The Phase Space in Quantum Field Theory

How Small? How Large?

Martin Porrmann

II. Institut für Theoretische Physik
Universität Hamburg

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Overview

Relativistic Quantum Field Theory

Standard Assumptions of Relativistic Quantum Field Theory

- Poincaré Covariance
- Local Commutativity
- Causality
- Spectrum Condition

Not sufficient to single out theories with reasonable physical behaviour

Deficiencies

- Complete Particle Interpretation
- Thermodynamical Equilibrium States

Supplementary Assumption

Idea

Implement the uncertainty relations by way of putting appropriate restrictions on the phase space

How many states fit into a given volume of phase space?

- As a consequence of the uncertainty principle in Quantum Mechanics, for a particle in 3 space dimensions: $\frac{V}{\hbar^3}$ independent states in phase space volume V.
- Quantum Field Theory: No simple expression of the above kind available!
  But nonetheless, in theories with short range forces and particle interpretation, one expects the number of localised states with bounded energy to be limited.

Upper Bounds

Different formulations of this additional assumption in terms of *upper bounds* on the number of independent states led to the following

Results

- Existence of thermodynamical equilibrium states
- Split property (statistical independence of local algebras)
- Characterisation of the type of local algebras

Energy Conditions in General Relativity

Energy conditions in classical General Relativity are used to ensure causality, *e. g.*,

Weak Energy Condition

Let T_{ab} denote the energy-momentum tensor, then one has for any future-directed, timelike vector u^a:

$$T_{ab} u^a u^b \geqslant 0.$$

This is known to be violated in quantum field theory. Instead, weighted averages of the energy-momentum tensor have been shown to be bounded from below (Quantum Energy Inequalities—QEIs).

Quantum Energy Inequalities and Phase Space

Interpretation of QEIs

Quantum energy inequalities restrict the violation of the classical energy conditions to a magnitude compatible with the uncertainty relations of quantum theory.

Example

For all physical states ω and suitable sampling functions f^{ab}

$$\int d\,vol(x) \langle T_{ab}(x) \rangle_\omega f^{ab}(x) \geqslant -\mathscr{Q}(f).$$

Connection with phase space properties?

Need for *lower bounds* in the phase space criteria to establish an equivalence of phase space criteria and quantum energy inequalities

Compact Operators

Definition (Total Boundedness, Precompactness)

A subset T of a metric space (M, d) is called *totally bounded (precompact)* if for every $\epsilon > 0$ there is a finite cover of T by sets of diameter $\leqslant \epsilon$.

Definition (Compact Operators)

Let $\mathfrak{E}$ and $\mathfrak{F}$ be normed vector spaces. An operator $\Theta : \mathfrak{E} \to \mathfrak{F}$ is called *compact* if it maps bounded subsets of $\mathfrak{E}$ onto totally bounded subsets of $\mathfrak{F}$.

Remark

This means that, given $\epsilon > 0$, a compact operator Θ is characterised by a number $N_\Theta(\epsilon) \in \mathbb{N}$, defined as the minimal number of ϵ-balls needed to cover the image of the unit ball, $\Theta(\mathfrak{E}_1)$.

The Compactness Criterion of Haag and Swieca

Compactness Criterion (Haag and Swieca 1965)

In a local quantum field theory the operators

$$\Theta_{\beta,\mathscr{O}} : \mathfrak{A}(\mathscr{O}) \to \mathscr{H} \quad A \mapsto \exp(-\beta H)A\Omega$$

mapping the local algebras $\mathfrak{A}(\mathscr{O})$ into the Hilbert space $\mathscr{H}$ ought to be compact for any $\mathscr{O}$ and any $\beta > 0$.

- Ω the vacuum;
- H the Hamiltonian, β an inverse temperature.

Remark

Restriction in phase space in two steps:

1. Localisation—local operator $A \in \mathfrak{A}(\mathscr{O})$ applied to Ω.
2. Energy damping—smooth energy cut-off via $\exp(-\beta H)$.

The Compactness Criterion of Fredenhagen and Hertel

Compactness Criterion (Fredenhagen and Hertel 1979)

In a local quantum field theory the operators

$$\Pi_{\beta,\mathscr{O}} : \mathscr{T}_\beta \to \mathfrak{A}(\mathscr{O})^* \quad \phi \mapsto \phi \restriction \mathfrak{A}(\mathscr{O})$$

mapping the normal linear functionals with bounded energy into the dual of the local algebras ought to be compact.

