Excercise Sheet 7 to General Relativity

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Discussion on 14.12.2012 in the exercise classes

1. **The Generators of** $SO(3)$

   In cartesian coordinates (x, y, z) on $\mathbb{R}^3$ the components of the Killing vector field generators of $SO(3)$ are given by

$$\begin{aligned} R^\mu &= (-y, x, 0)\,, \\ S^\mu &= (z, 0, -x)\,, \\ R^\mu &= (0, -z, y)\,. \end{aligned} \tag{1}$$

   (a) Show that in polar coordinates (r, θ, ϕ)

$$\begin{aligned} R^\mu &= \partial_\phi\,, \\ S^\mu &= \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi\,, \\ R^\mu &= -\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi\,. \end{aligned} \tag{2}$$

   (b) Show that these three Killing vector fields satisfy the following algebra

$$\begin{aligned} [R, S] &= T\,, \\ [S, T] &= R\,, \\ [T, R] &= S\,. \end{aligned} \tag{3}$$

   please turn over

2. **Kruskal Coordinates for the Schwarzschild Metric**

In terms of the standard Schwarzschild coordinates t, r the Kruskal coordinates are defined by

$$\begin{aligned} T &= \left(\frac{r}{2G_{\rm N}M} - 1\right)^{1/2} \exp\left(\frac{r}{4G_{\rm N}M}\right) \sinh\left(\frac{t}{4G_{\rm N}M}\right) , \\ R &= \left(\frac{r}{2G_{\rm N}M} - 1\right)^{1/2} \exp\left(\frac{r}{4G_{\rm N}M}\right) \cosh\left(\frac{t}{4G_{\rm N}M}\right) . \end{aligned} \tag{4}$$

(a) Show that expressed in terms of the Kruskal coordinates the Schwarzschild metric reads

$$ds^2 = \frac{32G_{\rm N}^3M^3}{r} \exp\left(-\frac{r}{2G_{\rm N}M}\right) \left(dT^2 - dR^2\right) - r^2 d\Omega^2 \, , \tag{5}$$

where r is defined implicitly by

$$T^2 - R^2 = \left(1 - \frac{r}{2G_{\rm N}M}\right) \exp\left(\frac{r}{2G_{\rm N}M}\right) . \tag{6}$$

(b) Determine the range of values that T and R can take.

(c) Determine the curves of constant r in terms of T and R.

(d) Determine the curves of constant t in terms of T and R.

(e) Determine the location of the Schwarzschild radius, $r = 2G_{\rm N}M$, in terms of T and R.

(f) What happens at $r < 2G_{\rm N}M$?