α_s in e^+e^- annihilation

a) Lowest order. Quark-hadron duality

$$\Pi(q^2)$$

$$-i\int dx\, e^{iqx}\langle 0|T(j_\mu(x)j_\nu(0)|0\rangle = (q_\mu q_\nu - g_{\mu\nu}q^2)\,\Pi(q^2)$$

$$R(q^2) = \frac{\sigma(e^+e^-\to \text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)} = 12\pi\, \mathrm{Im}\,\Pi\left(\frac{q^2}{\mu^2}, \alpha_s(\mu)\right)$$

Everywhere $N_c \to \infty$ (no quark loops).

$$\Pi(Q^2) = \Pi^{(0)}(Q^2) + \alpha_s \Pi^{(1)}(Q^2) + \alpha_s^2\Pi^{(2)}(Q^2) + \ldots$$

$$\Pi^{(0)}(Q^2) = \frac{1}{12\pi^2}\sum_{n=0}^{\infty}\frac{C_n}{Q^2+M_n^2}$$

$$H^{(0)}\Psi_n = M_n\Psi_n\ ;$$

$$H^{(0)} = 2\sqrt{p^2+m^2} + \sigma r + \text{const.}$$ Dubin, Kaidalov + Yu.S

$$M_n^2 = 2\pi\sigma(2n_r + \ell) + M_0^2$$

Preparata, Nardulli et al $\quad C_n(\ell=0) = \frac{2}{3}Q_f^2 N_c m_0^2\,;\quad \boxed{m_0^2 \equiv 4\pi\sigma}$

$$C_n(\ell=2) = \frac{1}{3}Q_f^2 N_c m_0^2\,;$$

$$\Pi^{(0)}(Q^2) = \frac{1}{12\pi^2}\sum_{n=0}^{n_0-1}\frac{C_n}{M_n^2+Q^2} - \frac{Q_f^2 N_c}{12\pi^2}\,\psi\left(\frac{Q^2+M_0^2+n_0 m_0^2}{m_0^2}\right) +$$

$+$ divergent constant.

where $\psi(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$

$$M_n^2(f) \cong 4\pi\sigma n + \Delta \equiv m_0^2 n + \Delta \qquad n \equiv (n_r, L)$$

Δ is weakly depending on n

$$C_n^f(L=0) = \frac{2}{3} Q_f^2 N_c m_0^2$$

$$m_0^2 \equiv 4\pi\sigma$$

$$C_n^f(L=2) = \frac{1}{3} Q_f^2 N_c m_0^2$$

Introducing the radial quantum number n_r

$$M_n^2 = 2\pi\sigma(2n_r + L) + \Delta$$

$\left\{ \begin{matrix} n_r, L=0 \\ n_r-1, L=2 \end{matrix} \right\}$ degenerate (when dep. $\Delta(n,L)$ is neglected).

$$\boxed{\sum_{n=n_0}^{\infty} \frac{1}{M_n^2+Q^2} = -\frac{1}{m_0^2}\psi\left(\frac{Q^2+\Delta+n_0 m_0^2}{m_0^2}\right) + \text{diverg. const.}}$$

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)};$$

$$\boxed{\psi(z\to\infty) = \ln z - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k\, z^{2k}}}$$

B_n – Bernulli numbers, $B_2 = 1/6$

Finally:

$$\boxed{\Pi_{NP}^{f(0)}(Q^2) = \frac{1}{12\pi^2}\sum_{n=0}^{n_0-1} \frac{C_n^f}{M_n^2(f)+Q^2} - \frac{Q_f^2 N_c}{12\pi^2}\psi\left(\frac{Q^2+\Delta+n_0 m_0^2}{m_0^2}\right)}$$

$+$ diverging const.

