1. a) i) $\left[-\frac{\hbar^2}{2m}\Delta + V(\vec{r})\right]\psi_\epsilon(\vec{r}) = \epsilon\psi_\epsilon(\vec{r})$

ii) $\left[\frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c_0}\vec{A}(\vec{r})\right)^2 + V(\vec{r})\right]\psi'_\epsilon(\vec{r}) = \epsilon\psi'_\epsilon(\vec{r})$

$$\psi'_\epsilon(\vec{r}) = e^{i\alpha(\vec{r})}\psi_\epsilon(\vec{r}) \longrightarrow \alpha(\vec{r})\,?$$

$$\left[-i\hbar\vec{\nabla} - \frac{e}{c_0}\vec{A}(\vec{r})\right]e^{i\alpha(\vec{r})}\psi_\epsilon(\vec{r}) = e^{i\alpha(\vec{r})}\left[-i\hbar\vec{\nabla}\psi_\epsilon(\vec{r})\right]$$

$$\Leftrightarrow\quad -i\hbar\cdot i\vec{\nabla}\alpha(\vec{r})\,e^{i\alpha(\vec{r})}\psi_\epsilon(\vec{r}) - i\hbar e^{i\alpha(\vec{r})}\vec{\nabla}\psi_\epsilon(\vec{r}) - \frac{e}{c_0}\vec{A}(\vec{r})e^{i\alpha(\vec{r})}\psi_\epsilon(\vec{r})$$

$$= e^{i\alpha(\vec{r})}\left[-i\hbar\vec{\nabla}\psi_\epsilon(\vec{r})\right]$$

$$\Leftrightarrow\quad \hbar\vec{\nabla}\alpha(\vec{r}) = \frac{e}{c_0}\vec{A}(\vec{r}) \quad\Leftrightarrow\quad \vec{\nabla}\alpha(\vec{r}) = \frac{e}{c\hbar}A(\vec{r})$$

$$\Rightarrow\quad \psi'_\epsilon(\vec{r}) = e^{\frac{ie}{\hbar c}\int_{\vec{r}_0}^{\vec{r}} d\vec{s}\cdot\vec{A}(\vec{r})}\psi_\epsilon(\vec{r})$$

Check ii)

$$\frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c_0}\vec{A}(\vec{r})\right)\left(-i\hbar\vec{\nabla} - \frac{e}{c_0}\vec{A}(\vec{r})\right)e^{i\alpha(\vec{r})}\psi_\epsilon(\vec{r}) + V(\vec{r})e^{i\alpha(\vec{r})}\psi_\epsilon(\vec{r}) =$$

$$= \epsilon\, e^{i\alpha(\vec{r})}\psi_\epsilon(\vec{r})$$

$$\Rightarrow \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c_0}\vec{A}(\vec{r})\right)e^{i\alpha(\vec{r})}\left(-i\hbar\vec{\nabla}\psi_\epsilon(\vec{r})\right) + V(\vec{r})e^{i\alpha(\vec{r})}\psi_\epsilon(\vec{r}) = \epsilon e^{i\alpha(\vec{r})}\psi_\epsilon(\vec{r})$$

$$\Rightarrow \frac{1}{2m}\Big[-i\hbar\, i\vec{\nabla}\alpha(\vec{r})\, e^{i\alpha(\vec{r})}\left(-i\hbar\vec{\nabla}\psi_\varepsilon(\vec{r})\right) - i\hbar\, e^{i\alpha(\vec{r})}\left(-i\hbar\vec{\nabla}^2\psi_\varepsilon(\vec{r})\right)$$

$$-\frac{e}{c}\vec{A}(\vec{r})\, e^{i\alpha(\vec{r})}\left(-i\hbar\vec{\nabla}\psi_\varepsilon(\vec{r})\right)\Big] + V(\vec{r})\, e^{i\alpha(\vec{r})}\psi_\varepsilon(\vec{r}) = \varepsilon\, e^{i\alpha(\vec{r})}\psi_\varepsilon(\vec{r})$$

$$\Rightarrow \left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{r})\right]\psi_\varepsilon(\vec{r}) = \varepsilon\,\psi_\varepsilon(\vec{r})$$

1.b)

i) $$i\hbar \frac{\partial \psi(\vec r,t)}{\partial t} = \left[\frac{1}{2m} \left(\frac{\hbar}{i}\vec\nabla - \frac{e}{c}\vec A(\vec r,t)\right)^2 + e\phi(\vec r,t)\right]\psi(\vec r,t)$$

$$\vec A(\vec r,t) \longrightarrow \vec A'(\vec r,t) = A(\vec r,t) + \vec\nabla \Lambda(\vec r,t)$$

$$\phi(\vec r,t) \longrightarrow \phi'(\vec r,t) = \phi(\vec r,t) - \frac{1}{c}\frac{\partial \Lambda(\vec r,t)}{\partial t}$$

$$\psi(\vec r,t) \longrightarrow \psi'(\vec r,t) = e^{i f(\Lambda)}\psi(\vec r,t) \quad ?$$

ii) $$i\hbar \frac{\partial \psi'(\vec r,t)}{\partial t} = \left[\frac{1}{2m}\left(\frac{\hbar}{i}\hat\nabla - \frac{e}{c}\hat A(\vec r,t) - \frac{e}{c}\vec\nabla\Lambda(\vec r,t)\right)^2 + \right.$$

$$\left. + e\phi(\vec r,t) - \frac{e}{c}\frac{\partial\Lambda(\vec r,t)}{\partial t}\right]\psi'(\vec r,t)$$

