (-1)^{\eta_i\eta_j}\beta_j\alpha_i$, where *grading* $\eta_i = 0(1)$ for complex (Grassmann) α_i.

Graded generator t_j obeys $\alpha_i t_j = (-1)^{\eta_i\eta_j} t_j\alpha_i$, where $\eta_i = 0(1)$ for bosonic (fermionic) generator t_i.

For transformations $\mathscr{O} \to e^{I\alpha_i t_i}\mathscr{O}e^{-I\alpha_i t_i}$ to preserve grading of any operator $\mathscr{O}$, α_i has same grading as t_i.

Graded Lie algebra: $[(-1)^{\eta_i\eta_j} t_i t_j - t_j t_i] = IC_{ijk}t_k$.

Repeating steps in derivation of Lie algebra on page 7 gives

$\frac{1}{2}IC_{ijk}\alpha_i\beta_j t_k = \frac{1}{2}\left[\alpha_i t_i \beta_j t_j - \beta_j t_j \alpha_i t_i\right] = \frac{1}{2}\left[(-1)^{\eta_i\eta_j}\alpha_i\beta_j t_i t_j - \alpha_i\beta_j t_j t_i\right]$.

Fermionic generators Q_i: $U(\Lambda)Q_i U^\dagger(\Lambda) = C_{ij}(\Lambda)Q_j$, so Q_i furnishes representation of Lorentz group.

Choose $Q_i = Q_{ab}^{(A,B)}$ in (A,B) representation: $[\boldsymbol{A}, Q_{ab}^{(A,B)}] = -\boldsymbol{J}_{aa'}^{(A)}Q_{a'b}^{(A,B)}$ and $[\boldsymbol{B}, Q_{ab}^{(A,B)}] = -\boldsymbol{J}_{bb'}^{(B)}Q_{ab'}^{(A,B)}$.

Anticommutators of fermionic generators can be used to build bosonic generators of various (A,B).

Coleman-Mandula theorem puts limits on allowed bosonic generators, and hence allowed (A,B) for the $Q_{ab}^{(A,B)}$.

Coleman-Mandula theorem: Only bosonic generators are of internal + Poincaré group symmetries.

Simple argument: Additional conserved additive rank ≥ 1 tensors constrain scattering amplitude too much.

Only 1 4-vector, P^μ: Consider 2$\rightarrow$2 scattering, c.m. frame.

Conservation of momentum and angular momentum $\rightarrow$ amplitude depends on scattering angle θ.

Second conserved additive 4-vector R^μ gives additional constraints unless $R^\mu \propto P^\mu$.

Only 1 2nd rank tensor, $J^{\mu\nu}$: Assume rank 2 conserved additive tensor $\Sigma_{\mu\nu}$.

Additive property means $[\Sigma_{\mu\nu}, a^\dagger_\sigma(\boldsymbol{p})] = C_{\mu\nu\ \sigma}(\boldsymbol{p}) a^\dagger_\sigma(\boldsymbol{p})$. Then $C_{\mu\nu}(\boldsymbol{p},\sigma) = \alpha_\sigma(m^2) p_\mu p_\nu + \beta_\sigma(m^2) g_{\mu\nu}$.

> Lorentz transformation of RHS is $U(\Lambda)[\Sigma_{\mu\nu}, a^\dagger_\sigma(\boldsymbol{p})]U^\dagger(\Lambda) = [U(\Lambda)\Sigma_{\mu\nu}U^\dagger(\Lambda), U(\Lambda)a^\dagger_\sigma(\boldsymbol{p})U^\dagger(\Lambda)]$
>
> $= [\Lambda^\rho{}_\mu \Lambda^\delta{}_\nu \Sigma_{\rho\delta}, D_{\sigma'\sigma}(W(\Lambda,p)) a^\dagger_{\sigma'}(\boldsymbol{\Lambda p})] = D_{\sigma'\sigma}(W(\Lambda,p)) \underline{\Lambda^\rho{}_\mu \Lambda^\delta{}_\nu C_{\rho\delta}(\boldsymbol{\Lambda p},\sigma)} a^\dagger_{\sigma'}(\boldsymbol{\Lambda p})$,
>
> and of LHS is $\underline{C_{\mu\nu}(\boldsymbol{p},\sigma)} D_{\sigma'\sigma}(W(\Lambda,p)) a^\dagger_{\sigma'}(\boldsymbol{\Lambda p})$, so $\Lambda_\mu{}^\rho \Lambda_\nu{}^\delta C_{\rho\delta}(\boldsymbol{p},\sigma) = C_{\mu\nu}(\boldsymbol{\Lambda p},\sigma)$. Only candidates are $p_\mu p_\nu$ and $g_{\mu\nu}$.

In 2$\rightarrow$2 scattering, conservation of $\Sigma_{\mu\nu}$ implies $\alpha_{\sigma_1}(m_1^2) p_1^\mu p_1^\nu + \alpha_{\sigma_2}(m_2^2) p_2^\mu p_2^\nu = \alpha_{\sigma_1}(m_1^2) p_1'^\mu p_1'^\nu + \alpha_{\sigma_2}(m_2^2) p_2'^\mu p_2'^\nu$.

P^μ conservation: $p_1^\mu + p_2^\mu = p_1'^\mu + p_2'^\mu \implies p_{1,2}^\mu = p_{1,2}'^\mu$, i.e. no scattering (allowed $p_{1,2}^\mu = p_{2,1}'^\mu$ if $\alpha_{\sigma_1}(m_1^2) = \alpha_{\sigma_2}(m_2^2)$).

No higher rank tensors: Generalize last argument to higher rank tensors.

Allowed representations for fermionic generators $Q_{ab}^{(A,B)}$: $A + B = \frac{1}{2}$, i.e. $\left(\frac{1}{2}, 0\right)$ or $\left(0, \frac{1}{2}\right)$, and $j = \frac{1}{2}$.

$\{Q_{C,-D}^{(C,D)}, Q_{C,-D}^{(C,D)\dagger}\} = X_{C+D,-C-D}^{(C+D,C+D)}$: Firstly, $Q^{(C,D)\dagger}$ is of type $(A, B) = (D, C)$ because $\boldsymbol{B}^\dagger = \boldsymbol{A}$.

Writing $Q_{C,-D}^{(C,D)\dagger} = \widetilde{Q}_{a,b}^{(D,C)}$, we find $b = -C$ because $[B_-, \widetilde{Q}_{a,b}^{(D,C)}] = -[A_+, Q_{C,-D}^{(C,D)}]^\dagger = 0$ using $B_- = A_+^\dagger$. Similarly $a = D$.

Now $\{Q_{C,-D}^{(C,D)}, Q_{C,-D}^{(C,D)\dagger}\} = \{Q_{C,-D}^{(C,D)}, \widetilde{Q}_{D,-C}^{(D,C)}\}$, must have $A_3 = C + D$ and $B_3 = -C - D$, i.e. $A, B \geq C + D$.

But since A must be $\leq (C + D)$ (from triangle inequality), it must be $= C + D$. Similarly for B.

Since $X^{(C+D,C+D)}$ is bosonic, CM theorem means it must be P^μ $((\frac{1}{2}, \frac{1}{2}))$, or internal symmetry generator $((0, 0))$.

Latter implies $C = D = 0$, not possible by spin-statistics connection. Final result is relation with momentum on page 78.

Take Q_a to be $\left(\frac{1}{2}, 0\right)$ spinor, i.e. $[\boldsymbol{A}, Q_a] = -\frac{1}{2}\sigma_{ab}Q_b$, $[\boldsymbol{B}, Q_a] = 0$ ($Q_a^\dagger$ will be $\left(0, \frac{1}{2}\right)$).

Q_a not ruled out by reasoning of CM theorem, because no similar conservation law:

Take $|i\rangle$, $|j\rangle$ to have definite particle number. $\langle j|i\rangle \neq 0 \implies$ even difference in fermion numbers, so $\langle j|Q_a|i\rangle = 0$.

Can have multiple generators Q_{ar}, $r = 1, ..., N \longrightarrow$ *simple* SUSY is $N = 1$, *extended* SUSY is $N \geq 2$.

Summary:

CM theorem: only bosonic generators are $(0, 0)$ (internal symmetry), $\left(\frac{1}{2}, \frac{1}{2}\right)$ (P^μ), and $(1, 0)$ and $(0, 1)$ ($J^{\mu\nu}$).

HLS theorem: only fermionic generators are $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$ (Q_{ar}).

Relation with momentum for any N: $\{Q_{ar}, Q^\dagger_{\dot{b}s}\} = 2\delta_{rs}\sigma^\mu_{a\dot{b}}P_\mu$.

From allowed representations for fermionic generators on page 82, $\{Q_{ar}, Q^\dagger_{\dot{b}s}\} = 2N_{rs}\sigma^\mu_{a\dot{b}}P_\mu$. N_{rs} is Hermitian,

because $\{Q_{ar}, Q^\dagger_{\dot{b}s}\}^\dagger = \{Q_{bs}, Q^\dagger_{\dot{a}r}\} = 2N^*_{rs}\sigma^\mu_{b\dot{a}}P_\mu$, but $\{Q_{bs}, Q^\dagger_{\dot{a}r}\} = 2N_{sr}\sigma^\mu_{b\dot{a}}P_\mu$. So N diagonalized by unitary matrix W.

Writing $Q'_{ar'} = W_{r'r}Q_{ar}$ gives $\{Q'_{ar}, Q'^\dagger_{\dot{b}s}\} = 2n_r\delta_{rs}\sigma^\mu_{a\dot{b}}P_\mu$ (no sum over r on RHS), where n_r are eigenvalues of N_{rs}.

Writing $Q_{ar} = Q'_{ar}/\sqrt{n_r}$ gives result if $n_r > 0$ (otherwise we have a factor -1):

Taking $Q'^\dagger_{\dot{b}s} = (Q'_{ar})^\dagger$ and operating from right and left with $|X(p)\rangle$ and $\langle X(p)|$ gives

on LHS: $\langle X(p)|\{Q'_{ar}, (Q'_{ar})^\dagger\}|X(p)\rangle = |\,(Q'_{ar})^\dagger\}|X(p)\rangle|^2 + |Q'_{ar}\}|X(p)\rangle|^2 > 0$, and

on RHS: $2n_r(p^0 \pm p^3)$, where $\pm$ for $a = 1, 2$. If $p^0 \geq \mp p^3$, then $n_r > 0$ as required.

SUSY implies $\langle 0|H|0\rangle = 0$ for supersymmetric vacuum ($Q_{ar}|0\rangle = Q^\dagger_{\dot{a}r}|0\rangle = 0$).

Commutation with momentum: $[Q_a, P^\mu] = 0$.

$[Q_a, P^\mu]$ is $\left(\frac{1}{2}, 0\right) \times \left(\frac{1}{2}, \frac{1}{2}\right) = \left(1, \frac{1}{2}\right) + \left(0, \frac{1}{2}\right)$. No $\left(1, \frac{1}{2}\right)$ generator, but $Q^{\dagger\dot{a}}$ is $\left(0, \frac{1}{2}\right)$.

So $[Q_b, P^\mu] = k\sigma^\mu_{b\dot{a}}Q^{\dagger\dot{a}}$ and therefore $\left[Q^\dagger_{\dot{a}}, P^\mu\right] = -k^*Q^b\sigma^\mu_{b\dot{a}}$, or, using ϵ matrix, $\left[Q^{\dagger\dot{a}}, P^\mu\right] = k^*\overline{\sigma}^{\mu\ \dot{a}b}Q_b$.

Jacobi identity: $0 = [[Q_a, P^\mu], P^\nu] + [[P^\mu, P^\nu], Q_a] + [[P^\nu, Q_a], P^\mu] = k\sigma^\mu_{a\dot{b}}[Q^{\dagger\dot{b}}, P^\nu] - k\sigma^\nu_{a\dot{b}}[Q^{\dagger\dot{b}}, P^\mu] = |k|^2[\sigma^\mu, \overline{\sigma}^\nu]_a{}^bQ_b$.

Since $[\sigma^\mu, \overline{\sigma}^\nu]_a{}^b \neq 0$ for all μ, ν, must have $|k|^2 = k = 0$.

Anticommuting generators: $\{Q_{ar}, Q_{bs}\} = \epsilon_{ab} Z_{rs}$, with $(0,0)$ generators $Z_{rs} = -Z_{sr}$.

$\{Q_{ar}, Q_{bs}\}$ is $(1,0)+(0,0)$. From CM theorem, only $(1,0)$ generator is $J^{\mu\nu}$.

But $\{Q_{ar}, Q_{bs}\}$ commutes with P^μ from commutation with momentum on page 83, while (linear combinations of) $J^{\mu\nu}$ doesn't.

So only possibility is $(0,0)$ generators, which in general commute with P^μ from CM theorem.

Lorentz invariance requires $\epsilon_{ab}(=-\epsilon_{ba})$, but whole expression must be symmetric under $ar \leftrightarrow bs$ so $Z_{rs} = -Z_{sr}$.

Antisymmetry of $Z_{rs} \rightarrow$ vanish for $N = 1$. Z_{rs} are *central charges* due to following commutation relations:

Commutation with central charges: $[Z_{rs}, Q_{at}] = [Z_{rs}, Q^\dagger_{\dot{a}t}] = 0$.

Jacobi identity $0 = [\{Q_{ar}, Q_{bs}\}, Q^\dagger_{\dot{c}t}] + [\{Q_{bs}, Q^\dagger_{\dot{c}t}\}, Q_{ar}] + [\{Q^\dagger_{\dot{c}t}, Q_{ar}\}, Q_{bs}]$.

2nd, 3rd terms vanish from commutation with momentum on page 83. Thus $[Z_{rs}, Q^\dagger_{\dot{a}t}] = 0$.

Generalized Jacobi identity $0 = -[Z_{rs}, \{Q_{at}, Q^\dagger_{\dot{b}u}\}] + \{Q^\dagger_{\dot{b}u}, [Z_{rs}, Q_{at}]\} - \{Q_{at}, [Q^\dagger_{\dot{b}u}, Z_{rs}]\}$.

1st, 3rd terms vanish because Z_{rs} commutes with P^μ and $Q^\dagger_{\dot{b}u}$.

Commutator in 2nd term must be $[Z_{rs}, Q_{at}] = M_{rstv} Q_{av}$, so $0 = \{Q^\dagger_{\dot{b}u}, [Z_{rs}, Q_{at}]\} = 2M_{rstu}\sigma^\mu_{a\dot{b}} P_\mu$, i.e. $M_{rstu} = 0$ so $[Z_{rs}, Q_{at}] = 0$.

Commuting central charges: $[Z_{rs}, Z_{tu}] = [Z_{rs}, Z^\dagger_{tu}] = 0$. $[Z_{rs}, Z_{tu}] = [\{Q_{1r}, Q_{2s}\}, Z_{tu}] = 0$, etc.

***R*-symmetry**: $Z_{rs} = 0$ gives U(N) symmetry $Q_{ar} \rightarrow V_{rs} Q_{as}$ ($N = 1$ case: $Q_a \rightarrow e^{I\phi} Q_a$, always true).

Consider (anti)commutation relations of all internal symmetry generators t_i with SUSY algebra.

CM theorem: $[t_i, P^\mu] = [t_i, J^{\mu\nu}] = 0$.

Since t_i is $(0,0)$, must have $[t_i, Q_{\frac{1}{2}r}] = -(a_i)_{rs} Q_{\frac{1}{2}s}$. Matrices a_i represent G.

Jacobi identity $0 = [[t_i, t_j], Q_{\frac{1}{2}r}] + [[Q_{\frac{1}{2}r}, t_i], t_j] + [[t_j, Q_{\frac{1}{2}r}], t_i] = IC_{ijk}[t_k, Q_{\frac{1}{2}r}] + (a_i)_{rt}[Q_{\frac{1}{2}t}, t_j] - (a_j)_{rt}[Q_{\frac{1}{2}t}, t_i]$

$= -IC_{ijk}(a_k)_{rs}Q_{\frac{1}{2}s} + (a_i)_{rt}(a_j)_{ts}Q_{\frac{1}{2}s} - (a_j)_{rt}(a_i)_{ts}Q_{\frac{1}{2}s}$, i.e. $[a_i, a_j]_{rs} = IC_{ijk}(a_k)_{rs}$.

Simple SUSY and internal symmetry generators commute. If N=1, numbers a_i cannot represent G unless $a_i = 0$.

3.3 Supermultiplets

Supermultiplets: Particles that mix under SUSY transformations, furnish representation of SUSY.

Irreducible representations of SUSY: doesn't contain 2+ supermultiplets separately mixing under SUSY.

Action of Q_{ar} or $Q^\dagger_{\dot{a}r}$ converts one particle into another of the same irreducible supermulitplet (superpartners).

Since $[Q_{ar}, P^\mu] = [Q^\dagger_{\dot{a}r}, P^\mu] = 0$, all particles in supermultiplet have same P^μ (and hence same mass P^2).

Equal number of bosonic and fermionic degrees of freedom in supermultiplet: $n_B = n_F$.

Convention: $\sum_X$ over supermultiplet states, $\sum_{\text{all } X}$ over complete basis. Choose given r for $Q_{ar} \equiv Q_a$, $Q^\dagger_{\dot{a}r} \equiv Q^\dagger_{\dot{a}}$ below.

Supermultiplet's states $|X\rangle$ have same p^μ, and $(-1)^{2s}|X\rangle = \pm 1|X\rangle$ for spin s bosonic/fermionic $|X\rangle$,

so $P'_\mu \equiv \sum_X \langle X|(-1)^{2s} P_\mu |X\rangle = p_\mu(n_B - n_F)$. We show $P'_\mu = 0$: From relation with momentum for any N on page 83,

$2\sigma^\mu_{a\dot{b}} P'_\mu = \sum_X \langle X|(-1)^{2s} Q_a Q^\dagger_{\dot{b}}|X\rangle + \underline{\sum_X \langle X|(-1)^{2s} Q^\dagger_{\dot{b}} Q_a|X\rangle} = \sum_X \langle X|(-1)^{2s} Q_a Q^\dagger_{\dot{b}}|X\rangle + \underline{\sum_{X,\text{all } Y} \langle X|(-1)^{2s} Q^\dagger_{\dot{b}}|Y\rangle\langle Y|Q_a|X\rangle}$.

Since $Q_a|X\rangle$, $|X\rangle$ in same supermultiplet, limit $\sum_{\text{all } Y} \to \sum_Y$. Conversely, extend $\sum_X \to \sum_{\text{all } X}$ (so $\sum_{\text{all } X} |X\rangle\langle X| = 1$).

$2\sigma^\mu_{a\dot{b}} P'_\mu = \langle X|(-1)^{2s} Q_a Q^\dagger_{\dot{b}}|X\rangle + \underline{\sum_{\text{all } X,Y} \langle Y|Q_a|X\rangle\langle X|(-1)^{2s} Q^\dagger_{\dot{b}}|Y\rangle} = \sum_X \langle X|(-1)^{2s} Q_a Q^\dagger_{\dot{b}}|X\rangle + \underline{\sum_Y \langle Y|Q_a(-1)^{2s} Q^\dagger_{\dot{b}}|Y\rangle}$.

But $(-1)^{2s} Q^\dagger_{\dot{b}}|Y\rangle = -Q^\dagger_{\dot{b}}(-1)^{2s}|Y\rangle$, so $2\sigma^\mu_{a\dot{b}} P'_\mu = \sum_X \langle X|(-1)^{2s} Q_a Q^\dagger_{\dot{b}}|X\rangle + \underline{-\sum_Y \langle Y|(-1)^{2s} Q_a Q^\dagger_{\dot{b}}|Y\rangle} = 0$.

Massless supermultiplets: In frame where $p^1 = p^2 = 0$, $p^3 = p^0$, $\begin{pmatrix} \{Q_{\frac{1}{2}r}, Q^\dagger_{\frac{1}{2}s}\} = 4p^0\delta_{rs} & \{Q_{\frac{1}{2}r}, Q^\dagger_{-\frac{1}{2}s}\} = 0 \\ \{Q_{-\frac{1}{2}r}, Q^\dagger_{\frac{1}{2}s}\} = 0 & \{Q_{-\frac{1}{2}r}, Q^\dagger_{-\frac{1}{2}s}\} = 0 \end{pmatrix}$.

$Q_{-\frac{1}{2}r}$, $Q^\dagger_{-\frac{1}{2}r}$ annihilate supermultiplet states, so Z_{rs} annihilate supermultiplet states.

For any $|X\rangle$ in supermultiplet, $0 = \langle X|\{Q_{-\frac{1}{2}r}, Q^\dagger_{-\frac{1}{2}r}\}|X\rangle = |\left(Q_{-\frac{1}{2}r}\right)^\dagger |X\rangle|^2 + |Q_{-\frac{1}{2}r}|X\rangle|^2$ (no sum over r).

All supermultiplet states reached by acting on maximum helicity state $|\lambda_{\text{max}}\rangle$ with the $Q_{\frac{1}{2}r}$.

$Q^\dagger_{\frac{1}{2}r}$ give no new states: Consider $|X\rangle$ not containing $Q_{\frac{1}{2}r}$. Then $Q^\dagger_{\frac{1}{2}r}|X\rangle = 0$ ($Q^\dagger_{\frac{1}{2}r}$ commutes across to act directly on $|\lambda_{\text{max}}\rangle$).

Then $Q^\dagger_{\frac{1}{2}r}Q_{\frac{1}{2}r}|X\rangle = \{Q_{\frac{1}{2}r}, Q^\dagger_{\frac{1}{2}r}\}|X\rangle = 4E|X\rangle$ (no sum over r), i.e. $Q^\dagger_{\frac{1}{2}r}$ just removes $Q_{\frac{1}{2}r}$.

$Q^\dagger_{\frac{1}{2}r}|X\rangle$ (or $Q_{\frac{1}{2}r}|X\rangle$) has helicity greater (or less) than $|X\rangle$ by $\frac{1}{2}$. Recall $[J_3, Q^\dagger_{\frac{1}{2}r}] = \frac{1}{2}Q^\dagger_{\frac{1}{2}r}$.

Range of helicities in supermultiplet: $\frac{N!}{n!(N-n)!}$ helicity $\lambda_{\text{max}} - \frac{n}{2}$ states, $\lambda_{\text{min}} = \lambda_{\text{max}} - \frac{N}{2}$.

Obtain supermultiplet states $|X\rangle$ by acting on maximum helicity state $|\lambda_{\text{max}}\rangle$ with any n of $Q_{\frac{1}{2}1}$, $Q_{\frac{1}{2}2}$, ..., $Q_{\frac{1}{2}N}$.

