General kinetic description of relativistic mixed neutrinos

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Received 16 December 1992
(Revised 20 April 1993)
Accepted for publication 22 June 1993

We derive a general Boltzmann-type collision integral for mixed neutrinos interacting with each other and with a medium. Our treatment is fully relativistic in that antineutrino degrees of freedom are included. This collision integral allows one to account for the simultaneous effects of neutrino oscillations in a medium and for the effects of collisions. Our result generalizes previous attempts of unify the first- and second-order interaction effects in a single self-consistent equation. Most importantly, our equation includes effects non-linear in the neutrino density matrices (or occupation numbers) such as Pauli blocking of neutrino final states or neutrino refraction in a medium of neutrinos. We apply the definition of the entropy of a non-equilibrium Fermi gas to the case of mixed neutrinos, and we prove that our collision integrals obey the relevant thermodynamic inequalities.

1. Introduction

Although the neutrino interactions with matter are extremely weak, they play an important role in a variety of astrophysical sites. To first order of a perturbation expansion a modification of the neutrino dispersion relation occurs, allowing for the widely discussed effect of resonant neutrino oscillations which may solve the solar neutrino problem [1]. To second order neutrinos are scattered, absorbed, or produced by the medium. Therefore, they play a dominant thermal and dynamical role in certain phases of stellar evolution, notably in supernovae, and in the early universe. The second-order effects are usually treated in the form of a kinetic equation, a differential equation for the occupation numbers of the neutrino field modes which involves a Boltzmann collision integral. The first-order (or refractive) effects are usually treated on the amplitude level in the form of a Schrödinger equation for the single-particle wave functions for the mixing flavors.

Neither of these approaches can account quantitatively for a situation where neutrino oscillations and collisions are both important. This would be the case in the early universe where, for example, a sterile admixture to the standard neutrino-
nos could be populated by the simultaneous action of oscillations and collisions [2–4]. Another example is a supernova core immediately after collapse where the electron neutrinos have a large chemical potential while the other flavors do not: the action of oscillations and collisions could lead to flavor equilibrium, i.e. to identical chemical potentials for some or all neutrino flavors [5–7]. Also, the emission of sterile neutrinos could play an important role in this environment [8].

Early attempts to “marry” the first- and second-order effects in a single equation were pioneered by Dolgov [2] and by Stodolsky [9]. Dolgov started from a kinetic equation, replacing the neutrino occupation numbers for the $n$ mixing flavors by $n \times n$ density matrices in flavor space for each momentum mode. The diagonal entries of these matrices represent the usual occupation numbers while the off-diagonal elements contain more subtle phase information. For pure states the use of density matrices is equivalent to the wave-function approach of neutrino mixing. However, when the partial or complete loss of coherence between the mixing components of a neutrino state becomes important, which is unavoidable in the presence of collisions, an amplitude treatment is no longer adequate. Stodolsky [9] was mostly concerned with the gradual loss of coherence between the components of a two-level system by the interaction with an external heat bath; he derived a “Bloch equation” for the corresponding density matrix. This equation accounts for the effect of oscillations and also that of collisions in the form of a damping parameter which has to be calculated from the underlying scattering processes. However, in this treatment the kinetic degrees of freedom did not explicitly appear: only the evolution of the total “flavor spin” of the ensemble was considered. Manohar [10] and Thomson [11] also discussed simple equations to study the oscillation and damping of momentum-averaged density matrices.

Recently, Raffelt, Sigl, and Stodolsky [12] have made a first attempt at a systematic unification of these different approaches. In an essentially non-relativistic treatment where the antiparticle degrees of freedom do not appear they derived a rigorous Boltzmann-type collision integral for the neutrino density matrices. In the limiting case of negligible energy transfers this equation reduced to Stodolsky’s [9] previous result. In the limit of dilute neutrinos where Pauli blocking effects can be neglected it reduced to Dolgov’s [2] result. One of the main advances of the new equation was the rigorous inclusion of effects non-linear in the neutrino occupation numbers. Put another way, the matrix of neutrino coupling constants and the neutrino density matrices where rigorously combined to form “non-abelian Pauli blocking factors”. Although this equation does not include antiparticle degrees of freedom it is sufficient to treat the problem of neutrino flavor conversion in the interior of a supernova core, a problem to which we applied it in much detail in a separate paper [13].

A Boltzmann-type kinetic equation similar to that of ref. [12] had been previously derived by Rudzsky [14]. He included the neutrino spin degree of freedom and thus took spin-processes into account, but restricted his discussion to flavor-
conserving neutral currents. In the relevant limits the results of refs. [12,13] can be shown to agree with those of ref. [14].

We presently aim at a generalization of the previous works to include the antiparticle degrees of freedom as well as degeneracy effects. In ref. [12] only scattering processes of the form $\nu_p X \rightarrow X' \nu_p$ were included, where $X$ and $X'$ are configurations of the medium. In ref. [13] we also included the production and absorption by charged-current processes of the type $X \rightarrow X' \nu_p$ or $\nu_p X \rightarrow X'$. The medium states before and after the collision differ in their lepton numbers by one unit. In addition we now need to include the same processes for antineutrinos as well as the pair processes $X \rightarrow X' \bar{\nu}_p \bar{\nu}_p$ and $\nu_p \bar{\nu}_p X \rightarrow X'$. To this end we use the standard low-energy four-fermion interaction hamiltonian for neutral- and charged-current processes and provide a rigorous definition of the relevant structure functions of the medium. We then derive the kinetic terms corresponding to these processes. In particular we show how the terms for antineutrinos and those for pair processes can be obtained from the original scattering terms by simple "crossing relations".

Another generalization that we will discuss are neutrino interactions among themselves where we do away with the external medium. Neutrino refraction and scatterings are then only with other neutrinos. It was recently stressed by Pantaleone [7] and also discussed by Thomson and McKellar [15] that the previous discussions of this subject had been incomplete in that the refractive index should really be viewed as an $n \times n$ matrix: there are refractive indices "off-diagonal" in flavor space. We fully agree with this result, except that we find it more natural to express it in terms of density matrices where the "refractive index matrix" and the amplitude of the propagating "test neutrino" are both represented by one and the same object. We also derive the second-order terms for this case.

In the early universe the lepton-number asymmetry is of the order $10^{-9}$. Nötzold and Raffelt [16] pointed out that in this case one had to take a higher-order term in the refractive index into account which represents the low-energy tail of the intermediate gauge boson resonance. This extra term is proportional to the number density of leptons plus antileptons rather than to the difference. For the second-order or scattering terms, of course, the standard four-fermion hamiltonian remains sufficient. We generalize the higher-order refractive term to our matrix formalism.

It is not obvious that our collision integrals respect the thermodynamic requirement that in a closed system the entropy must never decrease. In order to prove this property we apply the standard definition of the entropy of a non-equilibrium Fermi gas to our case of mixed neutrinos. We then prove that our collision integral involving the neutrino self-interactions does indeed fulfill the required inequality. For the effective neutral-current interactions which lead to scattering and pair processes (but which conserve lepton number in the neutrino sector) we prove that the free energy of the neutrino ensemble never increases if the external medium
can be treated as an ideal heat bath whose temperature remains unaffected by the interactions with the neutrinos. For the charged-current processes which change the lepton number as well as the total energy of the neutrino ensemble we prove that the thermodynamic grand potential never increases, assuming that the temperature and chemical potentials which characterize the medium represent kinetic and chemical equilibrium and do not change by the interaction with the neutrinos.

In sect. 2 we begin with a formal derivation of the first- and second-order terms of a kinetic equation for the neutrino and antineutrino density matrices on the basis of a general current-current interaction Hamiltonian. We then proceed to make the formal expressions explicit for three assumed interaction types: In sect. 3 charged-current interactions which change both lepton number and energy of the neutrino ensemble. In sect. 4 effective neutral-current interactions which conserve the lepton number but change the total energy of the neutrino ensemble. In sect. 5 neutrino self-interactions which conserve both the lepton number and energy of the ensemble. For each case we prove the validity of the relevant thermodynamic inequalities. For the neutral-current and self-interactions we also extend the refractive terms to include a higher-order gauge boson propagator contribution. A brief summary is provided in sect. 6.

2. Equation of motion

2.1. MATRIX OF DENSITIES

For unmixed neutrinos one usually studies a kinetic equation for the evolution of particle and antiparticle occupation numbers \( f_p \) and \( \tilde{f}_p \) of a momentum mode \( p \). In order to justify and generalize this approach for mixed neutrinos in the framework of field theory we begin with a momentum expansion of the left-handed massless neutrino field (notation \( dp \equiv d^3p/(2\pi)^3 \))

\[
\psi(x) = \int dp (a_p(t)u_p + b_{-p}^\dagger(t)v_{-p}) e^{ip\cdot x},
\]

(2.1)

where \( a_p \) is an annihilation operator for negative-helicity neutrinos of momentum \( p \) while \( b_{-p}^\dagger \) is a creation operator for positive-helicity antineutrinos. (Whenever possible we will write the momentum variable as a subscript.) The Dirac spinors \( u_p \) and \( v_p \) refer to massless negative-helicity particles and positive-helicity antiparticles, respectively; we take the spinor normalization to be unity. For \( n \) flavors \( a_p \) and \( b_{-p}^\dagger \) are column vectors of \( n \) particle annihilators \( a_i(p) \) and antiparticle creators \( b_i^\dagger(p) \), respectively. They satisfy the anticommutation relations \( \{a_i(p), a_j^\dagger(p')\} = \{b_i(p), b_j^\dagger(p')\} = \delta_{ij}(2\pi)^3\delta^{(3)}(p - p') \).
In the massless limit and when only left-handed interactions are present we may ignore the right-handed field entirely. However, in order to include flavor mixing we need to introduce an \( n \times n \) mass matrix \( M \) which is nondiagonal in the interaction basis. Even in this case we will continue to ignore the “wrong” helicity states because we will focus on the ultrarelativistic regime where spin-flip reactions are suppressed by an approximate factor \( (m_e/2E_e)^2 \ll 1 \). In this limit we may also ignore possible lepton-number violating effects if \( m_e \) were a Majorana mass term.

