QCD Lagrangian

$$L = i \sum_q{}' \bar{\psi}_q^a (\nabla_\mu \gamma_\mu + i m_q) \psi_q^a - \frac{1}{4} G_{\mu\nu}^n G_{\mu\nu}^n \quad (1) \quad q = u, d, s, \ldots$$

$$\nabla_\mu = \partial_\mu + i g \frac{\lambda^n}{2} A_\mu \qquad a = 1, 2, 3$$

$$G_{\mu\nu}^n = \partial_\mu A_\nu^n - \partial_\nu A_\mu^n - g f^{nml} A_\mu^m A_\nu^n \qquad n, m, l = 1, \ldots 8$$

Asymptotic freedom

$$\alpha_s(Q^2) \equiv \frac{g^2}{4\pi} = \frac{4\pi}{b \ln \frac{Q^2}{\Lambda^2}} + \text{higher order terms} \quad (2)$$

$$b = 11 - \frac{2}{3} n_f \qquad n_f = 3 \rightarrow b = 9$$

Effective one-loop $\Lambda_{eff} \approx 200$ MeV

$$\alpha_s(1\,GeV^2) \approx 0.44 \qquad \frac{\alpha_s}{\pi} \approx 0.14$$

Light quark masses.

Light quarks – u, d, s

$$m_q(M) = Z_q(M/\mu)\, m_q(\mu) \qquad q = u, d, s$$

For small masses $Z_q(M/\mu)$ is the same for $q = u, d, s$

$$\frac{m_{q_1}(M)}{m_{q_2}(M)} = \frac{m_{q_1}(\mu)}{m_{q_2}(\mu)} \quad \text{– scale indep.}$$

Axial currents

$$j^-_{\mu 5} = \bar{d}\gamma_\mu \gamma_5 u$$

$$j^3_{\mu 5} = [\bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d]/\sqrt{2}$$

$$j^{s-}_{\mu 5} = \bar{s}\gamma_\mu\gamma_5 u \qquad j^{0}_{\mu 5} = \bar{s}\gamma_\mu\gamma_5 d$$

Matrix elements between vacuum and π or K

$$\langle 0 | j^-_{\mu 5} | \pi^+ \rangle = i f_{\pi^+} p_\mu$$

$$\langle 0 | j^3_{\mu 5} | \pi^0 \rangle = i f_{\pi^0} p_\mu \qquad (3)$$

$$\langle 0 | j^{s-}_{\mu 5} | K^+ \rangle = i f_{K^+} p_\mu$$

$$\langle 0 | j^{s0}_{\mu 5} | K^0 \rangle = i f_{K^0} p_\mu$$

In $SU(3)$ limit $f_{\pi^+} = f_{\pi^0} = f_{K^+} = f_{K^0}$. By isotopical symmetry $f_{\pi^+} = f_{\pi^0} \equiv f_\pi$, $f_{K^+} = f_{K^0}$.

$f_K / f_\pi = 1.22$ $\qquad f_\pi = 131$ MeV. – pion decay const.

From QCD Lagrangian

$$\partial_\mu [\bar{q}_1(x)\gamma_\mu\gamma_5 q_2(x)] = i(m_{q_1} + m_{q_2})\bar{q}_1(x)\gamma_5 q_2(x)$$

Multiply (3) by p_μ

$$i(m_u + m_d)\langle 0 | \bar{d}\gamma_5 u | \pi^+ \rangle = f_{\pi^+} m^2_{\pi^+}$$

$$(i/\sqrt{2})[(m_u + m_d)\langle 0 | \bar{u}\gamma_5 u - \bar{d}\gamma_5 d | \pi^0 \rangle + (m_u - m_d)\langle 0 | \bar{u}\gamma_5 u + \bar{d}\gamma_5 d | \pi^0 \rangle]$$
$$= f_{\pi^0} m^2_{\pi^0} \qquad (4)$$

$$i(m_s + m_u)\langle 0 | \bar{s}\gamma_5 u | K^+ \rangle = f_{K^+} m^2_{K^+}; \quad i(m_s + m_d)\langle 0 | s\gamma_5 d | K^0 \rangle =$$
$$= f_{K^0} m^2_{K^0}$$

From isospin invariance (E.M. interaction neglected) L3

$$\langle 0|\bar{u}\gamma_5 u + \bar{d}\gamma_5 d|\pi^0\rangle = 0.$$

$$\langle 0|\bar{d}\gamma_5 u|\pi^+\rangle = \frac{1}{\sqrt{2}}\langle 0|\bar{u}\gamma_5 u - \bar{d}\gamma_5 d|\pi^0\rangle$$

$$\rightarrow \qquad \underline{m_{\pi^+}^2 = m_{\pi^0}^2}$$

$\rightarrow \Delta m_\pi = m_{\pi^+} - m_{\pi^0} = 4.6$ MeV is caused by E.M. interaction

