Seminar: QCD Lagrangian → effective quark interaction → NJL model → bosonization → → color superconductivity

$$L_{QCD} = \bar{q}(i\gamma_\mu D^\mu - m_q)q - \frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu}$$

$$D_\mu = \partial_\mu - igt^a A^a_\mu, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu$$

$N_c = 3$, $a = 1 \sim 8$, $N_f = 2$ at the present seminar

L_{QCD} is invariant under global chiral symmetry (in the $m_q \to 0$ limit)

$$q_L = \frac{1-\gamma_5}{2}q, \quad q_R = \frac{1+\gamma_5}{2}q, \quad q = q_L + q_R$$

$$\bar{q}(i\gamma_\mu D^\mu)q = \bar{q}_R(i\gamma_\mu D^\mu)q_R + \bar{q}_L(i\gamma_\mu D^\mu)q_L$$

Chiral transformation:

$$q_L \to e^{-i\theta_L \vec{\tau}/2} q_L, \quad q_R \to e^{-i\theta_R \vec{\tau}/2} q_R \qquad SU_L(2) \otimes SU_R(2)$$

Effective Theories of QCD

Simonov's lectures

$$Z=\int \mathcal{D}A\,\mathcal{D}\bar{q}\,\mathcal{D}q\,\exp(-S),\quad S=\int d^4x\,L_{QCD},$$

$$Z\to\int \mathcal{D}\bar{q}\,\mathcal{D}q\,\exp\left(-\int d^4x\,L_0-S_{eff}\right)$$

$$L_0=\bar{q}\,(i\gamma_\mu\partial_\mu-m)\,q$$

$$S_{eff}=\sum_{n=2}^{\infty}\frac{1}{n!}\langle\langle\theta^n\rangle\rangle,\quad \theta=\int d^4x\,\bar{q}\,g\gamma_\mu A_\mu^a t^a q.$$

Still too complicated. Steepest descent:

(i) QCD as it is

$\downarrow$

(ii) —"— for pedestrians

$\downarrow$

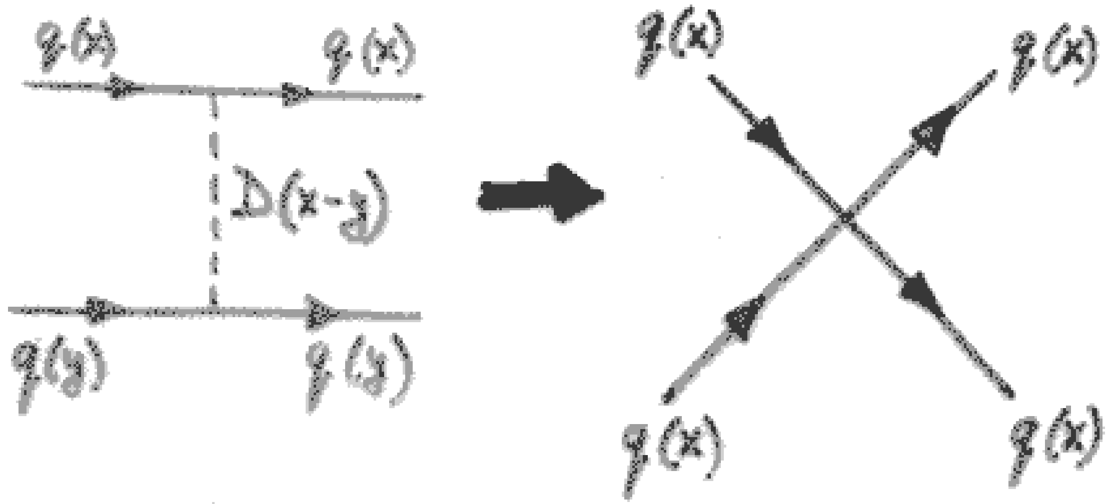
(iii) —"— for plumbers $\leftarrow$ **our destination**

Stage (ii) – one-gluon exchange motivated effective action

$$S_{eff} = \int d^4x\, d^4y \left(\bar{q}(x)\gamma_\mu t^a q(x)\right)\left(\bar{q}(y)\gamma_\mu t^a q(y)\right) D(x-y)$$

