The Accelerator Physics of Linear Collider Damping Rings

Notes for USPAS Course on Linear Colliders
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Andy Wolski
Lawrence Berkeley National Laboratory

1 Introduction

In these notes, we set out some of the basic accelerator physics underlying the functions of linear collider damping rings. Most of this is standard material for electron storage rings, but the emphasis here is appropriate to the special tasks for which damping rings are designed. Thus, we start by deriving expressions for the damping times, and proceed to consider some of the effects of the long damping wiggler, lattice designs based on the Theoretical Minimum Emittance cell for very low emittance storage rings, and discuss the generation of vertical emittance and some alignment issues. We assume that the reader is familiar with the basic structure of an electron storage ring, i.e. that a magnetic lattice consisting of dipoles and quadrupoles is used to contain the beam, and an RF cavity is used to replace the energy lost by synchrotron radiation. We also assume that the reader is familiar with the lattice functions used to describe the transverse motion (i.e. the beta function and dispersion). We try to derive everything from these fundamentals, though some results relating to the properties of the emitted radiation are quoted without derivation.

There are a number of good texts on accelerator physics. Much of the material in these notes draws heavily from

- "Accelerator Physics", S.Y. Lee, World Scientific, 1999.

We use the TESLA and NLC damping rings (designs as at December 2002) to illustrate the application of the principles we discuss. These designs are presently the most mature of the various linear collider proposals, and are described in

- "TESLA Technical Design Report", DESY 2001-11, March 2001.
- "2001 Report on the Next Linear Collider", SLAC-R-571.

Further material may be found on the web sites

- http://www.desy.de/~wdecking/dog/dogbone.html
- http://awolski.lbl.gov/

Other useful references for damping rings include:

- "Zeroth-Order Design Report for the NLC", SLAC Report 474, May 1996.
- "The Generation and Acceleration of Low Emittance Flat Beams for Future Linear Colliders", T.O. Raubenheimer, SLAC-R-387, February 1992.
- "A Systematic Approach to Damping Ring Design", P. Emma and T. Raubenheimer, PhysRevSTAB, Volume 4, 021001 (2001).

2 Radiation Damping and Equilibrium Emittance

Fundamental to the performance of a damping ring is the damping of the transverse and longitudinal emittances. The emittance in any plane at a time *t* after injection is given by:

$$\varepsilon(t) = \varepsilon_{\mathrm{inj}} e^{-2t/\tau} + \varepsilon_{\mathrm{equ}} \left(1\text{-}e^{-2t/\tau}\right) \tag{1}$$

where $\varepsilon_{\mathrm{inj}}$ is the injected emittance, $\varepsilon_{\mathrm{equ}}$ is the equilibrium emittance, and τ is the damping time. The values of the equilibrium emittances and damping times depend on the plane under consideration, and on a range of parameters related to the lattice design, the ring energy and the alignment of the magnets. Collective effects also need to be taken into account, though we shall neglect these for the present.

Equation (1) is usually the starting point for the design of a damping ring for a linear collider, so we shall spend some time to understand it. We deal with the synchrotron (longitudinal) motion first, and then tackle the betatron (transverse) motion.

2.1 Synchrotron Motion

Our aim will be to derive expressions for the equilibrium longitudinal emittance and the longitudinal damping time. We shall proceed in three stages:

- First, we consider the energy gain from the RF cavities and the energy loss from radiation in the classical limit, with the approximation that the energy loss is independent of the particle energy. We find that particles perform stable harmonic oscillations in longitudinal phase space.
- Second, we include the dependence of radiation energy loss on the particle energy. This introduces an extra term in the equation of motion, that leads to damping of synchrotron oscillations, and we find the damping time.
- Third, we include the fact that the radiation is not continuous, but energy is emitted in quanta. This leads to the phenomenon of quantum excitation, and we find an expression for the equilibrium longitudinal emittance (specifically, the equilibrium bunch length and energy spread).

2.1.1 Synchrotron Oscillations

We use the longitudinal phase space co-ordinates τ and δ. τ is the time separation between the particle and a nominal reference particle which always has the correct energy and zero betatron amplitude. We use the convention that a *positive* τ means that the particle is *ahead* of the reference particle[1]. δ is the energy deviation of the particle, given by:

$$\delta = \frac{E - E_0}{E_0}$$

where E is the particle energy, and E_0 the design energy of the lattice.

[1] With this convention, (τ, δ) form a canonical pair, with δ the conjugate momentum to the coordinate τ.

The closed orbit in a storage ring is a function of the particle's energy. The dispersion η is defined as the rate of change of the closed orbit with respect to the energy deviation, thus:

$$x_{\text{co}} = \eta\delta$$

Note that the transverse co-ordinate x of a particle is always with respect to the dispersive orbit.

The change in total orbit length (or circumference, C) is related to the energy deviation by the momentum compaction α_p:

$$\frac{\Delta C}{C_0} = \alpha_p \delta$$

The subscript 0 denotes the circumference is that of the design orbit. α_p is a property of the lattice, and is given by:

$$\alpha_p = \frac{1}{C_0} \oint \frac{\eta}{\rho} \mathrm{d}s$$

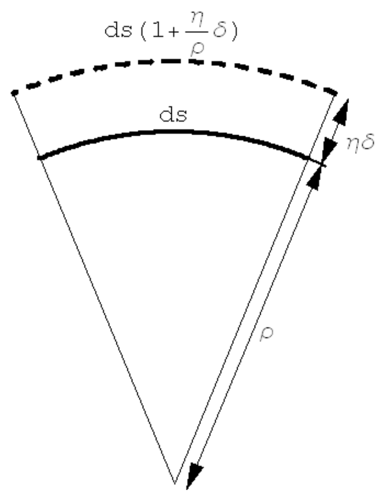

Figure 1

In a bending magnet, the reference trajectory is curved. A particle following a dispersive trajectory has a different path length in the magnet. Integrating the difference in path length per unit energy deviation through all the bending magnets in the lattice leads to the momentum compaction.

For a relativistic particle, the orbital period T is proportional to the circumference. Thus,

$$\frac{\Delta T}{T_0} = \frac{\Delta C}{C_0} = \alpha_p \delta$$

It follows that the change in the time separation τ from orbit n to orbit $n+1$ is given by:

$$\tau_{n+1} - \tau_n = -\alpha_p T_0 \delta$$

or:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = -\alpha_p \delta \qquad (2)$$

The energy of a particle will change because of synchrotron radiation, and because of the RF power supplied through the RF cavities. The change in energy deviation from one turn to the next is given by:

$$\delta_{n+1} - \delta_n = \frac{eV_{RF}}{E_0}\sin(\phi_s - \omega_{RF}\tau) - \frac{U}{E_0}$$

where V_{RF} is the peak RF voltage, ϕ_s the RF phase on which our reference particle passes through the cavity (the synchronous phase), $\omega_{RF}/2\pi$ is the RF frequency, and U the energy loss per turn of the particle of interest. For the moment, we shall pretend that U is a constant, and set it equal to U_0, the energy loss of a particle on the design orbit and with the correct energy. In fact, the energy loss is a function of the particle energy: this phenomenon will lead to damping of the synchrotron oscillations, as we shall see below. For now, we note that

$$eV_{RF}\sin(\phi_s) = U_0$$

and observe that for $\omega_{RF}\tau << 1$ we can write:

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} = -\frac{eV_{RF}}{E_0 T_0}\cos(\phi_s)\omega_{RF}\tau \qquad (3)$$

Combining equations (2) and (3), we find:

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}t^2} = -\omega_s^2\tau$$

where the synchrotron frequency ω_s is given by:

$$\omega_s^2 = -\frac{eV_{RF}}{E_0}\frac{\omega_{RF}}{T_0}\alpha_p\cos(\phi_s) \qquad (4)$$

For stable oscillations, we require

$$\omega_s^2 > 0$$

Since for most conventional lattices $\alpha_p > 0$, the synchronous phase must satisfy:

$$\frac{\pi}{2} < \phi_s < \frac{3\pi}{2}$$

The longitudinal phase space co-ordinates obey:

$$\tau = \hat{\tau}\cos(\omega_s t - \theta_s)$$
$$\delta = \frac{\omega_s}{\alpha_p}\hat{\tau}\sin(\omega_s t - \theta_s) \tag{5}$$

2.1.2 Radiation Damping of Synchrotron Oscillations

In our treatment of synchrotron oscillations, we simplified matters by making the energy loss per turn a constant, independent of the energy deviation. In fact, the energy loss depends on the particle energy for two reasons. First, higher energy particles radiate more power per se. Second, the closed orbit depends (through the dispersion) on the particle energy, so in a combined-function bending magnet, different energy particles will see different magnetic field strengths, and so will radiate different amounts of energy. We shall find that the dependence of energy loss on the particle's energy will lead to damping of the synchrotron oscillations. Our purpose is to find an expression that describes this phenomenon.

To take account of the energy dependence (to first order), we write:

$$U = U_0 + W E_0 \delta$$

where

$$W = \left.\frac{\mathrm{d}U}{\mathrm{d}E}\right|_{E=E_0}$$

Including the extra term, equation (3) becomes:

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} = -\frac{eV_{RF}}{E_0 T_0}\cos(\phi_s)\omega_{RF}\tau - \frac{W}{T_0}\delta$$

The longitudinal equation of motion is then:

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} + 2\alpha_E\frac{\mathrm{d}\delta}{\mathrm{d}t} + \omega_s^2\delta = 0$$

with an identical expression for τ. The solution (5) becomes:

$$\tau = \hat{\tau}\,\mathrm{e}^{-\alpha_E t}\cos(\omega_s t - \theta_s)$$
$$\delta = \frac{\omega_s}{\alpha_p}\hat{\tau}\,\mathrm{e}^{-\alpha_E t}\sin(\omega_s t - \theta_s)$$

where the damping rate α_E is given by:

$$\alpha_E = \frac{W}{2T_0} \tag{6}$$

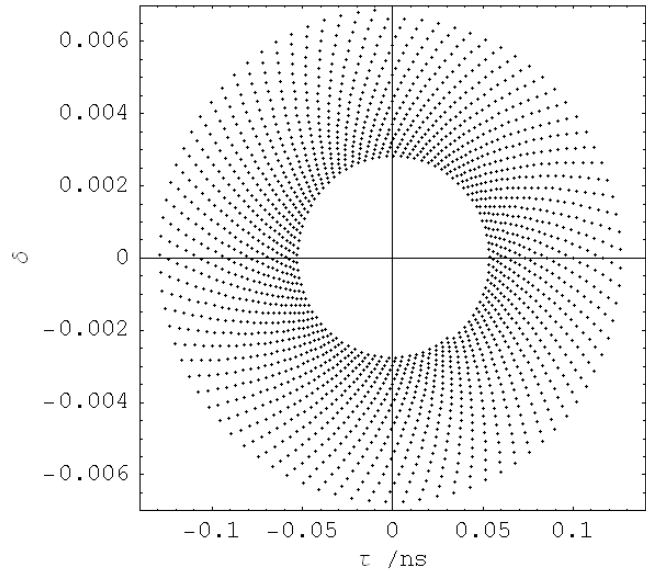

Figure 2

Longitudinal phase space damping in the NLC MDR. The points show the longitudinal phase space co-ordinates on successive turns around the ring. The synchrotron tune is 0.0118, the momentum compaction factor is 1.4×10^{-3}, the damping time is 2.2 ms, and the revolution period is 1 μs.

To complete our analysis, we need to find an explicit expression for W, the dependence of the energy loss on the particle energy. Remember that there are two effects to take into account: the direct dependence of radiated power on energy, and the dependence of radiated power on the magnetic field (which can vary with the orbit if the lattice uses combined-function bending magnets).

The rate at which a relativistic particle radiates energy in a magnetic field is given by:

$$P_\gamma = \frac{e^2 c^3 C_\gamma}{2\pi} E^2 B^2 = \frac{c C_\gamma}{2\pi} \frac{E^4}{\rho^2} \tag{7}$$

where

$$C_\gamma = \frac{e^2}{3\varepsilon_0 \left(m_e c^2\right)^4} \approx 8.846\times 10^{-5}\,\text{m/GeV}^3$$

This is a classical expression. There is no mention of Planck's constant, and hence no account is taken of the fact that the radiation is emitted in discrete quanta (i.e. photons). When we come to include the effect of radiation of individual photons, we shall find that this leads to an excitation of the synchrotron amplitude (quantum excitation).