Assumption $\psi'(\vec r,t) = e^{i\alpha\Lambda(\vec r,t)}\psi(\vec r,t)$

$$\Downarrow$$

$$i\hbar\, i\alpha \frac{\partial\Lambda(\vec r,t)}{\partial t} e^{i\alpha\Lambda(\vec r,t)}\psi(\vec r,t) + i\hbar e^{i\alpha\Lambda(\vec r,t)}\frac{\partial\psi(\vec r,t)}{\partial t} =$$

$$= \left[\frac{1}{2m}(\quad)^2 + e\phi(\vec r,t) - \frac{e}{c}\frac{\partial\Lambda(\vec r,t)}{\partial t}\right] e^{i\alpha\Lambda(\vec r,t)}\psi(\vec r,t)$$

$$\rightarrow i\hbar e^{i\alpha\Lambda}\frac{\partial\psi}{\partial t} = \left[\frac{1}{2m}(\,)^2 + e\phi - \left(\frac{e}{c} - \hbar\alpha\right)\frac{\partial\Lambda}{\partial t}\right] e^{i\alpha\Lambda}\psi(\vec r,t)$$

Let's set $\alpha = \frac{e}{c\hbar} \Rightarrow$

$$i\hbar e^{i\alpha\Lambda}\frac{\partial\psi}{\partial t} = \left[\frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A} - \frac{e}{c}\vec{\nabla}\Lambda\right)^2 + e\phi - 0\frac{\partial\Lambda}{\partial t}\right]e^{i\alpha\Lambda}\psi$$

(*)
$$= \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A} - \frac{e}{c}\vec{\nabla}\Lambda\right)\left(\cancel{-i\hbar\cdot i\alpha\vec{\nabla}\Lambda e^{i\alpha\Lambda}\psi} - i\hbar e^{i\alpha\Lambda}\vec{\nabla}\psi -\right.$$

$$\left.-\frac{e}{c}\vec{A}e^{i\alpha\Lambda}\psi - \cancel{\frac{e}{c}\vec{\nabla}\Lambda e^{i\alpha\Lambda}\psi}\right) + e\phi e^{i\alpha\Lambda}\psi =$$

$$= \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A} - \frac{e}{c}\vec{\nabla}\Lambda\right)e^{i\alpha\Lambda}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)\psi + e\phi e^{i\alpha\Lambda}\psi =$$

$$= \frac{1}{2m}\left[\cancel{-i\hbar\cdot i\alpha\vec{\nabla}\Lambda e^{i\alpha\Lambda}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)\psi} - i\hbar e^{i\alpha\Lambda}\vec{\nabla}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)\psi\right.$$

$$\left.-\frac{e}{c}\vec{A}e^{i\alpha\Lambda}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)\psi - \cancel{\frac{e}{c}\vec{\nabla}\Lambda e^{i\alpha\Lambda}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)\psi}\right] + e\phi e^{i\alpha\Lambda}\psi =$$

$$= \frac{1}{2m}e^{i\alpha\Lambda}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)\psi + e\phi e^{i\alpha\Lambda}\psi$$

$$\Downarrow$$

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 + e\phi\right]\psi$$

(*) $\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A} - \frac{e}{c}\vec{\nabla}\Lambda\right)e^{i\alpha\Lambda}\psi = e^{i\alpha\Lambda}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)\psi$

$$(i) = (ii) \Rightarrow \alpha = \frac{e}{c_0 \hbar} \Rightarrow \Psi'(\vec{r},t) = e^{\frac{ie}{c\hbar}\Lambda(\vec{r},t)} \Psi(\vec{r},t)$$

2. a) $$S_{\hat{n}} = \frac{\hbar}{2}(n_x \sigma_x + n_y \sigma_y + n_z \sigma_z) =$$

$\vec{n} = (n_x, n_y, n_z)$

$$= \frac{\hbar}{2}\left[n_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + n_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + n_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] =$$

$$= \frac{\hbar}{2} \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}$$

b) $$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \langle S_{\hat{n}} \rangle_{\chi_+} = \chi_+^\dagger S_{\hat{n}} \chi_+ = (1 \ 0) \frac{\hbar}{2} \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar}{2} n_z$$

