Übungsblatt 9

1.) $L_x = \frac{1}{2}(L_+ + L_-) \qquad L_y = \frac{1}{2i}(L_+ - L_-)$

$$\Rightarrow L_x^2 = \frac{1}{4}\left(L_+^2 + L_-^2 + L_+L_- + L_-L_+\right)$$

$$L_y^2 = -\frac{1}{4}\left(L_+^2 + L_-^2 - L_+L_- - L_-L_+\right)$$

$$\Rightarrow \langle l,m|L_x^2|l,m\rangle = \frac{1}{4}\langle l,m|L_+L_- + L_-L_+|l,m\rangle = \langle l,m|L_y^2|l,m\rangle \quad \Big| \text{ da } \langle l,m|L_\pm^2|l,m\rangle = 0$$

Andererseits ist nach Vorlesung

$$\vec{L}^2 = L_+L_- + L_z^2 - \hbar L_z = L_-L_+ + L_z^2 + \hbar L_z$$

$$\Rightarrow \frac{1}{4}\left(L_+L_- + L_-L_+\right) = \frac{1}{2}\left(\vec{L}^2 - L_z^2\right)$$

$$\Rightarrow \langle l,m|L_x^2|l,m\rangle = \langle l,m|L_y^2|l,m\rangle = \frac{\hbar^2}{2}\left[l(l+1) - m^2\right] \qquad (*)$$

$$L_{\vec{n}} = \cos\theta\, L_z + \sin\theta\, L_x$$

$$\Rightarrow \langle l,m|L_{\vec{n}}|l,m\rangle = \cos\theta \cdot \hbar m$$

$$\langle l,m|L_x|l,m\rangle = \frac{1}{2}\langle l,m|L_+ - L_-|l,m\rangle = 0$$

$$L_{\vec{n}}^2 = \cos^2\theta\, L_z^2 + \cos\theta\sin\theta\left(L_zL_x + L_xL_z\right) + \sin^2\theta\, L_x^2$$

$$\langle l,m|L_{\vec{n}}^2|l,m\rangle = \cos^2\theta\, \hbar^2 m^2 + \sin^2\theta\, \langle l,m|L_x^2|l,m\rangle$$

$$\langle l,m|L_zL_x + L_xL_z|l,m\rangle = 2\hbar m\, \langle l,m|L_x|l,m\rangle = 0$$

$$\underset{(*)}{=} \cos^2\theta\, \hbar^2 m^2 + \frac{\hbar^2}{2}\sin^2\theta\left[l(l+1) - m^2\right]$$

2.) In Kugelkoordinaten ist

$$\vec r = (x, y, z) = r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$\Rightarrow \frac{\partial \vec r}{\partial r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \; ; \; \left|\frac{\partial \vec r}{\partial r}\right| = 1 \; \cancel{= \vec e_r}$$

$$\Rightarrow \vec e_r = \frac{1}{|\partial \vec r / \partial r|}\frac{\partial \vec r}{\partial r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$\frac{\partial \vec r}{\partial \theta} = r(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \; ; \; \left|\frac{\partial \vec r}{\partial \theta}\right| = r$$

$$\Rightarrow \vec e_\theta = \frac{1}{|\partial \vec r / \partial \theta|}\frac{\partial \vec r}{\partial \theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$$

$$\frac{\partial \vec r}{\partial \varphi} = r(-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0) \; ; \; \left|\frac{\partial \vec r}{\partial \varphi}\right| = r\sin\theta$$

$$\Rightarrow \vec e_\varphi = \frac{1}{|\partial \vec r / \partial \varphi|}\frac{\partial \vec r}{\partial \varphi} = \frac{1}{\sin\theta}(-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0) = (-\sin\varphi, \cos\varphi, 0)$$

a) $\Rightarrow \vec\nabla = \sum_{i=1}^{3} \frac{1}{b_i}\vec e_i \frac{\partial}{\partial y_i} = \vec e_r \frac{\partial}{\partial r} + \frac{\vec e_\theta}{r}\frac{\partial}{\partial \theta} + \frac{\vec e_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$

b) $\vec L = \vec r \times \vec p = -i\hbar\, \vec r \times \vec\nabla \quad ; \; \vec r = r\vec e_r$