- $\mathscr{T}_\beta \doteq \{\phi \in \mathscr{T} : \exp(\beta H)\phi \exp(\beta H) \in \mathscr{T}\}$;
- Norm $\|\phi\|_\beta \doteq \|\exp(\beta H)\phi \exp(\beta H)\|$.

Remark

Restriction in phase space in two steps of opposite order:

1. Energy damping—considering only functionals in $\mathscr{T}_\beta$.
2. Localisation—restriction to the local algebras $\mathfrak{A}(\mathscr{O})$.

Nucliearity

Definition (Nuclearity)

Let $\mathfrak{E}$ and $\mathfrak{F}$ be normed vector spaces. An operator $\Theta : \mathfrak{E} \to \mathfrak{F}$ is called *nuclear* if there exist sequences $\{\varphi_k : k \in \mathbb{N}\} \subset \mathfrak{F}$ and $\{\ell_k : k \in \mathbb{N}\} \subset \mathfrak{E}^*$ such that

$$\Theta(x) = \sum_{k=1}^{\infty} \ell_k(x)\varphi_k, \quad x \in \mathfrak{E},$$

$$\text{and} \quad \sum_{k=1}^{\infty} \|\ell_k\|\|\varphi_k\| < \infty.$$

The nuclearity index of Θ is defined by

$$\|\Theta\|_1 \doteq \inf_{\substack{\text{nuclear} \\ \text{decompositions}}} \sum_{k=1}^{\infty} \|\ell_k\|\|\varphi_k\|.$$

Quantitative Description of Operators

Interpretation of Nuclearity

- A nuclear operator Θ maps the unit ball of its domain into a parallelepiped with summable edge lengths.
- The nuclearity index describes the minimal sum (best approximation) possible.
  Actually, $\|\Theta\|_1$ is half of this sum.

Formulating phase space criteria in terms of nuclearity thus yields *quantitative* information as opposed to the merely *qualitative* criteria considered up to this point.

The Nuclearity Criterion of Buchholz and Wichmann

Nuclearity Criterion (Buchholz and Wichmann 1986)

In a local quantum field theory the operators

$$\Theta_{\beta,\mathscr{O}} : \mathfrak{A}(\mathscr{O}) \to \mathscr{H} \quad A \mapsto \exp(-\beta H)A\Omega$$

mapping the local algebras $\mathfrak{A}(\mathscr{O})$ into the Hilbert space $\mathscr{H}$ ought to be nuclear for any $\mathscr{O}$ and any $\beta > 0$. The nuclearity index should satisfy

$$\|\Theta_{\beta,\mathscr{O}}\|_1 \leqslant \exp(cr^3\beta^{-n}).$$

- r the spatial radius of $\mathscr{O}$;
- c, n appropriate positive constants.

The Nuclearity Criterion of Buchholz and Porrmann

Nuclearity Criterion (Buchholz and Porrmann 1990)

1. In a local quantum field theory the operators

$$\Pi_{\beta,\mathscr{O}} : \mathscr{T}_\beta \to \mathfrak{A}(\mathscr{O})^* \quad \phi \mapsto \phi \restriction \mathfrak{A}(\mathscr{O})$$

   mapping the normal linear functionals with bounded energy into the dual of the local algebras ought to be nuclear for any bounded region $\mathscr{O}$ and any $\beta > 0$.

2. *Dual formulation*:
   In a local quantum field theory the operators

$$\Xi_{\beta,\mathscr{O}} : \mathfrak{A}(\mathscr{O}) \to \mathfrak{A} \quad A \mapsto \exp(-\beta H) A \exp(-\beta H)$$

   mapping the local algebras into the quasi-local algebra ought to be nuclear for any $\mathscr{O}$ and $\beta > 0$.

Relations between the Criteria

Lattice of Constraints

$$\begin{array}{ccc} \text{Buchholz/Porrmann} & \Longrightarrow & \text{Buchholz/Wichmann} \\ \Downarrow & & \Downarrow \\ \text{Fredenhagen/Hertel} & \Longrightarrow & \text{Haag/Swieca} \end{array}$$

No further relation can be added to this diagram!

