Quark in the external field:

$$G^{\alpha}_{\beta}(x,y) = \langle \psi^{\alpha}(x)\,\bar{\psi}_{\beta}(y)\rangle =$$

$$= \langle x \,|\, \int_0^\infty ds\, \exp(-(m^2 - \mathcal{D}_\mu^2)s)\,|y\rangle \qquad \mathcal{D}_\mu = \partial_\mu + ig B_\mu$$

$$= \int_0^\infty ds \int Dz\, \exp\Big(\underbrace{-m^2 s - \frac{1}{4}\int_0^s \dot{z}^2(\tau)\,d\tau}_{K}\Big)\; \phi_z{}^{\alpha}_{\beta}(x,y) \quad \text{F.-Sch. rep.}$$

$$\phi_z{}^{\alpha}_{\beta}(x,y) = \Big(P \exp ig \int_z B_\mu dz_\mu\Big)^{\alpha}_{\beta}$$

Gauge invariant $q\bar{q}$ configuration:

$$|q\bar{q}\rangle = |\,\psi^{\alpha}(x)\,\phi_{\alpha}^{\beta}(\bar{x},x)\,\bar{\psi}_{\beta}(\bar{x})\rangle$$

$q\bar{q}$ Green function:

$$G_M(x\bar{x}, x'\bar{x}') = \int_0^\infty ds \int_0^\infty d\bar{s} \int Dz \int D\bar{z}\,\cdot$$

$$\cdot \exp\{-K_q - K_{\bar{q}}\}\,\underbrace{\langle \phi_{\bar{z}}(\bar{x}',\bar{x})\,\phi(\bar{x},x)\,\phi_z(x,x')\,\phi(x'\bar{x}')\rangle_B}_{\displaystyle Sp\, P \exp ig \int_C B_\mu dz_\mu}$$

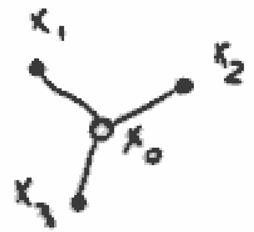
More complicated configurations

Baryon

- $qqq =$

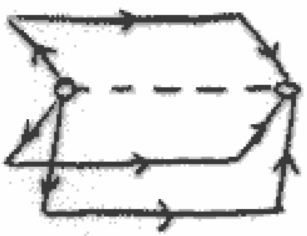
$$= \psi^{\alpha}(x_1)\, \psi^{\beta}(x_2)\, \psi^{\gamma}(x_3)\, \phi^{\alpha'}_{\alpha}(x_0, x_1)\, \phi^{\beta'}_{\beta}(x_0, x_2)\, \phi^{\gamma'}_{\gamma}(x_0, x_3)\, \varepsilon_{\alpha'\beta'\gamma'}$$

$x_0 \Rightarrow$ string junction



$$G_{3q}(x_1 x_2 x_3 x_0 ; y_1 y_2 y_3 y_0) =$$

$$= \int_0^\infty ds_1 \int_0^\infty ds_2 \int_0^\infty ds_3 \int Dz_1 Dz_2 Dz_3 \exp\{-K_1 - K_2 - K_3\} \langle W_{3q} \rangle_B$$



Problem: Prove the identity

$$1 = \frac{1}{3!} \varepsilon_{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \phi^{\alpha_1}_{\beta_1}(\Gamma)\, \phi^{\alpha_2}_{\beta_2}(\Gamma)\, \phi^{\alpha_3}_{\beta_3}(\Gamma)$$

where Γ is an arbitrary (open) path, and represent W as a product of closed Wilson loops

- States with perturbative gluons

$$A_\mu = B_\mu + a_\mu$$

under gauge transformation

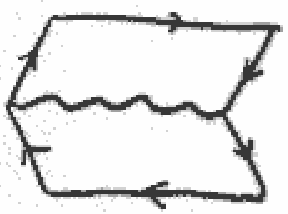
$$B_\mu \to u^+\left(B_\mu + \frac{i}{g}\partial_\mu\right)u \quad \Longrightarrow \quad a_\mu \to u^+ a_\mu u$$