In equation (7), we have made use of the relationship between the beam rigidity $B\rho$ and the energy E for a relativistic particle:

$$B\rho = \frac{p}{e} = \frac{E}{ec}$$

We shall continue to make frequent use of this relationship.

The total radiated energy in one turn is:

$$U = \oint P_\gamma \mathrm{d}t = \frac{1}{c}\oint P_\gamma \left(1 + \frac{\eta\delta}{\rho}\right)\mathrm{d}s$$

Substituting for P_γ from (7), and writing:

$$B = B_0 + \frac{\partial B}{\partial x}\eta\delta = B_0 + \frac{E_0}{ec^2}k_1\eta\delta$$

where k_1 is the quadrupole strength in the bending magnets, we find after some manipulation:

$$W = \left.\frac{\mathrm{d}U}{\mathrm{d}E}\right|_{E=E_0} = \mathcal{J}_E \frac{U_0}{E_0} \tag{8}$$

where $\mathcal{J}_E$ (the *longitudinal damping partition number*) is a function of the lattice, given by:

$$\mathcal{J}_E = 2 + \frac{I_4}{I_2}$$

and the second and fourth *synchrotron radiation integrals* are defined:

$$I_2 = \oint \frac{1}{\rho^2}\mathrm{d}s$$

$$I_4 = \oint \frac{\eta}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right)\mathrm{d}s$$

Using equations (6) and (8), we can write the longitudinal damping time:

$$\tau_E = \frac{2}{\mathcal{J}_E}\frac{E_0}{U_0}T_0 \tag{9}$$

Note that for a lattice that has no gradient in the bending magnets, $\mathcal{J}_E \approx 2$, and the longitudinal damping time is the time it would take a particle to lose all its energy, if the energy were to be lost at a constant rate.

The damping time τ_E is the exponential decay time for the *amplitude* of synchrotron oscillations. From equations (5), we note that a particle follows an ellipse in longitudinal phase space. If we define the longitudinal action as the area of this ellipse, we would write:

$$J = \pi \hat{\delta} \hat{\tau} = \pi \frac{\omega_s}{\alpha_p} \hat{\tau}^2$$

Of course, the action is proportional to the square of the oscillation amplitude. This means that the action will damp as:

$$J(t) = J_0 e^{-2t/\tau_E} \tag{10}$$

Remember that we have so far taken no account of quantum excitation. In the classical case, described by equation (10), the longitudinal emittance damps to zero; each bunch in the storage ring would eventually have zero bunch length and zero energy spread!

2.1.3 Quantum Excitation of Synchrotron Oscillations

Consider a particle that emits a photon of energy u when the synchrotron phase is $\omega_s t - \theta_s = \theta$. The phase space co-ordinates immediately before the emission are:

$$\tau = \frac{\alpha_p}{\omega_s} \hat{\delta} \cos(\theta)$$

$$\delta = \hat{\delta} \sin(\theta)$$

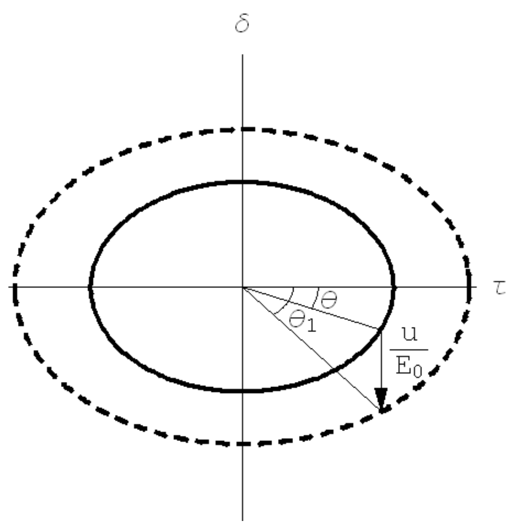

Figure 3

Change in longitudinal phase space co-ordinates of a particle with emission of a photon.

And immediately after the photon emission:

$$\tau_1 = \frac{\alpha_p}{\omega_s} \hat{\delta}_1 \cos(\theta_1) = \frac{\alpha_p}{\omega_s} \hat{\delta} \cos(\theta)$$

$$\delta_1 = \hat{\delta}_1 \sin(\theta_1) = \hat{\delta} \sin(\theta) - \frac{u}{E_0}$$

from which it follows:

$$\hat{\delta}_1^2 = \hat{\delta}^2 + \frac{u^2}{E_0^2} - 2\frac{u}{E_0}\hat{\delta}\sin(\theta)$$

The rate of emission of photons depends on the local curvature of the orbit, which varies widely around the ring. Since the damping time and the synchrotron period are generally long compared with the revolution period, we assume that we can average around the ring. The last term on the right hand side in the above equation vanishes in this average, so we see that the net effect of the photon emission is an average growth in the synchrotron amplitude. Including the effect of damping, and averaging the photon energy over the photon spectrum, gives the equation of motion:

$$\frac{\mathrm{d}\hat{\delta}^2}{\mathrm{d}t} = \frac{1}{E_0^2 C_0}\oint N\langle u^2\rangle \mathrm{d}s - 2\frac{\hat{\delta}^2}{\tau_E}$$

where N is the number of photons emitted per unit time. At equilibrium, we have

$$\hat{\delta}^2 = \frac{\tau_E}{2E_0^2 C_0}\oint N\langle u^2\rangle \mathrm{d}s$$

Since the synchrotron oscillations are sinusoidal, and the phase distribution of particles in the beam is uniform, the equilibrium mean square energy deviation of the beam is given by:

$$\sigma_\delta^2 = \frac{1}{2}\langle\hat{\delta}^2\rangle = \frac{\tau_E}{4E_0^2 C_0}\oint N\langle u^2\rangle \mathrm{d}s \tag{11}$$

To complete the analysis, we need another result from the synchrotron radiation theory. This is the second order moment of the photon energy at a point where the bending radius is ρ:

$$N\langle u^2\rangle = \int_0^\infty u^2 n(u)\mathrm{d}u = 2C_q\gamma^2 E_0\frac{P_\gamma}{\rho} \tag{12}$$

Here, $n(u)$ is the normalized number of photons in the energy range u to $u + \mathrm{d}u$ emitted per unit time. The constant C_q is given by:

$$C_q = \frac{55}{32\sqrt{3}}\frac{\hbar}{m_e c} \approx 3.832\times10^{-13}\,\mathrm{m}$$

In equation (11), we need the value of P_γ/ρ averaged around the ring. Since $P_\gamma \propto B^2$ and $B\rho = E_0/ec$ is a constant around the ring, we have:

$$\frac{1}{C_0}\oint\frac{P_\gamma}{\rho}\mathrm{d}s = \frac{I_3}{I_2}\cdot\frac{1}{C_0}\oint P_\gamma \mathrm{d}s$$

where the third synchrotron radiation integral I_3 is defined:

$$I_3 = \oint \frac{1}{|\rho|^3}\,\mathrm{d}s$$

Furthermore, for a particle following the design orbit, we can write the synchrotron radiation power averaged around the lattice as:

$$\frac{1}{C_0}\oint P_\gamma \mathrm{d}s = \frac{2E_0}{\mathcal{J}_E \tau_E}$$

and hence we find:

$$\frac{1}{C_0}\oint N\langle u^2\rangle \mathrm{d}s = 4C_q \frac{\gamma^2 E_0^2}{\mathcal{J}_E \tau_E}\cdot\frac{I_3}{I_2}$$

Substituting this into equation (11) gives for the square of the energy spread:

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{\mathcal{J}_E I_2}$$

and for the bunch length:

$$\sigma_\tau^2 = \left(\frac{\alpha_p}{\omega_s}\right)^2 C_q \gamma^2 \frac{I_3}{\mathcal{J}_E I_2}$$

We note that the energy spread is independent of the RF voltage and frequency, but increases linearly with the energy. The bunch length, on the other hand, depends on both the RF voltage and RF frequency through the synchrotron frequency. Explicitly:

$$\sigma_\tau^2 = -C_q \gamma^3 \frac{\alpha_p m_e c^2 T_0}{eV_{RF}\omega_{RF}\cos(\phi_s)}\frac{I_3}{\mathcal{J}_E I_2}$$

2.2 Betatron Motion

Particles in a storage ring perform transverse oscillations about the closed orbit. At any point around the orbit, the phase space co-ordinates of the particle, i.e. its position and momentum with respect to the (closed) design orbit, determine its state. Recall that the horizontal motion may be written:

$$\begin{aligned} x &= \sqrt{2\beta J}\cos(\phi) \\ x' &= -\sqrt{\frac{2J}{\beta}}\left[\sin(\phi)+\alpha\cos(\phi)\right] \end{aligned} \tag{13}$$

and similarly for the vertical. It is really the action J (a constant around the ring) that defines the amplitude of the oscillation, since α and β are functions of the lattice. Note that for fixed values of α and β (i.e. at a chosen observation point in the lattice), the phase space co-ordinates lie on an ellipse defined by:

$$2J = \gamma x^2 + 2\alpha x x' + \beta x'^2 \tag{14}$$

where γ is defined by:

$$\beta\gamma - \alpha^2 = 1$$

This is shown in Figure 4.

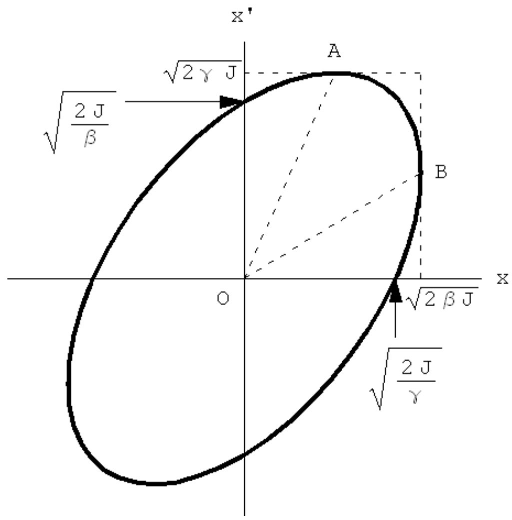

Figure 4

Phase space ellipse defined by equation (4). The area of the ellipse is $2\pi J$, and the centroid is at O. The line OA has slope $-\gamma/\alpha$ and OB has slope $-\alpha/\beta$.

For an ensemble of particles all with different action, the following relations follow directly from (13):

$$\langle x^2 \rangle = \beta \langle J \rangle$$
$$\langle x'^2 \rangle = \gamma \langle J \rangle$$
$$\langle x x' \rangle = -\alpha \langle J \rangle$$

It then follows that the emittance, defined as the determinant of the matrix of second order moments, is just the mean action:

$$\varepsilon = \det \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x x' \rangle & \langle x'^2 \rangle \end{pmatrix} = \langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2 = \langle J \rangle$$

The emittance defined in this way is sometimes called the rms emittance of the beam.

It is sometimes convenient to write the phase space co-ordinates in normalized form, using a symplectic transformation:

$$\begin{pmatrix} \tilde{x} \\ \tilde{x}' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \cos(\phi) \\ -\sin(\phi) \end{pmatrix}$$

In normalized co-ordinates, the transformation associated with any beamline may be written just as a rotation. It then follows that the transformation between two points can be constructed from a transformation to normalized co-ordinates, a rotation, and a transformation out of normalized co-ordinates:

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \mathbf{M}_{21} \cdot \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

$$\mathbf{M}_{21} = \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \cdot \begin{pmatrix} \cos(\Delta\phi) & -\sin(\Delta\phi) \\ \sin(\Delta\phi) & \cos(\Delta\phi) \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ \frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}$$

Conventionally, we parameterize the particle motion using s, the distance along the design orbit, rather than the time t. Note that where the design orbit is curved with radius ρ, the path length and the time are related by:

$$\mathrm{d}t = \frac{1}{c}\left(1 + \frac{x}{\rho}\right)\mathrm{d}s$$

For off-momentum particles, the co-ordinate x specifies the displacement of the particle from the dispersive orbit, i.e. the closed orbit for the appropriate momentum.

2.2.1 Radiation Damping of Vertical Betatron Motion

The vertical betatron motion is generally more straightforward than the horizontal, since lattices are usually designed with zero vertical dispersion[2]. We shall treat the spurious vertical dispersion introduced by magnet misalignments separately.

