Excercise Sheet 2 to General Relativity

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Discussion on 2.11.2012 in the exercise classes

1. **Contravariant and Covariant Vectors**

   The components a^μ of a contravariant vector transform as

   $$a'^\mu = \Lambda^\mu{}_\nu a^\nu$$

   under a Lorentz transformation $\Lambda^\mu{}_\nu$.

   (a) Show that the components

   $$a_\mu \equiv \eta_{\mu\nu} a^\nu$$

   of the corresponding covariant vector (often also called *dual vector*) transforms as

   $$a'_\mu = \Lambda_\mu{}^\nu a_\nu \,,$$

   where we introduced the notation

   $$\Lambda_\mu{}^\nu \equiv \left(\Lambda^{-1}\right)^\nu{}_\mu \,.$$

   (b) By introducing a set of *basis vectors* $e_{(\mu)}$ a (contravariant) vector can also by defined in an abstract way as a quantity

   $$a \equiv a^\mu e_{(\mu)}$$

   which is independent of the coordinate system. Show that the set of basis vectors then transforms as

   $$e'_{(\mu)} = \Lambda_\mu{}^\nu e_{(\nu)} \,.$$

   please turn over

2. **Commutators of Vector Fields**

For two smooth vector fields v and w the *commutator* is defined by

$$[v,w](f) \equiv v[w(f)] - w[v(f)]\,, \tag{1}$$

for all smooth (C^∞) functions $f: M \to \mathbb{R}$.

(a) Show that the definition Eq. (1) satisfies the linearity and Leibnitz properties and thus defines a vector field.

(b) Show that the components of the commutator of two smooth vector fields v and w are given by

$$[v,w]^\mu = v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \tag{2}$$

in any coordinate system.

(c) Show that any three smooth vector fields X, Y, Z satisfy the *Jacobi identity*,

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0\,. \tag{3}$$

3. **Coordinate Transformations of the Metric Tensor**

The metric of special relativity is given by the Lorentz metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{4}$$

Derive the components of the metric $g_{\mu\nu}$ and the inverse metric $g^{\mu\nu}$ in the rotating coordinate frame given by

$$\begin{aligned} t' &= t\,, \\ x' &= (x^2+y^2)^{1/2}\cos(\phi - \omega t)\,, \\ y' &= (x^2+y^2)^{1/2}\sin(\phi - \omega t)\,, \\ z' &= z\,, \end{aligned}$$

where $\phi = \arctan y/x$.