Excercise Sheet 9 to General Relativity

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Discussion on 18.01.2013 in the exercise classes

1. **Change of Light Propagation Time by a Gravitational Wave**

   The metric for a "plus-polarized" gravitational wave of frequency ω and amplitude h propagating in the $z-$direction can be written as

$$ds^2 = dt^2 - [1 + h\cos(\omega(t-z)+\phi_0)]\,dx^2 - [1 - h\cos(\omega(t-z)+\phi_0)]\,dy^2 - dz^2\,,$$

   where ϕ_0 is a phase.

   (a) Assuming that $h \ll 1$ calculate the travel time of a light ray over a distance $\mathbf{L} = L\mathbf{n}$, where $\mathbf{n}$ is a unit vector as a function of L, $\mathbf{n}$, ω and ϕ_0 to first order in the amplitude h. Assume the condition $\omega L \ll 1$ for the first order approximation to be good that we derived in the lecture.

   (b) Determine the orientations of $\mathbf{n}$ for which the change $\Delta t/t$ of the light travel time relative to the light travel time t in the absence of the gravitational wave is maximal and minimal. For which orientations does Δt vanish ?

   please turn over

2. **Linearized Gravity from a Variational Principle**

To first order in $h_{\mu\nu}$ the Ricci tensor is given by

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(-\Box\, h_{\mu\nu} + \partial_\sigma\partial_\mu h^\sigma_\nu + \partial_\sigma\partial_\nu h^\sigma_\mu - \partial_\mu\partial_\nu h^\sigma_\sigma\right) , \tag{1}$$

where all indices are raised and lowered with the Lorentz metric $\eta_{\mu\nu}$.

(a) Compute the Einstein tensor to first order $G^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)}$.

(b) Show that $G^{(1)}_{\mu\nu}$ can be obtained by varying the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu h^{\mu\nu})(\partial_\nu h) - (\partial_\mu h^{\rho\sigma})(\partial_\rho h^\mu{}_\sigma) + \frac{1}{2}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h)(\partial_\nu h)\right] , \tag{2}$$

with respect to $h_{\mu\nu}$, where $h \equiv h^\sigma{}_\sigma$.

3. **Fermat's Principle for Light Rays Propagating in a Newtonian Potential**

According to Fermat's principle the trajectories of light rays in a medium with refractive index $n(\mathbf{r})$ extremize the time

$$t = \int \left(d\mathbf{r}^2\right)^{1/2} n(\mathbf{r})\, , \tag{3}$$

with $d\mathbf{r}^2 = dx^2 + dy^2 + dz^2$. Show that for the refractive index $n(\mathbf{r}) = 1 - 2\Phi(\mathbf{r})$ this principle leads to the correct geodesic equation for photons in a Newtonian potential $\Phi(\mathbf{r})$.