One gluon hybrid:

$$q\bar{q}g = \bar{\Psi}_\alpha(\bar{x})\,\phi^\alpha_\beta(\bar{x}, x_g)\, a^\beta_\gamma(x_g)\, \phi^\gamma_\delta(x_g, x)\, \psi^\delta(x)$$

$$a^\alpha_\beta = a_a(\lambda_a)^\alpha_\beta$$

$$G_H = \int_0^\infty ds \int_0^\infty d\bar{s} \int_0^\infty ds_g \int Dz\, D\bar{z}\, Dz_g \exp\{-K_q - K_{\bar{q}} - K_g\} \langle W_H \rangle_B$$



$$W_H = (\lambda_a)^\alpha_\beta \left(\phi_z(y,x)\right)^\beta_\gamma (\lambda_b)^\gamma_\delta \left(\phi_{\bar{z}}(\bar{x},\bar{y})\right)^\delta_\alpha \left(\phi_{z_g}(\bar{y}, y)\right)_{ab}$$

ϕ_{z_g} is in adjoint rep

Problem: Prove the identity

$$\frac{1}{2}\left(\phi_\Gamma(x,y)\right)_{ab} = (\lambda_a)^\alpha_\beta \left(\phi_\Gamma(x,y)\right)^\beta_\gamma (\lambda_b)^\gamma_\delta \left(\phi_{\bar{\Gamma}}(y,x)\right)^\delta_\alpha$$

and represent W_H as the product of closed Wilson loops

All the information on Y.-M. theory

is contained in $\langle W \rangle_B$

(if the $N_c \to \infty$ limit is implied)

It is due to the vector nature of QCD coupling

The problem is reduced to the

relativistic quantum mechanical one

*   *   *

Back to classical action: (in Minkowski)

$$S = -\int_0^T d\tau \left\{ m_1 \sqrt{\dot{x}_1^2} + g\,\dot{x}_1 B(x_1) + m_2 \sqrt{\dot{x}_2^2} + g\,\dot{x}_2 B(x_2) \right\}$$

$$\Longrightarrow S = -\int_0^T d\tau \left\{ m_1 \sqrt{\dot{x}_1^2} + m_2 \sqrt{\dot{x}_2^2} + \mathcal{L}_{int}(x_1, x_2, \dot{x}_1, \dot{x}_2) \right\}$$

where $\mathcal{L}_{int}$ is read out of Wilson loop evaluation

Long range limit (confinement)

$$< W(C) > \longrightarrow N_C \exp(-\sigma S) \quad \text{in Euclidean}$$

$$S = S_{min}(C)$$

$$S_E \Longleftrightarrow S_M \text{ (string)}$$

$$\Rightarrow \quad \mathcal{L}_{int} = -\sigma \int_0^1 d\beta \sqrt{(\dot{w} w')^2 - \dot{w}^2 w'^2} \qquad \text{Nambu-Goto}$$

$w_\mu = w_\mu(\tau, \beta)$ is the world surface of minimal string

$$w_\mu(\tau, 0) = x_{1\mu}(\tau)$$
$$w_\mu(\tau, 1) = x_{2\mu}(\tau)$$
$$\dot{w}_\mu(\tau,\beta) = \frac{\partial w_\mu}{\partial \tau}, \quad w'_\mu(\tau,\beta) = \frac{\partial w_\mu}{\partial \beta}$$

$\Longrightarrow$ Quantum mechanical problem of quarks at the ends of minimal string

* * *

drastic simplifications:

- $w_\mu(\tau,\beta) = x_{1\mu}(1-\beta) + x_{2\mu} \cdot \beta$ straight line string

•• $x_{10}(\tau) = x_{20}(\tau) = \tau$: proper time $\equiv$ lab time

Neglecting backward motion !!