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \langle S_{\hat{n}} \rangle_{\chi_-} = \chi_-^\dagger S_{\hat{n}} \chi_- = (0 \ 1) \frac{\hbar}{2} \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} =$$

$$= -\frac{\hbar}{2} n_z$$

c) eigenvectors

$$\chi_{n+} = \frac{1}{\sqrt{2(1+n_z)}} \begin{pmatrix} 1+n_z \\ n_x + i n_y \end{pmatrix} , \quad \chi_{n-} = \frac{1}{\sqrt{2(1-n_z)}} \begin{pmatrix} 1-n_z \\ -n_x - i n_y \end{pmatrix}$$

$$|\langle \chi_+ | \chi_{n+}\rangle|^2 = \frac{(1+n_z)^2}{2(1+n_z)} = \frac{1+n_z}{2}$$

$$|\langle \chi_- | \chi_{n+}\rangle|^2 = \frac{n_x^2+n_y^2}{2(1+n_z)} = \frac{(1-n_z)(1+n_z)}{2(1+n_z)} = \frac{1-n_z}{2}$$

$$|\langle \chi_+ | \chi_{n-}\rangle|^2 = \frac{1-n_z}{2}$$

$$|\langle \chi_- | \chi_{n-}\rangle|^2 = \frac{1+n_z}{2}$$

3.
a)

$$U_{\vec{e}\theta} = e^{\frac{i}{2}\theta\,\vec{e}\cdot\vec{\sigma}} = \underset{\substack{\uparrow \\ x \equiv \frac{\theta}{2}\vec{e}\cdot\vec{\sigma}}}{e^{ix}}$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{i^{2n} x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i^{2n+1} x^{2n+1}}{(2n+1)!} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \frac{\theta^{2n}}{2^{2n}} (\vec{e}\cdot\vec{\sigma})^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i(-1)^n \frac{\theta^{2n+1}}{2^{2n+1}} (\vec{e}\cdot\vec{\sigma})^{2n+1}}{(2n+1)!} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \frac{\theta^{2n}}{2^{2n}} [(\vec{e}\cdot\vec{\sigma})(\vec{e}\cdot\vec{\sigma})]^n}{(2n)!} + i(\vec{e}\cdot\vec{\sigma}) \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\theta}{2}\right)^{2n+1} ((\vec{e}\cdot\vec{\sigma})(\vec{e}\cdot\vec{\sigma}))^n}{(2n+1)!}$$

$$(\vec{e}\cdot\vec{\sigma})(\vec{e}\cdot\vec{\sigma}) = e_i\sigma_i e_j\sigma_j = e_ie_j(\delta_{ij} + i\varepsilon_{ijk}\sigma_k) =$$

$$= e_ie_i + i\varepsilon_{ijk}e_ie_j\sigma_k = \vec{e}^{\,2} = 1$$

$$\Rightarrow e^{ix} = \underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\theta}{2}\right)^{2n}}{(2n)!}}_{\cos\frac{\theta}{2}} + i(\vec{e}\cdot\vec{\sigma})\underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\theta}{2}\right)^{2n+1}}{(2n+1)!}}_{\sin\frac{\theta}{2}}$$

$$\Rightarrow U_{\vec{e}\theta} = \mathbb{1}\cos\frac{\theta}{2} + i\vec{e}\cdot\vec{\sigma}\sin\frac{\theta}{2}$$

b) $U_{e_x\theta}\vec{\sigma}U_{e_x\theta}^{+}$

$$U_{e_x\theta}\sigma_i U_{e_x\theta}^{+} = \left(\cos\frac{\theta}{2} + i\sigma_x\sin\frac{\theta}{2}\right)\sigma_i\left(\cos\frac{\theta}{2} - i\sigma_x\sin\frac{\theta}{2}\right) =$$

$$= \cos^2\frac{\theta}{2}\sigma_i + \cos\frac{\theta}{2}\sigma_i\left(-i\sigma_x\sin\frac{\theta}{2}\right) + i\sigma_x\sin\frac{\theta}{2}\sigma_i\cos\frac{\theta}{2} +$$

$$+ i\sigma_x\sin\frac{\theta}{2}\sigma_i(-i)\sigma_x\sin\frac{\theta}{2} = \cos^2\frac{\theta}{2}\sigma_i + i\sin\frac{\theta}{2}\cos\frac{\theta}{2}[\sigma_x,\sigma_i]$$

$$+ \sin^2\frac{\theta}{2}\sigma_x\sigma_i\sigma_x = \cos^2\frac{\theta}{2}\sigma_i + i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot 2i\varepsilon_{xij}\sigma_j +$$

$$+ \sin^2\frac{\theta}{2}(\delta_{xi} + i\varepsilon_{xij}\sigma_j)\sigma_x = \cos^2\frac{\theta}{2}\sigma_i - \sin\theta\,\varepsilon_{xij}\sigma_j +$$

$$+ \sin^2\frac{\theta}{2}\delta_{xi}\sigma_x + i\sin^2\frac{\theta}{2}\varepsilon_{xij}(\delta_{jx} + i\varepsilon_{jxk}\sigma_k) =$$

$$= \cos^2\frac{\theta}{2}\sigma_i - \sin\theta\, \varepsilon_{xij}\sigma_j + \sin^2\frac{\theta}{2}\delta_{xi}\sigma_x - \sin^2\frac{\theta}{2}\varepsilon_{xij}\varepsilon_{jxk}\sigma_k$$

$$\Rightarrow U_{e_x,\theta}\vec{\sigma}U^\dagger_{e_x,\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$$