Large Q^2 :

$$\Pi^{f\,(0)}_{NP}(Q^2) = -\frac{Q_f^2 N_c}{12\pi^2} \ln \frac{Q^2+\Delta+n_0 m_0^2}{\mu^2} + O\left(\frac{m_0^2}{Q^2+\Delta+n_0 m_0^2}\right)$$

$$\underbrace{R^{(0)}(s) = 12\pi \,\mathrm{Im}\, \Pi^{f(0)}_{NP}(-s) = N_c Q_f^2}_{\text{Partonic result!}}$$

This is Quark - hadron duality :

$$\int_{\Delta s} R^f(s)\,ds = \sum_{\Delta n = \frac{\Delta s}{m_0^2}} C_n^f = (N_c Q_f^2 m_0^2)\,\Delta n = (N_c Q_f^2)\cdot \Delta s$$

Or in closure - theorem terms

$$\sum_{n\in\Delta n} \langle p_1 | n\rangle \langle n | p_2\rangle \approx \sum_{k_1 k_2} \langle p_1 | k_1 k_2\rangle \langle k_1 k_2 | p_2\rangle$$

bound states

free quarks.

Therefore

$$\alpha_s^n \Pi^{(n)}_{NP}(Q^2) \xrightarrow[Q^2\to\infty]{} \alpha_s^n \Pi^{(n)}_{free}(Q^2) + O\left(\frac{m_0^2}{Q^2}\right)$$

and one can define the renorm. scheme as

$$\boxed{\Pi(Q^2) = \Pi^{(0)}_{NP}(Q^2) + \frac{\sum_f Q_f^2}{4\pi^2}\,\frac{\alpha_s^{Ren.}(Q^2)}{\pi} + O\left(m_0^2/Q^2\right)}$$

to one loop one has for $Q^2 \to \infty$

$$\alpha_s^R(Q^2) = \alpha_s^{(0)}(\mu) - \frac{b_0}{4\pi}\left(\alpha_s^{(0)}(\mu)\right)^2 \ln\frac{Q^2}{\mu^2} + \ldots$$

Asymptotics:

$$\psi(z) = \ln z - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k\, z^{2k}}$$

B_{2k} - Bernoulli numbers

$$\Pi^{(0)}(Q^2) = -\frac{Q_f^2 N_c}{12\pi^2} \ln \frac{Q^2+M_0^2}{\mu^2} + O\left(\frac{m_0^2}{Q^2}\right)$$

$$\boxed{R(q^2) = 12\pi\, \mathrm{Im}\, \Pi^{(0)}(-s) = N_c Q_f^2}$$

This is result as for the free quark loop!

Thus is justified the quark-hadron duality.

New and old OPE expansion.

The old OPE (Shifman, Vainshtein, Zakharov)

$$\Pi(Q^2) = -\frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\ln\frac{Q^2}{\mu^2} + \frac{6m^2}{Q^2} + \frac{2m\langle\bar{q}q\rangle}{Q^4} + \frac{\alpha_s\langle FF\rangle}{12\pi Q^4} + \dots$$

$m = m_q \to 0$

The new OPE from expansion of $\psi\left(\frac{Q^2+M_0^2}{m_0^2}\right)$

$$\Pi(Q^2) = -\frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\ln\frac{Q^2}{\mu^2} + \sum_{n=1}^{\infty}\frac{(m_0^2)^n}{(Q^2)^n} b_n$$

where $m_0^2 = 4\pi\sigma$, $b_n \sim a_n n!$, $a_n \sim n^\alpha$

from Bernoulli coef. B_{2n}

Hence

old OPE	new OPE
$\frac{(\Lambda_{QCD})^n}{(Q^2)^n} n!$	$\frac{(4\pi\sigma)^n}{(Q^2)^n} n!$

But $m_0^2 \sim 30\,\Lambda_{QCD}^2$!