But: $\Delta m_K = m_{K^+} - m_{K^0} = -4.0$ MeV ! $\rightarrow m_d > m_u$

Assuming SU(3) invariance of matrix elements in (4)

$$\frac{m_u}{m_d} = \frac{\bar{m}_\pi^2 - (\bar{m}_{K^0}^2 - \bar{m}_{K^+}^2)}{\bar{m}_\pi^2 + (\bar{m}_{K^0}^2 - \bar{m}_{K^+}^2)}$$

$$\frac{m_s}{m_d} = \frac{\bar{m}_{K^0}^2 + \bar{m}_{K^+}^2 - \bar{m}_\pi^2}{\bar{m}_{K^0}^2 - \bar{m}_{K^+}^2 + \bar{m}_\pi^2}$$

$\bar{m}$ are the masses with switched of E.M. interaction. For E.M. correction is SU(3) limit (Dashen formula)

$$(m_{\pi^+}^2 - m_{\pi^0}^2)_{el} = (m_{K^+}^2 - m_{K^0}^2)_{el}$$

$$\underline{\frac{m_u}{m_d}} = \frac{2m_{\pi^0}^2 - m_{\pi^+}^2 - (m_{K^0}^2 - m_{K^+}^2)}{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2} = \underline{0.56}$$

$$\underline{\frac{m_s}{m_d}} = \frac{m_{K^0}^2 + m_{K^+}^2 - m_{\pi^0}^2}{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2} = \underline{20.1}$$

An estimation (from mass differences in baryon octet) $\underline{m_s(1\,GeV) \approx 150\ MeV}$

$\underline{m_u = 4.2\ MeV}$ $\quad$ $\underline{m_d = 7.5\ MeV.}$

Chiral symmetry and its spontaneous violation.

If u, d, s quark masses are neglected, then not only vector, but also axial currents are conserved.

Parameters: $(m_u+m_d)/M \sim \underline{0.01} \rightarrow \underline{SU(2)_L \times SU(2)_R\, U(1)}$

$M \sim 1$ GeV $\qquad m_s/M \sim \underline{1/5} \rightarrow \underline{SU(3)_L \times SU(3)_R\, U(1)}$

The singlet axial current $\sum_q' \bar{q}\gamma_\mu\gamma_5 q$ is not conserved because of anomaly.

But: baryons are massive!

$\Rightarrow$ Chiral symmetry is strongly violated at low energies.

The properties of the ground state - the vacuum are modified.

$$i q_\mu (m_u+m_d)\int d^4x\, e^{iqx}\langle 0|T\{j_{\mu 5}^-(x),\, \bar{u}(0)\gamma_5 d(0)\}|0\rangle \underset{\substack{m_u=0\\ m_d=0}}{=}$$

$$= -(m_u+m_d)\int d^4x\, e^{iqx}\langle 0|\delta(x_0)[j_{05}^-(x),\, \bar{u}(0)\gamma_5 d(0)]|0\rangle =$$

$$= (m_u+m_d)\langle 0|\bar{u}(0)u(0)+\bar{d}(0)d(0)|0\rangle$$

limit $q_\mu \to 0$, sum over intermediate states:

$$q_\mu \langle 0|j_{\mu 5}^-|\pi^+\rangle \frac{-1}{q^2}\langle \pi^+|(m_u+m_d)\bar{u}\gamma_5 d|0\rangle = -f_\pi^2 m_\pi^2$$

$q = u, d$

$$\boxed{\langle 0|\bar{q}q|0\rangle = -\frac{1}{2}\frac{f_\pi^2 m_\pi^2}{m_u+m_d}} \quad (5)$$

Gell-Mann Oakes Renner

$$\langle 0|\bar{q}q|0\rangle = -(240\,\mathrm{MeV})^3 \text{ at } 1\,\mathrm{GeV}$$

$$\langle 0|\bar{q}q|0\rangle = \langle 0|\bar{q}_L q_R + \bar{q}_R q_L|0\rangle$$ – quark condensate

$$q_L = \frac{1}{2}(1-\gamma_5)q, \quad q_R = \frac{1}{2}(1+\gamma_5)q$$

Chiral symmetry is spontaneously violated in QCD vacuum.

The special role of quark condensate: the violating chiral symmetry vacuum expectation value (v.e.v.) of lowest dimension, $d=3$.