$q\bar{q}$: $m_1 = m_2 = m$

In the c.m.f. frame $\quad (\vec{p}_1 + \vec{p}_2 = 0)$

$$\vec{r}_1 - \vec{r}_2 = \vec{r} \qquad \vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2) = 0$$

$$\mathcal{L} = -2m\sqrt{1 - \frac{\dot{r}^2}{4}} - \sigma r \int_0^1 d\beta \sqrt{1 - \left(\beta - \frac{1}{2}\right)^2 \frac{(\vec{r} \times \dot{\vec{r}})^2}{r^2}} \qquad (*)$$

Nonrelativistic limit:

$$\mathcal{L} = -2m + \frac{m\dot{r}^2}{4} - \sigma r$$

$$H = 2m + \frac{p^2}{m} + \sigma r \qquad (**)$$

Problems:

i) Derive the Lagrangian $(*)$; find the corrections to the Hamiltonian $(**)$ due to the term $\sim\left(\beta - \frac{1}{2}\right)^2$

ii) Establish the leading confining force for the qqq baryon made of heavy quarks

Light constituents:

$$\mathcal{L}_0 = -m\sqrt{1-\dot{r}^2}$$

Introducing **einbein**:

$$\mathcal{L}_0 = -\frac{\mu}{2}(1-\dot{r}^2) - \frac{m^2}{2\mu} \quad ; \quad \frac{\partial \mathcal{L}_0}{\partial \mu} = 0 \implies \mu = \frac{m}{\sqrt{1-\dot{r}^2}}$$

$$\vec{p} = \frac{\partial \mathcal{L}_0}{\partial \dot{\vec{r}}} = \mu \dot{\vec{r}}$$

$$H = \vec{p}\,\dot{\vec{r}} - \mathcal{L} = \frac{\mu}{2} + \frac{p^2+m^2}{2\mu} \quad ; \quad \frac{\partial H}{\partial \mu} = 0 \implies \mu = \sqrt{p^2+m^2}$$

$$H = \sqrt{p^2+m^2}$$

works for massless particle too!

Einbein form for the straight-line string

$$\mathcal{L} = -2m\sqrt{1-\frac{\dot{r}^2}{4}} - \sigma r \int_0^1 d\beta \sqrt{1-(\beta-\tfrac{1}{2})^2 \frac{(\vec{r}\times\dot{\vec{r}})^2}{r^2}} \implies$$

$$\mathcal{L} = -\mu\left(1-\frac{\dot{r}^2}{4}\right) - \frac{m^2}{\mu} -$$

$$-\int_0^1 d\beta \frac{\nu(\beta)}{2}\left(1 - (\beta-\tfrac{1}{2})^2 \frac{(\dot{\vec{r}}\times\vec{r})^2}{r^2}\right) -$$

$$-\frac{\sigma^2 r^2}{2}\int_0^1 d\beta \frac{1}{\nu(\beta)}$$

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}}$$

$$H = \vec{p}\,\dot{\vec{r}} - \mathcal{L} = \frac{p_r^2 + m^2}{\mu} + \mu +$$

$$+ \frac{L^2}{r^2\{\mu + 2\int_0^1 (\beta - \frac{1}{2})^2 \nu(\beta)\, d\beta\}} + \frac{1}{2}\int_0^1 \left(\nu + \frac{\sigma^2 r^2}{\nu}\right) d\beta\,,$$

$$\vec{L} = [\vec{r}\,\vec{p}]\,, \quad p_r = \frac{(\vec{p}\,\vec{r})}{r}\,;$$

The einbeins μ and $\nu(\beta)$ are to be defined from

$$\frac{\partial H}{\partial \mu} = 0, \quad \frac{\delta H}{\delta \nu(\beta)} = 0\,.$$

Very nonlinear, not well suited for quantization

(ordering prescription !!)