Order doesn't matter since $Q_{\frac{1}{2}a}$ anticommute, each generator cannot appear more than once since $Q^2_{\frac{1}{2}r} = 0$.

Constraint on particle spins on page 53 implies $\lambda_{\text{min}} \geq -2$ and $\lambda_{\text{max}} \leq 2$, i.e. $N \leq 8$.

SM particles probably belong to $\sim$ massless supermultiplets.

Superpartners (masses $\sim M$) of SM particles ($\sim m$) not seen, i.e. $M \gg m$, so SUSY is broken, at energy $< m_{\text{SUSY}}$.

SUSY restored at m_{SUSY} $\therefore$ $m_{\text{SUSY}} \gg M - m \sim M$ (superpartner has same mass) $\therefore$ supermultiplets $\sim$ massless.

Most likely scenario is simple SUSY: quark, lepton (spin $\frac{1}{2}$) superpartners are scalars (0): squarks, sleptons.

Higgs (0), gauge bosons (1), graviton (2) superpartners fermionic: Higgsino + gauginos ($\frac{1}{2}$), gravitino ($\frac{3}{2}$).

Alternatives: In simple SUSY, SM gauge bosons cannot be superpartners of SM fermions.

SM fermions, gauge bosons in different representations (recall simple SUSY and SM SU(3)$\times$SU(2) commute from page 85).

Quarks, leptons cannot be in same supermultiplet as any beyond-SM vector (gauge) bosons.

Gauge bosons are in adjoint representation of a group. If e.g. helicity $+\frac{1}{2}$ fermion, $+1$ gauge boson in same supermultiplet,

fermion in adjoint = real representation. But SM is chiral, i.e. helicity $+\frac{1}{2}$ fermions belong to complex representations.

Superpartners of gauge bosons must be helicity $\pm\frac{1}{2}$, not $\pm\frac{3}{2}$, fermions.

Helicity $\pm\frac{3}{2}$ particle couples only to Q_a (principle of equivalence on page 53: ± 2 only couples to P^μ).

Extended SUSY probably cannot be realised in nature.

In extended SUSY, helicity $\pm\frac{1}{2}$ fermions either in same supermultiplet as gauge bosons ($N \geq 3$), or each other ($N = 2$).

Massive supermultiplets: For particles with masses $M \gg m_{\rm SUSY}$, e.g. heavy gauge bosons in SU(5).

In frame where $p^1 = p^2 = p^3 = 0$, $p_0 = M$, $\begin{pmatrix} \{Q_{\frac{1}{2}r}, Q^\dagger_{\frac{1}{2}s}\} = 2M\delta_{rs} & \{Q_{\frac{1}{2}r}, Q^\dagger_{-\frac{1}{2}s}\} = 0 \\ \{Q_{-\frac{1}{2}r}, Q^\dagger_{\frac{1}{2}s}\} = 0 & \{Q_{-\frac{1}{2}r}, Q^\dagger_{-\frac{1}{2}s}\} = 2M\delta_{rs} \end{pmatrix}$,

so $Q_{\frac{1}{2}r}$, $Q^\dagger_{-\frac{1}{2}r}$ (or $Q_{-\frac{1}{2}r}$, $Q^\dagger_{\frac{1}{2}r}$) lower (or raise) spin 3 component by $\frac{1}{2}$.

$a_{A(a,r)} = \frac{1}{\sqrt{2M}} Q_{ar}$ are fermionic annihilation / creation operators: $\{a_A, a^\dagger_B\} = \delta_{AB}$, $\{a_A, a_B\} = \{a^\dagger_A, a^\dagger_B\} = 0$.

Define Clifford "vacuum" $|\Omega\rangle$: $a_A|\Omega\rangle = 0$. $|\Omega\rangle$ has given spin j, range of spin 3 is $-j \le \sigma \le j$.

Supermultiplet is all states $a^\dagger_{A_1} \ldots a^\dagger_{A_n}|\Omega\rangle$, spins ranging from $\mathrm{Max}(j - \frac{N}{2}, 0)$, …, $j + \frac{N}{2}$.

Simple SUSY: States with spin $j \pm \frac{1}{2}$, and 2 sets of states with spin j.

If $j = 0$, there are two bosonic spin 0 states and two fermionic states of spin $\frac{1}{2}$ (i.e. spin 3 is $\pm\frac{1}{2}$).

3.3.1 Field supermultiplets (the left-chiral supermultiplet)

Simplest case: $N = 1$, scalar $\phi(x)$ obeying $[Q^\dagger_{\dot{b}}, \phi(x)] = 0$: Write $[Q_a, \phi(x)] = -I\zeta_a(x)$, which is a $\left(\frac{1}{2}, 0\right)$ field.

$\{Q^\dagger_{\dot{a}}, \zeta_b(x)\} = 2\sigma^\mu_{b\dot{a}}\partial_\mu\phi(x)$. Thus $[Q, \phi] \sim \zeta$ and $\{Q^\dagger, \zeta\} \sim \phi$, i.e. ϕ and ζ are each others' superpartners.

$\{Q^\dagger_{\dot{a}}, -I\zeta_b\} = \{Q^\dagger_{\dot{a}}, [Q_b, \phi]\} = [\{Q^\dagger_{\dot{a}}, Q_b\}, \phi] = 2\sigma^\mu_{b\dot{a}}[P_\mu, \phi]$. Poincaré transformation for fields on page 24: $[P_\mu, \phi] = I\partial_\mu\phi$.

$\{Q_a, \zeta_b(x)\} = 2I\epsilon_{ab}\mathscr{F}(x)$.

$\{Q_a, -I\zeta_b\} = \{Q_a, [Q_b, \phi]\} = -\{Q_b, [Q_a, \phi]\} = -\{Q_b, I\zeta_a\} \propto \epsilon_{ab}$. $\mathscr{F}$ is $(0,0)$ in $\left(\frac{1}{2}, 0\right) \times \left(\frac{1}{2}, 0\right) = (0,0) + (1,0)$.

$[Q_a, \mathscr{F}(x)] = 0$. $\mathscr{F}$ has no superpartner, it is *auxiliary field.*

$2I\epsilon_{ab}[Q_c, \mathscr{F}] = [Q_c, \{Q_a, \zeta_b\}] = -[Q_a, \{Q_c, \zeta_b\}] = -2I\epsilon_{cb}[Q_a, \mathscr{F}]$. For $a = c \neq b$, $2I\epsilon_{cb}[Q_c, \mathscr{F}] = -2I\epsilon_{cb}[Q_c, \mathscr{F}]$, so $[Q_c, \mathscr{F}] = 0$.

$[Q^{\dagger\dot{a}}, \mathscr{F}(x)] = -\overline{\sigma}^{\mu\ \dot{a}b}\partial_\mu\zeta_b(x)$.

$2I\epsilon_{ab}[Q^\dagger_{\dot{c}}, \mathscr{F}] = [Q^\dagger_{\dot{c}}, \{Q_a, \zeta_b\}] = [\{Q^\dagger_{\dot{c}}, Q_a\}, \zeta_b] - [Q_a, \{Q^\dagger_{\dot{c}}, \zeta_b\}] = [2\sigma^\mu_{a\dot{c}}P_\mu, \zeta_b] - [Q_a, 2\sigma^\mu_{b\dot{c}}\partial_\mu\phi] = 2I\sigma^\mu_{a\dot{c}}\partial_\mu\zeta_b - 2I\sigma^\mu_{b\dot{c}}\partial_\mu\zeta_a$.

Then $\epsilon^{da}\epsilon_{ab}[Q^\dagger_{\dot{c}}, \mathscr{F}] = \delta^d{}_b[Q^\dagger_{\dot{c}}, \mathscr{F}] = \epsilon^{da}\sigma^\mu_{a\dot{c}}\partial_\mu\zeta_b - \epsilon^{da}\sigma^\mu_{b\dot{c}}\partial_\mu\zeta_a = \epsilon^{da}\sigma^\mu_{a\dot{c}}\partial_\mu\zeta_b - \sigma^\mu_{b\dot{c}}\partial_\mu\zeta^d$. Sum $d = b$: $\delta^d{}_d[Q^\dagger_{\dot{c}}, \mathscr{F}] = 2[Q^\dagger_{\dot{c}}, \mathscr{F}] = -2\sigma^\mu_{b\dot{c}}\partial_\mu\zeta^b$.

Then $[Q^{\dagger\dot{a}}, \mathscr{F}] = \epsilon^{\dot{a}\dot{c}}[Q^\dagger_{\dot{c}}, \mathscr{F}] = -\epsilon^{\dot{a}\dot{c}}\sigma^\mu_{b\dot{c}}\epsilon^{bd}\zeta_d = -\overline{\sigma}^{\mu\ \dot{a}d}\zeta_d$, using 4-vector σ matrices with raised indices on page 41.

Conjugation of everything above gives right-chiral supermulitplet.

To simplify algebra, use 4-component Majorana spinors, $\psi = \begin{pmatrix} X_a \\ X^{\dagger\dot{b}} \end{pmatrix}$, with notation of page 42.

Majorana conjugation: $\overline{\psi} = \psi^T \gamma_5 E$ where $E = \begin{pmatrix} \epsilon^{ab} & 0 \\ 0 & -\epsilon_{\dot{a}\dot{b}}(= \epsilon^{ab}) \end{pmatrix}$. From $\overline{\psi} = \psi^\dagger \beta = (X^a, X^\dagger_{\dot{b}})$.

(Anti)commutation relations: $\{Q, \overline{Q}\} = -2I\gamma^\mu P_\mu$ and $[Q, P^\mu] = 0$, where Majorana $Q = \begin{pmatrix} Q_a \\ Q^{\dagger\dot{b}} = (\epsilon^{bc} Q_c)^\dagger \end{pmatrix}$.

Directly from relations on page 83.

Infinitesimal transformation of operator $\mathscr{O}$: $\delta\mathscr{O} = [I\overline{\alpha}Q, \mathscr{O}]$,

where Grassmann spinor $\alpha = \begin{pmatrix} \alpha_a \\ \alpha^{\dagger\dot{b}} \end{pmatrix}$, so $\overline{\alpha}Q = \alpha^a Q_a + \alpha^\dagger_{\dot{b}} Q^{\dagger\dot{b}}$. Implies $\delta\mathscr{O}^\dagger = [I\overline{\alpha}Q, \mathscr{O}^\dagger]$.

$(\delta\mathscr{O})^\dagger = [I\overline{\alpha}Q, \mathscr{O}]^\dagger = [I\overline{\alpha}Q, \mathscr{O}^\dagger] = \delta(\mathscr{O}^\dagger)$, because, as for scalar from 2 spinors and Hermitian conjugate of scalar on page 40,

$(\overline{\alpha}Q)^\dagger = (\alpha^a Q_a + \alpha^\dagger_{\dot{b}} Q^{\dagger\dot{b}})^\dagger = (Q^\dagger_{\dot{a}} \alpha^{\dagger\dot{a}} + Q^b \alpha_b) = (-\alpha^{\dagger\dot{a}} Q^\dagger_{\dot{a}} - \alpha_b Q^b) = (\alpha^\dagger_{\dot{a}} Q^{\dagger\dot{a}} + \alpha^b Q_b) = \overline{\alpha}Q$.

Product rule for δ: $\delta(AB) = (\delta A)B + A\delta B$, where A, B can be fermionic / bosonic.

$[I\overline{\alpha}Q, AB] = I(\overline{\alpha}QAB - AB\overline{\alpha}Q) = I(\overline{\alpha}QAB - A\overline{\alpha}QB + A\overline{\alpha}QB - AB\overline{\alpha}Q) = [I\overline{\alpha}Q, A]B + A[I\overline{\alpha}Q, B]$.

If Abelian limit on page 7 obeyed, finite transformation is $\mathscr{O} \to e^{I\overline{\alpha}Q} \mathscr{O} e^{-I\overline{\alpha}Q}$.

(Anti)commutation relations on page 90 can be simplified with 4-component spinor notation

and definitions $\phi = \frac{1}{\sqrt{2}}(A + IB)$, $\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \zeta_a \\ \zeta^{\dagger\dot{b}} \end{pmatrix}$ and $\mathscr{F} = \frac{1}{\sqrt{2}}(F - IG)$:

$\boxed{1.\ \delta A = \overline{\alpha}\psi}$, $\boxed{2.\ \delta B = -I\overline{\alpha}\gamma_5\psi}$.

$$[Q,\phi] = \begin{pmatrix} [Q_a,\phi] = -I\zeta_a \\ [Q^{\dagger\dot{b}},\phi] = \epsilon^{\dot{b}\dot{a}}[Q^\dagger_{\dot{a}},\phi] = 0 \end{pmatrix} \text{ and } [Q,\phi^\dagger] = \begin{pmatrix} [Q_a,\phi^\dagger] = -[Q^\dagger_{\dot{a}},\phi]^\dagger = 0 \\ [Q^{\dagger\dot{b}},\phi^\dagger] = -\epsilon^{\dot{b}\dot{a}}[Q_a,\phi]^\dagger = -I\zeta_{\dagger\dot{b}} \end{pmatrix},$$

so $[Q,A] = [Q, \frac{1}{\sqrt{2}}(\phi + \phi^\dagger)] = -I\psi$, and $[Q,B] = [Q, \frac{I}{\sqrt{2}}(\phi^\dagger - \phi)] = -\gamma_5\psi$ from definition of γ_5 on page 43..

$\boxed{3.\ \delta F = \overline{\alpha}\gamma^\mu\partial_\mu\psi}$, $\boxed{4.\ \delta G = -I\overline{\alpha}\gamma_5\gamma^\mu\partial_\mu\psi}$.

$$[Q,\mathscr{F}] = \begin{pmatrix} [Q_a,\mathscr{F}] = 0 \\ [Q^{\dagger\dot{b}},\mathscr{F}] = -\overline{\sigma}^{\mu\ \dot{b}a}\partial_\mu\zeta_a \end{pmatrix} \text{ and, using } \overline{\sigma}^{\mu\ \dot{b}c*} = \overline{\sigma}^{\mu\ \dot{c}b},$$

$$[Q,\mathscr{F}^\dagger] = \begin{pmatrix} [Q_a,\mathscr{F}^\dagger] = -\epsilon_{ab}[Q^{\dagger\dot{b}},\mathscr{F}]^\dagger = \epsilon_{ab}\overline{\sigma}^{\mu\ \dot{b}c*}\partial_\mu\zeta^\dagger_{\dot{c}} = \epsilon_{ab}\overline{\sigma}^{\mu\ \dot{c}b}\partial_\mu\zeta^\dagger_{\dot{c}} = \epsilon_{ab}\overline{\sigma}^{\mu\ \dot{c}b}\epsilon_{\dot{c}\dot{d}}\partial_\mu\zeta^{\dagger\dot{d}} = -\sigma^\mu_{a\dot{d}}\partial_\mu\zeta^{\dagger\dot{d}} \\ [Q^{\dagger\dot{b}},\mathscr{F}^\dagger] = -\epsilon^{ba}[Q_a,\mathscr{F}]^\dagger = 0 \end{pmatrix}, \text{ so}$$

$$[Q,F] = [Q, \tfrac{1}{\sqrt{2}}(\mathscr{F} + \mathscr{F}^\dagger)] = -\frac{I}{\sqrt{2}}\begin{pmatrix} -I\sigma^\mu_{a\dot{b}}\partial_\mu\zeta^{\dagger\dot{b}} \\ -I\overline{\sigma}^{\mu\ \dot{b}d}\partial_\mu\zeta_d \end{pmatrix} = -I\gamma^\mu\partial_\mu\psi \text{ and}$$

$$[Q,G] = [Q, \tfrac{I}{\sqrt{2}}(\mathscr{F} - \mathscr{F}^\dagger)] = \frac{1}{\sqrt{2}}\begin{pmatrix} I\sigma^\mu_{a\dot{b}}\partial_\mu\zeta^{\dagger\dot{b}} \\ -I\overline{\sigma}^{\mu\ \dot{b}d}\partial_\mu\zeta_d \end{pmatrix} = -\gamma_5\gamma^\mu\partial_\mu\psi.$$

$\boxed{5.\ \delta\psi = \partial_\mu(A + I\gamma_5 B)\gamma^\mu\alpha + (F - I\gamma_5 G)\alpha}$.

$$[I\overline{\alpha}Q, \psi] = \frac{I}{\sqrt{2}}\begin{pmatrix} \alpha^c\{Q_c, \zeta_a\} + \alpha^\dagger_{\dot{d}}\{Q^{\dagger\dot{d}}, \zeta_a\} \\ \alpha^c\{Q_c, \zeta^{\dagger\dot{b}}\} + \alpha^\dagger_{\dot{d}}\{Q^{\dagger\dot{d}}, \zeta^{\dagger\dot{b}}\} \end{pmatrix} = \frac{I}{\sqrt{2}}\begin{pmatrix} \alpha^c\{Q_c, \zeta_a\} + \alpha^\dagger_{\dot{d}}\epsilon^{\dot{d}\dot{c}}\{Q^\dagger_{\dot{c}}, \zeta_a\} \\ \alpha^c\{Q^\dagger_{\dot{c}}, \zeta_d\}^\dagger\epsilon^{\dot{b}\dot{d}} + \alpha^\dagger_{\dot{d}}\epsilon^{\dot{d}\dot{c}}\epsilon^{\dot{b}\dot{a}}\{Q_c, \zeta_a\}^\dagger \end{pmatrix}$$

$$= \frac{I}{\sqrt{2}}\begin{pmatrix} \alpha^c(2I\epsilon_{ca}\mathscr{F}) + \alpha^\dagger_{\dot{d}}\epsilon^{\dot{d}\dot{c}}(2\sigma^\mu_{a\dot{c}}\partial_\mu\phi) \\ \epsilon^{ca}\alpha_a(2\sigma^\mu_{c\dot{d}}\partial_\mu\phi^\dagger)\epsilon^{\dot{b}\dot{d}} - \alpha^\dagger_{\dot{d}}\epsilon^{\dot{d}\dot{c}}\epsilon^{\dot{b}\dot{a}}(2I\epsilon_{\dot{c}\dot{a}}\mathscr{F}^\dagger) \end{pmatrix}$$

$$= I\sqrt{2}\begin{pmatrix} -\alpha_a(I\mathscr{F}) - \alpha^{\dagger\dot{c}}(\sigma^\mu_{a\dot{c}}\partial_\mu\phi) \\ -\alpha_a(\overline{\sigma}^{\mu\ \dot{b}a}\partial_\mu\phi^\dagger) - \alpha^{\dagger\dot{b}}(I\mathscr{F}^\dagger) \end{pmatrix} = \begin{pmatrix} \alpha_a(F - IG) - I\sigma^\mu_{a\dot{c}}\alpha^{\dagger\dot{c}}\partial_\mu(A + IB) \\ \alpha^{\dagger\dot{b}}(F + IG) - I\overline{\sigma}^{\mu\ \dot{b}a}\alpha_a\partial_\mu(A - IB) \end{pmatrix}.$$

For $\mathscr{L} = -\frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}\partial_\mu B\partial^\mu B - \frac{1}{2}\overline{\psi}\gamma^\mu\partial_\mu\psi + \frac{1}{2}(F^2 + G^2) + m(FA + GB - \frac{1}{2}\overline{\psi}\psi)$

$+g[F(A^2 + B^2) + 2GAB - \overline{\psi}(A + I\gamma_5 B)\psi]$, above transformations 1 — 5 leave action $\mathscr{A} = \int d^4x\mathscr{L}$ invariant.

For example, for $m = g = 0$, $\delta\mathscr{L} = -\partial_\mu\delta A\partial^\mu A - \partial_\mu\delta B\partial^\mu B - (\delta\overline{\psi})\gamma^\mu\partial_\mu\psi + F\delta F + G\delta G$.

We have replaced $-\frac{1}{2}\overline{\psi}\gamma^\mu\partial_\mu\delta\psi \to -\frac{1}{2}(\delta\overline{\psi})\gamma^\mu\partial_\mu\psi$, since difference with analogous term in hermitian conjugate (h.c.) of $\mathscr{L}$ (must be added to make $\mathscr{L}$ real) is total derivative $\partial_\mu f$ which doesn't contribute to $\mathscr{A}$:

$(\overline{X}\gamma^\mu\partial_\mu Z)^\dagger = (X^\dagger\beta\gamma^\mu\partial_\mu Z)^\dagger = \partial_\mu Z^\dagger\gamma^{\mu\dagger}\beta^\dagger X = -\partial_\mu Z^\dagger\beta\gamma^\mu X = -\partial_\mu\overline{Z}\gamma^\mu X = \overline{Z}\gamma^\mu\partial_\mu X - \partial_\mu(\overline{Z}\gamma^\mu X)$.

No derivatives of F, G (or $\mathscr{F}$) appear, so they are *auxiliary fields*,

i.e. can be expressed in terms of the other supermultiplet fields by solving equations of motion $\frac{\partial\mathscr{L}}{\partial F} = \frac{\partial\mathscr{L}}{\partial G} = 0$.

3.4 Superfields and Superspace

As P^μ generates translations in spacetime x on field $\phi_l(x)$ via $[P^\mu, \phi_l(x)] = I\partial^\mu \phi_l(x)$, find formalism where

Q generates translations in *superspace* $(x, \theta = \left(\theta_a, \theta^{\dagger\dot{b}}\right)^T)$ on *superfield* $S(x,\theta)$ via $[I\overline{\alpha}Q, S(x,\theta)] = \overline{\alpha}\mathscr{Q}S(x,\theta)$.

Definition of superfield: $\delta S = \overline{\alpha}\mathscr{Q}S$. From infinitesimal transformation of operator $\mathscr{O}$ on page 91.

Condition on $\mathscr{Q}$: $\{\mathscr{Q}_\alpha, \overline{\mathscr{Q}}_\beta\} = 2\gamma^\mu_{\alpha\beta}\partial_\mu$, where α, β run over the 4 spinor indices.

From (anti)commutation relations on page 91, $[\{Q_\alpha, \overline{Q}_\beta\}, S] = -2I\gamma^\mu_{\alpha\beta}[P_\mu, S]$, which from above reads $\{\mathscr{Q}_\alpha, \overline{\mathscr{Q}}_\beta\}S = 2\gamma^\mu_{\alpha\beta}\partial_\mu S$.

Definition of $\mathscr{Q}$: $\mathscr{Q} = -\frac{\partial}{\partial\overline{\theta}} + \gamma^\mu\theta\partial_\mu$ (explicitly, $\mathscr{Q}_\alpha = -\frac{\partial}{\partial\theta_\beta}(\gamma_5 E)_{\beta\alpha} + \gamma^\mu_{\alpha\beta}\theta_\beta\partial_\mu = (\gamma_5 E)_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} + \gamma^\mu_{\alpha\beta}\theta_\beta\partial_\mu$).

