Excercise Sheet 5 to General Relativity

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Discussion on 23.11.2012 in the exercise classes

1. **Geodesic Equation and Extremal Action**

   Show that the geodesic equation of motion in a gravitational field,

$$\frac{D}{d\tau}\frac{dx^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\rho}{d\tau} = 0\,,$$

   can also be derived from extremizing the action

$$S_{\rm particle} = \frac{1}{c_0}\int ds = \int d\tau = \int d\lambda \mathcal{L}_{\rm particle} = \int d\lambda \left(\frac{d\tau}{d\lambda}\right) = \int d\lambda \left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2}\,. \qquad (1)$$

   under small variations $x^\mu(\lambda) \to x^\mu(\lambda) + \delta x^\mu(\lambda)$ of the path connecting two space-time points such that the variations δx^μ vanish at the end points. Hint: Transform to proper time as integration variable, using

$$d\lambda = \left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{-1/2} d\tau\,,$$

   such that

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = g_{\mu\nu}u^\mu u^\nu = 1\,,$$

   and thus the variation of Eq. (1) has the form

$$\delta S_{\rm particle} = \frac{1}{2}\int d\tau \delta\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)\,,$$

   and leads to the equation of motion

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}_{\rm particle}}{\partial(dx^\mu/d\tau)} = 2\,\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) = \frac{\partial\mathcal{L}_{\rm particle}}{\partial x^\mu} = (\partial_\mu g_{\rho\sigma})\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\,.$$

   please turn over

2. **Torsion and Riemann Tensor as Multi-Linear Maps**

The covariant derivative with respect to a vector field X is defined by

$$\nabla_X \equiv X^\mu \nabla_\mu$$

and defines a linear map from second vector field Y to a third vector field $\nabla_X Y$. Using this notation show the following:

(a) The torsion tensor with the components $T^\lambda_{\mu\nu} \equiv \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}$ for an arbitrary connection $\Gamma^\lambda_{\mu\nu}$ defines a bilinear map from two vector fields $X = X^\mu \partial_\mu$ and $Y = Y^\nu \partial_\nu$ to a third vector field whose components are given by

$$T^\lambda_{\mu\nu} X^\mu Y^\nu \,.$$

Show that this bilinear map can be written as

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \,.$$

(b) The Riemann tensor defines a trilinear map from three vector fields $X = X^\mu \partial_\mu$, $Y = Y^\nu \partial_\nu$, and $Z = Z^\sigma \partial_\sigma$ to a forth vector field whose components are given by

$$R^\rho{}_{\sigma\mu\nu} X^\mu Y^\nu Z^\sigma \,.$$

Show that this trilinear map can be written as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \,.$$

To demonstrate this express the components $R^\rho{}_{\sigma\mu\nu}$ of the Riemann tensor in terms of general connection coefficients $\Gamma^\lambda_{\mu\nu}$ as derived in the lecture.

3. **Properties of the Riemann Tensor**

(a) Show that the sum over cyclic permutations of the last three indices of the Riemann tensor vanishes,

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0 \,.$$

(b) Show that this is equivalent to

$$R_{\rho[\sigma\mu\nu]} = 0 \,.$$

Hint: Use the expression for the Riemann tensor in a locally inertial frame.