Mappings of Type l^p

Definition (Approximation Numbers)

Let $\mathfrak{E}$ and $\mathfrak{F}$ be normed vector spaces. The k'th approximation number ($k \in \mathbb{N}_0$) for the continuous operator $\Theta : \mathfrak{E} \to \mathfrak{F}$ is defined as

$$\alpha_k(\Theta) \doteq \inf\{\|\Theta - \Theta_k\| : \Theta_k \text{ operator of rank } \leqslant k\}.$$

These approximation numbers can be used to define subspaces of operators.

Definition (Mappings of Type l^p)

A continuous operator $\Theta : \mathfrak{E} \to \mathfrak{F}$ is a mapping of type l^p, $0 < p < \infty$, if

$$\rho_p(\Theta)^p \doteq \sum_{k=0}^{\infty} \alpha_k(\Theta)^p$$

is a finite real number.

p-Nuclearity?

Mappings of type l^p have been used by Buchholz, D'Antoni, and Longo 1990 to further generalise phase space criteria, noting that, for $0 < p < 1$, these mappings are nuclear. Indeed,

Corollary

Each mapping $\Theta : \mathfrak{E} \rightarrow \mathfrak{F}$ of type l^p allows for a decomposition in terms of sequences $\left\{\varphi_k : k \in \mathbb{N}\right\} \subset \mathfrak{F}$ and $\left\{\ell_k : k \in \mathbb{N}\right\} \subset \mathfrak{E}^*$ such that

$$\Theta(x) = \sum_{k=1}^{\infty} \ell_k(x)\varphi_k, \quad x \in \mathfrak{E},$$

$$\text{and} \quad \sum_{k=1}^{\infty} \|\ell_k\|^p \|\varphi_k\|^p < \infty.$$

Note that *no* restriction on $0 < p < \infty$ is present here!

p-Nuclearity? (cont.)

Later authors have used this perception to define, without restriction on p,

Definition (p-Nuclearity)

A bounded operator $\Theta : \mathfrak{E} \to \mathfrak{F}$ is called *p-nuclear* if it allows for a decomposition in terms of sequences $\{\varphi_k : k \in \mathbb{N}\} \subset \mathfrak{F}$ and $\{\ell_k : k \in \mathbb{N}\} \subset \mathfrak{E}^*$ such that the p-nuclearity index is finite:

$$\|\Theta\|_p \doteq \inf_{\substack{p\text{-nuclear} \\ \text{decompositions}}} \left(\sum_{k=1}^{\infty} \|\ell_k\|^p \|\varphi_k\|^p \right)^{\frac{1}{p}} < \infty.$$

Is this definition truly meaningful?

The Problems with p-Nuclearity

- The definition of p-nuclearity seems to be a reasonable extension as

$$p\text{-nuclearity} \quad \Longrightarrow \quad p'\text{-nuclearity}$$

if $p \leqslant p'$.

But:

- There is a deviating mathematicians' definition of p-nuclearity for $p \geqslant 1$, only accidentally coinciding with the physicists' notion for $p = 1$.
- Even worse: For $p > 1$ the p-nuclearity index is always trivial!

The Dilution Argument

Triviality of the Nuclearity Index for $p > 1$

Assume that $\Theta : \mathfrak{E} \to \mathfrak{F}$ is p-nuclear, *i. e.*,

$$\Theta(x) = \sum_{k=1}^{\infty} \ell_k(x)\varphi_k, \quad \sum_{k=1}^{\infty} \|\ell_k\|^p \|\varphi_k\|^p < \infty.$$

Replace every term in this sum by m equal terms $\ell_k(x) \cdot m^{-1}\varphi_k$. This yields another decomposition of Θ with

$$\sum_{k=1}^{\infty} m\|\ell_k\|^p \|m^{-1}\varphi_k\|^p = m^{-(p-1)} \sum_{k=1}^{\infty} \|\ell_k\|^p \|\varphi_k\|^p.$$

This is an upper bound for the p'th power of the p-nuclearity index and, due to $p > 1$, can be made arbitrarily small by choosing large m. $\quad \Longrightarrow \quad \|\Theta\|_p = 0$.