Goldstone theorem, Goldstone bosons

Goldstone theorem in QCD:

The spontaneous violation of chiral symmetry results to appearance of massless pseudoscal bosons in the hadronic spectrum (triplet in $SU(2)_L \times SU(2)_R$ symmetry, octet in $SU(3)_L \times SU(3)_R$ symmetry).

$$\langle p|j^{+}_{\mu 5}|n\rangle = \bar{v}_p(p')\left[\gamma_\mu\gamma_5 F_1(q^2) + q_\mu\gamma_5 F_2(q^2)\right] v_n(p) \quad (6)$$

$$q = p' - p$$

Multiply (6) by q_μ and go to the limit $q^2 \to 0$ (but $q_\mu \neq 0$). Lhs $=0$, $F_1(0) = g_A = 1.26$

$$\left[2m g_A + q^2 F_2(q^2)\right] \bar{v}_p(p')\gamma_5 v_n(p) = 0$$

$$F_2(q^2)_{q^2\to 0} = -\frac{2m g_A}{q^2}$$

– the pole corresponds to massless particle – the pion.

$$\langle p | j^+_{\mu 5} | n \rangle = g_A \bar{v}_p(p') \left[\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right] \gamma_5 v_n(p)$$

- the axial current is conserved.

$$L_{\pi NN} = i g_{\pi NN} \bar{v}_N \gamma_5 \tau^a v_N \varphi^a \quad a=1,2,3$$

- low energy πN Lagrangian,

$$g^2_{\pi NN}/4\pi \approx 14$$

The second term in (6)

$$-\sqrt{2}\, g_{\pi NN} f_\pi \bar{v}_p \gamma_5 v_n \frac{q_\mu}{q^2}$$

valid with 5% accuracy

$$\boxed{g_{\pi NN} f_\pi = \sqrt{2}\, m g_A}$$

Goldberger-Treiman relation

General proof of Goldstone theorem

Lie group G, generators Q_j, $j=1,\dots n$

$$[Q_j, H] = 0$$

Spontaneously broken theory: ground state is not invariant under the action of the subset Q_m, $1 \le m \le n$

$$Q_m |0\rangle \neq 0$$

Denote
$$|B_m\rangle = Q_m |0\rangle$$

$$H |B_m\rangle = 0$$

The states $|B_m\rangle$ are massless - Goldstone bosons. The generators Q_i, $i=m+1,\dots n$ generate a subgroup. In case of QCD it is $SU(3)_V$ (or $SU(2)_V$).

In case of QCD an explicit proof of Goldstone theorem can be done, starting from

$$\langle 0|[Q_5^-, \bar{u}\gamma_5 d]|0\rangle = -\langle 0|\bar{u}u+\bar{d}d|0\rangle$$

Chiral effective theory at low energies

The gap in the hadronic spectrum: pseudoscalar mesons, massless in the limit of massless quarks are separated by the gap from other hadronic states. The effective low energy theory may be formulated. The theory is selfconsistent, but only in terms of the expansion in powers of particle momenta $|\vec{p}|/M$, $M \sim 1$ GeV. <u>The theory is not a model!</u>

The theory can be based on $SU(3)_L \times SU(3)_R$ or $SU(2)_L \times SU(2)_R$ symmetry.

Consider $SU(2)_L \times SU(2)_R$ case.

Field equations valid near pion mass shell, i.e. at low energies

$$\begin{cases} j^i_{\mu 5} = -(f_\pi/\sqrt{2})\,\partial_\mu \varphi^i_\pi \\ j^i_{\mu 5} = \bar{q}\gamma_\mu\gamma_5 \tau^i q, \quad q = u, d \\ \partial_\mu j^i_{\mu 5} = (f_\pi/\sqrt{2})\, m^2_\pi \varphi^i_\pi \end{cases}$$

PCAC equations (7)

Adler selfconsistency condition

$$A \to B + \pi(p) \qquad p \to 0$$

$$M_i\,(2\pi)^4\,\delta^4(p_A - p_B - p) = \int d^4x\, e^{ipx}(\partial^2_\mu + m^2_\pi)\langle B|\varphi^i_\pi|A\rangle =$$

$$= (p^2 - m^2_\pi)\frac{(2\pi)^4\,\delta^4(p_A - p_B - p)}{(f_\pi/\sqrt{2})\cdot m^2_\pi}\, p_\mu \langle B|j^i_{\mu 5}|A\rangle \to 0$$

$$\boxed{M(A \to B\pi)_{p\to 0} \to 0}$$

Chiral theory is based on the following principles:

1. The pion field transforms under some representation of the group $G = SU(2)_L \times SU(2)_R$.
2. The action is invariant under these transformations.
3. After breaking the group reduces to $SU(2)$.
4. In the lowest order PCAC field equations are fullfilled.