Renormalization of charge and gluon self-energy part

In light-like gauge (no ghosts)
$n_\mu A_\mu = 0$, $n^2 = 0$

$$\Pi^{\mu\nu}_{ab}(Q) = \delta_{ab} C_2(\mathrm{adj}) \frac{g^2}{32\pi^2} \frac{22}{3} \left(Q^2 \delta_{\mu\nu} - Q_\mu Q_\nu \right) \cdot$$

$$\times \left\{ N_\varepsilon - \log \frac{Q^2}{\nu_0^2} + \text{const terms} \right\}$$

$N_\varepsilon = \frac{2}{\varepsilon} - \gamma_E + \ln 4\pi$

When gluons obey mass terms

$$\log \frac{Q^2}{\nu_0^2} \rightarrow \int dx \, \varrho(x) \log \frac{m^2 + x(1-x)Q^2}{\nu_0^2}$$

singularity at $Q^2 = -4m^2$

More careful renormalization for $\Pi^{(0)}$ to be used later

$$\Pi^{(0)}(Q^2) \sim \sum_{n=0}^{\infty} \frac{m_0^2}{Q^2+M_0^2+m_0^2 n} = \sum_{n=0}^{N-1} \frac{m_0^2}{Q^2+M_0^2+m_0^2 n} +$$

$$+ \sum_{N}^{\infty} \frac{m_0^2}{Q^2+M_0^2+m_0^2 n} = \sum_{n=0}^{N-1} + \int_{m_0^2 N}^{\infty} \frac{dq^2}{Q^2+M_0^2+q^2}$$

$$q^2 = m_0^2 n$$

$$dq^2 = \frac{1}{\pi} d^2 q \to \frac{1}{\pi}\int d^{D-2} q\, (\nu_0^{\varepsilon}) \quad , \quad D = 4-\varepsilon$$

dimensional regularization.

$$\Pi^{(0)}_{reg}(Q^2) = -\,\psi\left(\frac{Q^2+M_0^2+m_0^2}{m_0^2}\right) + \ln\frac{\nu_0}{m_0^2}$$

this is equivalent to $\overline{MS}$

g_0^4 g_0^2

In free pert. theory one obtains in $\overline{MS}$

$$\alpha_s(Q^2) = \alpha_s(\mu)\left(1 - \frac{b_0}{4\pi}\alpha_s(\mu)\ln\frac{Q^2}{\mu^2}\right)$$

In backgr. pert. theory one has to replace $\ln\frac{Q^2}{\mu^2} \to \psi\left(\frac{Q^2+M_0^2+m_0^2}{m_0^2}\right) - \ln\frac{\mu^2}{m_0^2}$

as one will see in a moment. First discuss absence of Landau pole.

Discussion and conclusions

1. No Landau poles in α_s^B: in all Euclidean region $\alpha_s(Q^2)$ is finite and reasonable. (see Fig)

2. No IR renormalons (see Simonov hep-ph 9311247 and backgr. pert series is Borel summable (may be)

3. Is this α_s freezing new?

No. in QED one has for $\Pi_{\mu\nu}(Q^2)$

$$\ln\frac{\tilde{\mu}^2}{\Lambda^2} \to \ln\frac{m^2+\frac{1}{4}Q^2}{\mu_0^2}$$

in Weinberg - Salam Π_{WW}

$$\Pi_{WW}(Q^2) \sim \int dx \ln\frac{x(1-x)Q^2+m_W^2}{m_W^2}$$

This leads to branch points in α_s for $Q^2<0$.

4. Minkowskian region

$$\alpha_s^B = \frac{4\pi}{b_0 \ln\frac{Q^2+M^2}{\Lambda^2}} \to \frac{4\pi}{b_0\left[\psi\left(\frac{Q^2+M^2}{m_0^2}\right)+\ln\frac{m_0^2}{\Lambda^2}\right]}$$

No branch points in $(-Q^2)$, only poles!
No gluon mass — has no physical meaning.