Synchrotron radiation is emitted within a cone of angle $1/\gamma$ of the instantaneous path of the electron. This opening angle actually places a fundamental lower limit on the vertical emittance, which we shall consider later. But for now, we use the relativistic approximation that the emission of a photon changes neither the co-ordinate nor the angle of the betatron motion, and hence the amplitude of the betatron motion is not affected by the radiation. In an RF cavity, however, there is a change in the longitudinal momentum that does change the transverse angle y':

[2] This is not the case for the present design of the TESLA damping ring. The long straight sections are designed to share the tunnel with the main linac, which follows the curvature of the earth. However, the vertical dispersion introduced by this is small, and makes no significant contribution to the vertical emittance.

$$y_1 = y$$
$$y_1' = \frac{p_y}{p+\delta p} \approx \frac{p_y}{p}\left(1-\frac{\delta p}{p}\right) = y'\left(1-\frac{\delta p}{p}\right) \tag{15}$$

For a particle with zero synchrotron amplitude, the RF cavity replaces the (on-orbit) energy loss U_0 from synchrotron radiation. Thus:

$$\frac{\delta p}{p} = \frac{U_0}{E_0}$$

Using equation (14), it is straightforward to show that the change in the action resulting from the kick (15) averaged over many turns (i.e. all betatron phases) is, to first order in the energy loss:

$$\Delta J = -\frac{U_0}{E_0} J$$

and hence:

$$\frac{\mathrm{d}J}{\mathrm{d}t} = -\frac{U_0}{E_0 T_0} J \tag{16}$$

Thus, we find:

$$J(t) = J_0 \mathrm{e}^{-2t/\tau_y}$$

where the vertical damping time is

$$\tau_y = 2\frac{E_0}{U_0} T_0 \tag{17}$$

Note that we have introduced a factor 2 in the exponential for the decay of the vertical action. This is so the damping time refers to the damping of the vertical betatron *amplitude*, with the action scaling as the square of the amplitude. Recall the longitudinal damping time similarly referred to the damping of the synchrotron amplitude.

Compare the vertical damping time (17) with the expression for the longitudinal damping time (9). For a lattice with no gradient in the bends, the vertical damping time is twice the longitudinal damping time.

It is clear from the above analysis that transverse damping occurs as a result of the combination of energy loss from radiation, and the restoration of the energy in the RF cavity. Without an RF cavity, there is no damping. Without synchrotron radiation, we can still damp the emittance, but only if we simultaneously accelerate the beam (adiabatic damping). In this case, the normalized emittance $\gamma\varepsilon$ is constant.

2.2.2 Radiation Damping of Horizontal Betatron Motion

The horizontal motion is complicated by the fact that the orbit changes with the energy. As the particle radiates, the betatron amplitude changes because of the change of orbit.

This is still a classical effect: it can be described without any reference to Planck's constant, and although it modifies the damping rate, the horizontal emittance would still damp to zero if no other effects were included. Below, we shall see how quantum effects excite horizontal oscillations, and lead to a non-zero horizontal emittance. But for now, we ignore the quantum excitation.

When a particle radiates a small amount of energy u, the new phase space co-ordinates are given by:

$$x_1 = x + \frac{u}{E_0}\eta$$

$$x_1' = x' + \frac{u}{E_0}\eta'$$

Substituting into the standard expression (14) for the single-particle emittance, the change in action resulting from the radiation is, to first order in the energy loss:

$$\Delta J = \frac{u}{E_0}\left[\gamma x\eta + \alpha(x'\eta + x\eta') + \beta x'\eta'\right]$$

For the energy loss in time $\mathrm{d}t$ we write:

$$u = P_\gamma(x)\mathrm{d}t = \left(P_{\gamma 0} + x\left.\frac{\partial P_\gamma}{\partial x}\right|_{x=0}\right)\mathrm{d}t = P_{\gamma 0}\left(1 + \frac{2}{B}\frac{\partial B}{\partial x}x\right)\left(1 + \frac{x}{\rho}\right)\mathrm{d}s$$

This takes into account the variation in field strength with horizontal co-ordinate (recall that $P_\gamma \propto B^2$). We then find for the change in emittance in the path length $\mathrm{d}s$:

$$\Delta J = \frac{P_{\gamma 0}}{E_0 c}\left[\gamma x\eta + \alpha(x'\eta + x\eta') + \beta x'\eta'\right]\left(1 + \frac{2}{B}\frac{\partial B}{\partial x}x\right)\left(1 + \frac{x}{\rho}\right)\mathrm{d}s$$

At a given point in the lattice, the particle will be at a different betatron phase on each turn through the ring. Since the damping time is much larger than the revolution period, we can average over all betatron phases:

$$\Delta J = J\frac{1}{E_0 c}\left(P_{\gamma 0}\rho^2\right)\left[\frac{\eta}{\rho}\left(2k_1 + \frac{1}{\rho^2}\right)\right]\mathrm{d}s$$

Integrating over the lattice, and including the damping from the RF cavities (16), we find:

$$\frac{\mathrm{d}J}{\mathrm{d}t} = -\left(1 - \frac{I_4}{I_2}\right)\frac{U_0}{E_0 T_0}J = -\mathcal{J}_x\frac{U_0}{E_0 T_0}J$$

$\mathcal{J}_x$ is the horizontal damping partition number:

$$\mathcal{J}_x = 1 - \frac{I_4}{I_2}$$

For a lattice without a gradient in the bending magnets, $\mathcal{J}_x \approx 1$. The horizontal action evolves as:

$$J(t) = J_0 \mathrm{e}^{-2t/\tau_x}$$

where the horizontal damping time is

$$\tau_x = 2\frac{E_0}{\mathcal{J}_x U_0} T_0 \tag{18}$$

Note that the damping partition numbers satisfy some simple relationships:

$$\mathcal{J}_x + \mathcal{J}_y + \mathcal{J}_E = 4$$

$$\mathcal{J}_x \tau_x = \mathcal{J}_y \tau_y = \mathcal{J}_E \tau_E = 2\frac{E_0}{U_0} T_0$$

where we have defined (for symmetry!) $\mathcal{J}_y = 1$.

2.2.3 Quantum Excitation of Betatron Motion

The combination of the energy loss from (classical) radiation with the energy gain from the RF cavities in a storage ring leads to damping of the betatron oscillations. In the case of the horizontal motion, the variation of the closed orbit with energy leads to some excitation of the oscillations that reduces the damping rate. This is still a classical effect. Consideration of only these phenomena leads to formulae that suggest the transverse beam size damps eventually to zero. In the case of synchrotron motion, inclusion of the quantum effects resulting from the emission of photons led to a non-zero equilibrium longitudinal emittance. We shall show in this section that similar effects lead to non-zero transverse emittances where there is dispersion in the bend magnets (which is always the case in the horizontal plane). Our treatment is valid for both horizontal and vertical planes – it does not matter whether the dispersion occurs by design, or is generated by misalignment of the magnets.

It is easy to show that the change in the single-particle emittance resulting from the emission of a photon of energy u at a single point in the lattice is given by:

$$\Delta J = \frac{1}{2}\left(\frac{u}{E_0}\right)^2 \mathcal{H}$$

where the $\mathcal{H}$ -function is defined by:

$$\mathcal{H} = \gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2$$

Note that we have (as usual) averaged over all betatron phases, making the assumption that any excitation or damping is slow compared to the revolution period. We then have the rate of change of the transverse single-particle emittance, including both quantum excitation and damping:

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \frac{1}{2E_0^2 C_0} \oint N\left\langle u^2 \right\rangle \mathcal{H} \mathrm{d}s - \frac{2}{\tau} J \tag{19}$$

Using equation (12) and equations following, and noting that $\mathcal{J}_E \tau_E = \mathcal{J}_x \tau_x = \mathcal{J}_y \tau_y$, we find:

$$\frac{\mathrm{d}J}{\mathrm{d}t} = C_q \gamma^2 \frac{2}{\mathcal{J}\tau} \frac{I_5}{I_2} - \frac{2}{\tau} J$$

Where the fifth synchrotron radiation integral is defined by:

$$I_5 = \oint \frac{\mathcal{H}}{|\rho|^3} \mathrm{d}s$$

Hence, the equilibrium action is given by:

$$J_{equ} = C_q \gamma^2 \frac{I_5}{\mathcal{J} I_2}$$

The transverse emittance of the beam is defined as the action averaged taken over all particles in the beam. We have not taken into account betatron coupling, which will exchange emittance between the two transverse planes. In the simple linear theory, the sum of the two emittances is a constant, the *natural emittance* ε_0 of the lattice. From consideration of the uncoupled case, where the vertical emittance is zero, we have:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{\mathcal{J}_x I_2}$$

2.2.4 Radiation Limited Emittance

We noted above that the non-zero opening angle of the radiation led to an excitation of the betatron oscillations. Our analysis in the previous sections ignored this effect, and for most storage rings it is negligible. However, damping rings for a future linear collider will need to operate with lower vertical emittances than have so far been achieved, and the contribution to the vertical emittance from the opening angle of the radiation is not quite negligible. For completeness, we quote the result:

$$\varepsilon_{y,\min} = \frac{13}{55} \frac{C_q}{\mathcal{J}_y} \frac{\oint \beta_y / |\rho^3| \mathrm{d}s}{\oint 1/\rho^2 \mathrm{d}s}$$

2.3 Evolution of Transverse and Longitudinal Emittances

From the analysis of the previous sections, we observe that the time evolution of the emittance in transverse and longitudinal planes can be written in the general form:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{2}{\tau}\varepsilon_{equ} - \frac{2}{\tau}\varepsilon \tag{20}$$

The first term on the right comes from quantum excitation, i.e. the effect of emitting radiation in photons. The second term on the right is the damping term, and comes from a classical treatment of the radiation. In the longitudinal motion, damping comes from the fact that higher energy particles radiate more quickly. In the transverse motion, the damping comes from the fact that radiation occurs in a narrow cone about the direction of the instantaneous motion of the particle, whereas the energy gain from the RF cavities always leads to an increase in longitudinal momentum.

The general solution to equation (20) is equation (1):

$$\varepsilon(t) = \varepsilon_{\mathrm{inj}} \mathrm{e}^{-2t/\tau} + \varepsilon_{\mathrm{equ}} \left(1\text{-}\mathrm{e}^{-2t/\tau}\right) \tag{1}$$

and we can now give explicit expressions for the damping times and equilibrium emittances:

$$\mathcal{J}_x \tau_x = \mathcal{J}_y \tau_y = \mathcal{J}_E \tau_E = 2 \frac{E_0}{U_0} T_0$$

$$\mathcal{J}_x = 1 - \frac{I_4}{I_2}$$

$$\mathcal{J}_y = 1$$

$$\mathcal{J}_E = 2 + \frac{I_4}{I_2}$$

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2$$

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{\mathcal{J}_E I_2} \qquad \sigma_\tau = \frac{\alpha_p}{\omega_s} \sigma_\delta$$

$$\alpha_p = \frac{I_1}{C_0}$$

$$\omega_s^2 = -\frac{eV_{RF}}{E_0} \frac{\omega_{RF}}{T_0} \alpha_p \cos(\phi_s)$$

$$\sin(\phi_s) = \frac{U_0}{eV_{RF}}$$

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{\mathcal{J}_x I_2}$$

The synchrotron radiation integrals are defined:

$$I_1 = \oint \frac{\eta}{\rho} \mathrm{d}s$$

$$I_2 = \oint \frac{1}{\rho^2} \mathrm{d}s$$

$$I_3 = \oint \frac{1}{|\rho|^3} \mathrm{d}s$$

$$I_4 = \oint \frac{\eta}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right)\mathrm{d}s \qquad k_1 = \frac{1}{B\rho}\frac{\partial B_y}{\partial x}$$

$$I_5 = \oint \frac{\mathcal{H}}{|\rho|^3} \mathrm{d}s \qquad \mathcal{H} = \gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2$$

3 The Theoretical Minimum Emittance (TME) Lattice

A linear collider will achieve high luminosity by compressing the transverse beam size at the interaction point. Although a very small vertical emittance is essential, a small horizontal emittance is also required. We saw in section 2.2.3 that the natural emittance of the lattice is given by:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{\mathcal{J}_x I_2}$$

With very low coupling (of the order 0.5% or less) this is essentially the same as the horizontal emittance. In the simple case of a lattice without a magnetic gradient in the dipoles and without a wiggler, $\mathcal{J}_x \approx 1$ and $I_2 = 2\pi/\rho$. Assuming a fixed bending radius in the dipoles, the only control we then have over the emittance is through I_5. In this section, we shall see that under these conditions, there is a minimum emittance that can be achieved, and that this requires specific values for the dispersion and horizontal beta function in the dipoles.