For a (and b) Grassmann, $\frac{\partial}{\partial a}$ is *left derivative*: (anti)commute a left then remove it, e.g. $\frac{\partial}{\partial a}ba = -\frac{\partial}{\partial a}ab = -b$.

This choice satisfies condition on $\mathscr{Q}$: We use $(\gamma_5 E)^2 = -\mathbf{1}$, $(\gamma_5 E)^T = -\gamma_5 E$ (direct calculation), and $\left(\frac{\partial}{\partial\overline{\theta}}\right)_\alpha = \left(\frac{\partial}{\partial\theta}\right)_\beta (\gamma_5 E)_{\beta\alpha}$

from Majorana conjugation on page 91. Now $\overline{\mathscr{Q}}_\delta = \mathscr{Q}^\dagger_\alpha\beta_{\alpha\delta} = -\left(\frac{\partial}{\partial\theta^\dagger}\right)_\beta (\gamma_5 E)_{\beta\alpha}\beta_{\alpha\delta} + \gamma^{\mu *}_{\alpha\beta}\theta^\dagger_\beta\beta_{\alpha\delta}\partial_\mu$. But $\gamma^{\mu *} = -\beta\gamma^{\mu T}\beta$,

and $\beta^2 = \mathbf{1}$ so that $\gamma^{\mu *}_{\alpha\beta}\theta^\dagger_\beta\beta_{\alpha\delta} = -\overline{\theta}_\alpha\gamma^\mu_{\alpha\delta}$. Thus, using $\{\gamma_5 E, \beta\} = 0$, $\overline{\mathscr{Q}}_\delta = -\left(\frac{\partial}{\partial\theta}\right)_\alpha (\gamma_5 E)_{\alpha\beta}(\gamma_5 E)_{\beta\delta} - \theta_\alpha(\gamma_5 E)_{\alpha\beta}\gamma^\mu_{\beta\delta}\partial_\mu$.

So $\overline{\mathscr{Q}}_\delta = \left(\frac{\partial}{\partial\theta}\right)_\delta - \theta_\alpha(\gamma_5 E\gamma^\mu)_{\alpha\delta}\partial_\mu$. Then $\{\mathscr{Q}_\alpha, \overline{\mathscr{Q}}_\beta\} = \delta_{\sigma\beta}(\gamma_5 E)_{\beta\alpha}(\gamma_5 E\gamma^\mu)_{\sigma\delta}\partial_\mu + \gamma^\mu_{\alpha\delta}\partial_\mu = 2\gamma^\mu_{\alpha\delta}\partial_\mu$.

Superfields from superfields I: $S = S_1 + S_2$, $S = S_1 S_2$, etc. are superfields when S_1, S_2 are superfields.

1st case clearly obeys SUSY transformation using $\mathscr{Q}$ above.

2nd case: From product rule for δ on page 91, $\delta S_1 S_2 = (\delta S_1) S_2 + S_1 \delta S_2 = (\overline{\alpha}\mathscr{Q} S_1) S_2 + S_1 \overline{\alpha}\mathscr{Q} S_2 = \overline{\alpha}\mathscr{Q}(S_1 S_2)$.

Superfields from superfields II: $S' = \mathscr{D}_\beta S$, $S' = \overline{\mathscr{D}}_\beta S$,

where *superderivative* $\mathscr{D} = -\frac{\partial}{\partial\overline{\theta}} - \gamma^\mu \theta \partial_\mu$ obeys $\{\mathscr{D}_\beta, \mathscr{Q}_\gamma\} = \{\overline{\mathscr{D}}_\beta, \mathscr{Q}_\gamma\} = 0$.

First case: $\delta S' = \delta \mathscr{D}_\beta S = [I\overline{\alpha}Q, \mathscr{D}_\beta S] = \mathscr{D}_\beta [I\overline{\alpha}Q, S]$ since $\overline{\alpha}Q$ is a commuting object.

Then $\delta S' = \mathscr{D}_\beta \overline{\alpha}_\gamma \mathscr{Q}_\gamma S = -\overline{\alpha}_\gamma \mathscr{D}_\beta \mathscr{Q}_\gamma S$ since $\overline{\alpha}_\gamma$, $\mathscr{D}_\beta$ anticommute. Then $\delta S' = \overline{\alpha}_\gamma \mathscr{Q}_\gamma \mathscr{D}_\beta S = \overline{\alpha}\mathscr{Q} S'$.

Summary: Function of superfields and their superderivatives is a superfield.

***R* quantum number assignments for superspace**: $\theta_{L/R}$ has R quantum number $\mathscr{R} = \pm 1$.

Recall definition of R-symmetry on page 84: $Q_a \to e^{I\mathscr{R}_L \phi} Q_a$ and $Q_a^\dagger \to e^{I\mathscr{R}_R \phi} Q_a^\dagger$, where $\mathscr{R} = \mathscr{R}_{L/R} = \pm 1$.

Assignments for superspace follow from definition of superfield (i.e. $[Q, S(x,\theta)] = -I\mathscr{Q}S(x,\theta)$) and definition of $\mathscr{Q}$ on page 94.

Note $\theta_R \sim \theta_L^*$.

From now on, write e.g. $A_\mu\gamma^\mu = \not{A}$.

General form of superfield: $S(x,\theta) = C(x) - I[\overline{\theta}\gamma_5]\omega(x) - \frac{I}{2}\left[\overline{\theta}\gamma_5\theta\right] M(x) - \frac{1}{2}[\overline{\theta}\theta]N(x) + \frac{I}{2}[\overline{\theta}\gamma_5\gamma_\mu\theta]V^\mu(x)$

$$-I[(\overline{\theta}\gamma_5\theta)\overline{\theta}]\left(\lambda(x) + \tfrac{1}{2}\not{\partial}\omega(x)\right) - \tfrac{1}{4}[\overline{\theta}\gamma_5\theta]^2\left(D(x) + \tfrac{1}{2}\partial^2 C(x)\right),$$

4 (Pseudo)scalar fields: C, M, N, D, 2 Spinor fields: ω, λ, 1 Vector field: V^μ

(i.e. 8 bosonic and 8 fermionic degrees of freedom).

This is most general Taylor series in $\theta^T = (\theta_1, \theta_2, \theta_3, \theta_4)$, or $\overline{\theta} = \theta^T\gamma_5 E = (-\theta_2, \theta_1, \theta_4, -\theta_3)$, in manifestly Lorentz invariant form:

2nd term $-I\overline{\theta}\gamma_5\omega(x)$ most general quantity linear in θ_α.

Next 3 bilinears in θ_α is expansion for any bilinear $B = B_{12}\theta_1\theta_2 + B_{13}\theta_1\theta_3 + B_{14}\theta_1\theta_4 + B_{23}\theta_2\theta_3 + B_{24}\theta_2\theta_4 + B_{34}\theta_3\theta_4$:

$\overline{\theta}\gamma_5(\frac{1}{I}\gamma_0 \text{ or } \frac{1}{I}\gamma_3)\theta = \underline{2\theta_2\theta_3 \pm 2\theta_4\theta_1}$, $\overline{\theta}\gamma_5(\frac{1}{I}\gamma_1 \text{ or } \frac{1}{I}\gamma_2)\theta = \underline{2\theta_2\theta_4 \pm 2\theta_3\theta_1}$ (gamma matrices on page 43), $\overline{\theta}(\mathbf{1} \text{ or } \gamma_5)\theta = \underline{2\theta_1\theta_2 \pm 2\theta_4\theta_3}$.

6th term has all 4 possible cubics in θ_α: $(\overline{\theta}\gamma_5\theta)\overline{\theta} = 2(\theta_4\theta_3\theta_2, -\theta_4\theta_3\theta_1, \theta_1\theta_2\theta_4, -\theta_1\theta_2\theta_3)$. Last term $\propto [\overline{\theta}\gamma_5\theta]^2 = 8\theta_1\theta_2\theta_3\theta_4$.

Any higher products of θ_α vanish, since $\theta_\alpha^2 = 0$ (no sum). E.g. $\theta_1\theta_2\theta_3\theta_4\theta_3 = -\theta_1\theta_2\theta_3^2\theta_4 = 0$.

Superfield transformation: $\delta S = I\overline{\alpha}\gamma_5\omega + \overline{\theta}(-\partial\!\!\!/ C + I\gamma_5 M + N - I\gamma_5 V\!\!\!/)\alpha + \frac{I}{2}[\overline{\theta}\gamma_5\theta]\overline{\alpha}(\lambda + \partial\!\!\!/\omega)$

$$-\tfrac{I}{2}[\overline{\theta}\theta]\overline{\alpha}\gamma_5(\lambda + \partial\!\!\!/\omega) + \tfrac{I}{2}[\overline{\theta}\gamma_5\gamma^\mu\theta]\overline{\alpha}\gamma_\mu\lambda + \tfrac{I}{2}[\overline{\theta}\gamma_5\gamma^\nu\theta]\overline{\alpha}\partial_\nu\omega$$

$$+\tfrac{1}{2}[(\overline{\theta}\gamma_5\theta)\overline{\theta}]\left(I\partial\!\!\!/ M - \gamma_5\partial\!\!\!/ N - I\partial_\mu V\!\!\!/\gamma^\mu + \gamma_5(D + \tfrac{1}{2}\partial^2 C)\right)\alpha - \tfrac{I}{4}[\overline{\theta}\gamma_5\theta]^2\overline{\alpha}\gamma_5(\partial\!\!\!/\lambda + \tfrac{1}{2}\partial^2\omega).$$

From definition of superfield and of $\mathscr{Q}$, on page 94, i.e. $\delta S = \overline{\alpha}\left(-\frac{\partial}{\partial\overline{\theta}} + \gamma^\mu\theta\partial_\mu\right)S$.

Note for $M = \mathbf{1}$, $\gamma_5\gamma_\mu$ and γ_5, $\frac{\partial}{\partial\overline{\theta}}(\overline{\theta}M\theta) = 2M\theta$. See also Weinberg III (hardback), page 63.

Component field transformations: $\delta C = I(\overline{\alpha}\gamma_5\omega)$, $\delta\omega = (-I\gamma_5\partial\!\!\!/ C - M + I\gamma_5 N + V\!\!\!/)\alpha$,

$\delta M = -\overline{\alpha}(\lambda + \partial\!\!\!/\omega)$, $\delta N = I\overline{\alpha}\gamma_5(\lambda + \partial\!\!\!/\omega)$, $\delta V_\mu = \overline{\alpha}(\gamma_\mu\lambda + \partial_\mu\omega)$,

$\delta\lambda = (\frac{1}{2}[\partial_\mu V\!\!\!/, \gamma^\mu] + I\gamma_5 D)\alpha$ and $\delta D = I\overline{\alpha}\gamma_5\partial\!\!\!/\lambda$.

Compare superfield transformation above with general form of superfield on page 96.

D-component of general superfield (coefficient of $[\overline{\theta}\gamma_5\theta]^2$) is candidate for SUSY Lagrangian.

If $\mathscr{L} \propto [S]_D \propto D + \frac{1}{2}\partial^2 C$, then $\delta\mathscr{L} \propto \delta D + \frac{1}{2}\partial^2\delta C = \partial_\mu(I\overline{\alpha}\gamma_5\gamma^\mu\lambda + \frac{1}{2}\partial^\mu\delta C)$, i.e. a derivative,

so action $\mathscr{A} = \int d^4x\mathscr{L}$ obeys $\delta\mathscr{A} = 0$.

3.4.1 Chiral superfield

Definition of chiral superfield $X(x,\theta)$: $\lambda = D = 0$, $V_\mu = \partial_\mu B$.

This is superfield because these conditions are preserved by SUSY transformations: $\delta D = I\overline{\alpha}\gamma_5 \partial\!\!\!/ \lambda = 0$,

and $\delta\lambda = (\frac{1}{2}[\partial_\mu V\!\!\!/, \gamma^\mu] + I\gamma_5 D)\alpha = \frac{1}{2}[\partial_\mu V\!\!\!/, \gamma^\mu]\alpha$. So require $[\partial_\mu V\!\!\!/, \gamma^\mu] = 0$.

But $[\partial_\mu V\!\!\!/, \gamma^\mu] = \partial_\mu V_\nu[\gamma^\nu, \gamma^\mu] = (\partial_\mu V_\nu - \partial_\nu V_\mu)\{\gamma^\nu, \gamma^\mu\} = 2g^{\nu\mu}(\partial_\mu V_\nu - \partial_\nu V_\mu)$, so $\partial_\mu V_\nu - \partial_\nu V_\mu = 0$ which requires $V_\mu = \partial_\mu B$.

Finally, $\delta V^\mu = \overline{\alpha}(\gamma_\mu\lambda + \partial_\mu\omega) = \partial_\mu\overline{\alpha}\omega$, i.e $\delta B = \overline{\alpha}\omega$, i.e. $V_\mu = \partial_\mu B$ condition is preserved.

Chiral superfield decomposition: $X(x,\theta) = \frac{1}{\sqrt{2}}[\Phi_+(x,\theta) + \Phi_-(x,\theta)]$ with left / right-chiral superfields

$$\Phi_\pm(x,\theta) = \phi_\pm(x) - \sqrt{2}\,\overline{\theta}\psi_{L/R}(x) + [\overline{\theta}P_{L/R}\theta]\mathscr{F}_\pm(x) \pm \tfrac{1}{2}[\overline{\theta}\gamma_5\gamma_\mu\theta]\partial^\mu\phi_\pm(x) \mp \tfrac{1}{\sqrt{2}}(\overline{\theta}\gamma_5\theta)\overline{\theta}\partial\!\!\!/\psi_{L/R}(x) - \tfrac{1}{8}[\overline{\theta}\gamma_5\theta]^2\partial^2\phi_\pm(x),$$

with $\psi_{L/R} = P_{L/R}\psi$, $\phi_\pm = \frac{1}{\sqrt{2}}(A \pm IB)$, $\mathscr{F}_\pm = \frac{1}{\sqrt{2}}(F \mp IG)$. Recall $\psi^T = \frac{1}{\sqrt{2}}\left(\zeta_a, \zeta^{\dagger\dot{b}}\right)$ from page 92.

Write $C = A$, $\omega = -I\gamma_5\psi$, $M = G$, $N = -F$, $Z = B$ for later convenience in general form of superfield on page 96,

so $X = A - \overline{\theta}\psi + \frac{1}{2}\overline{\theta}\theta F - \frac{I}{2}\overline{\theta}\gamma_5\theta G + \frac{I}{2}\overline{\theta}\gamma_5\gamma_\mu\theta\partial^\mu B + \frac{1}{2}(\overline{\theta}\gamma_5\theta)\overline{\theta}\gamma_5\partial\!\!\!/\psi - \frac{1}{8}[\overline{\theta}\gamma_5\theta]^2\partial^2 A$.

Left / right transformation: $\delta\psi_{L/R} = \sqrt{2}(\partial_\mu\phi_\pm\gamma^\mu P_{R/L} + \mathscr{F}_\pm P_{L/R})\alpha$, $\delta\mathscr{F}_\pm = \sqrt{2}\overline{\alpha}\partial\!\!\!/\psi_{L/R}$, $\delta\phi_\pm = \sqrt{2}\overline{\alpha}\psi_{L/R}$.

So left-chiral supermultiplet here = that on page 90 ($\zeta_a = \psi_L$, $\phi = \phi_+$, $\mathscr{F} = \mathscr{F}_+$), right-chiral by conjugation.

Compact form: $\Phi_\pm(x,\theta) = \phi_\pm(x_\pm) \mp \sqrt{2}\theta^T_{L/R}E\ \psi_{L/R}(x_\pm) \pm \theta^T_{L/R}E\theta_{L/R}\ \mathscr{F}_\pm(x_\pm)$,

where $\theta_{L/R} = P_{L/R}\theta$, $x^\mu_\pm = x^\mu \pm \theta^T_R E\gamma^\mu\theta_L$.

This is left / right-chiral superfield on page 98, by Taylor expansion in $\theta^T_R E\gamma^\mu\theta_L$:

Make use of $\theta^T = -\overline{\theta}\gamma_5 E$ (Majorana conjugation on page 91) and $E^2 = (\gamma_5 E)^2 = -\mathbf{1}$.

Firstly, rewrite as $\underline{\Phi_\pm(x,\theta) = \phi_\pm(x_\pm) - \sqrt{2}\overline{\theta}P_{L/R}\psi(x_\pm) + \overline{\theta}P_{L/R}\theta\mathscr{F}_\pm(x_\pm)}$,

because $\theta^T_{L/R}E\psi_{L/R} = -\overline{\theta}\gamma_5 E\frac{1}{2}(1\pm\gamma_5)E\frac{1}{2}(1\pm\gamma_5)\psi = -\overline{\theta}\frac{1}{2}(-\gamma_5\mp 1)\frac{1}{2}(1\pm\gamma_5)\psi = \pm\overline{\theta}P_{L/R}\psi$. Similarly, $\underline{x^\mu_\pm = x^\mu \pm \frac{1}{2}\overline{\theta}\gamma_5\gamma^\mu\theta}$, so

1st term in underlined equation above: $\phi_\pm(x_\pm) = \phi_\pm(x) \pm \frac{1}{2}(\overline{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\phi_\pm(x) - \frac{1}{2}(\frac{1}{2}\overline{\theta}\gamma_5\gamma^\mu\theta)(\frac{1}{2}\overline{\theta}\gamma_5\gamma^\nu\theta)\partial_\mu\partial_\nu\phi_\pm(x)$

$= \phi_\pm(x) \pm \frac{1}{2}(\overline{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\phi_\pm(x) - \frac{1}{2}(\frac{1}{2}\overline{\theta}\gamma_5\theta)^2\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu\phi_\pm(x)$, then use $\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} = g^{\mu\nu}$.

2nd term: $-\sqrt{2}\overline{\theta}P_{L/R}\psi(x_\pm) = -\sqrt{2}\overline{\theta}P_{L/R}\psi(x) \mp \frac{1}{\sqrt{2}}(\overline{\theta}\gamma_5\gamma^\mu\theta)\overline{\theta}\partial_\mu P_{L/R}\psi(x) = -\sqrt{2}\overline{\theta}P_{L/R}\psi(x) \mp \frac{1}{\sqrt{2}}(\overline{\theta}\gamma_5\theta)\overline{\theta}\gamma^\mu\partial_\mu P_{L/R}\psi(x)$.

3rd term: $\overline{\theta}P_{L/R}\theta\mathscr{F}_\pm(x_\pm) = \overline{\theta}P_{L/R}\theta\mathscr{F}_\pm(x) \pm \frac{1}{2}\overline{\theta}\gamma_5\gamma^\mu\theta\overline{\theta}P_{L/R}\theta\partial_\mu\mathscr{F}_\pm(x) = \overline{\theta}P_{L/R}\theta\mathscr{F}_\pm(x) + [(\theta^T_R E\gamma^\mu\theta_L)(\theta^T_{L/R}E\theta_{L/R})\partial_\mu\mathscr{F}_\pm(x)]$,

term in square brackets vanishes due to 3 occurences of “2-component” $\theta_{L/R}$.

Compact form on page 99 is most general function of $x_\pm$ and $\theta_{L/R}$.

Conjugate of right-chiral superfield is left-chiral superfield.

Superspace "direction" of chiral superfield: $\mathscr{D}_\mp \Phi_\pm = 0$, where $\mathscr{D}_\mp = P_{R/L}\mathscr{D}$.

Can be used to define left / right-chiral superfield.

Follows from $\mathscr{D}_\mp x^\mu_\pm = 0$, using $\mathscr{D}_\mp = \mp\epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_{R/L\ \beta}} - (\gamma^\mu\theta_{L/R})_\alpha\frac{\partial}{\partial x^\mu}$.

Chiral superfield from superfield: For general superfield S, $\mathscr{D}_{\pm\alpha}\mathscr{D}_{\pm\beta}S$ (i.e. $\mathscr{D}^T_\pm E\mathscr{D}_\pm S$) is left / right-chiral.

$\mathscr{D}_{\pm\gamma}(\mathscr{D}_{\pm\alpha}\mathscr{D}_{\pm\beta}S) = 0$ because the $\mathscr{D}_{\pm\alpha}$ anticommute and there are only 2 of them.

$\mathscr{F}$-component of chiral superfield (coefficient of $\overline{\theta}P_L\theta$) is candidate for SUSY Lagrangian.

If $\mathscr{L} \propto [\Phi_\pm]_{\mathscr{F}} = \mathscr{F}_\pm$, then $\delta\mathscr{L} \propto \overline{\alpha}\partial\!\!\!/\psi_{L/R}$, i.e. a derivative, so action $\mathscr{A} = \int d^4x\mathscr{L}$ obeys $\delta\mathscr{A} = 0$.

3.4.2 Supersymmetric Actions

From now on, only work with left-chiral superfields, write $\Phi_+ = \Phi$, $\phi_+ = \phi$, $\mathscr{F}_+ = \mathscr{F}$.

Supersymmetric action from chiral superfields: $\mathscr{A} = \int d^4x[f]_{\mathscr{F}} + \int d^4x[f]^*_{\mathscr{F}} + \int d^4x\frac{1}{2}[K]_D$,

where f is left-chiral, K is real superfield.

Superpotential f is polynomial in left-chiral superfields:

Recall function of left-chiral superfields (but not complex conjugates thereof) is left-chiral superfield.

Superderivatives (and therefore derivatives) of left-chiral superfield is not left-chiral.

Left-chiral superfield of form $\mathscr{D}_{+\alpha}\mathscr{D}_{+\beta}S$ (see chiral superfield from superfield on page 100) allowed in f,

then can write $f = \mathscr{D}_+^T E\mathscr{D}_+ h$, since $\mathscr{D}_+$ annihilates all left-chiral superfields after differentiating with product rule.

But $\mathscr{D}_+^T E\mathscr{D}_+ h \propto$ coeff. of $\theta_R^T E\theta_R$ (neglecting spacetime derivatives with don't contribute to $\mathscr{A}$ from now on).

Then $[\mathscr{D}_+^T E\mathscr{D}_+ h]_{\mathscr{F}} \propto$ coeff. of $(\theta_L^T E\theta_L)(\theta_R^T E\theta_R) = (\overline{\theta}\gamma_5\theta)^2$ in h, so $\int d^4x[\mathscr{D}_+^T E\mathscr{D}_+ h]_{\mathscr{F}} \propto \int d^4x[h]_D$,

i.e. $\mathscr{D}_{+\alpha}\mathscr{D}_{+\beta}S$ terms not necessary in f, can be included in $[K]_D$.

