Excercise Sheet 8 to General Relativity

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Discussion on 11.01.2013 in the exercise classes

1. **Movement within the Schwarzschild Horizon**

   Consider a particle inside the event horizon $R_S = 2G_{\rm N}M$ of a Schwarzschild black hole, $r \leq R_S$.

   (a) Show that independently of whether or not the particle moves on a geodesic the minimal rate with which the radius has to decrease is given by

   $$\left|\frac{dr}{d\tau}\right| \geq \left(\frac{2G_{\rm N}M}{r} - 1\right)^{1/2}, \tag{1}$$

   where τ is the proper time of the particle.

   (b) Using this compute the maximal proper time it takes the particle to fall from $r = R_S$ to the singularity at $r = 0$. Express this in terms of seconds for a black hole mass measured in solar masses.

   (c) Show that this maximal proper lifetime is realized when falling freely along a geodesic with the constant of motion

   $$E \equiv \left(1 - \frac{2G_{\rm N}M}{r}\right)\frac{dt}{d\tau} = 0\,. \tag{2}$$

   please turn over

2. **Movement within the Schwarzschild Horizon**

Consider an observer located at fixed Schwarzschild coordinates (r_0, θ_0, ϕ_0) dropping a beacon into a Schwarzschild black hole along a radial direction. The beacon emits photons with a constant frequency ω_{em} in the beacon rest frame.

(a) Calculate the Schwarzschild coordinate speed dr/dt of the beacon as a function of r.

(b) Calculate the speed of the beacon as it passes an observer that is held at a fixed coordinate $r < r_0$ in the rest frame of that observer. What is that speed when r approches $R_S = 2G_{\mathrm{N}}M$?

(c) Calculate the frequency ω_{obs} of the radiation arriving at the observer at r_0 as a function of radial coordinate r at which the radiation was emitted by the beacon.

(d) Calculate the coordinate time elapsing between emission of the photons at r and detection at r_0.

(e) Show that at late times, $t \gg G_{\mathrm{N}}M$, the frequency decreases exponentially, $\omega_{\mathrm{obs}} \propto \omega_{\mathrm{em}} \exp(-t/T)$. Express the suppression time scale T in terms of the black hole mass M.