An isomagnetic lattice is one where every bending magnet has the same field, and bends in the same direction. For such a lattice, the fifth synchrotron radiation integral can be written:

$$I_5 = \frac{\langle \mathcal{H} \rangle_{\text{dipoles}}}{\rho^3}$$

Where the average is taken only over the dipoles. Writing out an explicit expression for the evolution of $\mathcal{H}$ through a dipole and then minimizing the average with respect to the lattice functions is a fairly straightforward procedure if one knows how the lattice functions themselves evolve through the dipole. We do not write out the complete analysis (which is not very enlightening), but give the expressions needed as the starting point of the calculation and then quote the final results.

The dispersion obeys the inhomogeneous equation:

$$\eta''(s) + K\eta(s) = \frac{1}{\rho} \qquad K = \frac{1}{\rho^2} + k_1 \tag{21}$$

The general solution can be written in terms of the same transfer matrix that applies to the phase space co-ordinates:

$$\begin{pmatrix} \eta(s) \\ \eta'(s) \end{pmatrix} = \mathbf{M} \cdot \begin{pmatrix} \eta(0) \\ \eta'(0) \end{pmatrix} + \frac{1}{K\rho} \begin{pmatrix} 1 - \cos\left(\sqrt{K}s\right) \\ \sqrt{K}\sin\left(\sqrt{K}s\right) \end{pmatrix}$$

where $\mathbf{M}$ is the transfer matrix from 0 to s. The evolution of the Twiss parameters may be found from:

$$\mathbf{A}(s) = \mathbf{M} \cdot \mathbf{A}(0) \cdot \mathbf{M}^{\mathrm{T}}$$

$$\mathbf{A}^{-1}(s) = \begin{pmatrix} \gamma(s) & \alpha(s) \\ \alpha(s) & \beta(s) \end{pmatrix}$$

The transfer matrix for a dipole (which generalizes to a drift space and a quadrupole) is:

$$\mathbf{M} = \begin{pmatrix} \cos\left(\sqrt{K}s\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}s\right) \\ -\sqrt{K}\sin\left(\sqrt{K}s\right) & \cos\left(\sqrt{K}s\right) \end{pmatrix}$$

After some lengthy and not very enlightening algebra, it is possible to use these results to show that with minimum lattice functions at the center of the dipole, the values required to minimize $\langle \mathsf{H} \rangle$ (and hence minimize the natural emittance) are:

$$\beta_{0,\min} = \frac{L}{2\sqrt{15}} + O\left(\theta^3\right) \qquad \eta_{0,\min} = \frac{L\theta}{24} + O\left(\theta^4\right)$$

and the minimum emittance itself is:

$$\varepsilon_{\min} = C_q \gamma^2 \frac{\theta^3}{\mathcal{J}_x 12\sqrt{15}} + O\left(\theta^5\right)$$

In these equations, θ is the total bending angle of a *single* dipole. We have written the final expression including the horizontal damping partition, even though in our isomagnetic lattice, this is close to unity. It is possible, by including a gradient in the dipole, to *raise* the horizontal damping partition, thus reducing the horizontal damping time and the natural emittance. If the gradient is small, the above expressions for the optimal lattice functions and the minimum emittance are still valid. In a practical damping ring the wiggler reduces any advantages of the gradient, and the only significant benefit in including a gradient in the dipole comes from the extra flexibility in matching the lattice functions through the arc cell.

For reference, in a dipole where the beta function and dispersion reach a minimum at the center of the dipole the mean value of the $\mathcal{H}$ function is given by:

$$\langle \mathcal{H} \rangle = \frac{\eta_0^2}{\beta_0} + \frac{1}{2\beta_0 K\rho^2}\left[4\left(\frac{2\sin\left(\frac{1}{2}\sqrt{K}L\right)}{\sqrt{K}L} - 1 \right)\left(\eta_0\rho - \frac{1}{K} \right) - \left(\frac{\sin\left(\sqrt{K}L\right)}{\sqrt{K}L} - 1 \right)\left(\beta_0^2 - \frac{1}{K} \right) \right]$$

A *Theoretical Minimum Emittance* (TME) lattice is one aiming to achieve the minimum possible emittance through control of the lattice functions in the dipoles. The TESLA Damping Ring and NLC Main Damping Ring (MDR) lattices are both based on arcs using TME cells, although the cells look very different. The lattice functions for the TESLA ring are shown in Figure 5, and those for the NLC are shown in Figure 6. Some relevant parameters are given in Table 1.

Table 1

Some parameters for the TESLA and NLC damping rings.

	TESLA	NLC MDR
Beam Energy /GeV	5.00	1.98
Circumference /m	17000	300
Dipole Field /T	0.1941	1.201
Dipole Length /m	9.00	0.96
Gradient in Dipole /m^{-2}	0	-1.00
β_x at Center of Dipole /m	2.342	0.3436
η_x at Center of Dipole /m	0.1576	8.410×10^{-3}

Why do the TME cells for the TESLA and NLC damping rings look so different from each other? A lot of the answer has to do with the bunch train that the rings are required to damp. In the case of TESLA, this drives the circumference of the damping ring to 17 km. Note that the natural emittance scales as the inverse third power of the number of cells; in the TESLA damping ring there is plenty of room for any number of cells. Furthermore, since the TESLA ring requires a very long wiggler to achieve the necessary damping rate irrespective of the dipole field, the design team has opted for a long dipole with a low field, to give a relatively large momentum compaction. A larger momentum compaction increases the bunch length, which helps reduce the impact of a variety of collective effects. By contrast, the NLC has bunch trains about 80 m long. A ring of this circumference cannot accommodate the required number of cells for meeting the target natural emittance, so the design allows for storing three bunch trains (with gaps for firing the injection/extraction kickers). The circumference still needs to be kept as short as possible, to reduce the damping time and thus minimize the length of damping wiggler needed. Using strong dipoles increases the energy loss from the dipoles, and further reduces the length of the wiggler. The strong dipoles have the disadvantage of giving a low value for the momentum compaction, which gives a short bunch length, and makes the beam vulnerable to a range of collective effects.

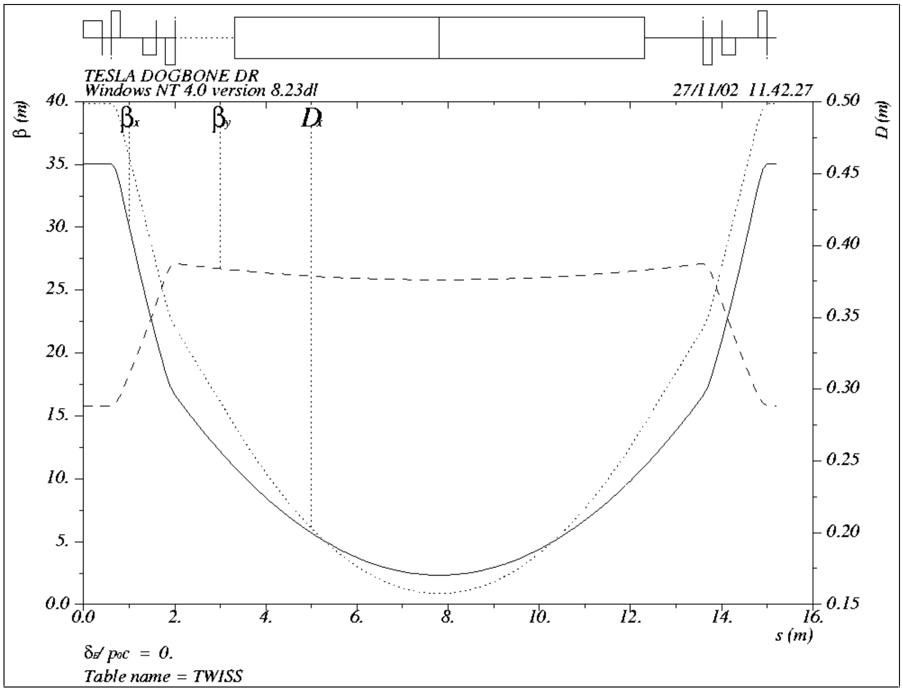

Figure 5

Lattice functions in a single TME arc cell in the TESLA Damping Ring.

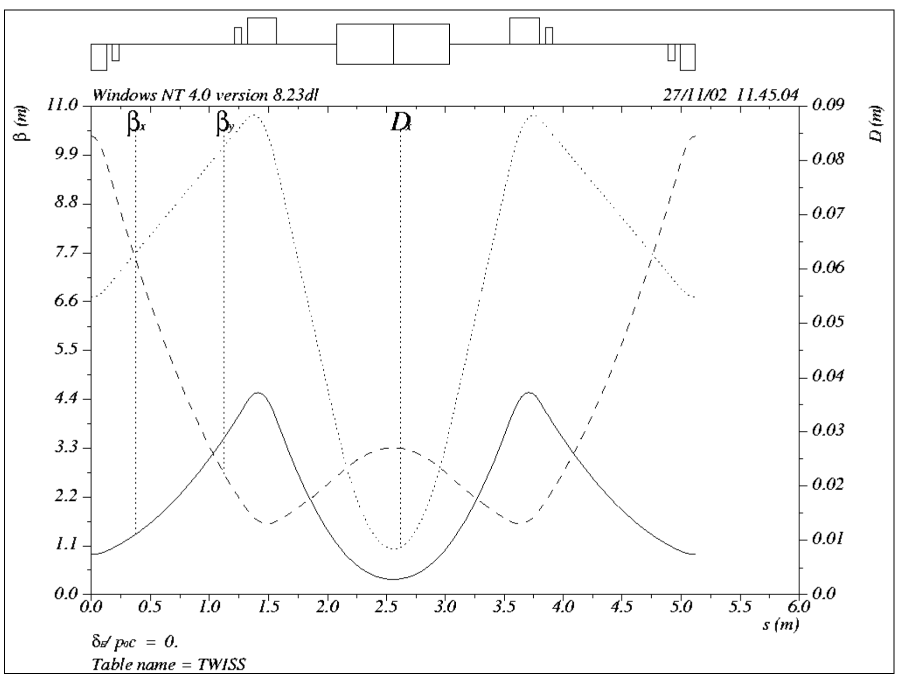

Figure 6

Lattice functions in a single TME arc cell in the NLC Main Damping Ring.

4 Damping Wiggler

It is difficult to achieve the required damping times for a future linear collider damping ring without use of a damping wiggler. The damping time (in any plane) is:

$$\mathcal{J}\tau = 2\frac{E_0}{U_0}T_0$$

where the energy loss per turn is

$$U_0 = \frac{C_\gamma}{2\pi}E_0^4 I_2 = \frac{e^2c^2C_\gamma}{2\pi}E_0^2\oint B^2\mathrm{d}s$$

It clearly helps to keep the circumference (and hence T_0) as short as possible consistent with the length of a bunch train. Beyond that, there are only two ways to reduce the damping time:

- increase the energy of the beam;
- increase the integrated magnetic field seen by the beam.

Increasing the energy also helps reduce the vulnerability to various collective effects. However, it has the undesirable effect of increasing the natural emittance (which scales as the square of the energy), so more cells are needed in the lattice, which drives up the circumference. Also, there is an impact on systems upstream and downstream of the damping ring. The favored method of achieving the short damping times required, therefore, is generally to use a damping wiggler.

By introducing extra bending, the wiggler has an effect on the values of all the synchrotron radiation integrals. In particular, if the wiggler is placed in a location where the dispersion is large, there is a significant growth in the emittance. The physical reason for this is clear: the quantum excitation rate depends on the dispersion through the $\mathcal{H}$ function. A large dispersion where there are large quantities of synchrotron radiation being produced therefore gives a rapid excitation, and a large equilibrium emittance. It is therefore desirable to place the wiggler in a section where the dispersion is nominally zero, although the wiggler itself generates some dispersion through bending the beam. With a proper lattice design, the increased damping rate from the energy loss in the wiggler dominates over the quantum excitation from the small amount of dispersion produced by the wiggler, and the natural emittance of the lattice is reduced to below the value found without the wiggler.

The vertical field component in a wiggler is generally approximated by a sine function:

$$B_y = B_w \sin(k_w z)$$

B_w is the peak field in the wiggler, $k_w = 2\pi/\lambda_w$ where λ_w is the wiggler period, and z is the distance along the wiggler axis. Note that z is distinct from s, the path length of the beam, since the beam trajectory does not follow the wiggler axis. The difference between the two variables, however, is generally small. It is easy to show that the amplitude of the orbit with respect to the wiggler axis is:

$$a_w = \frac{ecB_w}{k_w^2 E_0}$$

For reasonable damping ring parameters, $a_w k_w << 1$, and the path length along the orbit through one wiggler period can then be approximated by:

$$\int_{z=0}^{z=\lambda_w} \mathrm{d}s \approx \lambda_w \sqrt{1 + a_w^2 k_w^2}$$

We shall neglect this small difference, and proceed as though the path length through one wiggler period were the same as the wiggler period.