In general the neutrino ensemble is characterized by a hierarchy of all \( m \)-particle Green’s functions. They describe \( m \)-particle correlations and are given as expectation values of products of \( 2m \) operators \( a, a^\dagger, b \) and \( b^\dagger \), taken at different points in momentum space. In the absence of interactions they can be reduced to products of one-particle Green’s functions by using Wick’s theorem. If interactions are present the evolution of the exact \( m \)-particle Green’s functions can at least in principle be expanded perturbatively in terms of free one-particle Green’s functions at some fixed initial time where they can be taken to coincide with the exact ones. We will indeed perform such a perturbation expansion in the next section. Thus it is sufficient to consider only expectation values of all possible field bilinears.

The bilinears \( aa, a^\dagger a^\dagger, b^\dagger a \) and \( a^\dagger b \) violate lepton number by two units. If initially the ensemble was in an eigenstate of lepton number and if it is conserved by all interactions the expectation values of these operators are identically zero at all times. Of the remaining bilinears, \( a^\dagger b^\dagger \) and \( ab \) can be ignored because their expectation values oscillate “fast” around zero. This is seen if one remembers that for free massless fields \( a_p(t) = a_p(0)e^{-ip|t|} \) and \( b_p(t) = b_p(0)e^{-ip|t|} \). Therefore, even in the presence of small masses and weak interactions we have approximately, for example, \( \langle a_p(t)a_p(t) \rangle \approx e^{-2|p|t} \).

Hence, the only bilinears needed to describe the neutrino ensemble are the slowly-varying “density operators” \( a^\dagger a_p \) and \( b^\dagger b_p \). If we make the additional assumption of spatial homogeneity the expectation value of every physical observable constructed from the field \( \psi \) is independent of location. This implies that the expectation values of these densities contribute only for equal momenta \( p = p' \).

Therefore, a homogeneous neutrino ensemble is completely characterized by the dimensionless \( n \times n \) “matrices of densities” \( \rho_p(t) \) and \( \bar{\rho}_p(t) \), which are defined by

\[
\langle a_j^\dagger(p)a_i(p') \rangle = (2\pi)^3 \delta^{(3)}(p - p')(\rho_p)_{ij},
\]

\[
\langle b_j^\dagger(p)b_i(p') \rangle = (2\pi)^3 \delta^{(3)}(p - p')(\bar{\rho}_p)_{ij}.
\]

(The reversed order of the flavor indices in the definition of \( \bar{\rho}_p \) guarantees that

* Here and in the following all overbarred quantities refer to antineutrinos.
both matrices transform in the same way under a unitary transformation $\psi' = U\psi$ in flavor space.) The diagonal elements of $\rho_p$ and $\bar{\rho}_p$ are the usual particle and antiparticle occupation numbers, whereas the off-diagonal elements contain correlations between the mixing flavors. Thus, for a homogeneous ensemble of mixed particles these matrices are the relevant generalization of the usual occupation numbers as previously discussed in refs. [2,12].

On the operator level it is sufficient, therefore, to study the equations of motion of the $n \times n$ matrices

$$\mathcal{D}_{ij}(p, t) = a_i^*(p, t)a_j(p, t) \quad \text{and} \quad \mathcal{\bar{D}}_{ij}(p, t) = b_i^*(p, t)b_j(p, t). \quad (2.3)$$

According to eq. (2.2) the expectation values of $\mathcal{D}_p$ and $\mathcal{\bar{D}}_p$ diverge because they involve a factor $(2\pi)^3\delta^3(0)$ which is related to the infinite quantization volume necessary for our continuous momentum variables. In practice, this factor always drops out of final results so that we effectively set $(2\pi)^3\delta^3(0)$ equal to unity. This amounts to using $\rho_p = \langle \mathcal{D}_p \rangle$ and $\bar{\rho}_p = \langle \mathcal{\bar{D}}_p \rangle$ for all practical purposes.

In the absence of interactions $\psi$ satisfies the free Dirac equation, implying $a_p(t) = a_p(0)\exp(-i\Omega_p^0 t)$ and $b_p(t) = b_p(0)\exp(-i\Omega_p^0 t)$, where $\Omega_p^0 = (p^2 + M^2)^{1/2}$ is an $n \times n$ matrix of “vacuum oscillation frequencies”. In terms of $\Omega_p^0$ the free neutrino hamiltonian is

$$H_0 = \int dp \sum_{i,j=1}^n \left( a_i^*(p)\Omega_{ij}^0(p)a_j(p) + b_i^*(p)\Omega_{ij}^0(p)b_j(p) \right). \quad (2.4)$$

Under this hamiltonian the matrices of densities evolve according to

$$\dot{\rho}_p = -i\left[\Omega_p^0, \rho_p\right] \quad \text{and} \quad \dot{\bar{\rho}}_p = i\left[\Omega_p^0, \bar{\rho}_p\right]. \quad (2.5)$$

This equation represents the usual vacuum oscillation effects. Our goal is to derive a closed set of differential equations for the $\rho_p$ and $\bar{\rho}_p$ in the presence of interactions.

2.2. PERTURBATIVE EXPANSION

We now introduce an interaction hamiltonian $H_{int}(B, \psi)$ which is a functional of the neutrino field $\psi$ and a set $B$ of background fields. For example, assuming a four-fermion interaction model, neutral-current interactions with the medium correspond to $B$'s which are bilinears of nucleon or electron fields whereas for charged-current interactions they are trilinears of nucleon and electron fields. The equation of motion for $\mathcal{D}_p$ and $\mathcal{\bar{D}}_p$ is given by the Heisenberg equation. Taking on both sides the expectation value with regard to the initial state yields

$$\dot{\rho}_p(t) = -i\left[\Omega_p^0, \rho_p(t)\right] + i\langle H_{int}(B(t), \psi(t)), \mathcal{D}_p(t) \rangle \quad (2.6)$$
and an analogous equation for $\bar{\rho}_p(t)$. These equations are exact, but they are not a closed set of differential equations for the $\rho_p$ and $\bar{\rho}_p$.

In a first-order perturbative approximation we set the interacting fields $B(t)$ and $\psi(t)$ on the rhs of eq. (2.6) equal to the free fields $B_0(t)$ and $\psi_0(t)$ *`. Under the assumption that the original state contained no correlations between the neutrinos and the background the expectation value factorizes into a medium part and a neutrino part. With Wick’s theorem and ignoring $b^\dagger b^\dagger$ etc. as discussed in the previous section it can be reduced to an expression which contains only $\rho_p$'s and $\bar{\rho}_p$'s. The result gives the forward-scattering or refractive effect of the interaction.

In order to include non-forward collisions we need to go to second order in the perturbation expansion. We assume that the interactions described by $H_{\text{int}}$ can be taken as individual, isolated collisions where the neutrinos go from free states to free states as in ordinary scattering theory. Physically this amounts to the restriction that the neutrino collision rate is small enough that multiple-scattering effects can be ignored. Next, we express a general operator $\xi(t) = \xi(B(t), \psi(t))$ which is a functional of $B$ and $\psi$ at a given time $t$ to first order in $H_{\text{int}}$ by the corresponding free operator $\xi_0$ with the initial condition $\xi(0) = \xi_0(0)$,

$$\xi(t) = \xi_0(t) + i\int_0^t dt' \left[ H_{\text{int}}^0(t-t'), \xi_0(t) \right], \quad (2.7)$$

where $H_{\text{int}}^0(t) = H_{\text{int}}(B_0(t), \psi_0(t))$ is the interaction hamiltonian taken as a functional of the freely evolving fields. Applying this general iteration formula to the operator $\xi = [H_{\text{int}}(B, \psi), \mathcal{D}_p]$ which appears on the rhs of eq. (2.6), we arrive at

$$\dot{\rho}_p(t) = -i \left[ \Omega_p^0, \rho_p(t) \right] + i \left\{ \left[ H_{\text{int}}^0(t), \mathcal{D}_p^0 \right] \right\}$$

$$- \int_0^t dt' \left( \left[ H_{\text{int}}^0(t-t'), \left[ H_{\text{int}}^0(t), \mathcal{D}_p^0 \right] \right] \right) \quad (2.8)$$

and an analogous equation for $\bar{\rho}_p(t)$. The second term on the rhs is the first-order refractive part already discussed above; it is associated with forward-scattering. The second-order term in general contains both forward as well as non-forward scattering effects. Because all operators on the rhs are free the expectation value in the second-order term again factorizes between the neutrinos and the medium.

* These free operators are the solutions of the equations of motion in the absence of $H_{\text{int}}$; internal interactions of the medium such as nucleon–nucleon scattering are not excluded, however. Moreover, we take $\phi(0) = \psi_0(0)$ etc. as an initial condition for the interacting fields. We also ignore the mass term in the definition of $\psi_0$; its effect is included only in the first term on the rhs of eq. (2.6), the “vacuum oscillation term”. Therefore, the free creation and annihilation operators vary as $a_p^\dagger(0) = a_p^\dagger(0) e^{-i\theta_p \tau}$ etc. for all flavors. This implies that the operators $\mathcal{D}_p^0$ and $\mathcal{D}_p^0$ constructed from the free $a$'s and $b$'s are time dependent.
Therefore, we are led to equations for $\dot{\rho}_p(t)$ and $\tilde{\rho}_p(t)$ which on the rhs involve only $\rho_p(t)$ and $\tilde{\rho}_p(t)$ as well as $\langle \mathcal{D}_p^0 \rangle$ and $\langle \mathcal{G}_p^0 \rangle$ besides expectation values of $B$ operators.

We assume that the duration of one collision (the inverse of a typical energy transfer) is small relative to the time scale over which the density matrices vary substantially, i.e. small relative to the oscillation time and the inverse collision frequency. Furthermore, we will always assume that the medium is not changed substantially by the interactions with the neutrino ensemble, allowing us to neglect evolution equations for the medium variables which can thus be taken to be externally prescribed, typically by conditions of thermal equilibrium. If the medium is not stationary it is assumed that the time scale of variation is large compared to the duration of typical neutrino–medium collisions.