$\vec e_r \times \vec e_\theta = \vec e_\varphi$ und $\vec e_r \times \vec e_\varphi = -\vec e_\theta$ da $\vec e_r, \vec e_\theta, \vec e_\varphi$ = Orthonormal-Basis

$$\Rightarrow \vec L = -i\hbar\left(\vec e_\varphi \frac{\partial}{\partial\theta} - \frac{\vec e_\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$\Rightarrow L_x = \vec e_x \cdot \vec L = -i\hbar\left(\vec e_x \cdot \vec e_\varphi \frac{\partial}{\partial\theta} - \frac{\vec e_x \cdot \vec e_\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right) =$$

$$= i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$$

c) $L_y = \vec e_y \cdot \vec L = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$

d) $L_z = \vec e_z \cdot \vec L = i\hbar\frac{\vec e_z \cdot \vec e_\theta}{\sin\theta}\frac{\partial}{\partial\varphi} = -i\hbar\frac{\partial}{\partial\varphi}$

e) $L_\pm = \cancel{\hbar}(L_x \pm iL_y) = \hbar(i\sin\varphi \pm \cos\varphi)\frac{\partial}{\partial\theta} + i\hbar\cot\theta(\cos\varphi \pm i\sin\varphi)\frac{\partial}{\partial\varphi}$

$$= \hbar e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

f) Nach Vorlesung ist $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2 = L_+ L_- + L_z^2 - \hbar L_z$

Mit e) und $\frac{\partial \cot\theta}{\partial\theta} = -\frac{1}{\sin^2\theta}$ folgt

$$L_+ L_- = \hbar^2 e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$$= \hbar^2\Big[-\frac{\partial^2}{\partial\theta^2} - \frac{i}{\sin^2\theta}\frac{\partial}{\partial\varphi} + \cancel{i\cot\theta\frac{\partial^2}{\partial\theta\partial\varphi}} + \cot\theta\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$$- \cancel{i\cot\theta\frac{\partial^2}{\partial\theta\partial\varphi}} - \cot^2\theta\frac{\partial^2}{\partial\varphi^2}\Big]$$

$$= \hbar^2\left(-\frac{\partial^2}{\partial\theta^2} - i\frac{\partial}{\partial\varphi} - \cot\theta\frac{\partial}{\partial\theta} - \cot^2\theta\frac{\partial^2}{\partial\varphi^2}\right)$$

↓

$\frac{1}{\sin^2\theta} - \cot^2 = 1$

$L_z = -i\hbar\frac{\partial}{\partial\varphi} \qquad L_z^2 = -\hbar^2\frac{\partial^2}{\partial\varphi^2}$

$$\Rightarrow \vec{L}^2 = \hbar^2\Big[-\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \underbrace{(1+\cot^2\theta)}_{=\frac{1}{\sin^2\theta}}\frac{\partial^2}{\partial\varphi^2}\Big]$$

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

3.) a) Eigenfunktionen sind $Y_{lm}(\theta,\varphi)$ mit Eigenwerten

$E = \frac{\hbar^2 m^2}{2J}$. Die Eigenwerte sind entartet weil alle $l \geq m$ zu den gleichen Eigenwerten führen

b) $Y_{20}(\Omega) = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1)$

$$\Rightarrow \cos^2\theta = \frac{1}{3}\left(\sqrt{\frac{16\pi}{5}}\,Y_{20}(\Omega) + 1\right) = \frac{\sqrt{4\pi}}{3}\left[\frac{2}{\sqrt{5}}Y_{20}(\Omega) + Y_{00}(\Omega)\right]$$

$\Rightarrow$ $\psi(\theta,\varphi)$ ist sogar Eigenfunktion von L_z zum Eigenwert 0 (4)

$\Rightarrow$ Erwartungswert $= 0$ Standardabweichung $= 0$

Übrigens gilt dies auch für die Erwartungswerte der beiden anderen Komponenten L_x und L_y, da

$$\langle c_1 Y_{20} + c_2 Y_{00} | L_\pm | c_1 Y_{20} + c_2 Y_{00} \rangle = 0$$