We should like to calculate the contributions to the synchrotron radiation integrals from the wiggler. I_{2w} and I_{3w} are straightforward, since they do not involve the dispersion. For example, we can immediately write down the energy loss from the wiggler (assumed to consist of a whole number of periods) as:

$$U_w = \frac{e^2 c^2 C_\gamma}{4\pi} E_0^2 B_w^2 L_w$$

To evaluate the other synchrotron radiation integrals we need to know the dispersion generated by the wiggler. This is straightforward, if we use equation (21) and make some approximations. The dispersion in the wiggler satisfies:

$$\eta''(s) + \frac{\sin^2(k_w s)}{\rho_w^2} \eta(s) = \frac{\sin(k_w s)}{\rho_w}$$

If we assume that $|\eta|/\rho_w << 1$, we can drop the second term on the left, and write the solution:

$$\eta(s) \approx -\frac{\sin(k_w s)}{k_w^2 \rho_w}$$

This solution is valid if $k_w \rho_w >> 1$.

We can now write down the following expressions for the synchrotron radiation integrals in the wiggler:

$$I_{1w} \approx -\frac{L_w}{2k_w^2\rho_w^2}$$

$$I_{2w} \approx \frac{L_w}{2\rho_w^2}$$

$$I_{3w} \approx \frac{4L_w}{3\pi\rho_w^3}$$

$$I_{4w} \approx -\frac{3L_w}{8k_w^2\rho_w^4}$$

$$I_{5w} \approx \frac{4\langle\beta_x\rangle L_w}{15\pi k_w^2\rho_w^5}$$

Note that in the fifth integral, the horizontal beta function is averaged over the length of the wiggler. In deriving these expressions, we have treated the wiggler as a continuous sinusoidal magnetic field, with a whole number of periods. We have assumed that the field has no gradient, i.e. is independent of the transverse co-ordinates. For the cases of interest, the above expressions are generally good approximations. Since each synchrotron radiation integral is calculated by integrating around the entire circumference of the ring, it is possible to evaluate any synchrotron radiation integral simply by taking the sum of the contributions from the dipoles with the contribution from the wiggler.

Some parameters for the damping wiggler in the NLC Main Damping Ring and in the TESLA Positron Damping Ring are given in Table 2.

Table 2

Some parameters for the TESLA and NLC damping rings.

	TESLA e^+ DR	NLC MDR
Period /m	0.40	0.27
Peak Field /T	1.6	2.15
Total Length /m	473	46.3
Mean Beta Function $\langle\beta_x\rangle$ /m	11	6

5 Chromaticity, RF Voltage and Acceptance Issues

The average beam power injected into the damping rings is 55 kW for NLC, and 225 kW for TESLA. The injection efficiency may be defined as the fraction of particles lost within a few damping times after injection, and is limited by both physical and dynamic apertures. Injection efficiencies close to 100% have been achieved at machines such as the KEK-ATF, although third generation light sources typically do not suffer performance limitations if the injection efficiency is very much poorer. Because of the high average injected beam power, an injection efficiency that is not very close to 100% will lead to an unacceptable radiation load on components in the ring. The injection efficiency is limited by the physical apertures and by limits on the range of dynamic stability of the particles in the beam. Since the injection efficiency obtained in practice is usually significantly less than that predicted in simulations, it is necessary to design the damping rings with considerable margin in the physical and dynamic apertures.

The physical apertures are reasonably straightforward. Knowing the range of energy and betatron amplitude on the injected beam, and the beta functions and dispersion around the lattice, one can perform symplectic six-dimensional tracking of particles at the maximum injection amplitudes to determine the physical aperture requirements. The dynamic aperture is rather more complicated, since the dynamic stability of particles depends on details of the nonlinear magnetic fields present in the lattice. Since particles may appear to be stable over many hundreds of turns before being lost, it is necessary to perform symplectic tracking through at least this many turns, and preferably several damping times.

Where do the nonlinear magnetic fields come from? We must consider at least three significant sources:

- Sextupoles are needed to correct the chromaticity of the lattice.
- All magnets have higher-order multipole components arising from systematic and random errors.
- The damping wiggler has potentially strong nonlinear components intrinsic to the three-dimensional nature of its magnetic field.

We shall discuss only the first of these in any detail, since the sextupole scheme is a significant issue for the lattice design. Tolerances on the magnets and specification on the wiggler field are usually determined when the lattice design is nearing completion, and there is little that can be done to improve the situation beyond working harder on the designs of those components themselves.

In our treatment of synchrotron oscillations, we made a linear approximation for the time variation of the RF voltage. A more thorough treatment, using the correct sinusoidal variation, gives a definite stability limit on the energy deviation. This limit is the RF acceptance, and is an additional limit to the physical apertures and the dynamic aperture resulting from nonlinear magnetic fields. Since the RF acceptance is important for a number of reasons, we shall include a discussion of this in the present section.

5.1 Chromaticity and Chromatic Correction

The betatron tune is a function of the energy of the particle. The linear chromaticity is the first derivative of the tune with respect to the energy deviation:

$$\xi = \left.\frac{\partial \nu}{\partial \delta}\right|_{\delta=0}$$

The chromaticity is a problem for two reasons. First, particles with significant energy deviations may experience a tune shift that puts them on an integer resonance, where they will not be dynamically stable. Second, some collective phenomena (notably the head-tail instability) are sensitive to the chromaticity, and zero or slightly positive chromaticity is needed to minimize the adverse effects. As we shall see, a lattice consisting of only dipoles and quadrupoles always has large negative chromaticity. We therefore begin by deriving an expression for the chromaticity of a lattice, and then proceed to work out how to correct the chromatic effects using sextupoles.

5.1.1 An Expression for the Chromaticity

Consider first the horizontal plane. The single-turn map of a lattice with total phase advance $\mu_x = 2\pi\nu_x$, at a location where the Twiss parameters are α_x, β_x, γ_x, can be written:

$$\mathbf{M} = \begin{pmatrix} \cos(\mu_x) + \alpha_x \sin(\mu_x) & \beta_x \sin(\mu_x) \\ -\gamma_x \sin(\mu_x) & \cos(\mu_x) - \alpha_x \sin(\mu_x) \end{pmatrix} \tag{22}$$

A focusing error at this location will modify the single-turn matrix as follows:

$$\mathbf{M}' = \begin{pmatrix} 1 & 0 \\ -\Delta k_1 l & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\mu_x) + \alpha_x \sin(\mu_x) & \beta_x \sin(\mu_x) \\ -\gamma_x \sin(\mu_x) & \cos(\mu_x) - \alpha_x \sin(\mu_x) \end{pmatrix}$$

Multiplying out the matrices, and finding the new tune $\mu_x + \Delta\mu_x$ of the lattice gives to first order in the focusing error:

$$\Delta\nu_x = \frac{1}{4\pi}\beta_x \Delta k_1 l$$

Let us suppose that the focusing error comes from the variation in quadrupole focusing with the energy deviation of the particle:

$$k_1 l + \Delta k_1 l = \frac{1}{B\rho(1+\delta)}\frac{\partial B_y}{\partial x} l \approx k_1 l(1-\delta)$$

and hence:

$$\Delta\nu_x = -\frac{1}{4\pi}\beta_x k_1 l \delta$$

Every quadrupole contributes to the tune shift, so to find the total tune shift, we must integrate around the lattice. Thus we find that the horizontal chromaticity is given by:

$$\xi_x = -\frac{1}{4\pi}\oint \beta_x k_1 \mathrm{d}s$$

Quadrupoles that are horizontally focusing will contribute negative chromaticity, while horizontally defocusing quadrupoles will contribute positive chromaticity. However, the beta function is inevitably largest in horizontally focusing quadrupoles; hence the negative chromaticity wins out.

The same arguments apply in the vertical plane, except that the focusing occurs with the opposite sign. The vertical chromaticity is given by:

$$\xi_y = \frac{1}{4\pi}\oint \beta_y k_1 \mathrm{d}s$$

The vertical beta function is largest at quadrupoles with negative k_1, so the natural vertical chromaticity is also negative.

Some values for quadrupole strengths and lattice functions in arc cells of the NLC and TESLA damping rings (from which the chromaticities may be calculated) are given in Table 3. Note that the chromaticity of the full lattice includes the straight sections.

Table 3

Quadrupole parameters for arc cells of the TESLA and NLC damping rings. Note that the TESLA cell includes two horizontally focusing (QF) quadrupoles, and two horizontally defocusing (QD) quadrupoles, while the NLC cell includes two horizontally focusing quadrupoles, and just one horizontally defocusing quadrupole.

	TESLA e^+ DR				NLC MDR			
	$k_1 l$ /m^{-1}	β_x /m	β_y /m	η_x /m	$k_1 l$ /m^{-1}	β_x /m	β_y /m	η_x /m
QF	0.254	34.3	16.2	0.455	1.41	4.15	1.78	0.0833
QD	-0.209	17.6	26.6	0.352	-0.945	0.911	10.33	0.0548
Dipole	0	-	-	-	-0.96	0.746	2.99	0.0158

5.1.2 Chromatic Correction Using Sextupoles

A pure sextupole has only a second field derivative on the closed orbit:

$$k_2 = \frac{1}{B\rho} \left. \frac{\partial^2 B_y}{\partial x^2} \right|_{x=0}$$

Thus, a particle with some horizontal offset in its closed orbit through the sextupole sees a focusing (or defocusing) field:

$$k_1(x_{\mathrm{co}}) = \frac{1}{B\rho} \frac{\partial^2 B_y}{\partial x^2} x_{\mathrm{co}}$$

The horizontal offset may arise from a combination of dispersion with an energy deviation of the particle, $x_{\mathrm{co}} = \eta\delta$. In this case, the sextupole will contribute its own chromaticity to the lattice, and the expression for the horizontal chromaticity becomes:

$$\xi_x = -\frac{1}{4\pi} \oint \beta_x k_1 - \eta_x \beta_x k_2 \mathrm{d}s$$

and for the vertical chromaticity:

$$\xi_y = \frac{1}{4\pi} \oint \beta_y k_1 + \eta_x \beta_y k_2 \mathrm{d}s$$

Thus sextupoles with a positive k_2 compensate the horizontal chromaticity, and sextupoles with a negative k_2 compensate the vertical chromaticity. By placing positive k_2 sextupoles at locations where $\beta_x > \beta_y$, and negative k_2 sextupoles where $\beta_y > \beta_x$, it is possible to compensate simultaneously both horizontal and vertical chromaticities.

The drawback to this use of sextupoles is that geometric aberrations are introduced, i.e. the betatron oscillations become nonlinear, and possibly unstable, for particles with zero or non-zero energy deviation. The dynamic aperture is the range of betatron amplitudes over which the oscillations are stable. A large dynamic aperture is necessary for good injection efficiency. Optimization of the dynamic aperture while maintaining chromatic correction is a challenging task. Some general guidelines include the following:

- The magnitude of the natural chromaticity should be as small as possible in both planes. Generally, one aims for a normalized chromaticity (the chromaticity divided by the tune) of less than 3. This is achieved by keeping the beta functions small, and controlling the phase advance over different parts of the lattice.
- Locations for efficient use of sextupoles should be provided. These locations will have good separation of the beta functions, and large dispersion (see Figure 5 and Figure 6, for example). Note that the need for large dispersion is in conflict with the need for low dispersion to keep the emittance small.
- The phase advance between the sextupoles should be controlled to try and minimize the generation of terms driving betatron resonances.
- The tunes of the lattice should be as far as possible from resonance.

Although it is possible to correct the linear chromaticity with an appropriate sextupole scheme, there exist higher-order chromaticities (the higher order derivatives of tune with respect to energy deviation) that can be difficult to control, and can lead to large variations in tune for off-energy particles. We do not discuss the effects of higher-order chromaticity here.