Kahler potential K depends on left / right-chiral superfields but is more general.

R quantum number assignments for potentials f and K: $\mathscr{R}_f = 2$, $\mathscr{R}_K = 0$.

First case: Want $\int d^4x[f]_\mathscr{F}$ to have $\mathscr{R}_\mathscr{F} = 0$. From chiral superfield decomposition on page 98, term in left-chiral superfield Φ

containing $\mathscr{F}$ component is $\overline{\theta} P_L \theta \mathscr{F} = \pm\theta_L^T E \theta_L \mathscr{F}$. Since $\mathscr{R}_{\theta_L^T E\theta_L} = 2$ and want $\mathscr{R}_\mathscr{F} = 0$, must have $\mathscr{R}_\Phi = 2$.

Second case: Want $\int d^4x[K]_D$ to have $\mathscr{R}_D = 0$. From general form of superfield on page 96,

term containing D component is $\propto [\overline{\theta}\gamma_5\theta]^2(D + \frac{1}{2}\partial^2 C)$. But $\mathscr{R}_{[\overline{\theta}\gamma_5\theta]^2} = 0$, so for $\mathscr{R}_D = 0$ must have $\mathscr{R}_S = 0$.

For <u>renormalizable theory of chiral superfields</u>:

Superpotential f is at most cubic in left-chiral superfields Φ_n.

Operators O in $\mathscr{L}$ must have mass dimension $d_M(O) \leq 4$ for renormalizability. From definition of $\mathscr{D}$, $d_M(\theta) = -\frac{1}{2}$.

So $d_M([S]_\mathscr{F}) = d_M(S) + 1$, $d_M([S]_\mathscr{D}) = d_M(S) + 2$, so operators (represented by S here) in f (K) have mass dimensionality $\leq 3(2$

Since $d_M(\Phi_\pm) = d_M(\phi_\pm) = 1$, f can only be cubic polynomial in $\Phi_{\pm n}$.

General form of Kahler potential: $K(\Phi, \Phi^*) = \sum_{mn} g_{mn} \Phi_m^* \Phi_n$, g_{mn} Hermitian.

As noted above, operators in K have mass dimensionality ≤ 2. $\Phi_m\Phi_n$ terms (or conjugates thereof) don't appear,

since they are left-chiral so have no D term.

Kahler potential part of $\mathscr{L}$: $\frac{1}{2}[K(\Phi,\Phi^*)]_D = -\partial_\mu\phi_n^*\partial^\mu\phi_n + \mathscr{F}_n^*\mathscr{F}_n - \frac{1}{2}\overline{\psi}_{nL}\gamma^\mu\partial_\mu\psi_{nL} + \frac{1}{2}(\partial_\mu\overline{\psi}_{nR})\gamma^\mu\psi_{nR}$.

Direct calculation gives $\frac{1}{2}[K(\Phi,\Phi^*)]_D = g_{mn}\left[-\partial_\mu\phi_m^*\partial^\mu\phi_n + \mathscr{F}_m^*\mathscr{F}_n - \frac{1}{2}\overline{\psi}_{mL}\gamma^\mu\partial_\mu\psi_{nL} + \frac{1}{2}(\partial_\mu\overline{\psi}_{mR})\gamma^\mu\psi_{nR}\right]$.

Take $\Phi_m = N_{mn}\Phi_n$, $g'_{mn} = (N^\dagger g N)_{mn}$. Since g_{mn} Hermitian, choose unitary N to diagonalize it.

Diagonal terms positive to get positive coefficient for $-\partial_\mu\phi_n'^*\partial^\mu\phi_n'$, absorb them into ϕ_n' then drop primes.

Superpotential part of $\mathscr{L}$: $[f(\Phi)]_{\mathscr{F}} = -\frac{1}{2}\frac{\partial^2 f(\phi)}{\partial\phi_n\partial\phi_m}\overline{\psi}_{nR}\psi_{mL} + \mathscr{F}_n\frac{\partial f(\phi)}{\partial\phi_n}$. Similar for f^*.

From direct calculation.

$\mathscr{L}$ from chiral superfields: $\mathscr{L} = -\partial_\mu\phi_n^*\partial^\mu\phi_n - \frac{1}{2}\overline{\psi}_{nL}\gamma^\mu\partial_\mu\psi_{nL} + \frac{1}{2}(\partial_\mu\overline{\psi}_{nR})\gamma^\mu\psi_{nR}$

$$-\frac{1}{2}\frac{\partial^2 f(\phi)}{\partial\phi_n\partial\phi_m}\overline{\psi}_{nR}\psi_{mL} - \frac{1}{2}\left(\frac{\partial^2 f(\phi)}{\partial\phi_n\partial\phi_m}\right)^*(\overline{\psi}_{nR}\psi_{mL})^* - V(\phi),$$

where $\mathscr{F}_n = -\left(\frac{\partial f(\phi)}{\partial\phi_n}\right)^*$ and *scalar field potential* $V(\phi) = \sum_n\left|\frac{\partial f(\phi)}{\partial\phi_n}\right|^2$.

Sum Kahler potential and superpotential parts (and conjugate) of $\mathscr{L}$ above, use field equations to get $\mathscr{F}_n = -\left(\frac{\partial f(\phi)}{\partial\phi_n}\right)^*$.

Minimized $V(\phi_0) \geq 0$, equality (tree level) only when $\left.\frac{\partial f(\phi)}{\partial \phi_n}\right|_{\phi=\phi_0} = 0$ (note $\phi_{n0} = \langle 0|\phi_n|0\rangle$).

Tree-level expansion of V: $V(\phi) = V(\phi_0) + (\mathscr{M}^\dagger \mathscr{M})_{mn} \Delta\phi_m^\dagger \Delta\phi_n$,

where $\Delta\phi_n = \phi_n - \phi_{n0}$, bosonic mass matrix $\mathscr{M}_{mn} = \left.\frac{\partial^2 f}{\partial \phi_m \partial \phi_n}\right|_{\phi=\phi_0}$.

From Taylor's theorem and $\sum_n \mathscr{M}_{mn} \left.\left(\frac{\partial f}{\partial \phi_n}\right)^*\right|_{\phi=\phi_0} = 0$ (see page 105),

$$V = \sum_n \left|\frac{\partial f}{\partial \phi_n}\right|^2 \simeq \sum_n \left|\left.\frac{\partial f}{\partial \phi_n}\right|_{\phi=\phi_0} + \left.\frac{\partial^2 f}{\partial \phi_m \partial \phi_n}\right|_{\phi=\phi_0} \Delta\phi_m\right|^2 = V(\phi_0) + \Delta\phi_m^\dagger (\mathscr{M}^\dagger \mathscr{M})_{mn} \Delta\phi_n.$$

Matrix $\mathscr{M}$ can be diagonalised because it is complex symmetric.

From $\mathscr{L}$ from chiral superfields on page 103, fermionic mass term is $-\frac{1}{2}\mathscr{M}_{mn}\overline{\psi}_{nR}\psi_{mL}$,

so fermions and bosons have same mass, as required in SUSY.

Free Lagrangian density: $\mathscr{L}_0 = \sum_n \big[-\partial_\mu \Delta\phi_n^* \partial^\mu \Delta\phi_n - m_n^2 \Delta\phi_n^* \Delta\phi_n$

$$-\tfrac{1}{2}\overline{\psi}_{nL}\gamma^\mu \partial_\mu \psi_{nL} + \tfrac{1}{2}(\partial_\mu \overline{\psi}_{nR})\gamma^\mu \psi_{nR} - \tfrac{1}{2}m_n \overline{\psi}_{nR}\psi_{nL} - \tfrac{1}{2}m_n(\overline{\psi}_{nR}\psi_{nL})^* \big].$$

$$\mathscr{L}_0 = -\partial_\mu \Delta\phi_n^* \partial^\mu \Delta\phi_n - (\mathscr{M}^\dagger \mathscr{M})_{mn} \Delta\phi_m^* \Delta\phi_n - \tfrac{1}{2}\overline{\psi}_{nL}\gamma^\mu \partial_\mu \psi_{nL} + \tfrac{1}{2}(\partial_\mu \overline{\psi}_{nR})\gamma^\mu \psi_{nR} - \tfrac{1}{2}\mathscr{M}_{mn}\overline{\psi}_{nR}\psi_{mL} - \tfrac{1}{2}\mathscr{M}_{mn}^*(\overline{\psi}_{nR}\psi_{mL})^*$$

from $\mathscr{L}$ from chiral superfields on page 103 and tree-level expansion of V above, then diagonalize $\mathscr{M}$.

3.5 Spontaneous supersymmetry breaking

Broken SUSY vacuum condition: $\langle 0|\mathscr{F}|0\rangle \neq 0$ or $\left.\frac{\partial f(\phi)}{\partial \phi_n}\right|_{\phi=\phi_0} \neq 0$ or $V(\phi_0) > 0$.

In left / right transformation on page 99, want to make one of $\langle 0|\delta\Psi_l|0\rangle \neq 0$, where $\Psi_l = \psi_{nL}$, ϕ_n or $\mathscr{F}_n$.

Cannot make $\langle 0|\psi_{nL}|0\rangle \neq 0$ since vacuum Lorentz invariant, nor $\langle 0|\partial_\mu \phi_n|0\rangle \neq 0$ since $\langle 0|\phi_n|0\rangle$ constant.

Only possibility is $\langle 0|\mathscr{F}_n|0\rangle \neq 0$ (whence $\langle 0|\delta\psi_{nL}|0\rangle = \sqrt{2}\overline{\alpha}\langle 0|\mathscr{F}_n|0\rangle$),

which from $\mathscr{L}$ from chiral superfields on page 103 is equivalent to last 2 statements.

Spontaneous SUSY breaking requires f such that $\left.\frac{\partial f(\phi)}{\partial \phi_n}\right|_{\phi=\phi_0} = 0$ has no solution.

Spontaneous SUSY breaking gives rise to massless spin $\frac{1}{2}$ *goldstino*.

$\left.\frac{\partial V}{\partial \phi_n}\right|_{\phi=\phi_0} = 0$. But $V = \sum_n \left|\frac{\partial f}{\partial \phi_n}\right|^2$, so $2\sum_n \frac{\partial^2 f}{\partial \phi_m \partial \phi_n}\left(\frac{\partial f}{\partial \phi_n}\right)^*\Big|_{\phi=\phi_0} = 0$, i.e. $\sum_n \mathscr{M}_{mn}\left(\frac{\partial f}{\partial \phi_n}\right)^*\Big|_{\phi=\phi_0} = 0$.

So $\mathscr{M}$ has at least one zero eigenvalue, so from free Lagrangian density on page 104, since $\mathscr{M}$ eigenvalues are the m_n,

there is at least one linear combination of ψ_{nL} with zero mass.

In local SUSY, goldstino absorbed into longitudinal component of gravitino, gives it a mass.

3.5.1 O'Raifeartaigh Models

Theories in which left-chiral fields X_n, Y_i have $\mathscr{R} = 0, 2$. Most general superpotential is $f(X,Y) = \sum_i Y_i f_i(X)$.

From R quantum number assignments for potentials f and K on page 102, i.e. $\mathscr{R}_f = 2$ (X^*, Y^* not allowed since f left-chiral).

SUSY broken when no. fields $X <$ no. fields Y.

Write scalar components of X, Y as x, y. Condition $\frac{\partial f(x,y)}{\partial y_i} = 0$ implies $f_i(x) = 0$, i.e. more conditions than fields X_i, only possible to satisfy by careful choice of the $f_i(x)$.

Scalar field potential: $V = \sum_i |f_i(x)|^2 + \sum_n \left| \sum_i y_i \frac{\partial f_i(x)}{\partial x_n} \right|^2$.

From general superpotential above and definition of scalar field potential in $\mathscr{L}$ from chiral superfields on page 103.

Simplest (renormalizable) model: Fields X, Y_1, Y_2.

Then choice $f_1(X) = X - a$, $f_2(X) = X^2$ is general, for which $V = |x|^4 + |x-a|^2 + |y_1 + 2xy_2|^2$.

Renormalizability allows f_i to be quadratic only. Then take linear combinations of Y_i and shift and rescale X.

Vacuum matrix elements: $x_0 = y_{10} = 0$, y_{20} arbitrary, or $\langle 0|\mathscr{F}_{y_1}|0\rangle = \langle 0|\mathscr{F}_{y_2}|0\rangle = 0$ and $\langle 0|\mathscr{F}_x|0\rangle = a$.

Solve $\mathscr{F}_w^\dagger = -\frac{\partial V}{\partial w} = 0$ where $w = x^{(\dagger)}$, $y_1^{(\dagger)}$, $y_2^{(\dagger)}$. Recall $w_0 = \langle 0|w|0\rangle$.

3.6 Supersymmetric gauge theories

Gauge transformation of left-chiral supermultiplet: $(\phi, \psi_L, \mathscr{F})_n(x) \to \left(\exp[It_A\Lambda^A(x)]\right)_{nm}(\phi, \psi_L, \mathscr{F})_m(x)$.

Fields in same supermultiplet have same transformation properties under internal symmetry transformation U:

Since $[Q, a^{(\dagger)}_{B/F}]_{\mp} \sim a^{(\dagger)}_{F/B}$, where $a^{(\dagger)}_{B/F}$ are annihilation (creation) operators for bosonic and fermionic superpartners,

general transformation of $a^{(\dagger)}_{\sigma}$ on page 9 is same for $a^{(\dagger)}_{B}$, $a^{(\dagger)}_{F}$ because $[U, Q] = 0$ from page 85.

So from complete field on page 26, transformation same for all fields in supermultiplet.

Gauge transformation of left-chiral superfields: $\Phi(x, \theta) \to \exp[It_A\Lambda^A(x_+)]\Phi(x, \theta)$ (column vector Φ).

Implied by gauge transformation of left-chiral supermultiplet above and compact form on page 99.

Gauge transformation of right-chiral superfields: $\Phi^\dagger(x, \theta) \to \Phi^\dagger(x, \theta)\exp[-It_A\Lambda^A(x_-)]$.

From conjugate of gauge transformation of left-chiral superfields above. Note $\Lambda^A(x)$ is real function of x and $\Lambda^{A*}(x_+) = \Lambda^A(x_-)$.

Global gauge invariance $\Longrightarrow$ local for $\int d^4x[f(\Phi)]_{\mathscr{F}}$ but not for $\int d^4x[K(\Phi, \Phi^\dagger)]_D$.

No derivatives / conjugates of Φ in f, so argument y of $\Lambda^A(y)$ irrelevant. But K also contains $\Phi^\dagger$,

so e.g. $\Phi^\dagger_n\Phi_n$ not invariant under gauge transformation of left and right-chiral superfield above because $\Lambda^A(x_+) \neq \Lambda^A(x_-)$.

Define *gauge connection* $\Gamma(x,\theta)$, with transformation property

Gauge transformation of gauge connection: $\Gamma(x,\theta) \to \exp[It_A\Lambda^A(x_-)]\Gamma(x,\theta)\exp[-It_A\Lambda^A(x_+)]$,

then "new" right-chiral superfield $\Phi^\dagger(x,\theta)\Gamma(x,\theta)$ has transformation property

Gauge transformation of new right-chiral superfield: $\Phi^\dagger(x,\theta)\Gamma(x,\theta) \to \Phi^\dagger(x,\theta)\Gamma(x,\theta)\exp[-It_A\Lambda^A(x_+)]$.

Global gauge invariance $\implies$ local for $\int d^4x[K(\Phi,\Phi^\dagger\Gamma)]_D$. Also implies invariance under

Extended gauge transformations: $\Phi(x,\theta) \to \exp[It_A\Omega^A(x,\theta)]\Phi(x,\theta)$,

$\Gamma(x,\theta) \to \exp[It_A\Omega^{A\dagger}(x,\theta)]\Gamma(x,\theta)\exp[-It_A\Omega^A(x,\theta)]$ with any left-chiral superfields $\Omega^A(x,\theta)$.

We are extending $\Lambda^A(x_+)$ to $\Omega^A(x,\theta) = \Lambda^A(x_+) - \sqrt{2}\theta_L^T E\,\psi_{\Lambda L}(x_+) + \theta_L^T E\theta_L\,\mathscr{F}_{\Lambda+}(x_+)$ in compact form on page 99.

$\Phi(x,\theta)$ is left-chiral (depends on x_+, θ_L only), so cannot introduce x_- or θ_R in $\Omega^A(x,\theta)$.

Hermitian connection: $\Gamma(x,\theta) = \Gamma^\dagger(x,\theta)$.

Take $\Gamma \to \frac{1}{2}(\Gamma+\Gamma^\dagger)$ or $\frac{1}{2I}(\Gamma-\Gamma^\dagger)$. Ordinary gauge transformation of Γ above intact, since Γ, $\Gamma^\dagger$ have same transformation.

Connection from gauge superfields: $\Gamma(x,\theta) = \exp[-2t_AV^A(x,\theta)]$, with real *gauge superfields* V_A.

Form preserved by gauge transformation of gauge connection above, new V^A independent of t_A representation.

Extended gauge transformation of gauge superfields: $V^A(x,\theta) \to V^A(x,\theta) + \frac{I}{2}[\Omega^A(x,\theta) - \Omega^{A\dagger}(x,\theta)] + \ldots$,

where "..." = commutators of generators, which vanish for zero coupling and in U(1) (Abelian).

From extended gauge transformations and definition of gauge superfields above.

Gauge superfields in terms of components: $V^A(x,\theta) = C^A(x) - I\bar{\theta}\gamma_5\omega^A(x) - \frac{I}{2}\bar{\theta}\gamma_5\theta M^A(x) - \frac{1}{2}\bar{\theta}\theta N^A(x)$

$$+\frac{I}{2}\bar{\theta}\gamma_5\gamma^\mu\theta V^A_\mu(x) - I\bar{\theta}\gamma_5\theta\bar{\theta}[\lambda^A(x) + \frac{1}{2}\not{\partial}\omega^A(x)] - \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2(D^A(x) + \frac{1}{2}\partial^2 C^A(x)).$$

From general form of superfield on page 96.

Transformation superfields in terms of components: $\Omega^A(x,\theta) = W^A(x) - \sqrt{2}\bar{\theta}P_L w^A(x) + \mathscr{W}^A(x)\bar{\theta}P_L\theta$

$$+\frac{1}{2}\bar{\theta}\gamma_5\gamma_\mu\theta\partial^\mu W^A(x) - \frac{1}{\sqrt{2}}\bar{\theta}\gamma_5\theta\bar{\theta}\not{\partial}P_L w^A(x) - \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\partial^2 W^A(x).$$

From chiral superfield decomposition on page 98, taking left-chiral part (Φ^+ there).

Transformation of supermultiplet fields: $C^A(x) \to C^A(x) - \mathrm{Im}W^A(x) + \ldots$,

$\omega^A(x) \to \omega^A(x) + \frac{1}{\sqrt{2}}w^A(x) + \ldots$, $V^A_\mu(x) \to V^A_\mu(x) + \partial_\mu \mathrm{Re}W^A(x) + \ldots$, $M^A(x) \to M^A(x) - \mathrm{Re}\mathscr{W}^A(x) + \ldots$,

$N^A(x) \to N^A(x) + \mathrm{Im}\mathscr{W}^A(x) + \ldots$, $\lambda^A(x) \to \lambda^A(x) + \ldots$, $D^A(x) \to D^A(x) + \ldots$.

From all 3 above results. Again, "..." = commutators of generators, which vanish for zero coupling and in U(1) (Abelian).

Wess-Zumino gauge: $C^A, \omega^A, M^A, N^A = 0$ → $V^A(x,\theta) = \frac{I}{2}\overline{\theta}\gamma_5\gamma^\mu\theta V^A_\mu(x) - I\overline{\theta}\gamma_5\theta\overline{\theta}\lambda^A(x) - \frac{1}{4}(\overline{\theta}\gamma_5\theta)^2 D^A(x)$.

Transformation of supermultiplet fields on page 109: take $\text{Im}W^A(x) = C^A(x)$, $w^A(x) = -\sqrt{2}\omega^A(x)$, $\mathscr{W}^A(x) = M^A(x) - IN^A(x)$.

Sufficient for Abelian case because "…" = 0. For non-Abelian case, add nth order (in gauge couplings) terms

to $\text{Im}W^A(x)$, $w^A(x)$, $\mathscr{W}^A(x)$ to cancel nth order terms from commutators of $m \leq n-1$th order terms.

Wess-Zumino gauge is not supersymmetric.

Ensuring $\delta C^A, \delta\omega^A, \delta M^A, \delta N^A = 0$ in component field transformations on page 97, requires also having $V^A_\mu, \lambda^A = 0$,

which requires $\delta V^A_\mu, \delta\lambda^A = 0$. $\delta V^A_\mu = 0$ satisfied, but $\delta\lambda^A = 0$ requires $D^A = 0$, i.e. $V^A = 0$.

Gauge invariant $\mathscr{L}$ for chiral supermultiplet: $\frac{1}{2}[\Phi^\dagger\Gamma\Phi]_D = -\frac{1}{2}(D_\mu\phi)^\dagger D^\mu\phi - \frac{1}{2}\overline{\psi}_L\gamma^\mu D_\mu\psi_L + \frac{1}{2}\mathscr{F}^\dagger\mathscr{F}$

$+I\sqrt{2}\overline{\psi}_L t_A\lambda^A\phi - \frac{1}{2}D^A\phi^\dagger t_A\phi + \text{h.c.}$, where covariant derivative $D_\mu = \partial_\mu - It_A V^A_\mu$

(gauge transformation for gauge superfield derived next on page 111).

In Wess-Zumino gauge, $\Gamma(x,\theta) = 1 - I\overline{\theta}\gamma_5\gamma_\mu\theta t_A V^A_\mu(x) - \frac{1}{2}\overline{\theta}\gamma_5\gamma^\mu\theta\overline{\theta}\gamma_5\gamma^\nu\theta t_A t_B V^A_\mu(x)V^B_\nu(x) + 2I\overline{\theta}\gamma_5\theta t_A\overline{\theta}\lambda^A(x) + \frac{1}{2}(\overline{\theta}\gamma_5\theta)^2 t_A D^A(x)$.

Then use left-chiral superfield in chiral superfield decomposition on page 98.

Note D term is coefficient of $-\frac{1}{4}(\overline{\theta}\gamma_5\theta)^2$ minus $\frac{1}{2}\partial^2\times$ first "ϕ" term.