2×2 unitary matrix $U(x)$, $U^{-1}(x) = U^{+}(x)$ depending on $\varphi^i(x)$.

$$U'(x) = V_L U(x) V_R^{+}$$

After breaking, when G reduces to $SU(2)$, $V_L = V_R = V$

$$U'(x) = V U(x) V^{-1} -$$

– the transformation induced by vector current.

The general form of the lowest order Lagrangian (of order p^2)

$$L_{eff} = \kappa\, Tr(\partial_\mu U \cdot \partial_\mu U^{+}) \qquad (7)$$

Noether currents:

$$j_\mu^i = i\kappa\, Tr(\tau^i [\partial_\mu U, U^{+}]) \qquad (8)$$

$$j_{\mu 5}^i = i\kappa\, Tr(\tau^i \{\partial_\mu U, U^{+}\})$$

Exponential realization

$$U(x) = \exp(i\alpha \vec{\tau} \vec{\varphi}_\pi(x))$$

The first two terms in the expansion over pionic fields

$$L_{eff} = 2\kappa\alpha^2(\partial_\mu\vec{\varphi})^2 + \frac{2}{3}\kappa\alpha^4\left[(\vec{\varphi}_\pi\partial_\mu\vec{\varphi}_\pi)^2 - \vec{\varphi}^2(\partial_\mu\vec{\varphi})^2\right]$$

Requirement: first term is a free Lagrangian

$$2\kappa\alpha^2 = \frac{1}{2}$$

Noether currents:

$$j^i_{\mu v} = i\varepsilon_{ike}\varphi^k_\pi\frac{\partial\varphi^e_\pi}{\partial x_\mu}$$

$$j^i_{\mu 5} = -2\sqrt{\kappa}\,\frac{\partial\varphi^i_\pi}{\partial x_\mu} \quad\rightarrow\quad \kappa = \frac{1}{8}f_\pi^2$$

$$\alpha^{-1} = \frac{f_\pi}{\sqrt{2}}$$

$$\boxed{L_{eff} = (f_\pi^2/8)\,Tr(\partial_\mu U\cdot\partial_\mu U^+)}$$

$$\underline{L_{eff} = \frac{1}{2}(\partial_\mu\vec{\varphi}_\pi)^2 + \frac{1}{3}\frac{1}{f_\pi^2}\left[(\vec{\varphi}_\pi\partial_\mu\vec{\varphi}_\pi)^2 - \vec{\varphi}_\pi^2(\partial_\mu\vec{\varphi})^2\right] + \ldots} \quad (9)$$

<u>Other realization</u>

$$U = \frac{\sqrt{2}}{f_\pi}(\sigma + i\vec{\tau}\vec{\varphi}_\pi) \quad UU^+ = 1 \quad \sigma^2 + \vec{\varphi}^2 = \frac{f_\pi}{2}$$

is equivalent on the mass shell. Any other realizations are also equivalent for physical processes.

<u>Symmetry breaking</u>

$$L_{s.Br} = \frac{f_\pi^2}{4}B\,Tr\left(\mathcal{M}(U+U^+)\right) \quad (10)$$

$$\mathcal{M} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

$$L_{s.Br} = \frac{f_\pi^2}{2}B(m_u+m_d)\Big(1 - \underbrace{\vec{\varphi}_\pi^2\frac{1}{f_\pi^2}}_{\text{mass term}} + \frac{1}{6}\vec{\varphi}_\pi^4\frac{1}{f_\pi^4} + \ldots\Big) \quad (11)$$

$$B(m_u+m_d)=m_\pi^2$$

From Gell-Mann, Oakes, Renner relation

$$B=-\frac{\langle 0|\bar{q}q|0\rangle}{f_\pi^2}\cdot 2$$

From the other side:

$$\frac{\partial}{\partial m_u}\langle 0|L|0\rangle=\underbrace{-\langle 0|\bar{u}u|0\rangle}=\frac{1}{2}f_\pi^2 B=\underbrace{\frac{1}{2}\frac{f_\pi^2 m_\pi^2}{m_u+m_d}}$$

Example: the amplitude of $\pi^+\pi^0\to\pi^+\pi^0$ scattering (Weinberg).

From (9), (11)

$$M=\frac{2}{f_\pi^2}\left\{(p_+'-p_+)^2-m_\pi^2\right\}$$

In c.m. $\quad T_{c.m.}=\frac{1}{16\pi E_+}M \qquad d\sigma=|T|^2 d\Omega$

In $SU(3)_L\times SU(3)_R$ symmetry $\tau_i\to\lambda_n$, $n=1,\dots 8$.

$$\mathcal{M}=\begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s\end{pmatrix}$$