5.2 RF Acceptance

In section 2.1.1, we showed that the longitudinal equations of motion (including energy gain from the RF cavities and energy loss from the dipoles and wiggler) are:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = -\alpha_p \delta$$

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} = \frac{eV_{RF}}{E_0 T_0}\left[\sin(\phi_s - \omega_{RF}\tau) - \sin(\phi_s)\right]$$

These may be derived from the Hamiltonian:

$$H = -\tfrac{1}{2}\alpha_p \delta^2 - \frac{eV_{RF}}{E_0 T_0 \omega_{RF}}\left[\cos(\phi_s - \omega_{RF}\tau) - \sin(\phi_s)\omega_{RF}\tau\right]$$

using Hamilton's equations:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\partial H}{\partial \delta}$$

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} = -\frac{\partial H}{\partial \tau}$$

The Hamiltonian is a constant of the motion, so in longitudinal phase space the trajectory of a particle appears as a line of constant H. The longitudinal phase space for the NLC MDR is shown in Figure 7. Note the separatrix passing through the unstable fixed point at $\tau \approx 340$ ps; the existence of this fixed point is directly related to the sinusoidal shape of the RF voltage. Trajectories outside this separatrix are unstable. This means that there exists a maximum energy deviation, δ_{RF}, beyond which particles are lost because they are outside the RF bucket height. This maximum energy deviation is the RF acceptance, and it is given by:

$$\delta_{RF}^2 = -\frac{4eV_{RF}}{E_0 T_0 \omega_{RF} \alpha_p}\left[\cos(\phi_s) + \left(\phi_s - \frac{\pi}{2}\right)\sin(\phi_s)\right]$$

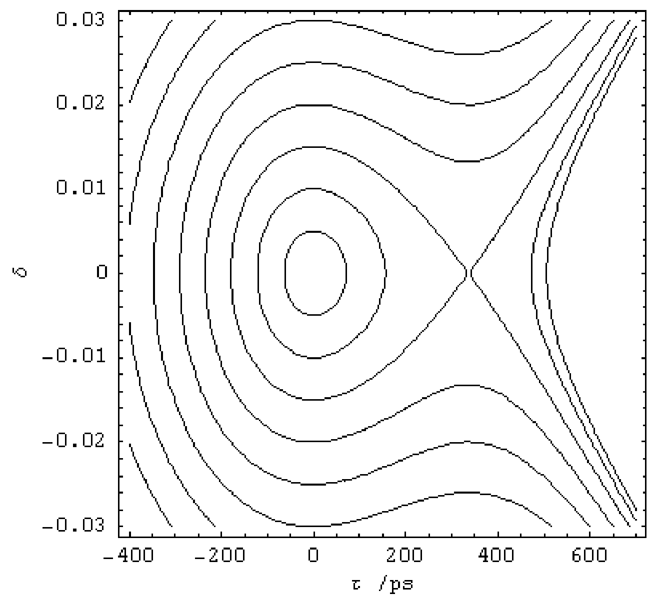

Figure 7

Longitudinal phase space portrait for the NLC MDR.

The RF voltage must be set so that the energy range on the injected beam is within the RF bucket height, with some margin. Usually, the energy acceptance is limited not by the RF voltage, but because the transverse dynamic aperture collapses as the energy deviation increases.

5.3 A Note on Injection Schemes for Damping Rings

Third-generation light sources generally use off-axis injection schemes. In off-axis injection, kicker magnets are used to give a local distortion to the closed orbit, so that the beam is brought close to the septum blade, in the zero-field region of the septum. At the same time, particles are injected into the ring through the region of the septum carrying a magnetic field, so that they arrive parallel to the stored beam, but with some horizontal offset. The kickers are turned off over several turns, but because of the betatron oscillations, the newly injected particles avoid collision with the septum blade. The kickers remain off during several damping times, while the trajectory of the newly injected particles damps down to the closed orbit. The kickers can then be turned on again with losing any particles. The advantage of this scheme is that it allows beam to be "stacked" in the storage ring, with particles being added to RF buckets already containing numbers of particles.

Unfortunately, this off-axis injection scheme cannot be used to stack current in a damping ring for a linear collider, since several damping times are required between each injection of current. This would limit the repetition rate of the collider, and allow only a fraction of the potential luminosity to be achieved. Instead, damping rings are designed with on-axis injection, where the kickers are used so that particles arriving close to, but at some angle to the closed orbit at the entrance to the kicker, are following the closed orbit at the exit. If there are any particles already in the buckets to be filled, the kicker will kick them out of the ring, so stacking current is not possible. Instead, the buckets must be filled in one shot, with the kickers turning on and off in the gap between bunches (TESLA) or bunch trains (NLC). The shortest rise/fall time that can be achieved with kicker technology (consistent with the required amplitude and stability) determines the length of the TESLA damping ring.

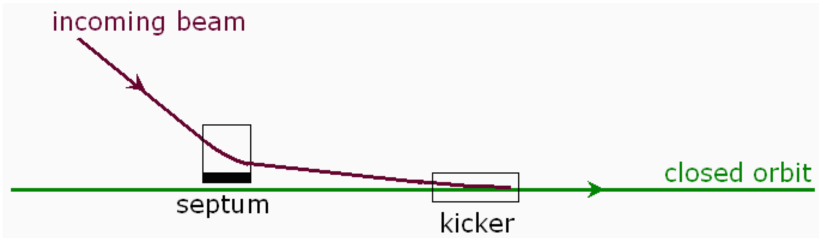

Figure 8

On-axis injection. The incoming beam is initially at a large angle to the closed orbit, and is deflected by the static field in the septum to be almost parallel to the closed orbit. Particles already on the closed orbit see no field from the septum. The small deflection from the kicker removes the remaining angle from the injected beam such that particles are on the closed orbit and parallel to it at the exit of the kicker. Any particles already on the closed orbit would be kicked out of the ring by the kicker, so stacking current is not possible.

The difference between the injection/extraction schemes in NLC and TESLA arises from the fact that the bunch train must be compressed in the TESLA damping ring. Figure 9 shows the scheme used in the NLC damping ring, where three trains are stored at any one time, and the injection and extraction kickers fire in the gap between two trains. The TESLA scheme is straightforward, and simply requires the injection/extraction of individual bunches.

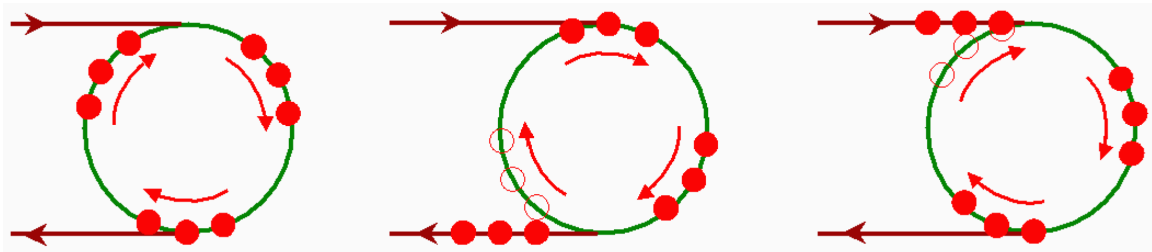

Figure 9

Injection/extraction scheme for the NLC damping rings. The ring stores three trains of 192 bunches (the figure shows just three bunches for clarity). (a) All bunch trains are damping, kickers are turned off. (b) The extraction kicker turns on in the gap between two bunch trains, and the train that has been damping for longest is extracted. (c) Less than one turn later, the injected train enters the ring, and is brought onto the closed orbit by the injection kicker, filling exactly the gap left by the bunch train that has just been extracted. The RF cavities follow the injection point, and are unaware that an injection/extraction event has taken place.

6 Alignment and Stability

The luminosity of a linear collider depends crucially on the vertical emittance extracted from the damping ring (and preserved through the rest of the machine). Both TESLA and NLC specify extracted normalized vertical emittances of 0.02 µm. In the case of the NLC, this requires an equilibrium geometric vertical emittance of 3.4 pm, and for TESLA, 1.4 pm. The lowest vertical emittances that have been achieved in any storage ring so far are of the order 10 pm. Calculation of the lower vertical emittance limit arising from the vertical opening angle of the synchrotron radiation gives values an order of magnitude below those required in the damping rings. So why are very small vertical emittances difficult to achieve? Essentially, there are three reasons:

- Horizontal betatron oscillations are coupled into the vertical plane in skew fields, that come (for example) from rotated quadrupoles or vertically offset sextupoles.
- Horizontal dispersion is coupled into the vertical plane by skew fields. This leads to a non-zero value for the vertical $\mathcal{H}$-function, resulting in vertical quantum excitation by the same process that gives horizontal quantum excitation.
- Collective effects can act in such a way as to drive vertical oscillations.

In this section, we shall concern ourselves with the first two phenomena, which are both related to alignment and orbit correction issues. We shall consider collective effects later. Generally collective effects reduce if the bunch charge is reduced, whereas the emittance growth from coupling (or vertical dispersion) is a single-particle effect. In this section, we are really referring to the vertical emittance in the limit of zero bunch charge.

6.1 Betatron Coupling

It is easy to understand where betatron coupling comes from, but rather more difficult to quantify its effects in any but the simplest cases. Let us start by considering a single skew quadrupole in an otherwise "ideal" (i.e. coupling-free) lattice. The relevant feature of a skew quadrupole is that it gives a particle a vertical kick depending on its horizontal position. For a thin skew quadrupole, we can write:

$$\Delta y' = -k_s l \cdot x \tag{23}$$

where:

$$k_s l = \frac{1}{B\rho}\int \frac{\partial B_x}{\partial x}\,\mathrm{d}s$$

and the integral is over the length (approaching zero) of the skew quadrupole. We see at once that a particle initially performing only horizontal betatron oscillations will, after passing through the skew quadrupole, be performing both horizontal and vertical betatron oscillations. Note that in consequence of Maxwell's equations, we must also have:

$$\Delta x' = -k_s l \cdot y$$

To see the equilibrium effect of a distribution of skew fields around a storage ring, we need to add up the skew kicks, taking the phase advance between them into account, and do some averaging. This is where it can get tricky. The approach we shall follow here uses Hamiltonian mechanics to construct the equations of motion expressed in action-angle variables. It is easy to see that the skew quadrupole kicks given above, may be derived from Hamilton's equations:

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\partial H}{\partial x'}$$
$$\frac{\mathrm{d}x'}{\mathrm{d}s} = -\frac{\partial H}{\partial x}$$

using the Hamiltonian:

$$H = k_s xy$$

We define action-angle variables J_x, ϕ_x, by:

$$x = \sqrt{2J_x\beta_x}\cos(\phi_x)$$
$$x' = -\sqrt{\frac{2J_x}{\beta_x}}\left[\sin(\phi_x) + \alpha_x\cos(\phi_x)\right]$$

and similarly for the vertical plane. Since this transformation is canonical, we may express the equations of motion directly in terms of the action-angle variables using the Hamiltonian:

$$\begin{aligned} H &= 2k_s\sqrt{\beta_x\beta_y}\sqrt{J_xJ_y}\cos(\phi_x)\cos(\phi_y) \\ &= k_s\sqrt{\beta_x\beta_y}\sqrt{J_xJ_y}\left[\cos(\phi_x - \phi_y) + \cos(\phi_x + \phi_y)\right] \end{aligned}$$

We now make our first approximation. We shall be interested only in the effects of the term involving the *difference* of the angle variables: this is the first-order difference resonance. The other term, the sum resonance, we shall not consider further.

The Hamiltonian for our system (a particle moving through a magnetic lattice) is a function of the independent variable s. This creates difficulties that the theory of accelerator optics (beta functions etc.) has been designed to solve. For our present discussion of coupling, we shall sweep these difficulties aside, and look for some "averaged" Hamiltonian that can be used to determine global properties of the beam. Thus, we shall find a ratio of the equilibrium emittances, assumed to be the same throughout the lattice, whereas in reality this quantity is a function of the position in the lattice.

In the simple case of uncoupled linear betatron motion, the change in the angle variable between any two points of the lattice is simply equal to the betatron phase advance. Thus, we can construct a Hamiltonian describing the dynamics in the storage ring:

$$\frac{C_0}{2\pi} H = \nu_x J_x + \nu_y J_y + \kappa(s)\sqrt{J_x J_y}\cos(\phi_x - \phi_y)$$

where the betatron tunes are ν_x and ν_y, and $\kappa(s)$ is given by:

$$\kappa(s) = \frac{C_0}{2\pi}\sqrt{\beta_x \beta_y}\, k_s$$

Our aim is to "average" the Hamiltonian so that it does not depend explicitly on the independent variable s. We can then simply construct the equations of motion, and investigate their solutions. To proceed, we note that $\kappa(s)$ is periodic, with period C_0, the circumference of the lattice. We may then write $\kappa(s)$ as a sum over Fourier modes, with an appropriate phase function $\chi_n(s)$:

$$\kappa(s) = \sum_n \tilde{\kappa}_n \mathrm{e}^{-\mathrm{i}\chi_n(s)} \tag{24}$$

What is an appropriate form for the phase function? There are three conditions that an appropriate function should satisfy:

- The coupling effects of skew quadrupoles should add coherently, depending on the phase advance between them. More explicitly, two skew quadrupoles will clearly add in phase if the phase advances horizontally and vertically satisfy $\mu_x - \mu_y = 0$.
- $\kappa(s)$ is periodic in s, so $\chi_n(s)$ must also be periodic (modulus 2π).
- The modes should be orthonormal.