We may then choose the time step of iteration $t$ in eq. (2.8) both small relative to the evolution time scale and large relative to the duration of one collision. Under these circumstances the time integral can be extended to infinity while setting $\rho_p(t)$ equal to $\rho_p(0) = \langle \mathcal{D}_p^0 \rangle$. Hence, we end up with

$$\dot{\rho}_p(0) = -i\left[\Omega_p^0, \rho_p(0)\right] + i\left\langle \left[H_{\text{int}}^0(0), \mathcal{D}_p^0\right]\right\rangle - \frac{1}{2} \int_{-\infty}^{+\infty} dt \left\langle \left[H_{\text{int}}^0(t), \left[H_{\text{int}}^0(0), \mathcal{D}_p^0\right]\right] \right\rangle$$

and analogously for $\tilde{\rho}_p$. In this equation we have replaced $\int_0^\sigma dt \langle \ldots \rangle$ by $\frac{1}{2} \int_{-\infty}^{+\infty} dt \langle \ldots \rangle$. The difference between these expressions corresponds to a principle-part integral which leads to a second-order correction to the refractive term which we may ignore. Moreover, it is understood that terms quadratic in the refractive index are to be ignored in evaluating the expectation value. In the form of eq. (2.9) the time integral leads to energy conservation in individual collisions.

An explicit evaluation of the rhs of eq. (2.9) for a given interaction model thus yields the desired set of differential equations for the $\rho_p$'s and $\tilde{\rho}_p$'s at time $t = 0$. It will be valid at all times if the correlations built up by neutrino collisions are “forgotten” before the next collision occurs. This assumption corresponds to “molecular chaos” in the derivation of the usual Boltzmann equation.

In general $H_{\text{int}}$ will be a sum of different interaction terms. Then the refractive part is just a sum of the different contributions because it is linear in $H_{\text{int}}$, whereas the collision term, which is quadratic in $H_{\text{int}}$, will in general include interference effects between these contributions. For example, the contributions of neutral-current scattering on protons and neutrons will not simply add incoherently if there are correlations between these targets. However, there will be no interference terms between the different types of interactions discussed in the introduction except for second-order forward-scattering effects which we ignore. Therefore, we may use eq. (2.9) to derive separate contributions to the kinetic equations from...
three neutrino interaction types: charged-current (CC) and effective neutral-current (NC) interactions with a medium and neutral-current neutrino self-interactions (S),
\[
\hat{\rho}_p = -i\left[\Omega_p^0, \rho_p\right] + \left(\hat{\rho}_{p,\text{CC}}\right) + \left(\hat{\rho}_{p,\text{NC}}\right) + \left(\hat{\rho}_{p,\text{S}}\right)
\]
and an analogous equation for \(\hat{\rho}_{\bar{p}}\). In sects. 3–5 we calculate these individual contributions and study their kinetic properties.

2.3. WEAK INHOMOGENEITIES

Although the main emphasis of our paper is on the kinetic terms (the rhs of eq. (2.10)) we briefly want to remark on the treatment of weak inhomogeneities within our formalism. Put another way, we will show how the Liouville terms which appear on the lhs of the usual Boltzmann equation have to be modified for mixed neutrinos. These terms describe the transport of energy or lepton number due to temperature or density gradients.

If the length scales of the inhomogeneities are large compared to the wavenumbers of the field quanta under consideration, the ensemble may still be described by “occupation numbers” which are functions of location \(x\) (Wigner distributions [17,18]). In our case they correspond to the neutrino density matrices

\[
\rho_{ij}(x, p) = \int \frac{d^3\Delta}{(2\pi)^3} e^{i\Delta \cdot x} \langle a_j^\dagger(x - \frac{1}{2}\Delta)a_i(x + \frac{1}{2}\Delta) \rangle
\]

where \(a(x)\) is the positive energy part of \(\psi(x)\) in eq. (2.1). An analogous definition holds for the antineutrino density matrices. A comparison with eq. (2.2) shows that in the homogeneous limit our Wigner distributions agree with the previously defined density matrices \(\rho_p\) and \(\hat{\rho}_p\).

In the limit of weak inhomogeneities the Wigner distributions obey the kinetic equation (2.10) with \(\rho_p \rightarrow \rho(x, p)\) and the following additional Liouville term on the lhs:

\[
\frac{1}{2}\left\{\nabla_x \rho(x, p), \nabla_p \Omega(x, p)\right\} - \frac{1}{2}\left\{\nabla_p \rho(x, p), \nabla_x \Omega(x, p)\right\}
\]

and similarly for \(\hat{\rho}(x, p)\). Here, \(\Omega(x, p)\) is the matrix of total energies which includes external potentials like gravity as well as the refractive energy shifts discussed in the following sections. Therefore, the lhs of eq. (2.10) now includes a drift term proportional to the neutrino velocity \(\nabla_x \Omega(x, p)\) and a term proportional to the force \(\nabla_x \Omega(x, p)\) acting onto the neutrino. Note that formally the matrices
\( \Omega(x, p) \) and \( \Omega'(x, p) \) are the variational derivatives of the averaged total energy \( \langle H \rangle \) of the interacting system with respect to \( \rho(x, p) \) and \( \bar{\rho}(x, p) \), respectively,

\[
\delta \langle H \rangle = \int \frac{d^3x \ d^3p}{(2\pi)^3} \text{Tr}[\Omega(x, p)\delta\rho(x, p) + \Omega'(x, p)\delta\bar{\rho}(x, p)].
\]

(2.13)

This corresponds to the definition of the single-particle energies in the Landau theory of Fermi liquids [19,20].

The Liouville terms are necessary to describe the effects of macroscopic density or temperature gradients that drive energy or particle fluxes. In addition there might be a statistical distribution of "mesoscopic" density inhomogeneities superimposed on those macroscopic variations. It is possible to include their effect in the collision terms on the rhs of the kinetic equation, an approach used by Sawyer [21]. In the absence of oscillations the cut between macroscopic and mesoscopic density fluctuations will be performed at the Compton wavelength of the neutrinos. In our case, however, the length scale of oscillations will be more appropriate [21].

3. Charged-current interactions

3.1. The Hamiltonian

We begin with the relatively simple case of the production and absorption of neutrinos and antineutrinos by CC interactions with a medium. The corresponding interaction hamiltonian can be written in the form

\[
H_{CC} = \frac{G_F}{\sqrt{2}} \int d^3x \ \chi^I(x) \psi(x) + \text{h.c.,}
\]

(3.1)

where \( G_F \) is the Fermi constant and the neutrino field \( \psi \) is a column vector in flavor space with the entries \( \psi_k \), \( k = 1, \ldots, n \). Moreover, \( \chi^I \) is a row of Dirac operators representing the medium. In the interaction basis \( \chi_k \) carries the lepton number corresponding to the flavor \( k \). For example, in a medium of nucleons and electrons the field \( \chi_e \) corresponding to the electron lepton number can be written for standard-model couplings as

\[
\chi_e = \gamma^\mu(1 - \gamma_5)\phi_e \bar{n}\gamma_\mu(C_V - C_A\gamma_5)\phi_p,
\]

(3.2)

where \( \phi_p \), \( \phi_n \) and \( \phi_e \) are the proton, neutron and electron Dirac fields, respectively, and \( C_V \) and \( C_A \) are the dimensionless vector and axial-vector nucleon coupling constants.
3.2. THE COLLISION INTEGRAL

We are now in a position to make explicit the rhs of eq. (2.9) in order to derive a kinetic equation for $\rho_p$ and $\bar{\rho}_p$. The operators $\chi_k$ violate the lepton number $L_k$ corresponding to the flavor $k$. Therefore, $\langle \chi_k \rangle = 0$ at all times if the medium is in an eigenstate of $L_k$, $k = 1, \ldots, n$. This assumption implies that the interaction eq. (3.1) does not contribute to refractive effects given by the first-order term in eq. (2.9).

In the second-order term $H_{\text{int}}$ appears quadratic so that we get expressions like $\langle \chi_i \chi_k \rangle$. However, because the medium is assumed to be in an eigenstate of $L_k$ they do not contribute for $i \neq k$. Thus, in the final result the contributions of different flavors can be added incoherently.

Defining $I_k$ to be a projector on the neutrino flavor $k$, we find for the CC collision integral

$$\left( \hat{\rho}_p \right)_{CC} = \frac{1}{2} \sum_{k=1}^{n} \left( \mathcal{P}_k(p) \{ I_k, (1 - \rho_p) \} - \mathcal{A}_k(p) \{ I_k, \rho_p \} \right),$$

where $p$ is the physical four-vector ($\omega = p_0 > 0$) of the neutrinos and $\{ , \}$ is an anticommutator. The rates of production $\mathcal{P}_k(\Delta)$ and absorption $\mathcal{A}_k(\Delta)$ are functions of the energy–momentum transfer $\Delta$ to the medium; it is defined for both positive and negative values of $\Delta_0$. We find

$$\mathcal{P}_k(\Delta) = \frac{1}{2} G_F^2 \int_{-\infty}^{+\infty} dt \exp(-i\Delta_0 t) \langle \chi_k(\Delta, t) \rho_v \Delta \chi_k(\Delta, 0) \rangle,$$

$$\mathcal{A}_k(\Delta) = \frac{1}{2} G_F^2 \int_{-\infty}^{+\infty} dt \exp(-i\Delta_0 t) \langle \text{Tr}(\gamma_\mu \Delta \chi_k(\Delta, 0) \chi_k(\Delta, t)) \rangle.$$ (3.4)

The rhs of eq. (3.3) is the difference between a gain term corresponding to the emission reaction $X \rightarrow X' \nu_p$ ($X$ and $X'$ denote medium states), and a loss term corresponding to the absorption reaction $\nu_p X \rightarrow X'$. For a single flavor they take on the familiar form $\mathcal{P}_\rho(1 - f_p)$ and $\mathcal{A}_p f_p$ where $(1 - f_p)$ is the usual Pauli blocking factor.