For $N_c \to \infty$ more exact formula for α_s^{ren}.

$$\alpha_s^{ren}(q) = \frac{4\pi}{b_0\left[\psi\left(\frac{q^2+M_0^2}{m^2}\right) + \ln\frac{m^2}{\mu^2}\right]}$$

In Euclidean region, $q^2>0$, when $\frac{q^2+M_0^2}{m^2} \gg 1$

one has $\alpha_s^{ren}(q) = \dfrac{4\pi}{b_0 \ln\frac{q^2+M_0^2}{\mu^2}}$

In Minkowskian region, $q^2<0$

$\psi\left(\frac{q^2+M_0^2}{m^2}\right)$ acquires poles at $\frac{q^2+M_0^2}{m^2} = 0, -1, -2, \ldots$

This corresponds to $N_c \to \infty$, only poles and summing

at $q^2<0$ and $|q^2| \to \infty$

$$-\frac{1}{m^2}\psi = \sum_n \frac{1}{M_n^2+q^2} \to \sum \frac{1}{M_n^2 - 2iM_n\Gamma_n + q^2}, \quad \Gamma_n \sim const$$

$$\Gamma_n = O(1/N_c)$$

$$\ln\frac{q^2+M_0^2}{\mu^2} \to \ln\left|\frac{q^2+M_0^2}{\mu^2}\right| - i\pi$$

Perturbation theory as a string orchestra

$e_+ e_-$ $\quad R(s) = \frac{11}{3}\left(1 + \frac{\alpha_R(s)}{\pi}\right)$

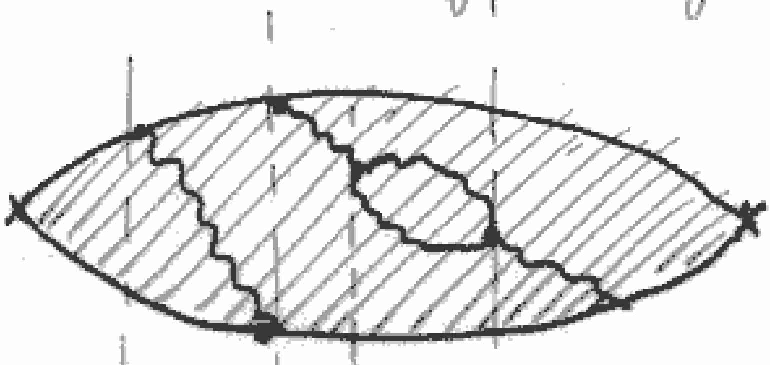
$$\alpha_R(s) = \alpha_0 + c_1 \alpha_0^2 + \dots$$

Usual def. without backgr.

With background:

a typical graph

$\Pi(Q^2)$



$$\sum \frac{c_n}{Q^2 + M_n^2}$$

$$\sum \frac{c'_{n\nu}}{Q^2 + M_{n\nu}^2}$$

$$\sum \frac{c_{n\nu_1\nu_2}}{Q^2 + M^2_{n\nu_1\nu_2}}$$

$c_n \sim \alpha_s^0 \qquad c'_{n\nu} \sim \alpha_s + \dots \qquad c_{n\nu_1\nu_2} \sim \alpha_s^2$

$$M_n^2 = 4\pi\sigma \cdot n + \Delta$$

$$M_{n\nu}^2 = 4\pi\sigma(n + \nu) + \Delta'$$

$$M^2_{n\nu_1\nu_2} = 4\pi\sigma(n + \nu_1 + \nu_2) + \Delta''$$

From OCS equation one can specify

$$\tilde{\mu}^2 = Q^2 + M_B^2$$

Hence to two-loop one obtains

$$\boxed{\alpha_s^B(Q) = \frac{4\pi}{b_0 \ln \frac{Q^2+M_B^2}{\Lambda_B^2}} \left(1 - \frac{b_1}{b_0^2} \frac{\ln\ln \frac{Q^2+M_B^2}{\Lambda_B^2}}{\ln \frac{Q^2+M_B^2}{\Lambda_B^2}}\right)}$$

For coordinate space (static potential) one similarly has $Q \to 1/R$

$$\boxed{\alpha_s^B(R) = \frac{4\pi}{b_0 \ln \frac{1/R^2+\tilde{M}^2}{\Lambda_R^2}} \left(1 - \frac{b_1}{b_0^2} \frac{\ln\ln \frac{1/R^2+\tilde{M}^2}{\Lambda_R^2}}{\ln \frac{1/R^2+\tilde{M}^2}{\Lambda_R^2}}\right)}$$

$\tilde{M} \neq M_B$

$\Lambda_R \neq \Lambda_B$

will be defined in next sections.