A suitable form for the phase function is:

$$\chi_n(s) = (\mu_x - \mu_y) - 2\pi(\nu_x - \nu_y - n)\frac{s}{C_0}$$

Using the orthonormality condition, we can write:

$$\tilde{\kappa}_n = \frac{1}{2\pi}\int_0^{C_0} \sqrt{\beta_x \beta_y}\, k_s \mathrm{e}^{\mathrm{i}\chi_n(s)} \mathrm{d}s$$

Let us suppose a single Fourier mode dominates over the others. Then we drop all except a single term in the summation in (24), and the Hamiltonian becomes:

$$\frac{C_0}{2\pi}H = \nu_x J_x + \nu_y J_y + |\tilde{\kappa}_n|\sqrt{J_x J_y}\cos(\phi_x - \phi_y) \tag{25}$$

By selecting a single Fourier mode driving the resonance, we have eliminated the explicit dependence of the Hamiltonian on the independent variable s. From now on, we drop the subscript n that indicates the selected Fourier mode. In action-angle variables, Hamilton's equations are:

$$\frac{\mathrm{d}J_x}{\mathrm{d}s} = -\frac{\partial H}{\partial \phi_x} \qquad \frac{\mathrm{d}\phi_x}{\mathrm{d}s} = \frac{\partial H}{\partial J_x}$$

and similarly for the vertical variables. It is then easy to write down the equations of motion:

$$\frac{\mathrm{d}J_x}{\mathrm{d}s} = |\tilde{\kappa}|\sqrt{J_x J_y}\sin(\phi_x - \phi_y) \qquad \frac{\mathrm{d}\phi_x}{\mathrm{d}s} = \frac{2\pi}{C_0}\left[\nu_x + \frac{|\tilde{\kappa}|}{2}\sqrt{\frac{J_y}{J_x}}\cos(\phi_x - \phi_y)\right]$$

$$\frac{\mathrm{d}J_y}{\mathrm{d}s} = -|\tilde{\kappa}|\sqrt{J_x J_y}\sin(\phi_x - \phi_y) \qquad \frac{\mathrm{d}\phi_y}{\mathrm{d}s} = \frac{2\pi}{C_0}\left[\nu_y + \frac{|\tilde{\kappa}|}{2}\sqrt{\frac{J_x}{J_y}}\cos(\phi_x - \phi_y)\right]$$

Don't panic, things are not as bad as they might appear. First, we notice straight away that the sum of the actions in the two planes is conserved:

$$J_0 = J_x + J_y$$

$$\frac{\mathrm{d}J_0}{\mathrm{d}s} = 0$$

Second, we find that there are fixed points, occurring at:

$$J_x = \frac{1}{2}J_0\left(1 + \frac{\Delta}{\sqrt{\Delta^2 + |\tilde{\kappa}|^2}}\right)$$
$$J_y = \frac{1}{2}J_0\left(1 - \frac{\Delta}{\sqrt{\Delta^2 + |\tilde{\kappa}|^2}}\right) \tag{26}$$

where the tune split is given by:

$$\Delta = |\nu_x - \nu_y|$$

The emittance ratio (sometimes loosely referred to as the coupling) is, for $|\tilde{\kappa}| << \Delta$:

$$\frac{J_y}{J_x} = \frac{|\tilde{\kappa}|^2}{|\tilde{\kappa}|^2 + 4\Delta^2}$$

Finally, we notice that in the presence of the coupling term, the tunes are now:

$$\frac{C_0}{2\pi}\frac{\mathrm{d}\phi}{\mathrm{d}s} = \tfrac{1}{2}\Sigma \pm \tfrac{1}{2}\sqrt{\Delta^2 + \left|\tilde{\kappa}\right|^2} \tag{27}$$

where

$$\Sigma = \nu_x + \nu_y$$

The most important equation is the second of equations (26). This tells us that to achieve a low vertical emittance, we need to maintain both a large tune split (i.e. stay away from the coupling resonance), and minimize the sources of coupling in the ring. In practice, this means correcting quadrupole rotations and vertical misalignments of sextupoles. Equation (27) is also important, since it provides a way for diagnosing the strength of the coupling that exists in the ring. The procedure is straightforward. One simply adjusts the betatron tunes to move across the coupling resonance, recording the tunes measured (for example) by the response to a beam shaker. The coupling strength is given simply by the closest approach of the measured tunes. Figure 10 shows the characteristic variation in coupled tunes as the uncoupled tune split is changed, in a lattice close to the coupling resonance, with some skew quadrupole at a location of zero dispersion. Figure 11 shows the corresponding changes in vertical emittance. The effect of the coupling resonance is clear. The width of the resonance peak is determined by the strength of the skew term(s) driving the coupling.

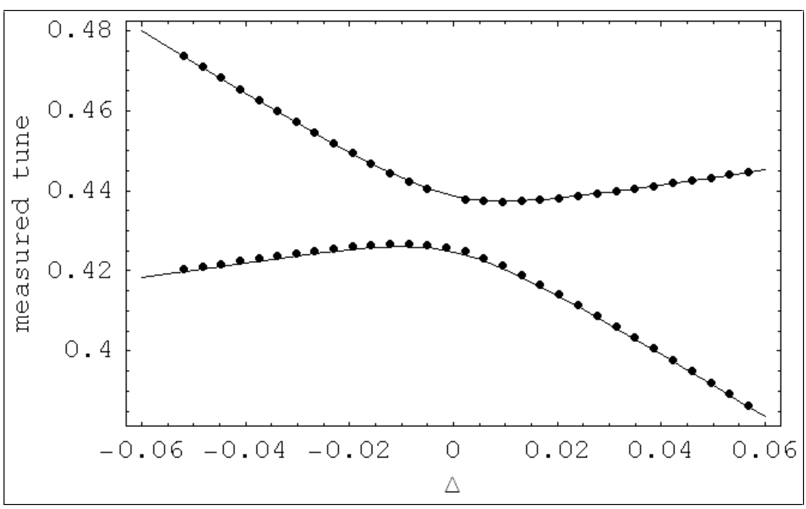

Figure 10

Measured tunes as a function of the uncoupled tune split. The lines show the expected variation from equation (27), while the points show the results from the beamline simulation code MERLIN. The lattice used was a modification of the NLC Main Damping Ring, with the wiggler omitted, and a single skew quadrupole inserted at a zero-dispersion location. The uncoupled tunes were controlled by making small adjustments to the focusing and defocusing quadrupoles in the arc cells.

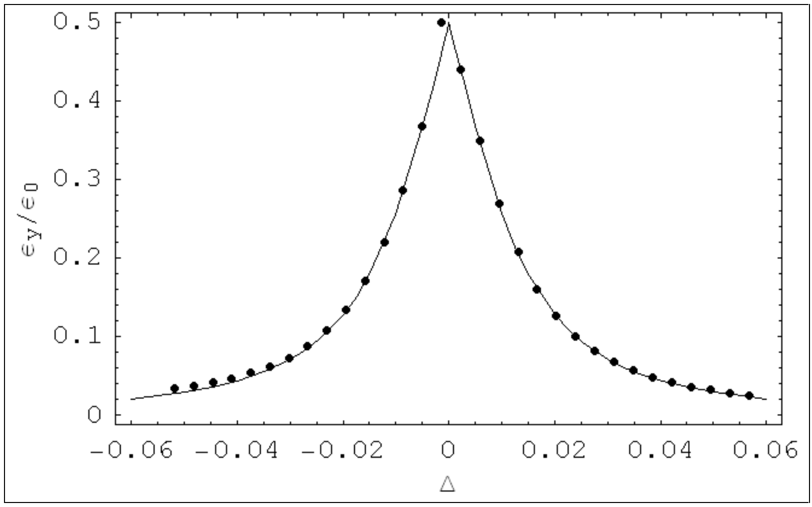

Figure 11

Variation of vertical emittance (normalized to the natural emittance) as a function of the uncoupled tune split, under the same conditions as the results shown in Figure 10. The line shows the expected variation from the second of equations (26).

In practice, one frequently finds that vertical sextupole misalignments dominate the coupling over quadrupole rotations. The Hamiltonian approach provides a simple way to find the strengths of the couplings introduced. The Hamiltonian for a quadrupole is:

$$H = \tfrac{1}{2}x'^2 + \tfrac{1}{2}y'^2 + \tfrac{1}{2}k_1\left(x^2 - y^2\right)$$

Rotating the quadrupole through an angle θ is equivalent to rotating the co-ordinates in the potential terms of the Hamiltonian through $-\theta$. The result is:

$$H \to \tfrac{1}{2}x'^2 + \tfrac{1}{2}y'^2 + \tfrac{1}{2}k_1\cos(2\theta)\left(x^2 - y^2\right) + k_1\sin(2\theta)xy$$

Thus, rotating a quadrupole through angle θ introduces a skew component with strength

$$k_s = k_1\sin(2\theta)$$

Similarly, we can write the Hamiltonian for a sextupole:

$$H = \tfrac{1}{2}x'^2 + \tfrac{1}{2}y'^2 + \tfrac{1}{6}k_2\left(x^3 - 3xy^2\right)$$

If the sextupole is displace vertically Δy, this is equivalent to the transformation $y \to y - \Delta y$ in the Hamiltonian. Thus, the Hamiltonian is transformed to first order in Δy:

$$H \to \tfrac{1}{2}x'^2 + \tfrac{1}{2}y'^2 + \tfrac{1}{6}k_2\left(x^3 - 3xy^2\right) + k_2\Delta y \cdot xy$$

Displacing a sextupole vertically by Δy introduces a skew component with strength

$$k_s = k_2\Delta y$$

6.2 Vertical Dispersion

The analysis of the vertical emittance generated by vertical dispersion is more straightforward than the analysis of betatron coupling. It is also not difficult to measure the vertical dispersion in an operating machine: one simply measures the change in orbit as the energy of the beam is varied. Usually, one controls the energy through the RF frequency. In practice, there are a number of technical details that limit the accuracy with which the dispersion may be measured, not the least of which is the BPM resolution. These considerations are important for damping rings, where, as we shall see, careful correction of the vertical dispersion is needed.

Vertical dispersion comes from two sources in a storage ring:

- Vertical quadrupole misalignments and dipole tilts generate horizontal magnetic fields that steer the beam vertically.
- Skew fields, for example from rotated quadrupoles or vertically misplaced sextupoles, can couple horizontal dispersion into the vertical plane.

Before we proceed to estimate the vertical emittance generated by vertical dispersion, let us consider each of these sources, and estimate how much dispersion we expect to find.

6.2.1 Vertical Dispersion from Vertical Steering

Let us start by understanding the closed orbit distortion and the dispersion resulting from a localized kick. We can then use linear superposition to write down general expressions for the result of a distribution of kicks around the ring.

If the single-turn transfer matrix at some point s_0 in the ring is $\mathbf{M}$, then the closed orbit condition is simply:

$$\mathbf{M} \cdot \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

If we introduce a magnet error at this point in the ring such that the beam is kicked by an angle θ, then the closed orbit condition becomes:

$$\mathbf{M} \cdot \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' - \theta \end{pmatrix}$$

The kick at one location in the lattice leads to a cusp in the closed orbit (Figure 12).

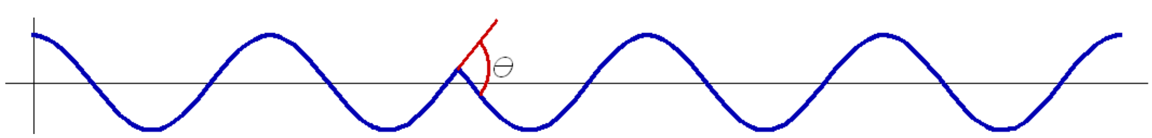

Figure 12

Orbit distortion from a kick through angle θ at one location in the lattice.