The kinetic term for the antineutrino density matrix can be found by direct calculation or from that for the neutrinos by “crossing”, i.e. by the transformation

$$p \rightarrow -p, \quad \rho_p \rightarrow (1 - \bar{\rho}_p)$$

everywhere in eq. (3.3). For example, the neutrino absorption reaction $\nu_p X \rightarrow X'$ transforms to the antineutrino emission reaction $X \rightarrow X' \bar{\nu}_p$ under this operation. We find explicitly

$$\left( \hat{\bar{\rho}}_p \right)_{CC} = \frac{1}{2} \sum_{k=1}^{n} \left( \mathcal{A}_k(-p) \{ I_k, (1 - \bar{\rho}_p) \} - \mathcal{P}_k(-p) \{ I_k, \bar{\rho}_p \} \right).$$ (3.6)
Therefore, $\mathcal{R}_k(-p)$ plays the role of an absorption rate for antineutrinos of four-momentum $p(p_0 > 0)$ while $\mathcal{A}_k(-p)$ that of a production rate. Put another way, $\mathcal{A}_k(\Delta)$ and $\mathcal{R}_k(\Delta)$ represent the rate of absorption or production of lepton number of type $k$, independently of the sign of $\Delta_0$.

A more compact result is achieved if we define the $n \times n$ matrices

$$\mathcal{R}_\Delta \equiv \frac{1}{2} \sum_{k=1}^{n} \mathcal{R}_k(\Delta) I_k,$$

$$\mathcal{A}_\Delta \equiv \frac{1}{2} \sum_{k=1}^{n} \mathcal{A}_k(\Delta) I_k.$$  \hspace{1cm} (3.7)

Our kinetic terms are then

$$\hat{\mu}_p(\text{CC}) = \{\mathcal{R}_P, (1 - \rho_p)\} - \{\mathcal{A}_P, \rho_p\},$$

$$\hat{\mu}_p(\text{NC}) = \{\mathcal{A}_{-P}, (1 - \rho_p)\} - \{\mathcal{R}_{-P}, \rho_p\}. \hspace{1cm} (3.8)$$

An evaluation of the expressions eq. (3.4) for $\mathcal{R}_k(\Delta)$ and $\mathcal{A}_k(\Delta)$ is very complicated if the medium is dense and strongly interacting as in a supernova core. However, in a sufficiently dilute medium they can be determined by the usual methods of perturbation theory. For example, in a medium of protons, neutrons, and electrons we may use eq. (3.2) and treat these medium constituents as free Dirac fields between collisions. Then eq. (3.4) is explicitly

$$\mathcal{R}_e(p) = \int dp' \int dq \int dq' (2\pi)^4 \delta^{(4)}(p + q - p' - q') \times \sum_{\text{spins}} |\mathcal{M}(q, q', p, p')|^2 n_e(p') n_p(q')(1 - n_n(q))$$

$$\mathcal{A}_e(p) = \int dp' \int dq \int dq' (2\pi)^4 \delta^{(4)}(p + q - p' - q') \times \sum_{\text{spins}} |\mathcal{M}(q, q', p, p')|^2 n_n(q)(1 - n_e(p'))(1 - n_p(q')),$$ \hspace{1cm} (3.9)

where $n_e(p')$, $n_n(q)$ and $n_p(q')$ are the electron, neutron, and proton occupation numbers, respectively, typically given by the Fermi–Dirac distributions at a temperature $T$ and the relevant chemical potentials. $\mathcal{M}$ is the usual weak matrix element for the process $e(p') + p(q') \leftrightarrow n(q) + \nu_e(p)$.

Strictly speaking, the CC and NC interactions with the medium do not fulfill our assumption that the medium is not influenced substantially by the neutrinos. For example, in a supernova core a change of the $\nu_e$ density due to oscillations into other neutrino flavors will also change the electron and nucleon chemical poten-
tials. However, we assume that the medium components always stay in kinetic equilibrium so that the impact of the interaction on the medium consists of a slow time variation of the temperature and chemical potentials. These parameters can be calculated from the relevant conservation laws. For example, the total lepton number $L_{\text{tot}}$ is conserved by all interactions discussed here. It can be written as a sum $L_{\text{tot}} = L_m + L_\nu$ of a medium contribution $L_m$ and a neutrino contribution $L_\nu = \text{Tr}(\rho - \bar{\rho})$ where $\rho = \int \text{d}p \, \rho_p$ and $\bar{\rho} = \int \text{d}p \, \bar{\rho}_p$ are the integrated neutrino and antineutrino density matrices.

3.3. EVOLUTION OF THERMODYNAMIC POTENTIALS

The main motivation for discussing kinetic equations for mixed neutrinos is a quantitative understanding of the approach to chemical equilibrium, i.e. the question of damping of flavor polarization. An average measure for this polarization is contained in the integrated density matrices for particles and antiparticles $\rho$ and $\bar{\rho}$. However, as already discussed in ref. [12], the relaxation in flavor space is in general coupled to kinetic relaxation. For example, with the NC interactions to be discussed in the next section it is indeed possible to build up a transient polarization even if one starts with an initially unpolarized ensemble which is not in kinetic equilibrium. Therefore, the deviation of the polarization from chemical equilibrium will not shrink at all times. Which quantity, then, is truly damped at all times by the action of the collision terms? Clearly we should consider thermodynamic potentials such as the entropy or the free energy. It is an important test of self-consistency that the appropriate thermodynamic potential decreases or increases at all times under the action of our collision terms.

The internal energy of the neutrino ensemble $U_\nu$, its total lepton number $L_\nu$, and its entropy $S_\nu$ are given by

$$U_\nu = \int \text{d}p \, \omega \, \text{Tr}(\rho_p + \bar{\rho}_p),$$

$$L_\nu = \int \text{d}p \, \text{Tr}(\rho_p - \bar{\rho}_p),$$

$$S_\nu = -\int \text{d}p \, \text{Tr}\left[\rho_p \ln \rho_p + (1 - \rho_p) \ln(1 - \rho_p)\right] + \bar{\rho}_p \ln \bar{\rho}_p + (1 - \bar{\rho}_p) \ln(1 - \bar{\rho}_p),$$

(3.10)

* Our collision terms eq. (3.8) are valid even if the medium is not in thermodynamic equilibrium as long as it is in a quasi-stationary state. However, in order to have well-defined thermodynamic potentials for the neutrino ensemble we need the medium to be a true “heat bath” in thermodynamic equilibrium. It is then characterized by a temperature $T$ and a set of chemical potentials for its constituents.
where \( \omega = |p| \) is the energy of a neutrino mode. The expression for \( S_\nu \) is an extension of the usual definition of the entropy of a one-component Fermi gas not necessarily in equilibrium [22]. \( S_\nu \) is well-defined because the \( \rho_p \)'s and \( (1 - \rho_p) \)'s are positive semi-definite matrices. For the medium we may also define \( U_m, L_m \) and \( S_m \); \( U_m + U_\nu \) and \( L_m + L_\nu \) are conserved while \( S_m + S_\nu \) will increase if the combined system was originally not in equilibrium.

In general the backreaction on the medium of the evolution of the neutrino ensemble cannot be neglected. In special cases it may be possible, however, to regard the medium as an external heat bath whose properties remain unaffected by the evolution of the neutrinos so that the medium may be thought of as absorbing or producing energy and lepton number without changing its temperature or chemical potentials. In equilibrium the neutrinos will then take on the same temperature \( T \) while all mixing flavors will be characterized by the same chemical potential \( \mu \) which is given in terms of medium properties. For a medium of nucleons and charged leptons, for example, \( \mu = \mu_1 + \mu_\mu - \mu_\nu \). Since we study the approach to true thermal equilibrium and since we allow for the violation of individual flavor lepton numbers by neutrino oscillations we must assume that the medium is characterized by chemical potentials for the charged leptons which are identical, \( \mu_1 = \mu_\mu = \mu_\nu = \mu_\tau \). Of course, if we study oscillations between, say, \( \nu_e \) and a sterile species \( \nu_x \) this requirement does not apply and the role of \( \mu_1 \) is played by \( \mu_\mu \) without any restriction on \( \mu_\nu \) and \( \mu_\tau \).

Under these circumstances the appropriate thermodynamic potential to characterize the neutrino ensemble is the grand potential

\[
\Omega_\nu \equiv U_\nu - TS_\nu - \mu L_\nu, \tag{3.11}
\]

not to be confused with the matrix of oscillation frequencies. (We stress that this definition applies even away from equilibrium where \( T \) and \( \mu \) are given by the medium properties as discussed above.) From this definition follows without further assumptions

\[
\dot{\Omega}_\nu = T \int dp \ Tr \left[ \dot{\rho}_p \ ln \left( \frac{e^{(\omega - \mu)/T} \rho_p}{1 - \rho_p} \right) + \dot{\bar{\rho}}_p \ ln \left( \frac{e^{(\omega + \mu)/T} \bar{\rho}_p}{1 - \bar{\rho}_p} \right) \right]. \tag{3.12}
\]

Here \( 1/(1 - \rho_p) \) is the inverse of the matrix \( 1 - \rho_p \) which commutes with \( \rho_p \) so that \( \rho_p/(1 - \rho_p) \) is a well-defined expression.