M_B^2 or $\tilde{M}^2$ is the new element which makes α_s^B finite for all $Q^2 \geq 0$.

α_s freezing

Conclusions

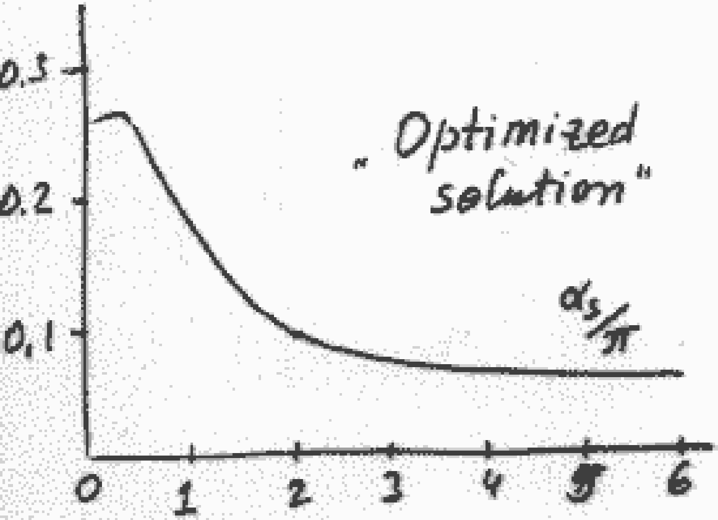
This freezing of α_s is welcome phenomenologically

Was introduced in '80–'82 Cornwall; Parisi & Petronzio

$$\frac{\alpha_s(Q^2)}{\pi} = \frac{12}{27}\,\frac{1}{\ln\left[(Q^2+4m_g^2)/\Lambda_0^2\right]}\,; \quad m_g \text{ – „gluon mass"}$$

Gluon mass violates gauge invariance $\quad m_g \sim 0.5$ GeV

$e^+e^- \to$ hadrons



for $R_{e^+e^-}$

Mattingly, Stevenson PR D49 (94)

$$\left(\frac{\alpha_s}{\pi}\right)_{max} = 0.26$$

Q (GeV)

Hadron spectroscopy (Isgur et al)

„hard-freeze" form $\quad \alpha_s(Q^2) \to \alpha_s(r) = \dfrac{4\pi}{b_0 \ln \frac{1}{r^2\Lambda_0^2}}\,, \; r < r_0$

