Excercise Sheet 6 to General Relativity

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Discussion on 30.11.2012 in the exercise classes

1. **Bianchi Identity for the Ricci Tensor**

   Show that the Einstein tensor
   $$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R$$
   is locally conserved,
   $$\nabla_\nu G^{\nu\mu} = \nabla_\nu \left(R^{\nu\mu} - \frac{1}{2} R\, g^{\nu\mu} \right) = 0\,. \tag{1}$$

2. **Another Form of Einstein's Equations**

   Show that Einstein's equations are equivalent to
   $$R_{\mu\nu} = 8\pi G_{\rm N} \left(T_{\mu\nu} - \frac{1}{2} T\, g_{\mu\nu} \right)\,, \tag{2}$$
   where $T = T^\lambda_\lambda$ is the trace of the energy-momentum tensor.

   please turn over

3. **Why the Equation $R_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu}$ would be Unphysical**

Show that the alternative to Einstein's equations,

$$R_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu} \,,$$

would imply that the trace $T = T^{\mu}_{\mu}$ of the energy-momentum tensor would be constant, $\partial_\mu T = 0$. Hint: Use the Bianchi identity for the Ricci tensor.

4. **Conformal Transformations and the Weyl Tensor**

(a) Show that the Weyl tensor defined by

$$\begin{aligned} C_{\rho\lambda\mu\nu} \;\equiv\;& R_{\rho\lambda\mu\nu} + \frac{1}{n-2}\left(g_{\rho\nu}R_{\mu\lambda} - g_{\rho\mu}R_{\nu\lambda} + g_{\lambda\mu}R_{\nu\rho} - g_{\lambda\nu}R_{\mu\rho}\right) + \\ +\;& \frac{R}{(n-1)(n-2)}\left(g_{\rho\mu}g_{\nu\lambda} - g_{\rho\nu}g_{\mu\lambda}\right) \,. \end{aligned} \tag{3}$$

satisfies the properties of the Riemann tensor and in addition has only vanishing contractions,

$$C^{\lambda}{}_{\sigma\lambda\nu} = 0 \,. \tag{4}$$

(b) Consider the conformal transformation

$$g'_{\mu\nu}(x) = \omega(x) g_{\mu\nu}(x) \,, \tag{5}$$

and assume the metric compatible Christoffel connection. Express the Riemann tensor $R'^{\rho}{}_{\lambda\mu\nu}(x)$ for the metric $g'_{\mu\nu}(x)$ in terms of the Riemann tensor $R^{\rho}{}_{\lambda\mu\nu}(x)$ for the metric $g_{\mu\nu}(x)$, derivatives of $\omega(x)$, and the old metric $g_{\mu\nu}(x)$.

(c) If you are very bold, using the result of (b) try to show that the Weyl tensor $C'^{\rho}{}_{\lambda\mu\nu}$ is invariant under conformal transformations Eq. (5).