In equilibrium we have \( \rho_p = \eta_p^0 \) and \( \bar{\rho}_p = \bar{\eta}_p^0 \) (times the \( n \times n \) unit matrix) with the equilibrium Fermi-Dirac distributions

\[
\eta_p^0 = \frac{1}{e^{(\omega - \mu)/T} + 1}, \quad \bar{\eta}_p^0 = \frac{1}{e^{(\omega + \mu)/T} + 1}. \tag{3.13}
\]
Moreover, in equilibrium $\dot{\rho}_p = \dot{\bar{\rho}}_p = 0$ so that detailed balance in eq. (3.8) allows us to write

\[
(\dot{\rho}_p)_{CC} = \left\{ \mathcal{R}_p, \left( 1 - \frac{\rho_p}{\bar{n}^0_p} \right) \right\},
\]

\[
(\dot{\bar{\rho}}_p)_{CC} = \left\{ \mathcal{A}_p, \left( 1 - \frac{\bar{\rho}_p}{\bar{n}^0_p} \right) \right\}. \tag{3.14}
\]

In order to obtain the CC contribution to $\dot{\Omega}_\nu$, we insert these expressions into eq. (3.12) and find

\[
(\dot{\Omega}_\nu)_{CC} = \int d\mathbf{p} \text{Tr} \left[ \mathcal{R}_p \left( 1 - \frac{\rho_p}{\bar{n}^0_p} \right), \ln \left( \frac{1 - \bar{n}^0_p}{\bar{n}^0_p} \frac{\rho_p}{1 - \rho_p} \right) \right] + \int d\mathbf{p} \text{Tr} \left[ \mathcal{A}_p \left( 1 - \frac{\bar{\rho}_p}{\bar{n}^0_p} \right), \ln \left( \frac{1 - \bar{n}^0_p}{\bar{n}^0_p} \frac{\bar{\rho}_p}{1 - \bar{\rho}_p} \right) \right]. \tag{3.15}
\]

Evaluating the traces in a basis where $\rho_p$ or $\bar{\rho}_p$ is diagonal, and using that $\mathcal{R}_p$ and $\mathcal{A}_p$ are positive semi-definite matrices it is now easy to see that always $(\dot{\Omega}_\nu)_{CC} \leq 0$: If an eigenvalue $n_i(p)$ of $\rho_p$ is smaller than $\bar{n}_i^0$, the logarithm is negative and $1 - n_i(p)/\bar{n}_i^0$ is positive, and vice versa if $n_i(p) \geq \bar{n}_i^0$. Hence, our kinetic terms are indeed consistent with the thermodynamic requirement $\dot{\Omega}_\nu \leq 0$.

4. Neutral-current interactions

4.1. THE HAMILTONIAN

We now consider that part of the interaction hamiltonian which is bilinear in the left-handed neutrino field $\psi$ and, after a suitable Fierz transformation, can be written as an effective NC interaction with an external medium consisting of various particle species $a$. We assume that all neutrino flavors $i$ scatter on a given species $a$ in the same way apart from an overall amplitude factor $g_a$. To be specific, we use the standard $V$–$A$ four-fermion interaction \(*\)

\[
H_{NC} = \sqrt{2} \, G_F \sum_a \int d^3 x \, B_a^\mu(x) \bar{\psi}(x) \gamma_\mu \, G^a \psi(x). \tag{4.1}
\]

\(*\) Note that $\psi$ is taken to be a left-handed field so that the usual $(1 - \gamma_3)$ factor does not appear explicitly.
Here, $B_a^\mu$ typically is also a bilinear of the form $\bar{\phi}_a \gamma^\mu \phi_a$ or $\bar{\phi}_a \gamma_5 \gamma^\mu \phi_a$, where $\phi_a$ is a Dirac field which describes nucleons or charged leptons in the medium. Furthermore, in the flavor basis $G^a = \text{diag}(g_1^a, \ldots, g_n^a)$. Therefore, in general it is a hermitian $n \times n$ matrix of dimensionless coupling constants. For the calculations it is convenient to write eq. (4.1) in momentum space,

$$H_{\text{NC}} = \sqrt{2} G_F \sum_a \int \frac{d\mathbf{p} \, d\mathbf{p}'}{2\pi} B_a^\mu(\mathbf{p} - \mathbf{p'}) \bar{\psi}_\mu G^\mu \psi_{\mu'},$$

where $B_a^\mu(\Delta) = \int d^3x \, B_a^\mu(x)e^{-i\Delta \cdot x}$ is the spatial Fourier transform of $B_a^\mu$, and $\psi_{\mu} = a_\mu u_\mu + b_\mu^* v_{-\mu}$ in terms of the annihilation and creation operators of eq. (2.1). The assumption of homogeneity implies that $\langle B_a^\mu(\Delta) \rangle = (2\pi)^3 \delta^{(3)}(\Delta) \langle B_a^\mu(x) \rangle$, where $\langle B_a^\mu(x) \rangle$ actually does not depend on the location $x$.

To give a concrete example let us consider oscillations between $\nu_e$ and $\nu_\mu$ or $\nu_\tau$ in a medium consisting of free ultrarelativistic electrons with the usual standard-model couplings to the two neutrino flavors. These oscillations can be described in the form of eq. (4.2) if we divide the electrons into a left-handed and a right-handed “species”, $a = L$ or $R$. Then we have $B_a^L = \bar{\phi}_e \gamma^\mu (1 + \gamma_5) \phi_e$ and $G^L = \text{diag}(\sin^2 \theta_W + \frac{1}{2}, \sin^2 \theta_W - \frac{1}{2})$ and $G^R = \text{diag}(\sin^2 \theta_W, \sin^2 \theta_W)$ for the dimensionless matrices of coupling constants in terms of the Weinberg angle $\theta_W$.

4.2. THE KINETIC TERMS

In order to evaluate eq. (2.9) we first note that the refractive term is just a sum over different medium components, whereas the collision term in general contains interference terms between different target species. At this point we assume the different target species to be uncorrelated, corresponding to $\langle B_a^\mu B_b^\nu \rangle = \langle B_a^\mu \rangle \langle B_b^\nu \rangle$ for $a \neq b$. Then these interference terms will only contribute to second-order forward-scattering effects which we continue to neglect. The collision term will be an incoherent sum over all target species so that in the following we may suppress the superscript $a$ everywhere for simplicity.

After a lengthy but straightforward calculation we arrive at the NC collision term

$$\left(\rho_\mu \right)_{\text{NC}} = -i\sqrt{2} G_F \rho_m [G, \rho_\mu]$$

$$+ \frac{1}{2} \int \frac{d\mathbf{p}'}{2\pi} \left[ W(\mathbf{p}', \mathbf{p}) (1 - \rho_\mu) G \rho_\mu G - W(\mathbf{p}, \mathbf{p}') \rho_\mu G (1 - \rho_\mu) G ight]$$

$$+ W(\mathbf{p}', \mathbf{p}) (1 - \rho_\mu) G (1 - \rho_\mu) G - W(\mathbf{p}, \mathbf{p}') \rho_\mu G \rho_\mu G + \text{h.c.},$$

where $W(\mathbf{p}, \mathbf{p}') = \int d^3x \, e^{i\mathbf{p}\cdot\mathbf{x}} B_a^\mu(x) e^{-i\mathbf{p}'\cdot\mathbf{x}}$.
where \( p_0 = |p| \) and \( p'_0 = |p'| \) are physical (positive) energies, isotropy of the medium was assumed \(^* \) and \( p_m = \langle B^\beta(x) \rangle \) is a kind of baryon or charged-lepton number density of the medium. The non-negative transition probabilities \( W(k', k) \) are Wick contractions of medium operators of the form

\[
W(k', k) = 2G_f^2 \mathcal{M}^{\mu\nu}(k' - k) \mathcal{A}^{\mu\nu}(k', k),
\]

(4.4)

where \( k \) and \( k' \) correspond to neutrino four-momenta with \( k_0, k'_0 \) positive or negative. The tensorial “medium structure function” is given by

\[
\mathcal{M}^{\mu\nu}(\Delta) \equiv \int_{-\infty}^{+\infty} dt \exp(i\Delta_0 t)\langle B^\mu(t, \Delta) B^\nu(0, -\Delta) \rangle,
\]

(4.5)

where \( \Delta_0 \) can be both positive and negative. In the ultrarelativistic limit the neutrino tensor can be written as \([23,24]\)

\[
\mathcal{A}^{\mu\nu}(k', k) = \frac{2}{2k_0 k'_0} \left( k^{\mu} k'^{\nu} + k'^{\mu} k^{\nu} - k \cdot k' g^{\mu\nu} - i\epsilon^{\mu\nu\alpha\beta} k_\alpha k'_\beta \right).
\]

(4.6)

For \( k_0 > 0 \) and \( k'_0 > 0 \) this tensor is the neutrino part of the squared, spin-summed matrix element for the scattering process \( \nu_k X \to X' \nu_{k'} \) where \( X \) and \( X' \) are, again, medium configurations. Note that \( \mathcal{A}^{\mu\nu} \) is an even function of \( k \) and \( k' \).

The integral in eq. (4.3) is the collision term. The first two terms in brackets are due to neutrino scattering off the medium, the positive term being a gain term corresponding to the scattering process \( \nu_p X \to \nu_p X' \) and the negative one being a loss term corresponding to the inverse reaction. We have already discussed these contributions in some detail in an earlier paper \([12]\). The third and fourth expressions in the integral account for pair processes with the positive term being a gain term from pair creations by the medium \( X \to X' \nu_p \bar{\nu}_p \) while the negative one is a loss term from pair annihilations \( \nu_p \bar{\nu}_p X \to X' \). One obtains these terms by direct calculation or from the scattering integral by “crossing” the neutrino \( \nu_p \) in analogy to eq. (3.5). For example, the reaction \( \nu_p X \to X' \nu_p \) transforms to \( \bar{\nu}_p X \to X' \nu_p \nu_p \) under this operation. If there is only one flavor present, the oscillation term vanishes and the collision integrals reduce to the usual Boltzmann collision terms \([25]\).

The corresponding equation for \( \bar{\rho}_p \) is found by direct calculation or by applying the crossing operation eq. (3.5) to all neutrinos and antineutrinos appearing in eq.

\(^* \) For a non-isotropic medium we would get a more general refractive term of the form

\[
-iv^2 G_f \langle B^\mu(x) \rangle (p_\mu / \omega) [G, \rho_p].
\]
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(4.3). Under this operation the neutrino gain terms transform to the antineutrino loss terms and vice versa. We find

\[
\left( \hat{\rho}_\nu \right)_{\text{NC}} = -i\sqrt{2} \, G_F \rho_m \left[ G, \bar{\rho}_\nu \right]
\]

\[+ \frac{1}{2} \int \text{d}p' \left[ W(-p, -p')(1 - \bar{\rho}_\nu)G\bar{\rho}_\nu G - W(-p', -p)\bar{\rho}_\nu G(1 - \bar{\rho}_\nu')G \right]
\]

\[+ W(-p, p')(1 - \bar{\rho}_\nu)G(1 - \rho_{\nu'})G - W(p', -p)\bar{\rho}_\nu G\rho_{\nu'} G + \text{h.c.}, \tag{4.7} \]

where because of eq. (2.5) the relative sign between the vacuum and the medium oscillation terms has changed as is well known from discussions of the MSW effect. \(W(k', k)\) can be interpreted as the probability per unit time of destroying lepton number in mode \(k'\) and producing lepton number in \(k\), independently of the sign of \(k_0\) and \(k'_0\).