$$0.4 \div 0.5 \approx \alpha_s(max)\,, \; r \geq r_0$$

Inclusive heavy quark production spectra

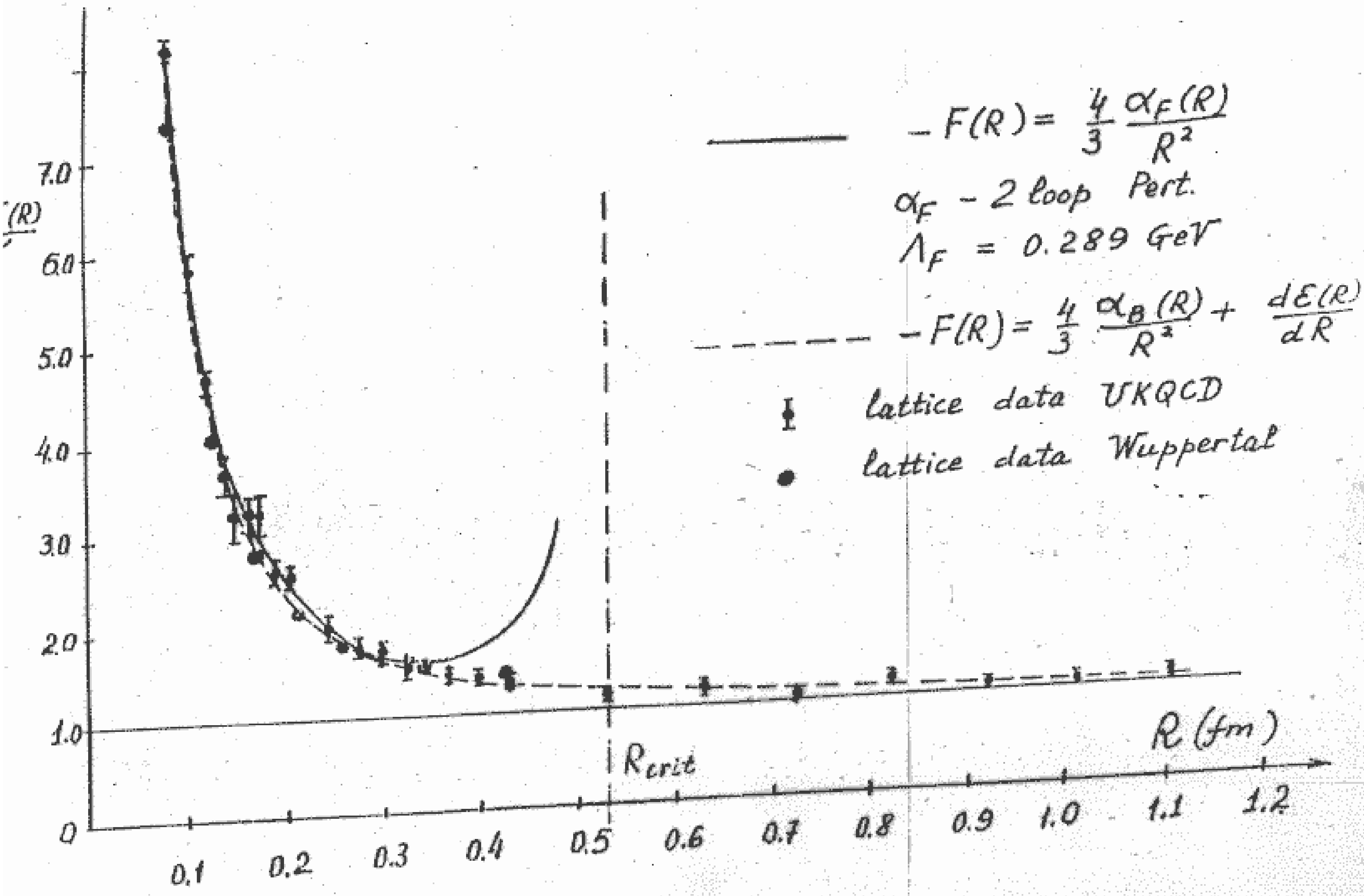
$e^+e^- \to Q(x) + \bar{Q} + \text{light part.}$ $\quad$ Dokshitzer, Khose, Troyan

$$\int_0^{1\,\mathrm{GeV}} dk \, \frac{\alpha_s(k^2)}{\pi} \approx 0.2 \text{ GeV} \quad \text{from exp. } \langle x_{c,B}(W)\rangle$$

corresponds to Matt.-Stev.

• „Lipatov pomeron": $\quad j(t) - 1 \sim \alpha_s(t)$

needed $\alpha_s(0)$ finite and not large.



(R)
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
R (fm)
R_{crit}
$-F(R) = \frac{4}{3}\frac{\alpha_F(R)}{R^2}$
α_F - 2 loop Pert.
Λ_F = 0.289 GeV
$-F(R) = \frac{4}{3}\frac{\alpha_B(R)}{R^2} + \frac{d\mathcal{E}(R)}{dR}$
lattice data UKQCD
lattice data Wuppertal

Fig 26