Sheet 8, Problem 1

$$d\tau^2 = \left(\frac{2GM}{r}-1\right)^{-1} dr^2 - \left(\frac{2GM}{r}-1\right) dt^2 \cancel{-r}^2 d\Omega^2 \geq 0$$

$$\Rightarrow \left|\frac{d\tau}{dr}\right| \leq \left(\frac{2GM}{r}-1\right)^{-1/2}$$

approaching the singularity

$$\frac{d\tau}{dr} = -\frac{1}{\sqrt{\frac{2GM}{r}-1}}$$

$$\Rightarrow \tau = \sqrt{(2GM-r)r} + GM \arctan\left[\frac{GM-r}{\sqrt{(2GM-r)r}}\right] + \text{const.}$$

proof:

$$\frac{d\sqrt{(2GM-r)r}}{dr} = \frac{GM-r}{\sqrt{(2GM-r)r}}$$

$$\Rightarrow \frac{d\tau}{dr} = \frac{GM-r}{\sqrt{(2GM-r)r}} + \frac{GM}{1+\frac{(GM-r)^2}{(2GM-r)r}}\left[-\frac{1}{\sqrt{(2GM-r)r}} - \frac{(GM-r)^2}{[(2GM-r)r]^{3/2}}\right]$$

$$\frac{GM}{1+\frac{(GM-r)^2}{(2GM-r)r}} = \frac{GM(2GM-r)r}{\cancel{2GMr} \cancel{-r^2} + (GM)^2 \cancel{-2GMr} \cancel{+r^2}}$$

$$\Rightarrow \frac{d\tau}{dr} = \frac{GM-r}{\sqrt{(2GM-r)r}} - \frac{1}{GM}\left[\sqrt{(2GM-r)r} + \frac{(GM-r)^2}{\sqrt{(2GM-r)r}}\right]$$

$$= \frac{1}{\sqrt{(2GM-r)r}}\left[GM-r-\frac{1}{GM}(2GM-r)r - \frac{1}{GM}(GM-r)^2\right]$$

$$= -\frac{r}{\sqrt{(2GM-r)r}} = -\frac{1}{\sqrt{\frac{2GM}{r}-1}}$$

$r \to 0 \Rightarrow \tau \to GM \arctan +\infty = \frac{\pi}{2} GM$

$r \to 2GM \Rightarrow \tau \to GM \arctan -\infty = -\frac{\pi}{2} GM$

$\Rightarrow \tau_{max} = \pi GM$

Now conside geodesic

$$\Rightarrow \left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau} = E = \text{const.}$$

$$d\tau^2 = -\left(\frac{2GM}{r}-1\right)dt^2 + \left(\frac{2GM}{r}-1\right)^{-1}dr^2 = \cancel{\frac{1}{E^2}\left(2GM\right)\left(\frac{2GM}{r}-1\right)^2 dt^2}$$

$$= -\left(\frac{2GM}{r}-1\right)^{-1}E^2\,d\tau^2 + \left(\frac{2GM}{r}-1\right)^{-1}dr^2$$

$$\Rightarrow \left(\frac{dr}{dt}\right)^2 = E^2 + \left(\frac{2GM}{r}-1\right)$$

$$\Rightarrow \left|\frac{d\tau}{dr}\right| = \frac{1}{\sqrt{E^2+\left(\frac{2GM}{r}-1\right)}}$$ maximum for $E \to 0$

a) Sheet 8, Problem 2

at $r = r_0$ with initial speed zero, $\frac{dr}{dt} = 0$

$$d\tau = \sqrt{1-\frac{2GM}{r_0}}\,dt \quad \text{at } r = r_0$$

$$\Rightarrow \left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau} = \sqrt{1-\frac{2GM}{r_0}} = E = \text{const.}$$

$$\Rightarrow d\tau^2 = \frac{\left(1-\frac{2GM}{r}\right)^2}{1-\frac{2GM}{r_0}}\,dt^2$$

$$\Rightarrow \left[\frac{\left(1-\frac{2GM}{r}\right)^2}{1-\frac{2GM}{r_0}} - \left(1-\frac{2GM}{r}\right)\right]dt^2 = -\left(1-\frac{2GM}{r}\right)^{-1}dr^2$$

$$\Rightarrow \frac{dr}{dt} = -\left(1-\frac{2GM}{r}\right)\sqrt{1-\frac{1-\frac{2GM}{r}}{1-\frac{2GM}{r_0}}}$$