Gauge transformation of gauge supermultiplet fields: $\delta_{\text{gauge}} V_\mu^A(x) = C_{ABC} V_\mu^B(x)\Lambda^C(x) + \partial_\mu \Lambda^A(x)$

(which is infinitesimal version of transformation of gauge fields on page 62), and

$\delta_{\text{gauge}}\lambda^A(x) = C_{ABC}\lambda^B(x)\Lambda^C(x)$, $\delta_{\text{gauge}} D^A(x) = C_{ABC} D^B(x)\Lambda^C(x)$ (i.e. λ^A, D^A in adjoint representation).

Firstly $\Lambda^A(x_+) = \Lambda^A(x) + \frac{1}{2}(\overline{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu\Lambda^A(x) - \frac{1}{8}(\overline{\theta}\gamma_5\theta)^2\partial^2\Lambda^A(x)$. Then gauge transformation of gauge connection on page 108

(using connection from gauge superfields) reads $\exp[-2t_A V^{A\prime}(x,\theta)] = \exp[It_A\Lambda^{A\dagger}(x_+)]\exp[-2t_A V^A(x,\theta)]\exp[-It_A\Lambda^A(x_+)]$.

Write as $\underline{\exp[-2t_A V^{A\prime}(x,\theta)] = e^a e^X e^b}$ where $\boxed{X = -2t_A V^A(x,\theta) = -2t_A[\frac{I}{2}\overline{\theta}\gamma_5\gamma_\mu\theta V_\mu^A(x) - I\overline{\theta}\gamma_5\theta\overline{\theta}\lambda^A(x) - \frac{1}{4}(\overline{\theta}\gamma_5\theta)^2 D^A(x)]}$, and

small quantities $\boxed{b + a = 2t_A\text{Im}\Lambda^A(x_+) = -I\overline{\theta}\gamma_5\gamma_\mu\theta t_A\partial^\mu\Lambda^A(x)}$, $\boxed{b - a = -2It_A\text{Re}\Lambda^A(x_+) = -2It_A[\Lambda^A(x) - \frac{1}{8}(\overline{\theta}\gamma_5\theta)^2\partial^2\Lambda^A(x)]}$.

To first order in a, b, $\underline{-2t_A V^{A\prime}(x,\theta) = X + \frac{1}{2}[X, b-a] + b + a + \ldots}$ and "$\ldots$" means higher order in a, b,

as well as terms of $O(X^n)O(b-a)$ with integer $n \geq 2$ which vanish: since $(\mathbf{1}, \gamma_5) = P_L \pm P_R$,

$X \sim \theta_L\theta_R(\times$ further θ factors), while $b + a \sim \theta_L\theta_R$, so $O(X^2)O(b-a) \sim \theta_L^3\theta_R^3(\times\ldots) = 0$.

So $\underline{V^{A\prime}(x,\theta) = V^A(x,\theta) + C_{ABC}V^B(x,\theta)\Lambda^C(x) + \frac{I}{2}\overline{\theta}\gamma_5\gamma_\mu\theta\partial^\mu\Lambda^A(x)}$,

compare superfield expansion in Wess-Zumino gauge on page 110.

Wess-Zumino gauge is gauge invariant (i.e. C^A, ω^A, M^A, N^A remain zero) under ordinary gauge transformations.

Note θ dependence of $V^{A\prime}(x,\theta)$ same as $V^A(x,\theta)$, i.e. $C^{A\prime}, \omega^{A\prime}, M^{A\prime}, N^{A\prime} = 0$.

3.6.1 Gauge-invariant Lagrangians

Construct gauge-invariant Lagrangian from

Gauge-covariant spinor superfield: $2t_A W^A_{L\alpha}(x,\theta) = \mathscr{D}_-^T E \mathscr{D}_- \exp[2t_A V^A(x,\theta)] \mathscr{D}_{+\alpha} \exp[-2t_A V^A(x,\theta)]$

which is left-chiral because $\mathscr{D}_- W_L = 0$ ($\mathscr{D}_{\alpha-}\mathscr{D}_{\beta-}\mathscr{D}_{\gamma-} = 0$), i.e.

Extended gauge transformation of W^A_L: $2t_A W^A_{L\alpha}(x,\theta) \to \exp[It_A\Omega^A(x,\theta)] 2t_A W^A_{L\alpha}(x,\theta) \exp[-It_A\Omega^A(x,\theta)]$.

From extended gauge transformations and connection from gauge superfields on page 108,

$2t_A W^A_{L\alpha} \to \mathscr{D}_-^T E \mathscr{D}_- \exp[It_A\Omega^A] \exp[2t_A V^A] \exp[-It_A\Omega^{A\dagger}]\ \mathscr{D}_{+\alpha}\ \exp[It_A\Omega^{A\dagger}] \exp[-2t_A V^A] \exp[-It_A\Omega^A]$.

Use product rule for the $\mathscr{D}_\pm$ here. Since Ω^A is left-chiral, $\mathscr{D}_-\Omega^A = \mathscr{D}_+\Omega^{A\dagger} = 0$.

Then $2t_A W^A_{L\alpha} \to \exp[It_A\Omega^A] \mathscr{D}_-^T E \mathscr{D}_- \exp[2t_A V^A] \exp[-It_A\Omega^{A\dagger}] \exp[It_A\Omega^{A\dagger}] \mathscr{D}_{+\alpha} \exp[-2t_A V^A] \exp[-It_A\Omega^A]$

$= \exp[It_A\Omega^A] \mathscr{D}_-^T E \mathscr{D}_- \exp[2t_A V^A] \underline{\mathscr{D}_{+\alpha}} \exp[-2t_A V^A] \underline{\exp[-It_A\Omega^A]}$. Not finished, because $\underline{\mathscr{D}_{+\alpha} \exp[-It_A\Omega^A]} \neq 0$

(however, $\mathscr{D}_-^T E \mathscr{D}_- \exp[-It_A\Omega^A]$ does vanish). But $\mathscr{D}_-^T E \mathscr{D}_- \mathscr{D}_{+\alpha} = \mathscr{D}_{+\alpha} \mathscr{D}_-^T E \mathscr{D}_- - 4\,[P_L \not{\partial} \mathscr{D}_-]_\alpha$, so $\mathscr{D}_-^T E \mathscr{D}_- \mathscr{D}_{+\alpha} \exp[-It_A\Omega^A] = 0$.

So $2t_A W^A_{L\alpha} \to \exp[It_A\Omega^A] \left\{ \mathscr{D}_-^T E \mathscr{D}_- \exp[2t_A V^A] \mathscr{D}_{+\alpha} \exp[-2t_A V^A] \right\} \exp[-It_A\Omega^A]$,

where all derivatives in quantity between $\Big\{$ and $\Big\}$ evaluated before multiplying on left / right with $\exp[\pm It_A\Omega^A]$.

Form of W_L^A: $W_L^A(x,\theta) = \lambda_L^A(x_+) + \frac{1}{2}\gamma^\mu\gamma^\nu\theta_L F_{\mu\nu}^A(x_+) + \theta_L^T E\theta_L \not{D}\lambda_R^A(x_+) - I\theta_L D^A(x_+)$.

Note $\exp[-2t_A V^A(x,\theta)] = 1 - I\overline{\theta}\gamma_5\gamma_\mu\theta t_A V_\mu^A(x) - \frac{1}{2}\overline{\theta}\gamma_5\gamma^\mu\theta\overline{\theta}\gamma_5\gamma^\nu\theta t_A t_B V_\mu^A(x) V_\nu^B(x) + 2I\overline{\theta}\gamma_5\theta t_A\overline{\theta}\lambda^A(x) + \frac{1}{2}(\overline{\theta}\gamma_5\theta)^2 t_A D^A(x)$.

After performing all $\mathscr{D}_\pm$ in gauge-covariant spinor superfield on page 112, choose gauge where $V_A(X) = 0$ at given $x = X$,

gives $W_L^A(X,\theta) = \lambda_L^A(x_+) + \frac{1}{2}\gamma^\mu\gamma^\nu\theta_L(\partial_\mu V_\nu^A(X_+) - \partial_\nu V_\mu^A(X_+)) + \theta_L^T E\theta_L \not{\partial}\lambda_R^A(X_+) - I\theta_L D^A(X_+)$.

Then convert to gauge covariant form consistent with this,

i.e. $\partial_\mu V_\nu^A - \partial_\nu V_\mu^A \to F_{\mu\nu}^A$ (the non-Abelian field strength of page 63) and $\not{\partial} \to \not{D}$.

Gauge supermultiplet Lagrangian: $\mathscr{L}_{\text{gauge}} = -\frac{1}{2}\text{Re}([W_L^{AT} E W_L^A]_{\mathscr{F}}) = -\frac{1}{4}F_{\mu\nu}^A F^{A\mu\nu} - \frac{1}{2}\overline{\lambda}^A(\not{D}\lambda)^A + \frac{1}{2}D^A D^A$.

From form of W_L^A above.

Additional pieces:

Additional non perturbative part: $\mathscr{L}_\theta = -\frac{g^2\theta}{16\pi^2}\text{Im}\left(W_L^{AT} E W_L^A\right) = -\frac{g^2\theta}{16\pi^2}\left(I\overline{\lambda}^A\not{D}\gamma_5\lambda^A - \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}F^{A\mu\nu}F^{A\rho\sigma}\right)$,

where g is coupling appearing in t_A.

Fayet-Iliopolis term: $\mathscr{L}_{\text{FI}} = \xi D$, where ξ is arbitrary constant, D is for Abelian supermultiplet..

Corresponding action is supersymmetric, because $\delta D = I\overline{\alpha}\gamma_5\not{\partial}\lambda$ is derivative.

Explicit check that action from $\mathscr{L}_{\text{gauge}}$ is supersymmetric.

In $V_A(X) = 0$ gauge, component field transformations on page 97 at $x = X$: $\delta V^A_\mu = \overline{\alpha}\gamma_\mu\lambda^A$, $\delta D^A = I\overline{\alpha}\gamma_5\partial\!\!\!/\lambda^A$, and

$\delta\lambda^A = (\frac{1}{4}F^A_{\mu\nu}[\gamma^\nu,\gamma^\mu] + I\gamma_5 D^A)\alpha$ where $F^A_{\mu\nu} = \partial_\mu V^A_\nu - \partial_\nu V^A_\mu + C_{ABC}V^B_\mu V^C_\nu$ is non-Abelian field strength of page 63.

Then using $\delta(F^A_{\mu\nu}F^{A\mu\nu}) = 2F^A_{\mu\nu}\delta F^{A\mu\nu}$ etc., $\underline{\delta(F^A_{\mu\nu}F^{A\mu\nu}) = 2F^{A\mu\nu}\overline{\alpha}(\gamma_\nu\partial_\mu - \gamma_\mu\partial_\nu)\lambda^A}$, $\underline{\delta(D^A D^A) = 2ID^A\overline{\alpha}\gamma_5\partial\!\!\!/\lambda^A}$

and $\underline{\delta(\overline{\lambda}^A\partial\!\!\!/\lambda^A) = 2\overline{\alpha}[\frac{1}{4}F^A_{\mu\nu}[\gamma^\mu,\gamma^\nu] + I\gamma_5 D]\partial\!\!\!/\lambda^A + \delta(C_{ABC}\overline{\lambda}^A V\!\!\!/^B\lambda^C)}$ up to derivatives

(we will show $\delta(C_{ABC}\overline{\lambda}^A V\!\!\!/^B\lambda^C) = 0$ later). For this last expression, use $[\gamma^\mu,\gamma^\nu]\gamma^\rho = -2g^{\mu\rho}\gamma^\nu + 2g^{\nu\rho}\gamma^\mu - 2I\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5$

(expand in 16 matrices on page 44, Lorentz and space inversion invariance limits this to these three terms.

1st coefficient by taking $\mu\nu\rho = 121$: $[\gamma^1,\gamma^2]\gamma^1 = -2g^{11}\gamma^2$, correct because from anticommutation relations for γ^μ on page 43,

$\gamma^1\gamma^2 = -\gamma^2\gamma^1$ and $\gamma^{1\ 2} = 2$. Similarly for $\mu\nu\rho = 211$ to get 2nd coefficient. 3rd coefficient from $\mu\nu\rho = 123$:

$[\gamma^1,\gamma^2]\gamma^3 = -2I\epsilon^{1230}\gamma_0\gamma_5$, LHS is $2\gamma^1\gamma^2\gamma^3 = -2\gamma_0\gamma^0\gamma^1\gamma^2\gamma^3 = -2I\gamma_0\gamma_5$).

So $\delta(\overline{\lambda}^A\partial\!\!\!/\lambda^A) = -F^{A\mu\nu}\overline{\alpha}(\gamma_\nu\partial_\mu - \gamma_\mu\partial_\nu)\lambda^A - I\epsilon^{\mu\nu\rho\sigma}F^A_{\mu\nu}\overline{\alpha}\gamma_\sigma\gamma_5\partial_\rho\lambda^A + 2ID^A\overline{\alpha}\gamma_5\partial\!\!\!/\lambda^A$. 2nd term replaceable, after integration by parts,

by $I\epsilon^{\mu\nu\rho\sigma}(\partial_\rho F^A_{\mu\nu})\overline{\alpha}\gamma_\sigma\gamma_5\lambda^A$, which vanishes since e.g. $\epsilon^{\mu\nu\rho\sigma}\partial_\rho\partial_\mu V^A_\nu = 0$, 1st and 3rd terms cancel with $\delta(F^A_{\mu\nu}F^{A\mu\nu})$ and $\delta(D^A D^A)$.

Thus we are left with $\delta(C_{ABC}\overline{\lambda}^A V\!\!\!/^B\lambda^C) = C_{ABC}\overline{\lambda}^A(\delta V\!\!\!/^B)\lambda^C = C_{ABC}\overline{\lambda}^A\gamma_\mu\lambda^C\overline{\alpha}\gamma^\mu\lambda^B = 0$,

where antisymmetry of C_{ABC} used in 1st step. Last step requires explicit calculation.

Lagrangian for chiral superfield interacting with gauge fields:

$$\mathscr{L} = -(D_\mu\phi)^\dagger_n(D^\mu\phi)_n - \tfrac{1}{2}\overline{\psi}_{nL}\gamma^\mu(D_\mu\psi_L)_n + \tfrac{1}{2}D_\mu\overline{\psi}_{nL}\gamma^\mu\psi_{nL} - \frac{1}{2}\frac{\partial^2 f}{\partial\phi_n\partial\phi_m}\psi^T_{nL}E\psi_{mL} - \frac{1}{2}\left(\frac{\partial^2 f}{\partial\phi_n\partial\phi_m}\right)^*\left(\psi^T_{nL}E\psi_{mL}\right)^*$$

$$-V(\phi) + I\sqrt{2}\overline{\psi}_{nL}(t_A)_{nm}\lambda^A\phi_m - I\sqrt{2}\phi^\dagger_n\overline{\lambda}^A(t_A)_{nm}\psi_{mL} - \tfrac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} - \tfrac{1}{2}\overline{\lambda}^A(\not{D}\lambda)^A + \frac{g^2\theta}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{A\mu\nu}F^{A\rho\sigma},$$

where potential $V(\phi) = \frac{\partial f(\phi)}{\partial\phi_n}\left(\frac{\partial f(\phi)}{\partial\phi_n}\right)^* + \frac{1}{2}\left(\xi^A + \phi^*_n(t_A)_{nm}\phi_m\right)\left(\xi^A + \phi^*_k(t_A)_{kl}\phi_l\right)$.

Sum of gauge invariant $\mathscr{L}$ for chiral supermultiplet on page 110, superpotential part of $\mathscr{L}$ on page 103 and 3 Lagrangians on page 113 gives $\mathscr{L} = \frac{1}{2}\left[\Phi^\dagger\exp\left(-2t_AV^A\right)\Phi\right]_D + 2\mathrm{Re}[f(\Phi)]_{\mathscr{F}} - \frac{1}{2}\mathrm{Re}\left(W^{AT}_L EW^A_L\right)_{\mathscr{F}} - \xi^A D^A - \frac{g^2\theta}{16\pi^2}\mathrm{Im}(W^{AT}_L EW^A_L)$, or explicitly (using Majorana conjugation on page 91, e.g. $\overline{\psi}_R = \psi^\dagger P_R\beta = \overline{\psi}P_L = \psi^T E\gamma_5 P_L = \psi^T P_L E\gamma_5 = \psi^T_L E\gamma_5$)

$$\mathscr{L} = -(D_\mu\phi)^\dagger_n(D^\mu\phi)_n - \tfrac{1}{2}\overline{\psi}_n\gamma^\mu(D_\mu\psi)_n + \mathscr{F}^\dagger_n\mathscr{F}_n - \mathrm{Re}\frac{\partial^2 f}{\partial\phi_n\partial\phi_m}\psi^T_n E\psi_m + 2\mathrm{Re}\frac{\partial f(\phi)}{\partial\phi_n}\mathscr{F}_n - 2\sqrt{2}\mathrm{Im}(t_A)_{nm}\overline{\psi}_{nL}\lambda^A\phi_m$$

$$+2\sqrt{2}\mathrm{Im}(t_A)_{mn}\overline{\psi}_{nR}\lambda^A\phi^\dagger_m - \phi^\dagger_n(t_A)_{nm}\phi_m D^A - \xi^A D^A + \tfrac{1}{2}D^A D^A - \tfrac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} - \tfrac{1}{2}\overline{\lambda}^A(\not{D}\lambda)^A + \frac{g^2\theta}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{A\mu\nu}F^{A\rho\sigma}.$$

Then use field equations for auxiliary fields: $\mathscr{F}_n = -\left(\frac{\partial f(\phi)}{\partial\phi_n}\right)^*$ and $D^A = \xi^A + \phi^*_n(t_A)_{nm}\phi_m$.

Perturbative non-renormalization theorem for $\mathscr{F}$ term:

For general SUSY gauge theory with coupling g absorbed into gauge superfields,

$$\mathscr{L} = \frac{1}{2}\left[\Phi^\dagger \exp\left(-2t_A V^A\right)\Phi\right]_D + 2\mathrm{Re}\left[f(\Phi)\right]_{\mathscr{F}} + \frac{1}{2g^2}\mathrm{Re}\left[W_L^{AT} E W_L^A\right]_{\mathscr{F}}.$$

For supersymmetric, gauge-invariant cut-off λ,

$$\mathscr{L}_\lambda = \frac{1}{2}\left[A_\lambda(\Phi, \Phi^\dagger, V, \mathscr{D}, ...)\right]_D + 2\mathrm{Re}\left[f(\Phi)\right]_{\mathscr{F}} + \frac{1}{2g_\lambda^2}\mathrm{Re}\left[W_L^{AT} E W_L^A\right]_{\mathscr{F}},$$

where one-loop effective gauge coupling $g_\lambda^{-2} = \mathrm{const} - 2b\ln\lambda$.

Implies coefficients in terms of superpotential unchanged by renormalization,

i.e. these coefficients receive no radiative corrections.

In MSSM, Higgs mass term appears in superpotential, so no hierarchy problem to all orders.

Tree-level vacuum is minimum of $V(\phi)$. Lorentz invariance: $\langle 0|(\psi_{nL}, \lambda^A, F^A_{\mu\nu})|0\rangle = 0$, so $\langle 0|\mathscr{L}|0\rangle = -\langle 0|V|0\rangle$.

Unbroken SUSY vacuum: $\mathscr{F}_{n0} = -\left[\frac{\partial f(\phi)}{\partial \phi_n}\right]_{\phi=\phi_0} = 0$, $D^A_0 = \xi^A + \phi^*_{n0}(t_A)_{nm}\phi_{m0} = 0 \iff V(\phi) = 0$.

Can write $V = \mathscr{F}^*_n\mathscr{F}_n + D^AD^A > 0$, so $V = 0$ (if allowed) is a minimum. In this case, $\mathscr{F}_{n0} = D^A_0 = 0$.

From left / right transformation on page 99, $\delta\psi_L = \delta\mathscr{F} = \delta\phi = 0$. Argument holds in reverse. Note: no overconstraining on N ϕ components: $\mathscr{F}_{n0} = D^A_0 = 0$ is N conditions, *not* $N + D$, where D is dimensionality of group, i.e. $A = 1, \ldots, D$, because only $N - D$ conditions needed to satisfy $\mathscr{F}_{n0} = \left.\frac{\partial f(\phi)}{\partial \phi}\right|_{\phi=\phi_0} = 0$: $f(\Phi)$ invariant under extended gauge transformations on page 108, $\frac{\partial f(\Phi)}{\partial \Omega^A} = 0 = \frac{\partial f(\Phi)}{\partial \Phi_n}\left.\frac{\partial([e^{It_B\Omega^B}]_{nm}\Phi_m)}{\partial \Omega^A}\right|_{\Omega^C=0} = \frac{\partial f(\Phi)}{\partial \Phi_n}(It_A\Phi)_n$, i.e. $\frac{\partial f(\phi)}{\partial \phi_n}(t_A\phi)_n = 0$, which is already D conditions.

Existence of any supersymmetric field configuration $\implies$ unbroken SUSY vacuum.

From above, SUSY field configuration has $V = 0$, which is absolute minimum so lower than V for non SUSY field configuration.

$\xi^A = 0$: To check vacuum is unbroken SUSY, enough to check that $\frac{\partial f(\phi)}{\partial \phi_n} = 0$ can be satisfied.

$f(\phi)$ has no ϕ^*, so in invariant under $\phi \to e^{I\Lambda^A t_A}\phi$ with Λ^A complex. If $\frac{\partial f(\phi)}{\partial \phi_n} = 0$ true for $\widetilde{\phi}$, true for $\phi^\Lambda = e^{I\Lambda^A t_A}\widetilde{\phi}$.

Choose Λ^A such that $\phi^{\Lambda\dagger}\phi^\Lambda$ minimum (which exists because it is real and positive), i.e. $\frac{\partial}{\partial \Lambda^A}(\phi^{\Lambda\dagger}\phi^\Lambda) \propto \phi^{\Lambda *}_n(t_A)_{nm}\phi^\Lambda_m = 0$, i.e. $D^A = 0$, so unbroken SUSY vacuum condition obeyed.

3.6.2 Spontaneous supersymmetry breaking in gauge theories

So break SUSY by

1. Making $\frac{\partial f}{\partial \phi} = 0$ impossible (already considered on page 105)

or

2. Fayet-Iliopoulos term ξD. Simple example:

2 left-chiral $\Phi^{\pm}$ with U(1) quantum numbers $\pm e$,

so spinor components are left-handed parts of electron / positron.

Then only possibility is $f(\Phi) = m\Phi^{+}\Phi^{-}$,

so from V defined on page 115, $V(\phi^{+}, \phi^{-}) = m^2|\phi^{+}|^2 + m^2|\phi^{-}|^2 + (\xi + e^2|\phi^{+}|^2 - e^2|\phi^{-}|^2)^2$,

which cannot vanish for $\xi \neq 0$, so SUSY broken.

