$|\text{Matrixelement}|^2$ für $e^-\mu^- \to e^-\mu^-$ Streuung: Spinmittelung

$$\overline{|M|^2} = \frac{1}{4}\sum_{s_1,s_3}\sum_{s_2,s_4}\left(\frac{e^4}{q^4}\right)[\bar{u}(p_3)\gamma_\mu u(p_1][\bar{u}(p_3)\gamma_\nu u(p_1]^+[\bar{u}(p_4)\gamma^\mu u(p_2][\bar{u}(p_4)\gamma^\nu u(p_2]^+$$

$$\overline{|M|^2} = \left(\frac{e^4}{q^4}\right) L_{\mu\nu}\cdot M^{\mu\nu}$$

Leptontensor des e^-: $L_{\mu\nu} = \frac{1}{2}\sum_{s_1,s_3}[\bar{u}(p_3)\gamma_\mu u(p_1)][\bar{u}(p_3)\gamma_\nu u(p_1)]^+$

Leptontensor des μ^-: $M^{\mu\nu} = \frac{1}{2}\sum_{s_2,s_4}[\bar{u}(p_4)\gamma^\mu u(p_2][\bar{u}(p_4)\gamma^\nu u(p_2]^+$

Jetzt Berechnung von $L_{\mu\nu}$ ($M^{\mu\nu}$ analog):

$$[\bar{u}(p_3)\gamma_\nu u(p_1]^+ = u^+(p_1)\gamma_\nu^+\bar{u}^+(p_3) \underbrace{=}_{(\gamma_\nu)^+=\gamma_0\gamma_\nu\gamma_0} u^+(p_1)\gamma_0\gamma_\nu\gamma_0\underbrace{\gamma_0^+}_{=\gamma_0}u(p_3) = \bar{u}(p_1)\gamma_\nu u(p_3)$$

$$L_{\mu\nu} = \frac{1}{2}\sum_{s_1,s_3}\bar{u}(p_3)\gamma_\mu \boxed{u(p_1)\bar{u}(p_1)}\gamma_\nu u(p_3)$$

Behauptung: $\sum u(p_1)\bar{u}(p_1) = u_1(p_1)\bar{u}_1(p_1) + u_2(p_1)\bar{u}_2(p_1) = (\not{p}+m)\cdot\mathbf{1}$

Berechnung von $\sum u(p_1)\bar{u}(p_1)$:

Wir wählen Koord.syst. so, dass $\vec{p}||z$: $\Rightarrow u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ 0 \end{pmatrix}$, $\quad u_2(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p_z}{E+m} \end{pmatrix}$

Verifizieren Sie, dass: $u_1\bar{u}_1 + u_2\bar{u}_2 = \not{p} + m = \gamma^0 E - \gamma_3 p_z + m$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot E \quad - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot p_z \quad + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot m$$

Anfang der Rechnung:

$$u_1\bar{u}_1 = \begin{pmatrix} E+m & 0 & -p_z & 0 \\ 0 & 0 & 0 & 0 \\ p_z & 0 & \frac{-p_z^2}{E+m} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; u_2\bar{u}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & E+m & 0 & p_z \\ 0 & 0 & 0 & 0 \\ 0 & -p_z & 0 & \frac{-p_z^2}{E+m} \end{pmatrix} \Rightarrow u_1\bar{u}_1 + u_2\bar{u}_2 = \begin{pmatrix} E+m & 0 & -p_z & 0 \\ 0 & E+m & 0 & p_z \\ p_z & 0 & \frac{-p_z^2}{E+m} & 0 \\ 0 & -p_z & 0 & \frac{-p_z^2}{E+m} \end{pmatrix}$$

Anleitung: ersetze in der letzten Matrix $-p_z^2$ als Funktion von E und m:

$\Rightarrow u_1\bar{u}_1 + u_2\bar{u}_2 =$

Lösung: $p_z^2 = E^2 - m^2 = (E+m)(E-m)$

$$\rightarrow u_1\bar{u}_1 + u_2\bar{u}_2 = \begin{pmatrix} E+m & 0 & -p_z & 0 \\ 0 & E+m & 0 & p_z \\ p_z & 0 & -E+m & 0 \\ 0 & -p_z & 0 & -E+m \end{pmatrix}$$

Leptontensor $L_{\mu\nu}$

$$L_{\mu\nu} = \frac{1}{2} \sum_{s_3} \bar{u}(p_3) \underbrace{\gamma_\mu (\not{p}_1 + m_1) \gamma_\nu}_{\text{Matrix } A_{jk}} u(p_3)$$

$$L_{\mu\nu} = \tfrac{1}{2} \sum_{j,k=1}^{4} \sum_{s_3} \bar{u}(p_3)_j A_{jk} u(p_3)_k = \tfrac{1}{2} \sum_{jk} A_{jk} \sum_{s_3} u(p_3)_k \bar{u}(p_3)_j$$

↑ komplexe Zahlen, vertausche Reihenfolge

$$\begin{aligned} L_{\mu\nu} &= \tfrac{1}{2} \sum_{jk} A_{jk} (\not{p}_3 + m_3)_{kj} \\ &= \tfrac{1}{2} \sum \left[\gamma_\mu (\not{p}_1 + m_1) \gamma_\nu\right]_{jk} \left[\not{p}_3 + m_3\right]_{kj} \end{aligned}$$

$$\sum_{jk} A_{jk} B_{kj} = Spur(A \cdot B)$$

Spur = Summe der Diagonalelemente

$$L_{\mu\nu} = \frac{1}{2} Spur\left[\gamma_\mu (\not{p}_1 + m_1) \gamma_\nu (\not{p}_3 + m_3)\right]$$

$$M^{\mu\nu} = \frac{1}{2} Spur\left[\gamma^\mu (\not{p}_2 + m_2) \gamma^\nu (\not{p}_4 + m_4)\right]$$