In our above example where the medium consisted of free electrons, the right-handed ones play no role in the refractive term because the commutator vanishes whereas the term for the left-handed ones is proportional to \(\rho^L_m \equiv n^L_e - \bar{n}^L_e\), where \(n^L_e\) and \(\bar{n}^L_e\) are the number densities of left-handed electrons and positrons, respectively. For an unpolarized medium we thus recover the well-known result that the medium-induced energy shift is \(\sqrt{2} \, G_F\) times the electron lepton number density of the medium. Furthermore, the left-handed and right-handed medium structure functions can be simply expressed in terms of the electron and positron occupation numbers \(n_e(q)\) and \(\bar{n}_e(q)\),

\[
\mathcal{M}_{L,R}^{\mu \nu}(\Delta) = \int \text{d}q \int \text{d}q' \left( 2\pi \right)^4 \mathcal{M}_{L,R}^{\mu \nu}(q, q') \times \left\{ \delta^{(4)}(\Delta + q - q')n_e(q)(1 - n_e(q')) + \delta^{(4)}(\Delta + q + q')n_e(q)\bar{n}_e(q') \right. \]

\[+ \delta^{(4)}(\Delta - q + q')\bar{n}_e(q')(1 - n_e(q')) + \delta^{(4)}(\Delta - q - q')(1 - \bar{n}_e(q))(1 - n_e(q')) \right\}, \tag{4.8} \]

where \(\mathcal{M}_{L,R}^{\mu \nu}(q, q') \equiv \mathcal{N}_{L,R}^{\mu \nu}(q, q')\) and \(\mathcal{M}_{R}^{\mu \nu}(q, q') \equiv (\mathcal{N}_{R}^{\mu \nu}(q, q'))^*\), with \(\mathcal{N}_{L,R}^{\mu \nu}(q, q')\) given by eq. (4.6). In eq. (4.8) the first two terms account for scattering on electrons and positrons whereas the third and fourth terms are due to \(e^+e^-\) annihilation and creation in analogy to the structure of eqs. (4.3) and (4.7).
We may also study the total neutrino lepton number density matrix \( \rho_L = \rho - \bar{\rho} \). Integrating eqs. (4.3) and (4.7), we obtain a commutator structure:

\[
(\rho_L)_{NC} = -i\sqrt{2} \, G_F \rho_m [G, \rho_L] \\
+ \frac{i}{2} \int dp \, dp' [G, W(p, p') \rho_p G(1 - \rho_p)] \\
+ W(-p, -p')(1 - \bar{\rho}_p) G \bar{\rho}_p' - \text{h.c.} \\
+ \frac{i}{2} \int dp \, dp' [G, W(-p, p')(1 - \bar{\rho}_p) G(1 - \rho_p)] \\
+ W(p, -p') \rho_p G \bar{\rho}_p' - \text{h.c.}.
\]  

(4.9)

The first term is the refractive part, the second term corresponds to scattering processes and the last one is due to pair processes. From this we immediately see that the total lepton number density \( L = \text{Tr} \rho_L \) of the neutrino ensemble is conserved as expected. Of course, eq. (4.9) is not a closed differential equation for \( \rho_L \) but really depends on all density matrices \( \rho_p \).

4.3. NON-LOCAL GAUGE BOSON EFFECTS IN THE EARLY UNIVERSE

In eq. (4.1) we have taken the usual low-energy local four-fermion coupling. The full standard-model effective interaction between leptons and neutrinos, however, must include gauge boson propagator effects. It can be written in the form

\[
H_{NC} = g^2 \int dp \int dp' \int dq \int dq' (2\pi)^3 \delta^{(3)}(q + p - q' - p') \\
\times \sum_{k=1}^n \left( \frac{i}{2} D_{\mu\nu}^{W}(\Delta(q', p)) \bar{\phi}_k(q) \gamma^\mu \psi_k(p') \bar{\psi}_k(p) \gamma^\nu \phi_k(q') \\
- \frac{1}{8 \cos^2 \theta_W} D_{\mu\nu}^{Z}(\Delta(p', p)) \bar{\phi}_k(q) \gamma^\mu \psi_k(p') \bar{\psi}_k(p) \gamma^\nu \phi_k(q') \right),
\]

(4.10)

where \( \phi_k \) is the Dirac field corresponding to the charged lepton of flavor \( k \), and \( g = (4\sqrt{2} \, G_F)^{1/2} m_W \) is the weak coupling constant. In terms of the W and Z masses the propagators are

\[
D_{\mu\nu}^{W,Z}(\Delta) = \left( g_{\mu\nu} - \frac{\Delta_{\mu\nu}}{m_{W,Z}^2} \right) \frac{1}{m_{W,Z}^2 - \Delta^2}.
\]

(4.11)
In eq. (4.10) the boson four-momentum appears as a function $\Delta(p, p') = p \pm p'$ of the physical four-momenta $p$ and $p'$ of the incoming particles, where the minus sign has to be taken for $t$-channel processes whereas the plus sign appears for $s$-channel processes. Then the effective “hamiltonian” eq. (4.10) gives the correct weak interaction matrix elements for all relevant tree-level processes.

We are typically interested in energies in the MeV range or below whereas $m_{\text{W},Z}$ are of order 100 GeV. Therefore, it will be a good approximation to expand eq. (4.11) in powers of $\Delta / m_{\text{W}}^2$ and use only the zeroth-order term which leads to the local four-fermion coupling eq. (4.2). Furthermore, the refractive terms are then proportional to a kind of baryon or charged-lepton number density $\rho_m$. However, the relative baryon and lepton number densities in the early universe are of order $10^{-9}$ so that $\rho_m$ is suppressed by that factor relative to the particle densities. Therefore, non-local effects caused by CC scattering on charged leptons become important in forward-scattering [16]. These effects are described by the higher-order terms in the W propagator. They are negligible in the second-order term of eq. (2.9); it is sufficient to include them in the refractive term.

For an unpolarized isotropic medium of free charged leptons the additional contributions to the refractive parts in eqs. (4.3) and (4.7) can be calculated. Writing out only the oscillation terms the kinetic equations are to first order in $\Delta / m_{\text{W}}^2$

$$\dot{\rho}_p = -i \left[ \left( \Omega_p^0 + \sqrt{2} G_F L - \frac{8\sqrt{2} G_F \rho_0}{3 m_{\text{W}}^2} E \right), \rho_p \right],$$

$$\dot{\bar{\rho}}_p = +i \left[ \left( \Omega_p^0 - \sqrt{2} G_F L - \frac{8\sqrt{2} G_F \rho_0}{3 m_{\text{W}}^2} E \right), \bar{\rho}_p \right],$$

where $L$ and $E$ are $n \times n$ matrices which are diagonal in the interaction basis. There, the diagonal entries of $L$ are the densities of lepton number of flavor $k$ carried by the medium while those of $E$ are the energy densities in charged leptons and antileptons of flavor $k$. Eq. (4.12) is a generalization of the results previously obtained by Nötzold and Raffelt [16].

4.4. EVOLUTION OF THE FREE ENERGY

We may now consider the evolution of the thermodynamic grand potential in the spirit of sect. 3.3 above, i.e. we may consider the contribution $(\dot{\Omega}_\nu)_{\text{NC}}$ to eq. (3.12). However, because NCs conserve the total lepton number $L_\nu$ of the neutrino ensemble so that $L_\nu = 0$ it is sufficient to consider the evolution of the free energy $F_\nu = U_\nu - TS_\nu$. If $(\dot{F}_\nu)_{\text{NC}}$ is found to be non-positive then $(\dot{\Omega}_\nu)_{\text{NC}}$ will have this property as well.
We may now verify that our kinetic terms eqs. (4.3) and (4.7) indeed lead to the inequality \((F_{\nu})_{NC} \leq 0\). To this end we note that setting \(\mu = 0\) in eq. (3.12) leads to the appropriate equation for \(\tilde{F}_{\nu}\). Again, the oscillation terms do not contribute under the trace so that we may divide \((\tilde{F}_{\nu})_{NC}\) into three parts \((\tilde{F})_{\nu\nu}\), \((\tilde{F})_{\nu\bar{\nu}}\), and \((\tilde{F})_{\nu\bar{\nu}}\) due to the collision integrals describing neutrino scattering, antineutrino scattering, and pair processes, respectively. Furthermore, we write \(n_i(p)\) and \(\bar{n}_i(p)\) for the \(i\)th eigenvalue of \(\rho_p\) and \(\bar{\rho}_p\) and define the non-negative numbers \(C_{ij}(p, p') = \text{Tr}(P_i(p)GP_j(p')G)\), where \(P_i(p)\) is the projector onto the \(i\)th eigenstate of \(\rho_p\). Using the symmetry of \(C\) in its two arguments and the detailed-balance result

\[
W(k', k) = \exp\left(-\frac{E_{\nu}}{T}\right) W(k, k'),
\]

we find

\[
(\tilde{F})_{\nu\nu} = -TG^2 \int dp\, dp' \sum_{i,j=1}^{n} W(p', p) e^{-\omega' / T} C_{ij}(\rho_p, \rho_{p'}) (1 - n_i(p))(1 - n_j(p'))
\]

\[
\times \left[ \ln \left( \frac{e^{\omega' / T} n_i(p)}{1 - n_i(p)} \right) - \ln \left( \frac{e^{\omega' / T} n_j(p')}{1 - n_j(p')} \right) \right] \left[ \frac{e^{\omega' / T} n_i(p)}{1 - n_i(p)} - \frac{e^{\omega' / T} n_j(p')}{1 - n_j(p')} \right].
\]

One obtains \((\tilde{F})_{\nu\bar{\nu}}\) from this expression by crossing both neutrinos \(\nu^i_p\) and \(\nu^j_{p'}\), whereas \((\tilde{F})_{\nu\bar{\nu}}\) is obtained by only crossing \(\nu^i_p\) and multiplying by two. Because of the monotony of the logarithm this indeed implies that all three time derivatives are non-positive, hence \((\tilde{F})_{\nu} \leq 0\).