Note U(1) symmetry intact for $|\xi| < \frac{m^2}{2e^2}$, since minimum at $\phi^{+} = \phi^{-} = 0$.

Tree-level mass sum rule: $\sum_{\text{spin }0}\text{mass}^2 - 2\sum_{\text{spin }\frac{1}{2}}\text{mass}^2 + 3\sum_{\text{spin }1}\text{mass}^2 = 0$.

Valid whether SUSY broken or not.

Take new scalar fields $\Delta\phi_n = \phi_n - \phi_{n0}$. Then quadratic part of V defined on page 115 in unitarity gauge $\phi^\dagger(t_A\phi_0) = 0$ is

$$V_{\text{quad}} = \tfrac{1}{2}\begin{pmatrix}\Delta\phi \\ \Delta\phi^*\end{pmatrix}^\dagger M_0^2 \begin{pmatrix}\Delta\phi \\ \Delta\phi^*\end{pmatrix}, \text{ where } \underline{M_0^2 = \begin{pmatrix}\mathscr{M}^*\mathscr{M} + (t_A\phi_0)(t_A\phi_0)^\dagger + D_0^A t_A & \cdots \\ \cdots & \mathscr{M}\mathscr{M}^* + ((t_A\phi_0)(t_A\phi_0)^\dagger)^* + D_0^A t_A^T\end{pmatrix}}$$

($\mathscr{M}$ defined in tree-level expansion of V on page 104). From Lagrangian for chiral superfield interacting with gauge fields on page 115, terms quadratic in fermion fields are $\mathscr{L}_{1/2} = -\frac{1}{2}\begin{pmatrix}\psi_L \\ \lambda\end{pmatrix}^T M \begin{pmatrix}\psi_L \\ \lambda\end{pmatrix}$, where $M = \begin{pmatrix}\mathscr{M} & I\sqrt{2}(t_A\phi_0)^* \\ I\sqrt{2}(t_A\phi_0)^* & 0\end{pmatrix}$,

so $\underline{M^\dagger M = \begin{pmatrix}\mathscr{M}^\dagger\mathscr{M} + 2(t_A\phi_0)(t_A\phi_0)^\dagger & \cdots \\ \cdots & 2\phi_0^\dagger t_B t_A\phi_0\end{pmatrix}}$. For gauge fields, $\mathscr{L}_V = -V_\mu^A(M_V^2)_{AB}V^{B\mu}$ where $\underline{(M_V^2)_{AB} = \phi_0^\dagger\{t_B, t_A\}\phi_0}$.

Underlined results above give $\text{Tr}M_0^2 = 2\text{Tr}(\mathscr{M}^*\mathscr{M}) + \text{Tr}M_V^2 + 2D_0^A\text{Tr}t_A$ and $\text{Tr}(M^\dagger M) = \text{Tr}(\mathscr{M}^*\mathscr{M}) + 2\text{Tr}M_V^2$.

But trace $= \sum$ eigenvalues, so $\sum_{\text{spin }0}\text{mass}^2 - 2\sum_{\text{spin }\frac{1}{2}}\text{mass}^2 + 3\sum_{\text{spin }1}\text{mass}^2 = 2D_0^A\text{Tr}t_A$.

In non-Abelian theories, $\text{Tr}t_A = 0$. Can have $\text{Tr}t_A \neq 0$ for U(1) but gives graviton-graviton-U(1) anomaly.

1 spin-3 component for scalar boson, 2 for spin-$\frac{1}{2}$ fermion, 3 for spin-1 gauge boson,

so sum rule above is just $\mathscr{C} = \sum_{\text{fermions}}\text{mass}^2 - \sum_{\text{bosons}}\text{mass}^2 = 0$.

$\mathscr{C}$ is coefficient of quadratic divergence in vacuum energy, i.e. breaking SUSY does not affect UV structure.

Recap of superfield formalism:

Superspace: extension of spacetime x to include Grassmann variable coordinates $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$,

which are components of Majorana field.

Superfield: Function of superspace, i.e. $S(x, \theta)$.

Since θ is Grassmann, expansion of S in θ components terminates at $\theta_1\theta_2\theta_3\theta_4$.

Schematically, $S(x, \theta) = \phi(x) + A(x)\theta + B(x)\theta^2 + C(x)\theta^3 + D(x)\theta^4$. Any function of superfields is a superfield.

Supermultiplet fields: These are the $\phi(x), A(x), \ldots, D(x)$, describing particles and their superpartners.

Note e.g. ϕ is a commuting scalar field, A a Grassmann spinor etc.

D is candidate for Lagrangian $\mathscr{L}$, because SUSY transformation $\delta D \propto \partial_\mu f \to$ invariant action $\mathscr{A} = \int d^4x \mathscr{L}(x)$.

Chiral superfield: Superfield subject to certain constraints.

Expansion is $\Phi(x,\theta) = \phi_L(x) + \psi_L(x)\theta + \mathscr{F}(x)\theta^2 + \ldots$.

(Only consider left-handed chiral superfields, right-handed by conjugation.)

Its component supermultiplet fields are candidate for

e.g. left-handed electron (ψ) and its superpartner the left-handed selectron (ϕ).

$\mathscr{F}$ is candidate for Lagrangian $\mathscr{L}$, since SUSY transformation $\delta\mathscr{F} \propto \partial_\mu f \to$ invariant action $\mathscr{A} = \int d^4x \mathscr{L}(x)$.

4 The Minimally Supersymmetric Standard Model

4.1 Left-chiral superfields

Assign left-handed SM fermions to left-chiral superfields according to table 4.1.1.

Table 4.1.1: MSSM equivalent of table 2.9.1 on page 64, for the left-chiral superfields. Supermultiplets for e.g. U are written u_L for the left-handed up quark and $\tilde{u}_L$ for its scalar superpartner, the up squark. Likewise for $\overline{U}$ we write $u_R^\dagger$ and $\tilde{u}_R^\dagger$.

Names	Label	Representation under $SU(3)_C \times SU(2)_L \times U(1)_Y$
(S)quarks	$Q = (U, D)$	$(\mathbf{3}, \mathbf{2}, \frac{1}{6})$
	$\overline{U}$	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})$
	$\overline{D}$	$(\overline{\mathbf{3}}, \mathbf{1}, \frac{1}{3})$
(S)leptons	$L = (N, E)$	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$
	$\overline{E}$	$(\mathbf{1}, \mathbf{1}, 1)$

Baryon / lepton number violating terms:

$$[Q_a^K (\epsilon L^M)_a \overline{D}^N]_{\mathscr{F}} = [(D^K N^M - U^K E^M)\overline{D}^N]_{\mathscr{F}},\ [(E^K N^M - N^K E^M)\overline{E}^N]_{\mathscr{F}},\ [\overline{D}^K \overline{D}^M \overline{U}^N]_{\mathscr{F}},$$

SU(3)×SU(2)×U(1) invariant terms not in SM,

allows proton decay $p \longrightarrow \pi^0 + e^+$ in ∼minutes, while experiment gives $> 10^{32}$ years.

Relevant processes are $u_R d_R \to (\tilde{s}_R^* \text{ or } \tilde{b}_R^*) \to \overline{e}_L \overline{u}_L$ and $u_L \to u_L$. (No $u_R d_R \to \tilde{d}_R^*$: coupling is $\lambda_{KMN} \overline{D}^K \overline{D}^M \overline{U}^N$ with λ_{KMN} antisymmetric in KM since $\overline{D}^K \overline{D}^M$ antisymmetric in colour indices to get colour singlet coupling.)

Rule out some / all baryon lepton number violating terms on page 120 by symmetry, e.g.:

1.) Rule out all by baryon number (B) and lepton number (L) assignments

$B_{U,\overline{U}} = B_{D,\overline{D}} = \pm\frac{1}{3}$, $L_{U,\overline{U}} = L_{D,\overline{D}} = 0$, $B_{N,\overline{N}} = B_{E,\overline{E}} = 0$, $L_{N,\overline{N}} = L_{E,\overline{E}} = \pm 1$, $B_{\theta_L} = B_{\theta_R} = L_{\theta_{L/R}} = 0$.

Or rule out some by requiring only conservation of linear combination, e.g. L, B, $B - L$ etc.

2.) Rule out $[\overline{D}^K \overline{D}^M \overline{U}^N]_{\mathscr{F}}$ by $L_N = L_E = 0$, $L_U = L_D = L_{\overline{U}} = L_{\overline{D}} = -1$, $L_{\overline{E}} = -2$.

"Conventional" L for quarks, leptons by taking $L_{\theta_{L/R}} = \pm 1$. L for squarks, sleptons is then unconventional.

3.) Rule out all by R parity, a discrete global symmetry, $= 1$ for SM particles and $= -1$ for their superpartners,

i.e. $\Pi_R = (-1)^{2s}(-1)^{3(B-L)}$. Lightest SUSY particle (LSP) (lightest particle with $\Pi_R = -1$) completely stable.

Cannot decay into other $\Pi_R = -1$ particles (heavier), or into $\Pi_R = 1$ particles only (violates R-parity conservation).

Colliders: sparticles produced in pairs.

Non-zero vacuum expectation values for scalar components of N^K $\longrightarrow$ charged lepton, charge $-\frac{e}{3}$ quark masses

via 1st, 2nd terms in baryon / lepton number violating terms on page 120, but charge $\frac{2e}{3}$ quarks remain massless.

Spontaneous breakdown of SU(2)×U(1) for massive quarks, leptons, $W^{\pm}$, Z by Higgs superfields in table 4.1.2:

Table 4.1.2: Required Higgs superfields in the MSSM.

Names	Label	Representation under $SU(3)_C \times SU(2)_L \times U(1)_Y$
Higgs(ino)	$H_1 = (H_1^0, H_1^-)$	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$
	$H_2 = (H_2^+, H_2^0)$	$(\mathbf{1}, \mathbf{2}, \frac{1}{2})$

Higgs-chiral superfield couplings in superpotential $[f(\Phi)]_{\mathscr{F}}$: $(K, M = \text{generation})$

$$h^D_{KM}[(D^K H_1^0 - U^K H_1^-)\overline{D}^M]_{\mathscr{F}},\ h^E_{KM}[(E^K H_1^0 - N^K H_1^-)\overline{E}^M]_{\mathscr{F}},\ h^U_{KM}[(D^K H_2^+ - U^K H_2^0)\overline{U}^M]_{\mathscr{F}}.$$

E.g. last term is just $Q_a(\epsilon H_2)_a\overline{U}$, leads to $-G_u^{KM}\overline{\mathscr{Q}}_{L\ a}^K(\epsilon\phi_H)_a^\dagger u_R^M$ term in $\mathscr{L}_{\text{Higgs-fermion}}$ on page 66.

Note second Higgs H_2 needed for last term, because we need a $(\mathbf{1}, \mathbf{2}, \frac{1}{2})$ left-chiral superfield. ($H_1^\dagger$ is $(\mathbf{1}, \mathbf{2}, \frac{1}{2})$, but is right-chiral.)

$\langle 0|\phi_{H_1^0}|0\rangle \neq 0$ gives mass to d-type quarks and charged leptons, $\langle 0|\phi_{H_2^0}|0\rangle \neq 0$ gives mass to u-type quarks.

In superpotential part of $\mathscr{L}$ on page 103, fermions get mass from first term $-\frac{1}{2}\frac{\partial^2 f(\phi)}{\partial\phi_n\partial\phi_m}\overline{\psi}_{nR}\psi_{mL}$.

So from e.g. first term of Higgs-chiral superfield couplings in superpotential $[f(\Phi)]_{\mathscr{F}}$ above,

mass term for d^K from $\frac{\partial^2 f(\phi)}{\partial\tilde{d}_L^K\partial\tilde{d}_R^{K*}}\overline{d}_R^K d_L^K = \phi_{H_1^0}\overline{d}_R^K d_L^K$.

For more Higgs superfields, number of H_1 and H_2 type superfields must be equal.

Higgsinos produce SU(2)-SU(2)-U(1) anomalies: For H_1, anomaly $\propto \sum t_3^2 y = \left(\frac{1}{2}g\right)^2\left(\frac{1}{2}g'\right) + \left(-\frac{1}{2}g\right)^2\left(\frac{1}{2}g'\right) = \frac{1}{2}g^2 g'$,

for H_2, anomaly $\propto \sum t_3^2 y = \left(\frac{1}{2}g\right)^2\left(-\frac{1}{2}g'\right) + \left(-\frac{1}{2}g\right)^2\left(-\frac{1}{2}g'\right) = -\frac{1}{2}g^2 g'$. No anomalies from gauginos, in adjoint representation.

Most general renormalizable Lagrangian for a gauge theory with R parity or $B-L$ conserved consists of

1. sum of $[\Phi^* \exp(-V)\Phi]_D$ terms for quark, lepton and Higgs chiral superfields,

2. sum of $[\epsilon_{\alpha\beta} W_\alpha W_\beta]_{\mathscr{F}}$ for gauge superfields,

3. sum of Higgs-chiral superfield couplings in superpotential $[f(\Phi)]_{\mathscr{F}}$ on page 122, and

4. μ **term**: $\mathscr{L}_\mu = \mu \left[H_1^T \epsilon H_2\right]_{\mathscr{F}} = \mu \left[H_2^+ H_1^- - H_2^0 H_1^0\right]_{\mathscr{F}}$.

   μ has no radiative corrections due to perturbative non-renormalization theorem for $\mathscr{F}$ term on page 116.

Gauge hierarchy problem on page 76 explicitly solved as follows:

Unbroken SUSY: 1-loop correction to Higgs mass from any particle cancelled by that particle's superpartner.

For broken SUSY, replace Λ_{UV} with $\sim$ mass of particle,

so 1-loop correction to Higgs mass from top is $\frac{|\kappa_t|^2}{8\pi^2}\Delta m_s^2$, where Δm_s^2 is mass splitting between top and stop.

No fine-tuning if this is $\lesssim 1$ TeV, so since $|\kappa_t|^2 \sim 1$, stop mass is $< \sqrt{8\pi^2} \sim 10$ TeV.

Flavour changing processes suppressed to below experimental bounds if squark masses $\sim$ equal,

so if gauge hierarchy problem solved by SUSY, all squark masses < 10 TeV.

4.2 Supersymmetry and strong-electroweak unification

Mentioned in grand unification subsubsection, page 72. Assume SUSY unbroken in most of range $< M_X$.

1-loop modifications to SM $\beta_i(g_i(\mu_r)) = \mu_r \frac{d}{d\mu_r} g_i(\mu_r)$ from new SUSY particles, with n_s Higgs chiral superfields:

$$\beta_1 = \frac{5 n_g g'^3}{36\pi^2} \to \frac{g'^3}{4\pi^2}\left(\frac{5n_g}{6} + \frac{n_s}{8}\right) \implies \frac{1}{g'^2(\mu_r)} = \frac{1}{g'^2(M_X)} + \frac{1}{2\pi^2}\left(\frac{5n_g}{6} + \frac{n_s}{8}\right)\ln\left(\frac{M_X}{\mu_r}\right)$$

$$\beta_2 = \frac{g^3}{4\pi^2}\left(-\frac{11}{6} + \frac{n_g}{3}\right) \to \frac{g^3}{4\pi^2}\left(-\frac{9}{6} + \frac{n_g}{2} + \frac{n_s}{8}\right) \implies \frac{1}{g^2(\mu_r)} = \frac{1}{g^2(M_X)} + \frac{1}{2\pi^2}\left(-\frac{3}{2} + \frac{n_g}{2} + \frac{n_s}{8}\right)\ln\left(\frac{M_X}{\mu_r}\right)$$

$$\beta_3 = \frac{g_s^3}{4\pi^2}\left(-\frac{11}{4} + \frac{n_g}{3}\right) \to \frac{g_s^3}{4\pi^2}\left(-\frac{9}{4} + \frac{n_g}{2}\right) \implies \frac{1}{g_s^2(\mu_r)} = \frac{1}{g_s^2(M_X)} + \frac{1}{2\pi^2}\left(-\frac{9}{4} + \frac{n_g}{2}\right)\ln\left(\frac{M_X}{\mu_r}\right).$$

Take $\mu_r = m_Z$, set $\sqrt{\frac{5}{3}} g'(M_X) = g(M_X) = g_s(M_X)$, solution (where e defined on page 68, $\sin^2\theta_W$ on page 67)

1. $\sin^2\theta_W(m_Z) = \dfrac{18 + 3n_s + \frac{e^2(m_Z)}{g_s^2(m_Z)}(60 - 2n_s)}{108 + 6n_s}$

2. $\ln\left(\frac{M_X}{m_Z}\right) = \dfrac{8\pi^2}{e^2(m_Z)} \dfrac{1 - \frac{8e^2(m_Z)}{3g_s^2(m_Z)}}{18 + n_s}$.

For measured values $\sin^2\theta_W = 0.231$, $\frac{e^2(m_Z)}{4\pi} = (128)^{-1}$, $\frac{g_s^2(m_Z)}{4\pi} = 0.118$, $m_Z = 91.2$GeV,

equation 1. above gives $n_s = 2$, then equation 2. above gives $M_X = 2 \times 10^{16}$GeV.

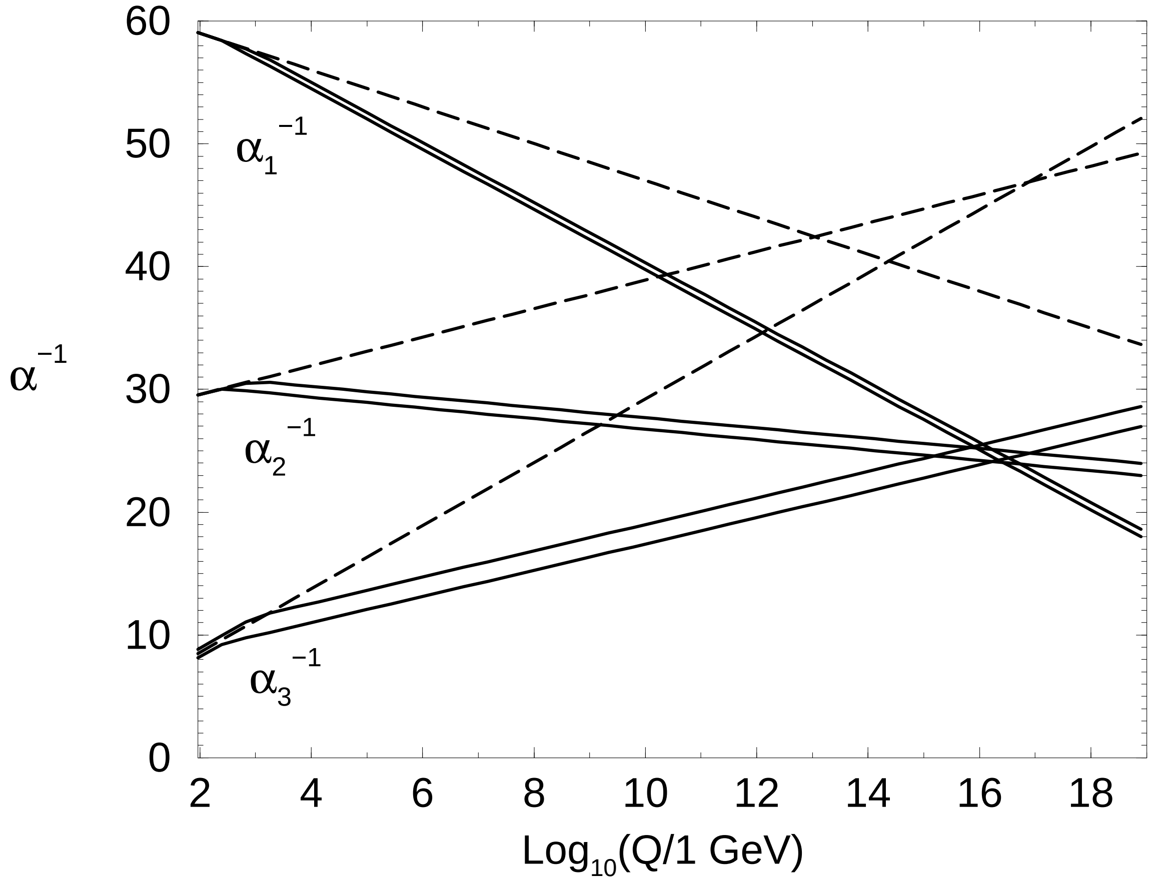

Figure 4.1: 2-loop RG evolution of inverse of $\alpha_i = \frac{g_i^2}{4\pi}$ in the SM (dashed) and MSSM (solid). $\alpha_3(m_Z)$ is varied between 0.113 and 0.123, sparticle mass thresholds between 250 GeV and 1 TeV.

4.3 Supersymmetry breaking in the MSSM

In effective MSSM Lagrangian, SUSY must be broken.

SUSY breaking terms must not reintroduce hierarchy problem, i.e. no $O(\Lambda_{UV}^2)$ corrections to δm_H^2.

Terms with coupling's mass dimension $\leq 0 \rightarrow |\kappa_\psi|^2 - \lambda_\phi = O(\ln \Lambda_{UV})$ (see page 76), so $\delta m_H^2 \sim \Lambda_{UV}^2 \ln \Lambda_{UV}$.

Superrenormalizable terms (coupling's mass dimension > 0, $\sim$ power of some M) $\rightarrow \delta m_H^2 \sim M^2 \ln \Lambda_{UV}$,

OK provided $M \sim m_{\mathrm{SUSY}} \lesssim 10\mathrm{TeV}$. So SUSY breaking terms must be superrenormalizable, called *soft terms*.