Quasiclassics

- $L \to \infty$, $n_r = 0$

Neglect radial motion; find extremum in einbein $\nu(\beta)$ directly in the Lagrangian:

$$\mathcal{L} = -\sigma r \int_0^1 d\beta \sqrt{1-(\beta-\tfrac{1}{2})^2 \frac{[\vec{r}\dot{\vec{r}}]^2}{r^2}}$$

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = \sigma r \int_0^1 d\beta \; \frac{(\beta-\frac{1}{2})^2 \left\{ \dot{\vec{r}} - \frac{(\dot{\vec{r}}\vec{r})\vec{r}}{r^2} \right\}}{\sqrt{1-(\beta-\frac{1}{2})^2 \frac{[\vec{r}\dot{\vec{r}}]^2}{r^2}}}$$

$$H = \vec{p}\dot{\vec{r}} - \mathcal{L} = \sigma r \int_0^1 d\beta \; \frac{1}{\sqrt{1-(\beta-\frac{1}{2})^2 \frac{[\vec{r}\dot{\vec{r}}]^2}{r^2}}}$$

$$\vec{L} = [\vec{r}\vec{p}] = \sigma r \int_0^1 d\beta \; \frac{(\beta-\frac{1}{2})^2 [\vec{r}\dot{\vec{r}}]}{\sqrt{1-(\beta-\frac{1}{2})^2 \frac{[\vec{r}\dot{\vec{r}}]^2}{r^2}}}$$

$$\begin{cases} H = \sigma r \arcsin l \\ L = -\dfrac{\sigma r^2}{4l}\sqrt{1-l^2} + \dfrac{\sigma r^2}{4l^2}\arcsin l \end{cases} \qquad l = \frac{1}{2}\left|\frac{[\vec{r}\dot{\vec{r}}]}{r}\right|$$

Lagrangian does not contain $\dot{r} = \frac{\partial}{\partial t}|\vec{r}| \Rightarrow$

r is the integral of motion

$$\begin{cases} \dfrac{\partial H}{\partial r} = 0 \\ \dfrac{\partial L}{\partial r} = 0 \end{cases} \Rightarrow l = 1 \Rightarrow \text{endpoints move with the}$$

<u>speed of light</u> !

$$\left.\begin{array}{l} M = \sigma z \frac{\pi}{2} \\ L = \sigma z^2 \frac{\pi}{8} \end{array}\right\} \quad M^2 = 2\pi\sigma L$$

To quantize say that L is integer:

$$M^2 = 2\pi\sigma L$$

(Canonical quantization $M^2 = 2\pi\sigma\sqrt{L(L+1)}$)

- $L = 0$, $n_z \to \infty$

Neglect orbital motion, take $V = \sigma z$

$$H = \frac{p_z^2 + m^2}{\mu} + \mu + \sigma z$$

$$\frac{\partial H}{\partial \mu} = 0 \Rightarrow \mu = \sqrt{p_z^2 + m^2} \Rightarrow H = 2\sqrt{p_z^2 + m^2} + \sigma z \quad (*)$$

Problem: find the quasiclassical spectrum of the Hamiltonian $(*)$.

(Calculate the Bohr-Zommerfeld integral with $m=0$)

Even more simple for low orbital momenta:

- take $V = \sigma r$, calculate integrals over β;
- • Consider the spectrum problem of the Hamiltonian

$$H = \frac{p_r^2 + m^2}{\mu} + \mu + \frac{L^2}{r^2(\mu + \sigma r)}$$

with μ being a c-number

- • • Find $\frac{\partial E_n}{\partial \mu}$ and define μ_0 from $\left.\frac{\partial E_n}{\partial \mu}\right|_{\mu=\mu_0} = 0$

$$E_n = E_n(\mu_0)$$

Problem: Consider in such a way the eigenvalue problem for TE $q\bar{q}g$ hybrid ($l_{q\bar{q}} = 0, l_g = 1$)