Excercise Sheet 3 to General Relativity

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Discussion on 9.11.2012 in the exercise classes

1. **Coordinate Transformations on the Sphere**

   Consider the sphere S^2 as a two dimensional differentiable manifold

   $$S^2 \equiv \{(x^1, x^2, x^3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\} \,.$$

   (a) Show that the charts $(U_\pm, \psi_\pm)$ given by

   $$\begin{aligned} U_+ &\equiv S^2 \setminus \{(0,0,1)\}\,; \quad \psi_+ : U_+ \to \mathbb{R}^2, (x^1, x^2, x^3) \mapsto (y^1, y^2) = \left(\frac{2x^1}{1-x^3}, \frac{2x^2}{1-x^3} \right) \,, \\ U_- &\equiv S^2 \setminus \{(0,0,-1)\}\,; \quad \psi_- : U_- \to \mathbb{R}^2, (x^1, x^2, x^3) \mapsto (z^1, z^2) = \left(\frac{2x^1}{1+x^3}, \frac{2x^2}{1+x^3} \right) \,, \end{aligned}$$

   represent the Mercator projection from the North and the South pole of the sphere, respectively. Show that they form a differentiable atlas for S^2.

   (b) Show that the coordinate transformation $\psi_- \circ \psi_+^{-1}$ is given by

   $$z^i = \frac{4y^i}{(y^1)^2 + (y^2)^2} \,, \quad i = 1, 2 \,, \quad 0 < (y^1)^2 + (y^2)^2 < \infty \,.$$

please turn over

2. **Transformation Properties of various Tensors**

   (a) Show that the symmetry and antisymmetry, $T_{\mu\nu} = \pm T_{\nu\mu}$, of an arbitrary $(0,2)$-tensor $T_{\mu\nu}$ is invariant under change of coordinates.

   (b) Let $y^\mu = \Lambda^\mu{}_\nu x^\nu$ be the Lorentz transformation between two inertial observers in special relativity. How do the components $F_{\mu\nu}$ of the electromagnetic field strength tensor change under this transformation?

3. **Curves and Tangent Vectors**

   Given is the parametrized curve $\gamma(t)$ in $M = \mathbb{R}^3$

$$\gamma(t) = [x(t), y(t), z(t)] = (\cos t, \sin t, t)\,.$$

   (a) Derive the parametrization of the curve in spherical coordinates

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

   (b) What are the components of its tangent vector in cartesian and spherical coordinates?