Soft SUSY breaking R parity / $B-L$ conserving SM invariant terms:

(Sum over SU(2), SU(3) indices and generations K, M)

1. $\mathscr{L}_{\rm SR} \supset \sum_S -M^{S2}_{KM}\phi^{K\dagger}_S\phi^M_S$, where $S = Q, \overline{U}, \overline{D}, L, \overline{E}$ superfields and ϕ_S their component scalars,

2. $\mathscr{L}_{\rm SR} \supset \sum_X \overline{\lambda}_X m_X \lambda_X$, where $X =$ gluino, wino, bino,

3. Trilinear terms: $\mathscr{L}_{\rm SR} \supset -A^D_{KM}h^D_{KM}(\phi^K_Q)^T\epsilon\phi_{H_1}\phi^M_{\overline{D}} - C^D_{KM}h^D_{KM}(\phi^K_Q)^T\phi^*_{H_2}\phi^M_{\overline{D}}$

$$-A^E_{KM}h^E_{KM}(\phi^K_L)^T\epsilon\phi_{H_1}\phi^M_{\overline{E}} - C^E_{KM}h^E_{KM}(\phi^K_L)^T\phi^*_{H_2}\phi^M_{\overline{E}}$$

$$-A^U_{KM}h^U_{KM}(\phi^K_Q)^T\epsilon\phi_{H_2}\phi^M_{\overline{U}} - C^U_{KM}h^U_{KM}(\phi^K_Q)^T\phi^*_{H_1}\phi^M_{\overline{U}}$$

where $h^{D,E,U}_{KM}$ defined in Higgs-chiral superfield couplings in superpotential $[f(\Phi)]_{\mathscr{F}}$ on page 122,

4. $\mathscr{L}_{\rm SR} \supset -\frac{1}{2}B\mu\phi^T_{H_2}\epsilon\phi_{H_1} - \frac{1}{2}m^2_{H_1}\phi^\dagger_{H_1}\phi_{H_1} - \frac{1}{2}m^2_{H_2}\phi^\dagger_{H_2}\phi_{H_2}$ where μ defined in μ term on page 123.

Recall Hermitian conjugate is added to $\mathscr{L}$. So $\mathscr{L}_{\rm SR} \supset -{\rm Re}\big\{B\mu\phi^T_{H_2}\epsilon\phi_{H_1}\big\} - m^2_{H_1}\phi^\dagger_{H_1}\phi_{H_1} - m^2_{H_2}\phi^\dagger_{H_2}\phi_{H_2}$.

Choose H_1, H_2 superfields' phases so $B\mu$ real, positive: $\mathscr{L}_{\rm SR} \supset -B\mu{\rm Re}\big\{\phi^T_{H_2}\epsilon\phi_{H_1}\big\} - m^2_{H_1}\phi^\dagger_{H_1}\phi_{H_1} - m^2_{H_2}\phi^\dagger_{H_2}\phi_{H_2}$

To respect approximate symmetries, choose A^S_{KM}, $B \sim 1$:

A^S_{KM} for chiral symmetry: Reflected by small Yukawa coupling of light quarks.

B for Peccei-Quinn symmetry: Reflected by small μ term on page 123.

C^S_{KM} terms involve scalar components of left- and right-chiral superfields,

$\rightarrow$ quadratic divergences $\Longrightarrow$ fine-tuning and hierarchy problems.

In fact these divergences from tadpole graphs which disappear into vacuum,

cannot occur since no SM invariant scalars.

Note SM superpartners acquire mass even if no electroweak symmetry breaking (i.e. if SM particles massless).

A^S_{KM}, B, C^S_{KM} are arbitrary and complex $\rightarrow > 100$ parameters even without C^S_{KM} terms.

But expect soft terms to arise from some underlying principle.

SUSY breaking at tree-level (see below and next page) ruled out:

Predicts squark mass(es) too small, would have effect in accurately measured $e^+e^- \longrightarrow$ hadrons.

This is good, otherwise get fine-tuning: tree-level SUSY breaking would require mass parameter M in Lagrangian.

Would affect all SM masses, so M must coincidentally be $\sim$ electroweak symmetry breaking scale $v = 246$ GeV.

No SUSY breaking at tree-level implies no SUSY breaking at all orders.

SUSY breaking at tree-level for 3 generations ruled out.

Tree-level mass sum rule on page 119 holds for each set of colour and charge values.

Gives e.g. $2(m_d^2 + m_s^2 + m_b^2) \simeq 2(5 \text{ GeV})^2 = \sum$ of all masses for bosonic degrees of freedom with charge $-e/3$.

So each squark mass is $< \sqrt{2}\ 5$ GeV $\simeq 7$ GeV.

SUSY breaking at tree-level for $\geq$3 generations ruled out.

General argument against tree-level SUSY breaking: Consider first underlined equation on page 119.

To conserve colour and charge, only allowed non-zero D_0^A terms are for y of U(1) (D_1) and t_3 of SU(2) (D_2).

Squarks are colour triplets, so have zero vacuum expectation values. So for $(\tilde{u}_L, \tilde{u}_R)$,

$M_{0U}^2 = \begin{pmatrix} \mathscr{M}_U^* \mathscr{M}_U - g'\frac{1}{6}D_1 + g\frac{1}{2}D_2 & \cdots \\ \cdots & \mathscr{M}_U \mathscr{M}_U^* + \frac{2}{3}g'D_1 \end{pmatrix}$. For $(\tilde{d}_L, \tilde{d}_R)$, $M_{0D}^2 = \begin{pmatrix} \mathscr{M}_D^* \mathscr{M}_D - g'\frac{1}{6}D_1 - g\frac{1}{2}D_2 & \cdots \\ \cdots & \mathscr{M}_D \mathscr{M}_D^* - \frac{1}{3}g'D_1 \end{pmatrix}$.

Second underlined equation on page 119 gives mass-squared matrix $\mathscr{M}_U^* \mathscr{M}_U$ for charge $\frac{2e}{3}$ quarks

and $\mathscr{M}_D^* \mathscr{M}_D$ for charge $-\frac{e}{3}$ quarks. Let v_x, $x = u$, d be unit eigenvector for quark of lowest mass, i.e. $\mathscr{M}_X^* \mathscr{M}_X v_x = m_x^2 v_x$.

Then mass^2 of lightest charge $\frac{2e}{3}$ squark $< \begin{pmatrix} 0 \\ v_u* \end{pmatrix}^\dagger M_{0U}^2 \begin{pmatrix} 0 \\ v_u* \end{pmatrix} = m_u^2 + \frac{2}{3}g'D_1$,

mass^2 of lightest charge $-\frac{e}{3}$ squark $< \begin{pmatrix} 0 \\ v_d* \end{pmatrix}^\dagger M_{0D}^2 \begin{pmatrix} 0 \\ v_d* \end{pmatrix} = m_d^2 - \frac{1}{3}g'D_1$,

so regardless of value / sign of D_1 there is at least one squark lighter than u or d quark.

In fact, since $D_1 \sim m_{\text{SUSY}}^2$, get squark with negative mass which breaks colour and charge conservation.

Possible solution is new U(1) gauge superfield, so mass^2 of lightest charge $(\frac{2e}{3}, -\frac{e}{3})$ squark $< (m_u^2 + \frac{2}{3}g'D_1, m_d^2 - \frac{1}{3}g'D_1) + \tilde{g}\tilde{D}$.

Alternative breaking: Introduce new strong-force-like gauge field, asymptotically free coupling $\frac{\mathscr{G}^2(\mu_r)}{8\pi^2} \simeq \frac{b}{\ln \mu_r}$.

$\mu_r \exp\left[-\frac{8\pi^2 b}{\mathscr{G}(\mu_r)}\right] \simeq 1 \implies$ force strong enough ($\mathscr{G}(m_{\text{SUSY}}) \sim 1$) at energy $m_{\text{SUSY}} = M_X \exp\left[-\frac{8\pi^2 b}{\mathscr{G}(M_X)}\right] \ll M_X$

to break SUSY, either non-perturbatively or by scalar field potential with non-zero vacuum expectation value.

To get $m_{\text{SUSY}} \sim 10\text{TeV} \ll M_X$, $\mathscr{G}(M_X)$ does not have to be very small (no fine-tuning).

New force clearly does not interact with SM particles,

so SUSY breaking occurs via $\langle 0|\mathscr{F}_{\text{hidden}}|0\rangle \neq 0$ in *hidden sector* of particles that interact via new force,

communicated to observed particles by interactions felt by both hidden and observed particles,

namely gravity (gravity-mediated SUSY breaking) or SM interactions (gauge-mediated SUSY breaking):

1. Gravity-mediated SUSY breaking: $m_{\text{SUSY}} \sim \frac{\langle 0|\mathscr{F}_{\text{hidden}}|0\rangle}{M_P}$ $\implies$ $\sqrt{\langle 0|\mathscr{F}_{\text{hidden}}|0\rangle} \sim 10^{11}$ GeV. (No C^S_{KM} terms.)

From dimensional analysis, subject to "no SUSY breaking conditions" $m_{\text{SUSY}} \to 0$ as $\langle 0|\mathscr{F}_{\text{hidden}}|0\rangle \to 0$ or $M_P \to \infty$.

2. Gauge-mediated SUSY breaking: Messenger particle of mass $M_{\text{messenger}}$ couples to $\langle 0|\mathscr{F}_{\text{hidden}}|0\rangle$

and to MSSM particles via SM interactions (loops). Then $m_{\text{SUSY}} \sim \frac{\alpha_i}{4\pi}\frac{\langle 0|\mathscr{F}_{\text{hidden}}|0\rangle}{M_{\text{messenger}}}$.

For $\sqrt{\langle 0|\mathscr{F}_{\text{hidden}}|0\rangle} \sim M_{\text{messenger}}$, $\implies$ $\sqrt{\langle 0|\mathscr{F}_{\text{hidden}}|0\rangle} \sim 10^5$ GeV. (Very small C^S_{KM} terms.)

Flavour blindness of above SUSY breaking mechanisms simplifies soft terms in Lagrangian at $\mu_r = M_X$.

E.g. in minimal supergravity (mSUGRA), K on page 102, which appears in supergravity Lagrangian,

is diagonal for hidden and observed sectors, i.e. $K = \sum_{i=\text{observed, hidden}} |\Phi_i|^2$.

Gives *organising principle* (for mSUGRA) (recall $S = Q, \overline{D}, \overline{U}, L, \overline{E}$)

$M^{S2}_{KM} = m_0^2 \delta_{KM}, \quad m^2_{H_1} = m^2_{H_2} = m_0^2, \quad m_{\text{gluino}} = m_{\text{wino}} = m_{\text{bino}} = m_{1/2}, \quad A^S_{KM} = A_0 \delta_{KM}, \quad C^S_{KM} = 0.$

This or similar principle expected: small experimental upper bounds on flavour changing, CP violating processes.

Flavour changing processes allowed because squark, slepton mass matrices not necessarily diagonalised

in same basis as quark, lepton mass matrices.

Most stringent limits for quarks are on $K^0 - \overline{K}^0$ transitions involving $d_L \rightarrow$ gluino + $\tilde{d}_L^K$, $\tilde{d}_L^K \rightarrow$ gluino + s_L,

most stringent limits for leptons are on $\mu \rightarrow e\gamma$ decays.

CP violating processes allowed due to many new phases in MSSM, can have large effect at low SM energies.

Upper bound on electric dipole moments of neutron and atoms and molecules

requires CP violating phases $\lesssim 10^{-2}$ or some superpartner masses $\gtrsim 1$ TeV.

4.4 Electroweak symmetry breaking in the MSSM

Consider Higgs H_1 and H_2 superfields' scalar components $\phi_{H_1} = (\phi_{H_1^0}, \phi_{H_1^-})^T$ and $\phi_{H_2} = (\phi_{H_2^+}, \phi_{H_2^0})^T$.

Scalar Higgs potential: $V = \frac{g^2}{2}\left|\phi^\dagger_{H_1}\phi_{H_2}\right|^2 + \frac{g^2+g'^2}{8}\left(\phi^\dagger_{H_1}\phi_{H_1} - \phi^\dagger_{H_2}\phi_{H_2}\right)^2$

$$+(m^2_{H_1} + |\mu|^2)\phi^\dagger_{H_1}\phi_{H_1} + (m^2_{H_2} + |\mu|^2)\phi^\dagger_{H_2}\phi_{H_2} + B\mu\mathrm{Re}(\phi^T_{H_1}\epsilon\phi_{H_2}).$$

From potential in Lagrangian for chiral superfield interacting with gauge fields on page 115,

with $\xi^A = 0$ and $f(H_1, H_2) = \mu H_1^T\epsilon H_2$ (the μ term on page 123),

together with part 4. of $\mathscr{L}_{\mathrm{SR}}$ in soft SUSY breaking terms on page 127.

V **bounded from below**: $2|\mu|^2 + m^2_{H_1} + m^2_{H_2} \geq B\mu$.

Terms in V which are quartic in ϕ_{H_1} and ϕ_{H_2} are positive or zero. When positive, V has a minimum.

However, quartic terms vanish when $\phi_{H_1} = (\phi, 0)^T$ and $\phi_{H_2} = (0, \phi)^T$, and then $V = (2|\mu|^2 + m^2_{H_1} + m^2_{H_2})|\phi|^2 - B\mu\mathrm{Re}(\phi^2)$,

which must not go to $-\infty$ as $|\phi| \to \infty$.

Definition of vacuum: $\langle 0|\phi_{H_2^+}|0\rangle = \langle 0|\phi_{H_1^-}|0\rangle = 0$, $\langle 0|\phi_{H_i^0}|0\rangle = v_i$ real,

so $(m_{H_1}^2 + |\mu|^2)v_1 + \frac{g^2+g'^2}{4}(v_1^2 - v_2^2)v_1 - \frac{1}{2}B\mu v_2 = 0$ and $1 \leftrightarrow 2$. Around vacuum, $\phi_{H_i^0} = v_i + \phi_i$.

Define $V^{\text{neutral}} = V$ when charged $\phi_{H_1^-} = \phi_{H_2^+} = 0$.

Choose minimum $\phi_{H_2^+} = 0$ by SU(2) rotation. $\frac{\partial V}{\partial \phi_{H_2^+}} = 0 \implies \phi_{H_1^-} = 0$ (or $B\mu = -\frac{g^2}{2}\phi_{H_1^0}^\dagger \phi_{H_2^0}^\dagger$, but then $v_i = 0$).

Then $V = V^{\text{neutral}}$, where $V^{\text{neutral}} = \frac{g^2+g'^2}{8}\left(|\phi_{H_1^0}|^2 - |\phi_{H_2^0}|^2\right)^2 + (m_{H_1}^2 + |\mu|^2)|\phi_{H_1^0}|^2 + (m_{H_2}^2 + |\mu|^2)|\phi_{H_2^0}|^2 - B\mu \text{Re}(\phi_{H_1^0}\phi_{H_2^0})$.

Let stationary point of V^{neutral} be at $\phi_{H_i^0} = v_i$, i.e. for $\phi_{H_i^0} = v_i + \phi_i$, must have $\left.\frac{\partial V^{\text{neutral}}}{\partial \phi_i}\right|_{\phi_i=0} = 0$,

i.e. coefficient of term $\propto \phi_i$ vanishes. Gives $(m_{H_1}^2 + |\mu|^2)v_1^* + \frac{g^2+g'^2}{4}(v_1^2 - v_2^2)v_1^* - \frac{1}{2}B\mu v_2 = 0$ and $1 \leftrightarrow 2$.

Last result implies v_2 real if v_1 real. Adjust relative phase between $\phi_{H_1^0}$ and $\phi_{H_2^0}$ so that v_1 is real.

Parameter relations with v_1 and v_2: $B\mu = m_A^2 \sin 2\beta$ where $\tan\beta = \frac{v_2}{v_1}$, $m_A^2 = 2|\mu|^2 + m_{H_1}^2 + m_{H_2}^2$, and

$m_{H_1}^2 + |\mu|^2 = \frac{1}{2}m_A^2 - \frac{1}{2}(m_A^2 + m_Z^2)\cos 2\beta$, $m_{H_2}^2 + |\mu|^2 = \frac{1}{2}m_A^2 + \frac{1}{2}(m_A^2 + m_Z^2)\cos 2\beta$ with $m_Z^2 = \frac{1}{2}(g^2 + g'^2)(v_1^2 + v_2^2)$.

From definition of vacuum above. Compare m_Z here with m_Z on page 67.

Limit on β: $0 \leq \beta \leq \frac{\pi}{2}$. $B\mu = m_A^2 \sin 2\beta$ above, but V bounded from below on page 134 is $m_A^2 \geq B\mu$, so $0 \leq \sin 2\beta \leq 1$.

Quadratic (mass) terms in V^{neutral}: $V^{\text{neutral}}_{\text{quad}} = \frac{1}{2}m_Z^2 \cos 2\beta \left(|\phi_1|^2 - |\phi_2|^2\right) + m_Z^2 \left(\text{Re}(\cos\beta\phi_1 - \sin\beta\phi_2)\right)^2$

$+\frac{1}{2}m_A^2(|\phi_1|^2 + |\phi_2|^2) - \frac{1}{2}(m_A^2 + m_Z^2)\cos 2\beta(|\phi_1|^2 - |\phi_2|^2) - m_A^2 \sin 2\beta \text{Re}(\phi_1\phi_2)$.

Neutral Higgs particle masses: m_A^2, 0, $m_H^2 = \frac{1}{2}\left(m_A^2 + m_Z^2 + \sqrt{(m_A^2 - m_Z^2)^2 + 4m_A^2 m_Z^2 \sin^2 2\beta}\right)$,

$m_h^2 = \frac{1}{2}\left(m_A^2 + m_Z^2 - \sqrt{(m_A^2 - m_Z^2)^2 + 4m_A^2 m_Z^2 \sin^2 2\beta}\right)$ (zero mass particle is Goldstone boson).

Real, imaginary parts of ϕ_i decouple in $V^{\text{neutral}}_{\text{quad}}$ (no terms like e.g. $\text{Re}(\phi_1)\text{Im}(\phi_2)$).

First two masses associated with $\text{Im}(\phi) = (\text{Im}(\phi_1), \text{Im}(\phi_2))$: Write $\text{Im}(\phi)$ dependence of $V^{\text{neutral}}_{\text{quad}}$ as $\text{Im}(\phi)^T M^2_{\text{Im}(\phi)} \text{Im}(\phi)$,

then $M^2_{\text{Im}(\phi)} = \begin{pmatrix} \frac{1}{2}m_A^2(1-\cos 2\beta) & \frac{1}{2}m_A^2 \sin 2\beta \\ \frac{1}{2}m_A^2 \sin 2\beta & \frac{1}{2}m_A^2(1+\cos 2\beta) \end{pmatrix}$. Eigenvalues are m_A^2 and 0, corresponding eigenstates C odd.

So m_A^2 must be positive to ensure $\phi_i = 0$ is local minimum, i.e. that eigenvalues of $\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}$ positive. Similarly,

last two masses are eigenvalues of $M^2_{\text{Re}(\phi)} = \begin{pmatrix} \frac{1}{2}m_A^2(1-\cos 2\beta) + \frac{1}{2}m_Z^2(1+\cos 2\beta) & -\frac{1}{2}(m_A^2 + m_Z^2)\sin 2\beta \\ -\frac{1}{2}(m_A^2 + m_Z^2)\sin 2\beta & \frac{1}{2}m_A^2(1+\cos 2\beta) + \frac{1}{2}m_Z^2(1-\cos 2\beta) \end{pmatrix}$.

Heaviest neutral scalar: $m_H^2 > m_A^2, m_Z^2$. **Lightest neutral scalar**: $m_h^2 < m_A^2, m_Z^2$.

Large top to bottom quark mass ratio suggests large $\tan\beta = \frac{v_2}{v_1} \to m_H^2 \simeq \text{Max}(m_A^2, m_Z^2)$, $m_h^2 \simeq \text{Min}(m_A^2, m_Z^2)$.

Define $V^{\text{charged}} = V$ when $\phi_1 = \phi_2 = 0$.

Quadratic (mass) terms in V^{charged}:

$V^{\text{charged}}_{\text{quad}} = \frac{1}{2}(m_W^2 + m_A^2)\left(|\phi_{H_1^-}|^2(1-\cos 2\beta) + |\phi_{H_2^+}|^2(1+\cos 2\beta) + 2\sin 2\beta \text{Re}(\phi_{H_1^-}\phi_{H_2^+})\right)$, with $m_W^2 = \frac{1}{2}g^2(v_1^2 + v_2^2)$.

Compare m_W here with m_W on page 67.

Charged Higgs particle masses: 0, $m_C^2 = m_W^2 + m_A^2$.

$V^{\text{charged}}_{\text{quad}} = (\phi_{H_1^-}, \phi_{H_2^+}) M_C^2 (\phi^*_{H_1^-}, \phi^*_{H_2^+})^T$, where $M_C^2 = \frac{1}{2}(m_W^2 + m_A^2)\begin{pmatrix} 1-\cos 2\beta & \sin 2\beta \\ \sin 2\beta & 1+\cos 2\beta \end{pmatrix}$.

Above masses are eigenvalues of M_C^2.

Results for neutral and charged scalar fields modified most significantly by radiative corrections from top quark, which has largest Yukawa couplings to Higgs. Most significant modification to above results are the *increases*

$$m_H^2 = \frac{1}{2}\left(m_A^2 + m_Z^2 + \Delta_t + \sqrt{((m_A^2 - m_Z^2)\cos 2\beta + \Delta_t)^2 + (m_A^2 + m_Z^2)^2 \sin^2 2\beta}\right)$$

$$m_h^2 = \frac{1}{2}\left(m_A^2 + m_Z^2 + \Delta_t - \sqrt{((m_A^2 - m_Z^2)\cos 2\beta + \Delta_t)^2 + (m_A^2 + m_Z^2)^2 \sin^2 2\beta}\right), \text{ where } \Delta_t = \frac{3\sqrt{2}m_t^4 G_F}{2\pi^2 \sin^2\beta}\ln\frac{M_{\text{st}}^2}{m_t^2}.$$

By taking stop mass $M_{\text{st}} > 300$ GeV, $\tan\beta > 10$, get $m_h > m_Z$.

Condition which leads to electroweak symmetry breaking: $4(m^2_{H_1}+|\mu|^2)(m^2_{H_2}+|\mu|^2) \leq (B\mu)^2$.

From parameter relations with v_1 and v_2 on page 135, $4(m^2_{H_1}+|\mu|^2)(m^2_{H_2}+|\mu|^2) = m^4_A \sin^2 2\beta - m^2_Z(m^2_Z+2m^2_A)\cos^2 2\beta$.

But $B\mu = m^2_A \sin 2\beta$, so $4(m^2_{H_1}+|\mu|^2)(m^2_{H_2}+|\mu|^2) = (B\mu)^2 - m^2_Z(m^2_Z+2m^2_A)\cos^2 2\beta$, so inequality follows from $\cos^2\beta \geq 0$.

These inequalities mean that second derivative matrix of V has negative eigenvalue at SU(2) respecting point $\phi_{H_1}=\phi_{H_2}=0$,

i.e. V unstable (not minimum) there: Quadratic part of scalar Higgs potential on page 134 can be written as four terms

$\tilde{\phi}^T M^2_\phi \tilde{\phi}$, where $\tilde{\phi}$ is separately the imainary and real parts of $(\phi_{H_1^-}, \phi_{H_2^+})$ and $(\phi_{H_1^0}, \phi_{H_2^0})$, and $M^2_\phi = \begin{pmatrix} m^2_{H_1}+|\mu|^2 & \pm\frac{1}{2}B\mu \\ \pm\frac{1}{2}B\mu & m^2_{H_2}+|\mu|^2 \end{pmatrix}$.