Banach Spaces with Schauder Basis

The same argument can be used to show

Proposition

Every bounded operator Θ mapping the Banach space $\mathfrak{E}$ with Schauder basis into the normed space $\mathfrak{F}$ is p-nuclear (in the physicists' sense) for every $p > 1$.

Remark

A Schauder basis of a Banach space is a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that every element $x \in \mathfrak{E}$ has a decomposition $x = \sum_{k=1}^{\infty} \alpha_k x_k$ with a *unique* set of coefficients $\{\alpha_k\}_{k\in\mathbb{N}}$.

Possible Ways out of this Problem

Actual usage of p-nuclearity is made in the literature with additional structure:

In the case of Hilbert spaces, nuclear decompositions in terms of *orthogonal* sets of vectors are used!

Possible Solution

Consider nuclear decompositions of operators

$$\Theta(x) = \sum_{k=1}^{\infty} \ell_k(x)\varphi_k$$

in terms of sequences obeying further restrictions:

- Linear independence of the vectors φ_k.
- Decomposition inside the range of Θ: $\varphi_k \in \operatorname{Ran}\Theta$.

The Special Case of Hilbert Spaces

In the case of Hilbert spaces this approach is known to work:

Proposition

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert spaces and let $\Theta : \mathscr{H}_1 \to \mathscr{H}_2$ be a continuous operator. Then Θ is a 2-nuclear operator with only decompositions in terms of orthonormal bases allowed if and only if $\Theta\Theta^* : \mathscr{H}_2 \to \mathscr{H}_2$ belongs to the trace-class. The corresponding nuclearity index is given by

$$\|\Theta\|_2{}^2 = \mathrm{Tr}\,(\Theta\Theta^*).$$

ϵ-Content

Another approach to get the phase space under control is the ϵ-content, in view of the fact that relevant sets describing bounded phase space in QFT are *compact*.

Definition (ϵ-Content)

Let $\Theta : \mathfrak{E} \rightarrow \mathfrak{F}$ be a compact linear operator. Then, given $\epsilon > 0$, $\Theta(\mathfrak{E}_1)$ can be covered by a finite number of ϵ-balls, and their minimal number $N(\epsilon)$ is called the ϵ-content of Θ.

The idea then is, in relation to phase space properties, to establish a physically reasonable dependence on β and $\mathscr{O}$ of the

Growth Orders (Akashi 1990)

- Upper growth order: $D_{\beta,\mathscr{O}}(\Theta) \doteq \limsup_{\epsilon \searrow 0} \frac{\ln \ln N_{\beta,\mathscr{O}}(\epsilon)}{\ln \epsilon^{-1}}$.
- Lower growth order: $d_{\beta,\mathscr{O}}(\Theta) \doteq \liminf_{\epsilon \searrow 0} \frac{\ln \ln N_{\beta,\mathscr{O}}(\epsilon)}{\ln \epsilon^{-1}}$.

Local Quantum Field Theory

The Assumptions of Local Quantum Field Theory

- Isotonous net of local algebras: $\mathscr{O} \mapsto \mathfrak{A}(\mathscr{O}) \subseteq \mathscr{B}(\mathscr{H})$, quasilocal algebra: $\mathfrak{A} = \overline{\bigcup_{\mathscr{O}} \mathfrak{A}(\mathscr{O})}^{C^*}$.
- Poincaré covariance: $\mathsf{P}_+^\uparrow \ni g \mapsto \alpha_g \in \operatorname{Aut}\mathfrak{A}$.
- Local Commutativity: $\left[\mathfrak{A}(\mathscr{O}_1), \mathfrak{A}(\mathscr{O}_2)\right] = 0$ if $\mathscr{O}_1$ and $\mathscr{O}_2$ are spacelike separated.
- Causality: $\mathfrak{A}(\mathscr{O}_2) \subseteq \mathfrak{A}(\mathscr{O}_1)$ if $\mathscr{O}_2$ belongs to the causal shadow of $\mathscr{O}_1$.
- Spectrum condition: $\mathbb{R}^{s+1} \ni x \mapsto U(x) = \exp(i\, P^\mu x_\mu)$ with spectrum of P^μ in the forward lightcone.
- Physical states are normalized positive functionals on $\mathfrak{A}$: $\omega(A) = \operatorname{Tr}(\rho_\omega A)$, $\rho_\omega :$ density matrix.

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