Next to stage (iii)

$$D_{\mu\nu}(x-y) = g^2\, \delta(x-y)\, g_{\mu\nu} \ , \quad [g^2] = 1/m^2$$



$$L_{eff} = g^2 \sum_{a=1}^{8} \left(\bar{q}\, t^a \gamma_\mu q\right)\left(\bar{q}\, t^a \gamma_\mu q\right)$$

$$t^a = \lambda^a/2$$

This is a prototype of the standard NJL (Nambu-Jona-Lasinio)

$$L_{eff} = g^2 (\bar{q} t^a \gamma_\mu q)(\bar{q} t^a \gamma_\mu q)$$

What has been preserved and what has been lost on our route from QCD?

Chiral symmetry is preserved (broken spontaneously)

Confinement is missing !!!

Reduction of L_{eff} to the standard NJL

$$L_{NJL} = \bar{q} i \gamma_\mu \partial^\mu q + g^2 [(\bar{q}q)^2 + (\bar{q} i \gamma_5 \vec{\tau} q)^2]$$

Two receiptes both based on Fierz transformation in Dirac, flavor and color spaces

Standard prescription:

$$(\bar{q}\lambda^a\gamma_\mu q)(\bar{q}\lambda^a\gamma_\mu q) \xrightarrow{\text{Fierz}} \mathcal{L}_{NJL} +$$

$$+ \text{(color octet terms)}$$

e.g. $(\bar{q}\vec{\tau}\lambda^a q)(\bar{q}\vec{\tau}\lambda^a q)$

Exercise: do this Fierz transform

Since spring 98' after (re)discovery by Wilczek, Shuryak et al. of color superconductivity another Fierz transform is in fashion

$$(\bar{q}\lambda^a\gamma_\mu q)(\bar{q}\lambda^a\gamma_\mu q) \Longrightarrow (\bar{q}\Gamma q)(\bar{q}\Gamma q) +$$

$$+ (\bar{q}\Lambda q^c)(\bar{q}^c\Lambda q)$$

diquarks

$$q^c = C\bar{q}^T = \gamma_2\gamma_4\bar{q}^T$$

$$\Gamma_{Dirac} = \left\{1,\ i\gamma_5,\ \frac{i}{\sqrt{2}}\gamma_\mu,\ \frac{i}{\sqrt{2}}\gamma_\mu\gamma_5\right\}$$

Exercise: show that

$$\left(\bar{q}^{\beta}\,\varepsilon_{\alpha\beta\gamma}\,\gamma_5\,\tau^{(2)}(q^{\gamma})^c\right) \text{ is } \begin{cases} {}^{2S+1}L_J = {}^1S_0 \\ P = +1 \\ \text{Flavor} = \bar{3} \end{cases}$$

Here a hint to the analogy with BCS comes

$$Z = \mathrm{Tr}\, e^{-\beta H} =$$

$$= \int \mathcal{D}q\,\mathcal{D}q^{+} \exp\left\{-\int_0^{\beta} d^3x\, d\tau \left[\sum_{\alpha=1}^{2} q_{\alpha}^{+}\left(\partial_{\tau} - \frac{\nabla^2}{2m} - \mu\right) q_{\alpha} - \right.\right.$$

$$\left.\left. - g\,\underbrace{(q_1^{+} q_2^{+})(q_2 q_1)}_{\text{diquarks}}\right]\right\}$$

Obvious simularity

Now back to L_{NJL}. Bosonization trick.

A way to deal with $(\bar{q}q)^{n\geq 2}$

$$\int \mathcal{D}q\,\mathcal{D}\bar{q}\; e^{\bar{q}Aq} = \det A$$

$$\int \mathcal{D}q\,\mathcal{D}\bar{q}\; e^{(\bar{q}Aq)^n} = ?$$

Hubbard-Stratonovich transformation Example:

$$\int \mathcal{D}q\,\mathcal{D}\bar{q} \exp\left\{\int \bar{q}(i\not{\partial})q - g^2(\bar{q}q)^2\right\} =$$