$$= -\frac{1-\frac{2GM}{r}}{\sqrt{1-\frac{2GM}{r_0}}}\sqrt{\frac{2GM}{r}-\frac{2GM}{r_0}} =$$

$$= -\frac{1-\frac{2GM}{r}}{\sqrt{(r_0-2GM)r}}\sqrt{2GM(r_0-r)}$$

note: for $r \to 2GM$ $\frac{dr}{dt} \to -\left(1-\frac{2GM}{r}\right)$ as for light

b) for $dr = 0$ $\quad d\tau_{obs} = \sqrt{1-\frac{2GM}{r}}\,dt$

$$\Rightarrow \frac{dr}{d\tau_{obs}} = \frac{dr}{dt}\frac{dt}{d\tau_{obs}} = \left[\frac{\left(1-\frac{2GM}{r}\right)2GM(r_0-r)}{(r_0-2GM)r}\right]^{1/2}$$

$$\frac{dr}{d\tau_{obs}} \to 0 \quad \text{for } r \to 2GM$$

~~Eliminate t~~

$$\frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} = -\sqrt{\frac{2GM}{r}-\frac{2GM}{r_0}}$$

c) $u_b^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, 0\right)$ of beacon is normalised to 1;

$$\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 = \frac{1-\frac{2GM}{r_0}}{1-\frac{2GM}{r}} - \frac{\frac{2GM}{r}-\frac{2GM}{r_0}}{1-\frac{2GM}{r}} = 1$$

$$\omega_{em} = u_{b\sigma}\frac{dx_\gamma^\sigma}{d\lambda} = \left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau}\frac{dt_\gamma}{d\lambda} - \left(1-\frac{2GM}{r}\right)^{-1}\frac{dr}{d\tau}\underbrace{\frac{dr_\gamma}{d\lambda}}$$

$$= \frac{dt_\gamma}{d\lambda}\frac{dr_\gamma}{dt_\gamma} = +\left(1-\frac{2GM}{r}\right)\frac{dt_\gamma}{d\lambda}$$

$$= \frac{dt_\gamma}{d\lambda}\left[\sqrt{1-\frac{2GM}{r_0}} + \sqrt{\frac{2GM}{r}-\frac{2GM}{r_0}}\right]$$

$$= \frac{E_\gamma}{1-\frac{2GM}{r}}[\ldots]$$

$$\omega_{obs} = u_{obs\,\sigma}\frac{dx_\gamma^\sigma}{d\lambda} = \left(1-\frac{2GM}{r_0}\right)^{1/2}\frac{E_\gamma}{\left(1-\frac{2GM}{r_0}\right)} = \frac{E_\gamma}{\sqrt{1-\frac{2GM}{r_0}}}$$

$$u^0_{obs} = \left(1-\frac{2GM}{r_0}\right)^{-1/2}$$

$$\Rightarrow \omega_{obs} = \omega_{em}\frac{1-\frac{2GM}{r}}{\sqrt{1-\frac{2GM}{r_0}}}\frac{1}{\sqrt{1-\frac{2GM}{r_0}}+\sqrt{\frac{2GM}{r}-\frac{2GM}{r_0}}}$$

for $r \to 2GM$

$$\omega_{obs} \to \frac{\omega_{em}}{2}\frac{1-\frac{2GM}{r}}{1-\frac{2GM}{r_0}}$$

null curve

$$\frac{dt}{dr} = \pm\left(1-\frac{2GM}{r}\right)^{-1}$$

$$\Rightarrow \quad t = \pm r_\gamma + \text{const.}$$

$$\Rightarrow \quad t_{obs} - t_{em} = r_{obs} - r_{em} + 2GM \ln\frac{\frac{r_{obs}}{2GM}-1}{\frac{r_{em}}{2GM}-1}$$

$$\cancel{= r_{obs} - r_{em} + 2GM \ln \frac{1-2GM/r}{}}$$

for $r_{em} \to 2GM$ this is $t_{obs} - t_{em} \simeq -2GM\ln\left(1-\frac{2GM}{r_{em}}\right)$

$$\Rightarrow \omega_{obs} \propto e^{-(t_{obs}-t_{em})/2GM} \simeq e^{-t_{obs}/4GM} \qquad t_{em} \simeq t_{obs} - t_{em},\; t_{em} \simeq \frac{t_{obs}}{2}$$