5. Self-interactions

5.1. THE HAMILTONIAN

Turning to neutrino self-interactions, the corresponding effective interaction hamiltonian is quartic in the neutrino field \(\psi\). We restrict ourselves to the usual V–A coupling, but in contrast to the standard model we allow for different coupling strengths for different neutrino flavors. Because self-interactions are mainly important in the early universe, a local four-fermion coupling would be too crude an approximation for the resulting refractive terms. Therefore, we include gauge boson propagator effects in the same way as discussed in sect. 4.3 above,

\[
H_S = \frac{1}{2} \left( \frac{g}{2 \cos \theta_W} \right)^2 \int dp \int dq \int dp' \int dq' (2\pi)^3 \delta^{(3)}(p + q - p' - q')
\]

\[
\times \bar{\psi} q^\mu G_S \psi q' D_{\mu\nu}^Z (\Delta(p, p')) \bar{\psi} p^\nu G_S \psi p'.
\]
Here $G_S$ is an $n \times n$ dimensionless hermitian matrix of NC coupling constants. The dependence on the $Z$-boson propagator $D^{Z}_{uv}(\Delta(p, p'))$ has been discussed after eq. (4.11). In eq. (5.1) the standard-model couplings correspond to $G_S = \text{diag}(1, \ldots, 1)$.

5.2. REFRACTIVE EFFECTS

We first turn to the refractive energy shifts $\Omega^S_p$ and $\overline{\Omega}^S_p$ due to neutrino self-interactions, i.e. to the first-order term in eq. (2.9). Carrying out the relevant contractions we arrive at *

$$\Omega^S_p = \sqrt{2} G_F \int dq \{ G_S(\rho_q - \overline{\rho}_q)G_S + G_S \text{Tr}\{(\rho_q - \overline{\rho}_q)G_S\} \}$$

and $\overline{\Omega}^S_p$ is given by the same formula with $\rho_q$ and $\overline{\rho}_q$ interchanged. Therefore, relative to the matrix of vacuum frequencies $\Omega^0_p$ the first term changes sign for antineutrinos whereas the second one remains unchanged. The trace expression in the first term implies the well-known result that neutrinos in a bath of their own flavor experience twice the energy shift relative to a bath of another flavor [16].

For two-flavor mixing the oscillation terms can be written in a more instructive form if one expands all $2 \times 2$ matrices in terms of the Pauli matrices $\tau$. We write $\rho_p = \frac{1}{2}(n_p + P_p \cdot \tau)$ and $\Omega_p = \frac{1}{2}(E_p + V_p \cdot \tau)$ where $\Omega_p$ is the total "oscillation matrix" which includes all medium and self-interaction effects. Here, $n_p$ is the total number of neutrinos in mode $p$, $P_p$ is a flavor polarization vector for this mode, and $E_p$ is twice the average energy including the medium-induced shifts. The oscillation terms are given be the well-known "precession formula"

$$\hat{P}_p = V_p \times P_p,$$

$$\overline{\hat{P}}_p = -\overline{V}_p \times \overline{P}_p,$$

where the overbarred vectors refer to antineutrinos. Recall that our sign-conven-

* If the neutrino ensemble is not isotropic one has to include an extra factor $(1 - \cos \theta_{pq})$ in the first term and $\frac{1}{2}(1 - \cos \theta_{pq})^2$ in the second term, where $\theta_{pq}$ is the angle between $p$ and $q$. 
tion is such that in vacuum $\mathbf{V}_p = \mathbf{V}_p^\dagger = \mathbf{V}_0^p$, which is given in terms of the mass matrix $M$ by $\frac{1}{2}(\epsilon^p_0 + \epsilon_0^p \cdot \tau) = (\mathbf{p}^2 + M^2)^{1/2}$.

It is now convenient to choose a basis where $G_S$ is diagonal so that it can be written as $G_S = \frac{1}{2}(g_0 + g_3 \tau_3)$. Furthermore, we write $\rho = \int d\mathbf{p} \rho_\mathbf{p} = \frac{1}{2}(n + \mathbf{P} \cdot \tau)$, where $n = \text{Tr} \rho$ is the total neutrino number density and $\mathbf{P}$ is the total neutrino flavor polarization density. This implies $n = \int d\mathbf{p} n_\mathbf{p}$ and $\mathbf{P} = \int d\mathbf{p} \mathbf{P}_\mathbf{p}$. Also, we introduce an “energy density matrix” $\rho_\mathbf{u} = \int d\mathbf{p} \mathbf{p}_0 \rho_\mathbf{p} = \frac{1}{2}(u + \mathbf{U} \cdot \tau)$ where $u = \text{Tr} \rho_u$ is the total neutrino energy density and $\mathbf{U}$ is the total neutrino “energy-polarization density”. Put another way, $u = \int d\mathbf{p} \mathbf{p}_0 n_\mathbf{p}$ and $\mathbf{U} = \int d\mathbf{p} \mathbf{p}_0 \mathbf{P}_\mathbf{p}$. Finally, we write $\mathcal{O}_p^S = \frac{1}{2}(E_p^S + V_p \cdot \tau)$ where $E_p^S = \text{Tr} \mathcal{O}_p^S$ for the oscillation matrix due to self-interactions. Eq. (5.2) yields

$$V_p^S = \sqrt{2} G_F \left\{ g_3 \left[ g_0 (n - \bar{n}) + g_3 (P_3 - \bar{P}_3) \right] e_3 + \frac{1}{2} \left[ g_0^2 - g_3^2 \right] (P - \bar{P}) \right\}$$

$$- \frac{8\sqrt{2} G_F p_0}{3m_\nu^2} \left\{ \frac{1}{2} g_3 \left[ g_0 (u + \bar{u}) + g_3 (U_3 + \bar{U}_3) \right] e_3 + \frac{1}{2} \left[ g_0^2 - g_3^2 \right] (U + \bar{U}) \right\}, \quad (5.4)$$

where $e_3$ is a unit vector in the 3-direction. For $\mathbf{V}_p^S$ the same formula applies where neutrino and antineutrino terms are interchanged so that the first expression changes sign.

It is interesting to discuss two limiting cases: First let us consider mixing between a neutrino with standard-model interactions and an inert flavor so that $G_S = \text{diag}(1, 0)$. Then eq. (5.4) simplifies to

$$V_p^S = \left[ 2\sqrt{2} G_F (n - \bar{n})_{\text{int}} - \frac{8\sqrt{2} G_F p_0}{3m_\nu^2} (u + \bar{u})_{\text{int}} \right] e_3, \quad (5.5)$$

where $n_{\text{int}}$ and $u_{\text{int}}$ and the corresponding antiparticle expressions are the number and energy density of the interacting neutrino type, respectively. Note that in this case $V_p^S$ has no “transverse components”. Therefore, this result can be obtained by the simple approach of refs. [4,16]; there is no “off-diagonal” refractive index.

Next, we consider oscillations between neutrinos with equal standard-model couplings, $G_S = \text{diag}(1, 1)$, so that eq. (5.4) reads

$$V_p^S = \sqrt{2} G_F (P - \bar{P}) - \frac{8\sqrt{2} G_F p_0}{3m_\nu^2} (U + \bar{U}). \quad (5.6)$$

It was first realized by Pantaleone [7] that the index of refraction caused by neutrino self-interactions can have nondiagonal elements in the flavor basis. In eq.
(5.6) this amounts to the transverse components of $V_p^S$. For the precession formula we find explicitly in this case

$$
\dot{P}_p = +V_p^0 \times P_p + \sqrt{2} G_F (P - \overline{P}) \times P_p - \frac{8\sqrt{2} G_F p_0}{3m^2} (U + \overline{U}) \times P_p,
$$

$$
\dot{\overline{P}}_p = -V_p^0 \times \overline{P}_p + \sqrt{2} G_F (P - \overline{P}) \times \overline{P}_p + \frac{8\sqrt{2} G_F p_0}{3m^2} (U + \overline{U}) \times \overline{P}_p. \tag{5.7}
$$

However, often the behaviour of a single mode will not be of interest so that we should rather consider the evolution of the total flavor polarization $P = \int dp \ P_p$,

$$
\dot{P} = + \int dp \ V_p^0 \times P_p - \sqrt{2} G_F \overline{P} \times P - \frac{8\sqrt{2} G_F p_0}{3m^2} U \times U,
$$

$$
\dot{\overline{P}} = - \int dp \ V_p^0 \times \overline{P}_p - \sqrt{2} G_F \overline{P} \times P - \frac{8\sqrt{2} G_F p_0}{3m^2} U \times U. \tag{5.8}
$$

If there are no antineutrinos present ($\overline{P} = \overline{U} = 0$) only the vacuum term in eq. (5.8) survives. The direction $\hat{V}_0$ of $V_p^0$ is fixed by the vacuum mixing angle while its magnitude is $\Delta m^2/2|p|$ so that we may write $V_p^0 = \hat{V}_0 \Delta m^2/2|p|$. Therefore, in the absence of antineutrinos we have

$$
\dot{P} = \hat{V}_0 \times \int dp \ \frac{\Delta m^2}{2|p|} P_p, \tag{5.9}
$$

so that $P$ changes only in the plane perpendicular to $\hat{V}_0$. Put another way, the projection of $P$ on the direction $\hat{V}_0$ given by the vacuum mixing angle is conserved. In the presence of antineutrinos the same argument applies to $P - \overline{P}$ whose projection on $\hat{V}_0$ also remains constant. The transverse part, of course, will evolve differently than in the vacuum because it depends on the polarizations in each mode for which the self-interaction contributions do not generally vanish as can be seen from eq. (5.7).