$$= \int \mathcal{D}q\,\mathcal{D}\bar{q}\,\underline{\underline{\mathcal{D}M}}\; \delta(M - g(\bar{q}q)) \exp\{\dots\} =$$

$$= \int \mathcal{D}q\,\mathcal{D}\bar{q}\,\underline{\mathcal{D}M}\,\underline{\underline{\mathcal{D}L}} \exp\left\{\int \bar{q}(i\not{\partial})q - M^2 - iL(M - g(\bar{q}q))\right\} =$$

$$= \underbrace{\int \mathcal{D}M \exp(-M^2 - iLM)}_{\exp(-L^2/4)} \int \mathcal{D}q\,\mathcal{D}\bar{q} \exp\left\{\int \bar{q}(i\not{\partial})q + g\,\bar{q}iLq\right\} =$$

$$= \int \mathcal{D}L \exp\left\{-\frac{L^2}{4} + \mathrm{tr}\ln(i\not{\partial} + igL)\right\}$$

Bosonization of the standard NJL

$$Z = \int \mathcal{D}\bar{q}\, \mathcal{D}q \exp\left\{ \int \bar{q}(i\gamma\partial)q - g\left[(\bar{q}q)^2 + (\bar{q}i\gamma_5\vec{\tau}q)^2\right] \right\} =$$

$$= \int \mathcal{D}\sigma\, \mathcal{D}\vec{\pi} \exp\left\{ \int d^4x \left[\mathrm{tr}\ln\left(i\gamma\partial - (\sigma + i\vec{\tau}\vec{\pi}\gamma^5\right) - \right.\right.$$

$$\left.\left. - \frac{1}{4g}(\sigma^2 + \pi^2)\right] \right\}$$

CSB manifests itself in nonzero v.e.v. of σ

$$\sigma = \langle \sigma \rangle , \quad \vec{\pi} = 0$$

$$\frac{\delta S}{\delta \sigma} = 0 \Rightarrow \sigma + g\,\mathrm{tr}\frac{\sigma}{\partial^2 + g^2\sigma^2} = 0$$

Gap equation

Reminder of BCS again

Here a link to chiral Lagrangians from Ioffe's lectures

$$\mathrm{tr}\,\ln\left(i\gamma\partial - (\sigma + i\vec{\tau}\vec{\pi}\gamma_5)\right) = \mathrm{tr}\sum_{n=1}^{\infty}\frac{1}{n}\left[(i\gamma\partial)^{-1}(\sigma + i\vec{\tau}\vec{\pi}\gamma_5)\right]^n$$

$$L_{eff}(\sigma,\pi) = \frac{1}{2}\left[(\partial_\mu\sigma)^2 + (\partial_\mu\vec{\pi})^2\right] - \frac{1}{2}m_\pi^2\pi^2 - \frac{1}{2}m_\sigma^2\sigma^2 -$$

$$- \frac{2m^2}{f_\pi}\sigma(\sigma^2 + \pi^2) - \frac{m^2}{2f_\pi^2}\pi(\sigma^2+\pi^2)^2 .$$

Color superconductivity

Only a sketch

$$L_{eff} = g^2 \sum_{a=1}^{8} (\bar{q} t^a \gamma_\mu q)(\bar{q} t^a \gamma_\mu q)$$

$$Z = \int \mathcal{D}\varphi \, \mathcal{D}\Delta \, \mathcal{D}\Delta^+ \exp\left\{ \int d^4x \left[\frac{\varphi^2}{4g_1^2} + \frac{\Delta\Delta^+}{g_\Delta^2} \right] - \right.$$

$$- \operatorname{tr} \ln \begin{pmatrix} i\hat{\partial} + i\varphi - i\gamma_4 \mu & 2\Gamma C \Delta \\ 2\Delta^+ C^{-1} \Gamma^+ & i\hat{\partial}^T - i\varphi + i\gamma_4 \mu \end{pmatrix}$$

$\varphi \sim \bar{q}q$, $\Delta \sim qq$ $\Gamma \sim \varepsilon_{\alpha\beta\gamma} \gamma_5 \tau^{(2)}$

μ is the chemical potential

Interplay of the two condensates:

(i) chiral condensate φ

(ii) diquark condensate Δ

Phase Diagram