Einige nützliche Rechenregeln: (Griffith S. 252/253)

1.) $Spur(A+B) = Spur(A) + Spur(B)$ $\quad A,\ B$ Matrizen

2.) $Spur(\alpha \cdot A) = \alpha \cdot Spur(A)$ $\quad \alpha =$ Zahl

3.) $Spur(AB) = Spur(BA)$

4.) $g_{\mu\nu} g^{\mu\nu} = 4$ $\quad g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

5.) $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \rightarrow \not{a}\not{b} + \not{b}\not{a} = 2ab$

6.) $\gamma_\mu \gamma^\mu = 4$

7.) $\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu \rightarrow \gamma_\mu \not{a} \gamma^\mu = -2\not{a}$

8.) ...

9.) ...

10.) $Spur$(Produkt einer ungeraden Anzahl von γ's) $= 0$! s. interaktive Übungsaufgabe

11.) $Spur(\mathbf{1}) = 4$

12.) $Spur(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$

13.) $Spur(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda})$

Berechnung des Leptontensors $L_{\mu\nu}$

$$2L_{\mu\nu} = Sp\underbrace{(\gamma_\mu m_1 \gamma_\nu m_3)}_{(1)} + Sp\underbrace{(\gamma_\mu \not{p}_1 \gamma_\nu m_3)}_{(2)} + Sp\underbrace{(\gamma_\mu m_1 \gamma_\nu \not{p}_3)}_{(3)} + Sp\underbrace{(\gamma_\mu \not{p}_1 \gamma_\nu \not{p}_3)}_{(4)}$$

$$(1) \quad Sp(\gamma_\mu m_1 \gamma_\nu m_3) = m_1 m_3 Sp(\gamma_\mu \gamma_\nu) = m_1 m_3 4 g_{\mu\nu} \underbrace{=}_{m_1=m_3=m} m^2 4 g_{\mu\nu}$$

$$(2) \quad Sp(\gamma_\mu \not{p}_1 \gamma_\nu m_3) = m Sp(\gamma_\mu \gamma_\alpha p_1^\alpha \gamma_\nu) = m p_1^\alpha Sp(\gamma_\mu \gamma_\alpha \gamma_\nu) = 0$$

$(3) \quad Sp(\gamma_\mu m \gamma_\nu \not{p}_3) = 0$ analog zu (2)

$$(4) \quad Sp(\gamma_\mu \not{p}_1 \gamma_\nu \not{p}_3) = Sp(\gamma_\mu \gamma_\alpha p_1^\alpha \gamma_\nu \gamma_\beta p_3^\beta) = p_1^\alpha p_3^\beta Sp(\gamma_\mu \gamma_\alpha \gamma_\nu \gamma_\beta)$$

$$= p_1^\alpha p_3^\beta 4 \left[g_{\mu\alpha} g_{\nu\beta} - g_{\mu\nu} g_{\alpha\beta} + g_{\mu\beta} g_{\alpha\nu}\right]$$

$$= 4 \left[p_{1\mu} p_{3\nu} + p_{1\nu} p_{3\mu} - g_{\mu\nu}(p_1 p_3)\right]$$

$$\Rightarrow L_{\mu\nu} = 2p_{1\mu} p_{3\nu} + 2p_{1\nu} p_{3\mu} - 2(p_1 p_3) g_{\mu\nu} + 2m^2 g_{\mu\nu}$$

$\rightarrow$ Analog für $M^{\mu\nu}$

Finales Ergebnis für $|M|^2$ der $e^- \mu^- \to e^- \mu^-$ Streuung

$$\overline{|M|^2} = \frac{e^4}{q^4} L_{\mu\nu} M^{\mu\nu}$$

Def. $m :=$ Elektronmasse, $\mathcal{M} :=$ Myonmasse

$$= \frac{4e^4}{q^4} \left\{ p_{1\mu} p_{3\nu} + p_{1\nu} p_{3\mu} + g_{\mu\nu} \left[m^2 - (p_1 p_3) \right] \right\} \left\{ p^{2\mu} p^{4\nu} + p^{2\nu} p^{4\mu} + g^{\mu\nu} \left[\mathcal{M}^2 - (p_2 p_4) \right] \right\}$$

mit $g_{\mu\nu} g^{\mu\nu} = 4$ $\Rightarrow$ *Empfehlung: folgenden Schritt nachrechnen:*

$$\overline{|M|^2} = \frac{8e^4}{q^4} \left[(p_1 p_2)(p_3 p_4) + (p_1 p_4)(p_2 p_3) - m^2 (p_2 p_4) - \mathcal{M}^2 (p_1 p_3) + 2 m^2 \mathcal{M}^2 \right]$$

exaktes Ergebnis! (noch keine Näherung gemacht)

Hochenergiefall: $m,\ M \to 0$

$$\overline{|M|^2} = \frac{8e^4}{q^4} \left[\underbrace{(p_1 p_2)}_{\frac{s}{2}} \underbrace{(p_3 p_4)}_{\frac{s}{2}} + \underbrace{(p_1 p_4)}_{-\frac{u}{2}} \underbrace{(p_2 p_3)}_{-\frac{u}{2}} \right]$$

Beweis für $p_1 p_2$:

$$s = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2 p_1 p_2$$
$$= m^2 + \mathcal{M}^2 + 2 p_1 p_2 \sim 2 p_1 p_2$$

$$\boxed{\overline{|M|^2} = \frac{2e^4}{t^2} \left[s^2 + u^2 \right] = 2\, e^4 \frac{s^2 + u^2}{t^2}}$$

mit $t = q^2$

"t-Kanal" Prozess