5.3. THE COLLISION INTEGRALS

In eq. (5.1) the terms of higher order in $\Delta^2/m_Z^2$ are negligible in the collision integral for energies in the MeV range or below. Therefore, for the calculation of
the collision terms we will only go to zeroth order with $D_{\mu \nu}^Z = g_{\mu \nu} / m_Z^2$ in eq. (5.1).
Furthermore, we introduce the non-negative numbers

$$f_{\text{scatt}}(p, q, p') = \frac{p \cdot q \cdot p' \cdot (p + q - p')}{p_0 q_0 p'_0 (p + q - p')_0},$$

$$f_{\text{pair}}(p, q, p') = \frac{p \cdot p' \cdot q \cdot (p + q - p')}{p_0 q_0 p'_0 (p + q - p')_0},$$

(5.10)

which are functions of the physical four-momenta $p$, $q$, and $p'$ and only depend on scattering angles. Then the result can be written as

$$\left(\tilde{\rho}_p\right)_S = -i \left[\Omega^S_{p'}, \rho_p\right]$$

$$+ G^2_F \int dq \int dp' \int dq' (2\pi)^4 \delta(4)(p + q - p' - q') f_{\text{scatt}}(p, q, p')$$

$$\times \left\{ (1 - \rho_p) G_S \rho_p G_S \left[ (1 - \rho_q) G_S \rho_q G_S + \text{Tr}(\ldots) \right] - \rho_p G_S (1 - \rho_p) G_S \left[ \rho_q G_S (1 - \rho_q) G_S + \text{Tr}(\ldots) \right] + \text{h.c.} \right\}$$

$$+ G^2_F \int dq \int dp' \int dq' (2\pi)^4 \delta(4)(p + q - p' - q') f_{\text{pair}}(p, q, p')$$

$$\times \left\{ (1 - \rho_p) G_S (1 - \tilde{\rho}_q) G_S \left[ \tilde{\rho}_q G_S \rho_p G_S + \text{Tr}(\ldots) \right] - \rho_p G_S \tilde{\rho}_q G_S \left[ (1 - \tilde{\rho}_q) G_S (1 - \rho_p') G_S + \text{Tr}(\ldots) \right]$$

$$+ (1 - \rho_p) G_S \rho_p' G_S \left[ \tilde{\rho}_q G_S (1 - \tilde{\rho}_q) G_S + \text{Tr}(\ldots) \right]$$

$$- \rho_p G_S (1 - \rho_p') G_S \left[ (1 - \tilde{\rho}_q') G_S \tilde{\rho}_q G_S + \text{Tr}(\ldots) \right] + \text{h.c.} \right\},$$

(5.11)

where the first term is the refractive contribution already discussed in the previous section. In the collision integrals $\text{Tr}(\ldots)$ means the trace of the term in front of it. The first phase space integral represents the gain and loss term due to the scattering reaction $\nu_p \nu_q \rightarrow \nu_p \nu_q$ and the inverse reaction, respectively, whereas the second integral describes gain and loss terms due to the pair reaction $\nu_p \tilde{\nu}_q \rightarrow \nu_p \tilde{\nu}_q$ and its inverse. The first two terms of the second integral can be obtained from the first integral by crossing $\nu_p$ and $\nu_q$ and relabeling $p' \leftrightarrow q$, $p' \leftrightarrow q'$ in this order. The last two terms arise from crossing $\nu_q$ and $\nu_q$ and relabeling $q' \leftrightarrow q$, $p' \leftrightarrow q'$. As usual, the antineutrino equation is obtained by crossing all neutrinos on both
sides of the equation which amounts to interchanging all \( \rho \)'s and \( \tilde{\rho} \)'s and substituting \( \Omega_{p}^{S} \rightarrow -\tilde{\Omega}_{p}^{S} \).

The total scattering cross section for two incoming neutrinos is proportional to the rate of change of occupation numbers in the incoming momentum modes. Therefore, the trace terms in our collision integral eq. (5.11) reproduce the well-known result that the total two-particle scattering cross section is twice as large for neutrinos of the same flavor as compared to neutrinos of different flavors, assuming standard-model couplings.

\subsection*{5.4. EVOLUTION OF THE ENTROPY}

Under the interaction hamiltonian eq. (5.1) the total lepton number in the neutrino ensemble \( L_{\nu} \) is conserved. Similarly, the energy delta functions in our kinetic terms show that the total internal energy \( U_{\nu} \) of the ensemble is also conserved. Thus, the self-interactions should have a non-negative contribution \((\dot{S}_{\nu})_{S}\) to the time derivative of the entropy \( S_{\nu} \). In order to prove this inequality we note that

\begin{equation}
\dot{S}_{\nu} = - \int dp \text{Tr} \left[ \dot{\rho}_{p} \ln \left( \frac{\rho_{p}}{1 - \rho_{p}} \right) + \dot{\tilde{\rho}}_{p} \ln \left( \frac{\tilde{\rho}_{p}}{1 - \tilde{\rho}_{p}} \right) \right].
\end{equation}

Moreover, we use the definitions of sect. 4.4 and define the non-negative numbers

\begin{equation}
C_{ijkl}(\rho_{p}, \rho_{q}, \rho_{p'}, \rho_{q'}) = \text{Tr} \left[ P_{i}(\rho_{p})G_{S}P_{j}(\rho_{p'})G_{S}\left[ P_{k}(\rho_{q})G_{S}P_{l}(\rho_{q'})G_{S} + \text{Tr}(\ldots) \right] \right] + (\rho_{p'} \leftrightarrow \rho_{q'}, k \leftrightarrow l),
\end{equation}

which were symmetrized with respect to the last two arguments. We may divide \((\dot{S}_{\nu})_{S}\) into three parts \((\dot{S})_{\nu\nu}\), \((\dot{S})_{\nu\bar{\nu}}\), and \((\dot{S})_{\bar{\nu}\bar{\nu}}\), caused by scattering processes involving only neutrinos, only antineutrinos, and pair processes, respectively. Then it can be shown that

\begin{equation}
(\dot{S})_{\nu\nu} = \frac{1}{4} G_{F}^{2} \int dp \int dq \int dp' \int dq' (2\pi)^{4} \delta^{(4)}(p + q - p' - q') f_{\text{scatt}}(p, q, p') \times \sum_{i,j,k,l=1}^{n} C_{ijkl}(\rho_{p}, \rho_{q}, \rho_{p'}, \rho_{q'})(1 - n_{i}(p)) \times (1 - n_{j}(q))(1 - n_{k}(p'))(1 - n_{l}(q')) \times \left[ \ln \left( \frac{n_{i}(p)}{1 - n_{i}(p)} \right) - \ln \left( \frac{n_{i}(q)}{1 - n_{i}(q)} \right) \right] \times \left[ \frac{n_{i}(p)}{1 - n_{i}(p)} - \frac{n_{i}(q)}{1 - n_{i}(q)} \right].
\end{equation}
We have used that the numbers \( C_{ijkl}(\rho_{p'}, \rho_{q'}, \rho_{p}, \rho_{q}) \) have the same symmetry with respect to their arguments as the delta function with respect to its corresponding arguments. The monotony of the logarithm implies that \((\hat{S})_{\nu\nu} \geq 0\). The other two parts, \((\hat{S})_{\nu\bar{\nu}}\) and \((\hat{S})_{\bar{\nu}\bar{\nu}}\), are obtained by the crossing operations discussed after eq. (5.11) and by multiplication with a factor four for \((\hat{S})_{\bar{\nu}\bar{\nu}}\). Therefore, \((\hat{S})_{\nu}\) \(\geq 0\) as expected. In the most general case where all three types of interactions are present, CC and NC interactions with a medium as well as self-interactions, we have proven that \(\hat{\Omega}_{\nu} \leq 0\).

### 6. Summary

In general a homogeneous ensemble of mixed neutrinos and antineutrinos (\(n\) flavors) is characterized by two \(n \times n\) density matrices for each momentum mode, one for the neutrinos and one for the antineutrinos. These matrices replace the usual occupation numbers for a single or several unmixed flavors. On the basis of the usual low-energy current–current interaction Hamiltonian between neutrinos and a “back-ground medium” we have derived a formal expression which, when made explicit, yields a differential equation for the density matrices. This kinetic equation is complete to second order in the interaction Hamiltonian and thus contains all effects due to the medium-induced energy shifts (first order) and those due to non-forward collisions (second order). It also contains all effects non-linear in the neutrino density matrices such as Pauli blocking of final states by mixed neutrinos or the refractive effects of neutrinos in a bath of neutrinos. Our work unifies and generalizes various previous attempts at the simultaneous treatment of neutrino oscillations and collisions.

We have made the kinetic equation explicit for three typical cases: (i) Charged-current interactions which allow the medium and the neutrino ensemble to exchange lepton number. (ii) Effective neutral-current interactions which conserve each flavor lepton number separately for the medium and the neutrino ensemble, but which allow for the exchange of energy. (iii) Neutrino self-interactions by neutral currents without need for any medium.

The properties of the medium were carefully separated from those of the neutrinos. The medium is characterized by densities (for the first-order effects) and by structure functions (second order) which need not be specified for the present purposes except through a formal definition. In practice they must be calculated by perturbative methods; for a dilute medium they are given by the squared matrix elements for given reactions, properly integrated over the phase space of the medium constituents.

When the medium is a true heat bath which can absorb energy and particles without changing its properties we have proven that our explicit collision integrals obey the relevant thermodynamic inequalities. To this end we had to extend the
usual expression for the entropy of a Fermi gas to the case of mixed neutrinos. Thus, our collision integrals, indeed, push the neutrino ensemble toward kinetic and chemical equilibrium. Because of the coupling between the flavor and kinetic degrees of freedom it is no contradiction that a neutrino ensemble initially in chemical but not in kinetic equilibrium can exhibit a transient flavor polarization on its way to full thermal equilibrium.

The generality of our explicit kinetic equations is limited by the assumption that different species of medium particles are uncorrelated with each other so that their effects can be added incoherently. In a supernova core this assumption is probably a poor approximation to the actual situation. However, as the medium structure functions in a nuclear medium are very poorly known anyway, formal expressions including these correlations would not seem to be of much use at the present time.

We thank Leo Stodolsky for discussions and encouragement during the course of this work. This paper is based, in part, on work to be submitted by G.S. as a doctoral thesis to the Ludwig-Maximilians-Universität (Munich).

References

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