Excercise Sheet 4 to General Relativity

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Discussion on 16.11.2012 in the exercise classes

1. **Maxwell's Equations in Terms of Differential Forms**

   (a) Show that the homogeneous Maxwell equations

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0$$

   can be written as

$$\partial_{[\mu} F_{\nu\sigma]} = 0\,,$$

   or simply as

$$dF = 0$$

   in terms of the field strength tensor two form $F_{\mu\nu}$.

   (b) Show that the inhomogeneous Maxwell equations

$$\partial_\mu F^{\mu\nu} = j^\nu\,,$$

   with $j^\nu = (\rho_e, \mathbf{j})$ the electric current density four-vector, can be written as

$$d(*F) = *J\,,$$

   with the definition of the *Hodge star operator* given in the lectures and the current density one-form $J_\nu \equiv \eta_{\nu\rho} j^\rho$. Hints: Show in a first step that

$$[d(*F)]_{\mu_1\mu_2\mu_3} = \frac{3}{2}\,\partial_{[\mu_1} \epsilon^{\nu_1\nu_2}_{\mu_2\mu_3]} F_{\nu_1\nu_2}\,,$$

   and figure out over which values the indices ν_1 and ν_2 run for given mutually different indices μ_1, μ_2, μ_3. Finally, express the result in terms of the unique index $\nu \neq (\mu_1, \mu_2, \mu_3)$ and keep track of all the numerical pre-factors.

   please turn over

2. **Cross Product and Hodge Dual in 3 Dimensions**

   Show that the cross product $\mathbf{A} \times \mathbf{B}$ of two vectors $\mathbf{A}$ and $\mathbf{B}$ in three-dimensional Euclidean space can be understood as the Hodge product of the two corresponding one-forms $A_i dx^i$ and $B_i dx^i$, where A_i and B_i are the Euclidean components of $\mathbf{A}$ and $\mathbf{B}$, respectively.

3. **The Divergence of Vector Fields**

   (a) Show that the Christoffel symbols satisfy

$$\Gamma^{\mu}_{\mu\lambda} = \frac{1}{\sqrt{|g|}} \, \partial_\lambda \sqrt{|g|} \, , \tag{1}$$

   where $g = \det g_{\mu\nu}$. Hint: First show that for an infinitesimal variation δM of an invertible $n \times n$ matrix M one has

$$\delta \ln \det M = \mathrm{Tr}\left(M^{-1} \delta M\right) \, . \tag{2}$$

   (b) Using this result show that the divergence of a vector field V^μ can be written as

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \, \partial_\mu \left(\sqrt{|g|} V^\mu\right) \, , \tag{3}$$

4. **Paraboloidal Coordinates**

   In Euclidean three-space, paraboloidal coordinates (u, v, ϕ) are defined by

$$x = uv \cos\phi \, , \quad y = uv \sin\phi \, , \quad z = \frac{1}{2}\left(u^2 - v^2\right) \, . \tag{4}$$

   (a) Calculate the metric $g_{\mu\nu}$ and the inverse metric $g^{\mu\nu}$ in paraboloidal coordinates.

   (b) Calculate the Christoffel symbols for the paraboloidal coordinates.