So mass eigenvalues m^2 obey $(m^2_{H_1}+|\mu|^2-m^2)(m^2_{H_2}+|\mu|^2-m^2)-\frac{1}{4}(B\mu)^2=0$,

solutions are $2m^2 = 2|\mu|^2+m^2_{H_1}+m^2_{H_2} \pm \sqrt{(2|\mu|^2+m^2_{H_1}+m^2_{H_2})^2+(B\mu)^2-4(m^2_{H_1}+|\mu|^2)(m^2_{H_2}+|\mu|^2)}$.

Inequality above implies one of these solutions is negative.

Example: For $\tan\beta=\infty$ ($\beta=\frac{\pi}{2}$), parameter relations with v_1 and v_2 on page 135

imply $m^2_{H_1}+|\mu|^2>0$ and $m^2_{H_2}+|\mu|^2<0$, so electroweak symmetry broken.

Alternatively, radiative corrections give $\frac{d}{d\ln\mu_r}m^2_{H_2} = x\ h^U_{33}+\ldots$ (h^U_{33} is top quark Yukawa coupling), $x>0$.

(Similarly for stop masses, smaller x.) So although $m^2_{H_2}>0$ at $\mu_r=M_X$, may have $m^2_{H_2}<(\ll)0$ at $\mu_r=v$.

4.5 Sparticle mass eigenstates

Consider gauginos and higgsinos. Particles with different SU(2)×U(1)

but same $\mathrm{U}(1)_{\mathrm{e.m.}}$ transformation properties can mix after electroweak symmetry breaking.

Neutralinos: 4 neutral fermionic mass eigenstates $\widetilde{\chi}_i^0$, $i = 1$(lightest), ..., 4,

mixtures of bino, neutral wino, neutral higgsinos.

Bilinear terms in these fields appearing in Lagrangian can be written $-\frac{1}{2}(\lambda^0)^T M_{\widetilde{\chi}^0} \lambda^0$, where $\lambda^0 = (\lambda_{\text{bino}}, \lambda_{\text{neutralwino}}, \lambda_{H_1^0}, \lambda_{H_2^0})$,

$$M_{\widetilde{\chi}^0} = \begin{pmatrix} m_{\text{bino}} & 0 & -c_\beta s_W m_Z & s_\beta s_W m_Z \\ 0 & m_{\text{wino}} & c_\beta c_W m_Z & -s_\beta s_W m_Z \\ -c_\beta s_W m_Z & c_\beta c_W m_Z & 0 & -\mu \\ s_\beta s_W m_Z & -s_\beta c_W m_Z & -\mu & 0 \end{pmatrix}$$ ($s_\beta = \sin\beta$ etc.). μ dependent part from μ term on page 123.

m_Z dependent part from 1. for $\Phi = H_i$ and $V =$ SU(2)×U(1) fields after electroweak symmetry breaking on page 123.

m_{bino}, m_{wino} dependent terms from SUSY breaking terms for gauginos from 2. on page 127.

$M_{\widetilde{\chi}^0}$ symmetric, diagonalized by unitary matrix.

$\widetilde{\chi}_1^0$ can be LSP, and therefore candidate for cold dark matter.

Charginos: 4 charged fermionic mass eigenstates $\widetilde{\chi}_i^{\pm}$, $i = 1, 2$, mixtures of charged winos and charged higgsinos.

Bilinear terms in these fields appearing in Lagrangian can be written $-\frac{1}{2}(\lambda^+)^T M_{\widetilde{\chi}^c}\lambda^-$,

where $\lambda^+ = (\lambda^+_{\text{charged wino}}, \lambda_{H_2^+})$, $\lambda^- = (\lambda^-_{\text{charged wino}}, \lambda_{H_1^-})$, $M_{\widetilde{\chi}^c} = \begin{pmatrix} m_{\text{wino}} & I\sqrt{2}m_W s_\beta \\ I\sqrt{2}m_W c_\beta & \mu \end{pmatrix}$.

Define $\lambda^c = (\lambda^+, \lambda^-)^T$, $\mathscr{M}_{\widetilde{\chi}^c} = \begin{pmatrix} 0 & M^\dagger_{\widetilde{\chi}^c} \\ M_{\widetilde{\chi}^c} & 0 \end{pmatrix}$, so contribution to Lagrangian including hermitian conjugate

is $-\frac{1}{2}(\lambda^c)^T \mathscr{M}_{\widetilde{\chi}^c}\lambda^c$. Squared mass eigenvalues can be obtained from diagonalization of $\mathscr{M}^2_{\widetilde{\chi}^c} = \begin{pmatrix} M^\dagger_{\widetilde{\chi}^c}M_{\widetilde{\chi}^c} & 0 \\ 0 & M_{\widetilde{\chi}^c}M^\dagger_{\widetilde{\chi}^c} \end{pmatrix}$,

gives $m^2_{\widetilde{\chi}^+_{1,2}} = m^2_{\widetilde{\chi}^-_{1,2}} = \frac{1}{2}\left(m^2_{\text{wino}} + 2m_W^2 + |\mu|^2 \pm \sqrt{(m^2_{\text{wino}} - |\mu|^2)^2 + 4m_W^4\cos^2 2\beta + 4m_W^2(m^2_{\text{wino}} + |\mu|^2 - 2m_{\text{wino}}\text{Re}(\mu\sin 2\beta))}\right)$.

Slepton, squark mass matrix: $m^2_{\text{squark}} = \begin{pmatrix} m^2_{L\text{squark}} & m^{T2}_{LR\text{squark}} \\ m^2_{LR\text{squark}} & m^2_{R\text{squark}} \end{pmatrix}$,

where $m^2_{L/R\text{squark},KM} = M^{Q/\overline{U}\ 2}_{KM} + (m^2_{\text{quark},K} + m_Z^2(T_3 - Q\sin^2\theta_W)\cos 2\beta)\delta_{KM}$ (no K sum),

M^{S2}_{KM} from 1. on page 127, m_{quark} from unbroken SUSY condition $m_{\text{squark},KK} = m_{\text{quark},K}$ (no K sum),

and term proportional to m_Z^2 from Higgs-chiral superfield couplings in superpotential $[f(\Phi)]_{\mathscr{F}}$ on page 122.

$m^2_{LR\text{squark},KM} = (A^U_{KM} - \delta_{KM}m_{\text{quark}}\mu\tan\beta)m_{\text{quark}}$. First term from 3. on page 127, second term from $\left|\frac{\partial f}{\partial\phi_{H_1^0}}\right|^2$.

Further reading

Stephen P. Martin, hep-ph/9709356 Very topical (not much basics)

— helps you understand what people are talking about

Ian J. R. Aitchison, hep-ph/0505105 Complete derivations up to and including the MSSM

Michael E. Peskin, 0801.1928 [hep-ph] Recent, readable

Steven Weinberg, The Quantum Theory Of Fields III Very complete derivations, quite formal but clear

5 Supergravity

Construct coordinates x_μ over whole of curved spacetime with metric tensor $g_{\mu\nu}(x)$,

and locally inertial coordinate system at each point with "flat" metric tensor $\eta_{ab} = \mathrm{diag}(1, 1, 1, -1)$.

Vierbein $e^a{}_\mu(x)$: Defined by $g_{\mu\nu}(x) = \eta_{ab} e^a{}_\mu(x) e^b{}_\nu(x)$.

Curved space transformation of vierbein: $e'^a{}_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} e^a{}_\mu(x)$.

Lorentz transformation of vierbein: $e^a{}_\mu(x) \to \Lambda^a{}_b(x) e^b{}_\mu(x)$.

Infinitesimal curved space transformation: $x^\mu \to x^\mu + \xi^\mu(x)$.

Infinitesimal Lorentz transformation: $\Lambda^a{}_b(x) = \delta^a{}_b + \omega^a{}_b(x)$.

Work in weak field limit, i.e. $e^a{}_\mu(x) \simeq \delta^a_\mu$, then $a \to \mu$ etc.

Weak field representation: $e_{\mu\nu}(x) = \eta_{\mu\nu} + 2\kappa\phi_{\mu\nu}$, i.e. $g_{\mu\nu}(x) = \eta_{\mu\nu} + 2\kappa(\phi_{\mu\nu} + \phi_{\nu\mu})$, where $\kappa = \sqrt{8\pi G}$.

Particle content of supergravity: spin-2 *graviton* $h_{\mu\nu} = \phi_{\mu\nu} + \phi_{\nu\mu}$ and spin-$\frac{3}{2}$ *gravitino* ψ_μ.

Graviton transformation: $\phi_{\mu\nu}(x) \rightarrow \phi_{\mu\nu}(x) + \frac{1}{2\kappa}\left[-\frac{\partial\xi_\mu(x)}{\partial x_\nu} + \omega_{\mu\nu}(x)\right]$.

Follows from performing infinitessimal curved space and Lorentz transformations on page 141.

Gravitino transformation: $\psi_\mu(x) \rightarrow \psi_\mu(x) + \partial_\mu\psi(x)$.

Required for low energy interactions, similar to requirement of invariance under $A^\mu(x) \rightarrow A^\mu(x) + \partial^\mu\alpha(x)$ found on page 52.

Goal: Put $\phi_{\mu\nu}$, ψ_μ into one superfield, ξ_μ, $\omega_{\mu\nu}$, ψ into another.

Graviton, gravitino as functions of supermultiplet components of

Metric superfield: $H_\mu(x,\theta) = C^H_\mu(x) - I[\overline{\theta}\gamma_5]\omega^H_\mu(x) - \frac{I}{2}\left[\overline{\theta}\gamma_5\theta\right] M^H_\mu(x) - \frac{1}{2}[\overline{\theta}\theta]N^H_\mu(x) + \frac{I}{2}[\overline{\theta}\gamma_5\gamma_\nu\theta]V^{H\ \nu}_\mu(x)$

$$-I[(\overline{\theta}\gamma_5\theta)\overline{\theta}]\left(\lambda^H_\mu(x) + \frac{1}{2}\partial\!\!\!/\omega^H_\mu(x)\right) - \frac{1}{4}[\overline{\theta}\gamma_5\theta]^2\left(D^H_\mu(x) + \frac{1}{2}\partial^2 C^H_\mu(x)\right),$$

This is general form of superfield on page 96, but with extra spacetime index μ.

Specifically,

Graviton, gravitino components of metric superfield H_μ: $\phi_{\mu\nu}(x) = V^H_{\mu\nu}(x) - \frac{1}{3}\eta_{\mu\nu}V^{H\lambda}_{\ \ \lambda}(x)$,

$$\frac{1}{2}\psi_\mu(x) = \lambda^H_\mu - \frac{1}{3}\gamma_\mu\gamma^\rho\lambda^H_\rho(x) - \frac{1}{3}\gamma_\mu\partial^\rho\omega^H_\rho(x).$$

Graviton, gravitino transformations on page 142 equivalent to

Transformation of metric superfield: $H_\mu(x,\theta) \to H_\mu(x,\theta) + \Delta_\mu(x,\theta)$, where

Transformation superfield:

$$\Delta_\mu(x,\theta) = C^\Delta_\mu(x) - I[\overline{\theta}\gamma_5]\omega^\Delta_\mu(x) - \frac{I}{2}\left[\overline{\theta}\gamma_5\theta\right] M^\Delta_\mu(x) - \frac{1}{2}[\overline{\theta}\theta]N^\Delta_\mu(x) + \frac{I}{2}[\overline{\theta}\gamma_5\gamma_\nu\theta]V^{\Delta\ \nu}_\mu(x)$$

$$-I[(\overline{\theta}\gamma_5\theta)\overline{\theta}]\left(\lambda^\Delta_\mu(x) + \frac{1}{2}\partial\!\!\!/\omega^\Delta_\mu(x)\right) - \frac{1}{4}[\overline{\theta}\gamma_5\theta]^2\left(D^\Delta_\mu(x) + \frac{1}{2}\partial^2 C^\Delta_\mu(x)\right).$$

See later for dependence of these fields on ψ, ξ_μ, $\omega_{\mu\nu}$ of graviton, gravitino transformations on page 142.

Gravity-matter coupling: $\mathscr{A}_{\text{int}} = 2\kappa \int d^4x \, [H_\mu \Theta^\mu]_D$, where

Supercurrent from left-chiral superfields: $\Theta_\mu = \frac{I}{12}\left[4\Phi_n^\dagger \partial_\mu \Phi_n - 4\Phi_n \partial_\mu \Phi_n^\dagger + (\overline{\mathscr{D}}\Phi_n^\dagger)\gamma_\mu(\mathscr{D}\Phi_n)\right]$.

Supercurrent is a superfield, contains conserved current and energy momentum tensor:

Supercurrent conservation law: $\gamma^\mu \mathscr{D}\Theta_\mu = \mathscr{D}\frac{2}{3}\text{Im}\left[M\frac{\partial f(\Phi)}{\partial M}\right]$.

M here defined as follows: Each coupling constant λ_i in f is written $\lambda_i = M^{d_M(\lambda_i)}\tilde{\lambda}_i$, where $\tilde{\lambda}_i$ dimensionless.

SUSY current from supercurrent components: $S^\mu = -2\omega^{\Theta\mu} + 2\gamma^\mu\gamma^\nu\omega_\nu^\Theta$.

SUSY current conservation law: $\partial_\mu S^\mu = 0$. Follows from supercurrent conservation law above.

Energy-momentum tensor from supercurrent components: $T_{\mu\nu} = -\frac{1}{2}V_{\mu\nu}^\Theta - \frac{1}{2}V_{\nu\mu}^\Theta + \eta_{\mu\nu}V^{\Theta\lambda}_{\lambda}$, obeys

Energy-momentum tensor conservation law: $\partial_\mu T^{\mu\nu} = 0$, Follows from supercurrent conservation law above.

Relation between energy-momentum tensor and momentum: $\int d^3x T^{0\nu} = P^\nu$.

Follows from SUSY transformation of ω_μ^Θ, take time component and integrate over x: $I\{\int d^3x S^0, \overline{Q}\} = 2\gamma_\nu \int d^3x T^{0\nu}$.

But $\int d^3x S^0 = Q$ (as in bosonic generator case), then use relation with momentum for any N on page 83.

Further useful results: $\mathscr{R}^\mu = 2C^{\Theta\mu}$, $M_\mu^\Theta = \partial_\mu \mathscr{M}$, $N_\mu^\Theta = \partial_\mu \mathscr{N}$.

Constraint on transformation superfield Δ_μ: $\Delta_\mu = \overline{\mathscr{D}}\gamma_\mu \Xi$, where Ξ obeys $(\overline{\mathscr{D}}\mathscr{D})(\overline{\mathscr{D}}\Xi) = 0$.

Ensures $\mathscr{A}_{\text{int}}$ in gravity-matter coupling on page 144 is invariant under transformation of metric superfield on page 143, follows from supercurrent conservation law on page 144.

Recal transformation parameters ξ_μ, $\omega_{\mu\nu}$, ψ in graviton, gravitino transformations on page 142.

Transformation parameter components of transformation superfield:

$V^\Delta_{\mu\nu} + V^\Delta_{\nu\mu} = -\frac{1}{2\kappa}\left[\frac{\partial \xi_\mu}{\partial x_\nu} + \frac{\partial \xi_\nu}{\partial x_\mu} - 2\eta_{\mu\nu}\frac{\partial \xi^\lambda}{\partial x^\lambda}\right]$, $\lambda^\Delta_\mu - \frac{1}{3}\gamma_\mu\gamma^\rho\lambda^\Delta_\rho - \frac{1}{3}\gamma_\mu\partial^\rho\omega^\Delta_\rho = \frac{1}{2}\partial_\mu\psi$.

From constraint on transformation superfield Δ_μ above, and graviton, gravitino transformations on page 142.

Further constraints on components of transformation superfield: $-\frac{1}{2}\epsilon^{\nu\mu\kappa\sigma}\partial_\kappa V^\Delta_{\nu\mu} = D^{\Delta\sigma} + \partial^\sigma\partial^\rho C^\Delta_\rho$, $\partial M^\Delta_\mu = \partial N^\Delta_\mu = 0$.

Again, from constraint on transformation superfield Δ_μ above.

Auxiliary fields: $b^\sigma = D^{H\sigma} + \frac{1}{2}\epsilon^{\nu\mu\kappa\sigma}\partial_\kappa V^H_{\nu\mu} + \partial^\sigma\partial^\rho C^H_\rho$, $s = \partial^\mu M^H_\mu$, $p = \partial^\mu N^H_\mu$, invariant.

From further constraints on components of transformation superfield above.

Choose $C^H_\mu = V^H_{\mu\nu} - V^H_{\nu\mu} = \phi_{\mu\nu} - \phi_{\nu\mu} = \omega^H_\nu = 0$. Then $h_{\mu\nu} = 2\phi_{\mu\nu}$.

Can be done by suitable choice of C^Δ_μ, $V^\Delta_{\mu\nu} - V^\Delta_{\nu\mu}$, ω^Δ_ν in transformation of metric superfield on page 143.

To summarize, components of superfields are:

$\Theta \ni T^{\kappa\sigma}, S^\sigma, \mathscr{R}_\sigma, \mathscr{M}, \mathscr{N}.$ $\qquad H_\mu \ni h_{\kappa\sigma}, \psi_\sigma, b^\sigma, s, p.$

Gravity-matter coupling in terms of components: $\mathscr{A}_{\text{int}} = \kappa \int d^4x [T^{\kappa\sigma} h_{\kappa\sigma} + \frac{1}{2}\overline{S}^\sigma \psi_\sigma + \mathscr{R}_\sigma b^\sigma - 2\mathscr{M} s - 2\mathscr{N} p]$.

For dynamic part of gravitational action, use:

Einstein superfield E_μ: $C^E_\mu = b_\mu$, $\omega^E_\mu = \frac{3}{2}L_\mu - \frac{1}{2}\gamma_\mu\gamma^\nu L_\nu$, $M^E_\mu = \partial_\mu s$, $N^E_\mu = \partial_\mu p$,

$V^E_{\mu\nu} = -\frac{3}{2}E_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}E^\rho{}_\rho + \frac{1}{2}\epsilon_{\nu\mu\sigma\rho}\partial^\sigma b^\rho$, $\lambda^E_\mu = \partial_\mu \gamma^\nu \omega^E_\nu - \partial\!\!\!/ \omega^E_\mu$, $D^E_\mu = \partial_\mu\partial^\nu b_\nu - \partial^2 b_\mu$, where $L^\nu = I\epsilon^{\nu\mu\kappa\rho}\gamma_5\gamma_\mu\partial_\kappa\psi_\rho$,

and *linearized Einstein tensor* $E_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\nu h^\lambda{}_\lambda + \partial^2 h_{\mu\nu} - \partial_\mu\partial^\lambda h_{\lambda\nu} - \partial_\nu\partial^\lambda h_{\lambda\mu} - \eta_{\mu\nu}\partial^2 h^\lambda{}_\lambda + \eta_{\mu\nu}\partial^\lambda\partial^\rho h_{\lambda\rho}\right)$

$$= \frac{1}{2\kappa}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right).$$

Dynamic part of gravitational Lagrangian: $\mathscr{L}_E = \frac{4}{3}[E_\mu H^\mu]_D = E_{\mu\nu}h^{\mu\nu} - \frac{1}{2}\overline{\psi}_\mu L^\mu - \frac{4}{3}(s^2 + p^2 - b_\mu b^\mu)$.

Now put together matter Lagrangian $\mathscr{L}_M$, and $\mathscr{L}_E$ and integrand of $\mathscr{A}_{\text{int}}$ above and eliminate auxiliary fields:

Lagrangian of nature: $\mathscr{L} = \mathscr{L}_M + E_{\mu\nu}h^{\mu\nu} - \frac{1}{2}\overline{\psi}_\mu L^\mu + \kappa[T^{\kappa\sigma}h_{\kappa\sigma} + \frac{1}{2}\overline{S}^\sigma\psi_\sigma]$ $+\frac{3}{4}\kappa^2\left(\mathscr{M}^2 + \mathscr{N}^2 - \frac{1}{4}\mathscr{R}_\mu\mathscr{R}^\mu\right)$.

Everything except $\mathscr{L}_M$ of order κ^2.

Vacuum energy density: $\rho_{\rm VAC} = -\mathscr{L}_{\rm VAC} = -\mathscr{L}_{\rm M\ VAC} - \frac{3}{4}\kappa^2(\mathscr{M}^2 + \mathscr{N}^2)$.

Only s and p can acquire vacuum expectation values.

Solution to Einstein field equations ($\rho_{\rm VAC}$ uniform) for $\rho_{\rm VAC} \gtrless 0$ is de Sitter / anti de Sitter space:

spacetime embedded in 5-D space with $x_5^2 \pm \eta_{\mu\nu}x^\mu x^\nu = R^2$ and $ds^2 = \eta_{\mu\nu}x^\mu x^\nu \pm dx_5^2$ for $\rho_{\rm VAC} \gtrless 0$.

$\rho_{\rm VAC} < 0$ corresponds to O(3,2), which includes $N = 1$ SUSY,

but $\rho_{\rm VAC} > 0$ corresponds to O(4,1), which excludes unbroken $N = 1$ SUSY.

In unbroken SUSY, $\mathscr{L}_{\rm M\ VAC} = 0$ so from vacuum energy density above, $\rho_{\rm VAC} < 0$.

$\rho_{\rm VAC} < 0$ is unstable. However, anti de Sitter space cannot form

since positive energy $S_1 > |\rho_{\rm VAC}|$ is needed for its surface tension.

Local supersymmetry: α in definition of superfield on page 94 becomes dependent on x.

Only change is to matter action: $\delta \int d^4x \mathscr{L}_M = -\int d^4x \overline{S}^\mu(x)\partial_\mu \alpha(x)$ (usual definition of current).

Cancelled by term $\kappa \int d^4x \frac{1}{2}\overline{S}^\sigma \psi_\sigma$ in gravity-matter coupling in terms of components on page 146

if SUSY transformation of gravitino modified to: $\delta\psi_\mu \to \delta\psi_\mu + \frac{2}{\kappa}\partial_\mu\alpha$.

But $\psi_\mu \to \psi_\mu + \frac{2}{\kappa}\partial_\mu\alpha$ is of same form as gravitino transformation on page 142,

i.e. leaves term $-\frac{1}{2}\overline{\psi}_\mu L^\mu$ in dynamic part of gravitational Lagrangian $\mathscr{L}_E$ on page 146 unchanged.

So supersymmetric gravity is locally supersymmetric.