Using the general expression (22) for the single-turn transfer matrix in terms of the Twiss parameters, we find that the change in the closed orbit at the location of the kick is:

$$y_0 = \frac{\beta_{y0}\theta}{2\sin(\pi\nu_y)}\cos(\pi\nu_y)$$

$$y_0' = \frac{\theta}{2\sin(\pi\nu_y)}\left[\sin(\pi\nu_y) - \alpha_{y0}\cos(\pi\nu_y)\right]$$

The invariant action associated with this orbit distortion is found to be:

$$J_y = \frac{\beta_{y0}\theta^2}{8\sin^2(\pi\nu_y)}$$

Hence, at a point s_1 in the lattice where the betatron phase is ϕ_y, the orbit is:

$$y(s_1) = \frac{\sqrt{\beta_y(s_1)\beta_y(s_0)}}{2\sin(\pi\nu_y)}\theta\cos(\phi_y)$$

$$y'(s_1) = -\frac{\theta}{2\sin(\pi\nu_y)}\sqrt{\frac{\beta_y(s_0)}{\beta_y(s_1)}}\left[\sin(\phi_y) + \alpha_y(s_1)\cos(\phi_y)\right]$$

For consistency, we must fix the phase at s_0 to $\phi_y(s_0) = -\pi\nu_y$. If we choose instead the phase $\phi_y(s=0) = 0$, we must write:

$$y(s_1) = \frac{\sqrt{\beta_y(s_1)\beta_y(s_0)}}{2\sin(\pi\nu_y)}\theta\cos\left(\pi\nu_y - \left|\phi_y(s_1) - \phi_y(s_0)\right|\right) \tag{28}$$

where the modulus signs are required to ensure that the closed orbit is periodic:

$$y(C_0) = y(0)$$

A distribution of kicks around the ring will give a closed orbit that is just the linear superposition of the orbits generated by the individual kicks. Hence, we have for the general case:

$$y(s_1) = \frac{\sqrt{\beta_y(s_1)}}{2\sin(\pi\nu_y)}\oint\frac{\sqrt{\beta_y(s)}}{\rho(s)}\cos\left(\pi\nu_y - \left|\phi_y(s_1) - \phi_y(s)\right|\right)\mathrm{d}s \tag{29}$$

Now let us consider the dispersion associated with an orbit distortion. We start with Hill's equation for the vertical orbit:

$$y'' - k_1 y = \frac{1}{(1+\delta)\rho} \approx \frac{1-\delta}{\rho} \tag{30}$$

where k_1 is the horizontal focusing:

$$k_1 = \frac{1}{(1+\delta)B\rho}\frac{\partial B_y}{\partial x} \approx (1-\delta)\frac{1}{B\rho}\frac{\partial B_y}{\partial x}$$

Taking the derivative of equation (30) with respect to the energy deviation gives:

$$\eta_y'' - k_1\eta_y = -k_1 y - \frac{1}{\rho} \tag{31}$$

In other words, the dispersion obeys the same equation of motion as the orbit itself, namely equation (30), but with a modified driving term on the right hand side. Thus, looking at equation (29) we can immediately write down the solution:

$$\eta_y(s_1) = -\frac{\sqrt{\beta_y(s_1)}}{2\sin(\pi\nu_y)}\oint\sqrt{\beta_y(s)}\left(k_1 y + \frac{1}{\rho}\right)\cos\left(\pi\nu_y - \left|\phi_y(s_1) - \phi_y(s)\right|\right)\mathrm{d}s$$

6.2.2 Dispersion Coupling

We should also consider vertical dispersion that comes from the coupling of horizontal dispersion into the vertical plane. Using equation (23) with $x = \eta_x\delta$, an off-momentum particle receives a vertical kick in a skew quadrupole of integrated strength $k_s l$, given by:

$$\Delta y' = -k_s l \eta_x \delta$$

This is readily included as a driving term in equation (31), as is the skew quadrupole seen by a particle passing through a sextupole with some vertical offset. If we assume that the sextupole is correctly aligned with respect to the *design* orbit, then the vertical offset of the particle simply corresponds to the orbit distortion at the sextupole location:

$$k_s = -k_2 y$$

Including all these terms gives for the vertical dispersion:

$$\eta_y(s_1) = -\frac{\sqrt{\beta_y(s_1)}}{2\sin(\pi\nu_y)}\oint\sqrt{\beta_y(s)}\left[(k_1 - k_2\eta_x)y + k_s\eta_x + \frac{1}{\rho}\right]\cos\left(\pi\nu_y - \left|\phi_y(s_1) - \phi_y(s)\right|\right)\mathrm{d}s \tag{32}$$

6.2.3 Some Simple Sensitivity Indicators

In general, a lattice will include random misalignments of all components. It is important to understand the effects that errors within certain limits will have on the performance of the damping ring, and to devise algorithms for effective compensation of the errors. Rigorous studies require detailed simulations, but with the preceding analysis we are in a position to determine a variety of "sensitivity indicators" which will allow comparison with operating facilities with the aim of deciding whether the performance targets of the damping rings are realistic. Starting from the expressions for the closed orbit distortion (29) and vertical dispersion (32), and assuming that the errors are random and uncorrelated, we can write simple expressions relating the orbit and dispersion to errors in the lattice.

First, the closed orbit is related to vertical misalignments of the quadrupoles by:

$$\left\langle \frac{y^2}{\beta_y} \right\rangle = \frac{\left\langle Y_q^2 \right\rangle}{8\sin^2(\pi\nu_y)}\sum \beta_y (k_1 l)^2$$

where $\langle \ \rangle$ indicates an average around the lattice, $\langle Y_q^2 \rangle$ is the mean square vertical misalignment of the quadrupoles, and the summation extends over all the quadrupoles in the lattice.

Second, the vertical dispersion is related to the rotations of the quadrupoles by:

$$\left\langle \frac{\eta_y^2}{\beta_y} \right\rangle = \frac{\langle \Theta_q^2 \rangle}{2\sin^2(\pi\nu_y)} \sum \beta_y (k_1 l \eta_x)^2$$

$\langle \Theta_q^2 \rangle$ is the mean square rotation of the quadrupoles, and the summation again extends over all the quadrupoles in the lattice.

Finally, the vertical dispersion is related to the vertical misalignments of the sextupoles by:

$$\left\langle \frac{\eta_y^2}{\beta_y} \right\rangle = \frac{\langle Y_s^2 \rangle}{8\sin^2(\pi\nu_y)} \sum \beta_y (k_2 l \eta_x)^2$$

$\langle Y_s^2 \rangle$ is the mean square vertical displacement of the sextupoles, and the summation extends over all the sextupoles in the lattice.

Each of the three above expressions includes a summation over lattice functions, which is readily evaluated from the lattice design. Large beta functions and strong magnets tend to increase the sensitivity to errors. It is also clear that from this perspective, the ideal vertical tune is a half integer.

6.2.4 Vertical Dispersion and Vertical Emittance

The effect with which we are most concerned at present, is the growth in vertical emittance from vertical dispersion. Now that we have expressions for estimating the vertical dispersion from alignment errors in a given lattice, we need to make the final step and relate the vertical emittance to the vertical dispersion. This is reasonably straightforward. We begin by writing the general expression for the equilibrium vertical emittance generated by the balance between quantum excitation and radiation damping:

$$\varepsilon_y = C_q \gamma^2 \frac{I_5}{\mathcal{J}_y I_2}$$

Assuming that the dispersion is generated by random errors, there is nothing special about the vertical dispersion in the horizontal bending magnets (where the radiation is produced). Thus, we can write:

$$I_5 = \oint \frac{\mathcal{H}_y}{|\rho|^3} \mathrm{d}s \approx \langle \mathcal{H}_y \rangle I_3$$

and so:

$$\varepsilon_y \approx C_q \gamma^2 \frac{\langle \mathcal{H}_y \rangle I_3}{\mathcal{J}_y I_2} = \frac{\mathcal{J}_E}{\mathcal{J}_y} \langle \mathcal{H}_y \rangle \sigma_\delta^2$$

Next, we note that the $\mathcal{H}$ -function plays a similar role for the dispersion, as the invariant action does for the orbit. Specifically, where there is no driving term for the vertical dispersion, we can write:

$$\eta_y = \sqrt{\beta_y \mathcal{H}_y} \cos(\phi_y)$$

For a large lattice with random errors, we can take an average around the ring:

$$\left\langle \frac{\eta_y^2}{\beta_y} \right\rangle \approx \tfrac{1}{2} \langle \mathcal{H}_y \rangle$$

and thus we find:

$$\varepsilon_y \approx 2 \frac{\mathcal{J}_E}{\mathcal{J}_y} \left\langle \frac{\eta_y^2}{\beta_y} \right\rangle \sigma_\delta^2$$

6.3 Some Comments on Alignment and Emittance Tuning

The sensitivity of a damping ring is such that magnet misalignments of the order of a few microns (for the quadrupoles) or tens of microns (for the sextupoles) are sufficient to blow up the vertical emittance to above the required value. Modern survey techniques are capable of aligning magnets to the design orbit to within a few tens of microns over several meters. Fortunately, we do not need to rely on the initial alignment survey to achieve the desired performance of the damping ring. Instead, we design the ring with a correction system to compensate for the fact that none of the magnets are exactly where we desire them to be.

Generally, the emittance tuning system performs its task in three stages.

- The vertical orbit is determined from the BPM readings. Beam based alignment (BBA) techniques can be used to determine the offsets between the BPMs and the magnets, leading to the construction of a “golden orbit” passing through the center of each quadrupole. Orbit correction is achieved using steering magnets, or quadrupole movers.
- Vertical dispersion is measured by changing the energy of the beam (by adjusting the RF frequency) and measuring the corresponding change in orbit. Correction of the vertical dispersion is achieved by a combination of steering the beam and coupling at locations of horizontal dispersion.
- Betatron coupling is measured by kicking the beam horizontally, and measuring the corresponding change in the vertical orbit. Correction is achieved using skew quadrupoles.

There are many possible tuning algorithms using variations and combinations of the above stages. For example, steering is often applied with the aim of simultaneously minimizing the orbit distortion and the vertical dispersion. Tuning algorithms have been developed for both the NLC and TESLA damping rings that perform well in simulations.

It is possible to include a wide variety of errors, nonetheless it is not possible to reproduce the full operational conditions that may be expected in a control room. For this reason, there is great interest in the experience of storage rings already in operation. The machines closest to the damping rings are the KEK-ATF, and the third generation light sources. A number of machines have demonstrated a vertical emittance approaching that required for the damping rings. Unfortunately for damping ring research, third generation light sources prefer to operate with much larger vertical emittance, providing a compromise between photon beam brightness and electron beam lifetime. Thus, there is little direct motivation for existing facilities to pursue an experimental program in this area.

It is important to note that after successfully tuning a lattice for low emittance, significant errors may remain apparently uncorrected. For example, in simulations of the TESLA damping ring the beam offset through the sextupoles after correcting the vertical emittance below the operational limit is typically several hundred microns. If the errors are assumed to be uncorrelated, then with this magnitude offset in the sextupoles, the expected betatron coupling approaching 100%. This apparent contradiction is easily resolved by the observation that after tuning the lattice, the beam offsets in the magnets are in fact highly correlated. If one calculates the betatron coupling using the known beam offset in each sextupole from the simulation after successful tuning, then a value is always found consistent with an emittance ratio below 0.5%.

For practical calculations, one needs to know the magnet strength and lattice functions for the particular lattice design. Some values for the NLC and TESLA damping rings that will allow some sensitivity estimates to be made are given in Table 4 and Table 5.

Table 4

Some parameters for arc cells of the TESLA and NLC damping rings. Note that the TESLA cell includes two horizontally focusing (QF) quadrupoles, and two horizontally defocusing (QD) quadrupoles, while the NLC cell includes two horizontally focusing quadrupoles, and just one horizontally defocusing quadrupole. The TESLA cell includes a single SF sextupoles, and two SD sextupoles; the NLC cell includes two of each type of sextupole.

	TESLA e^+ DR				NLC MDR			
	$k_n l$ /m^{-1}	β_x /m	β_y /m	η_x /m	$k_n l$ /m^{-1}	β_x /m	β_y /m	η_x /m
QF	0.254	34.3	16.2	0.455	1.41	4.15	1.78	0.0833
QD	-0.209	17.6	26.6	0.352	-0.945	0.911	10.33	0.0548
SF	2.55	35.1	15.8	0.500	23.6	4.00	2.22	0.0857
SD	-1.54	23.2	22.6	0.405	-18.1	1.09	8.99	0.0585

Table 5

Tunes of the NLC and TESLA damping rings.

	TESLA e^+ DR	NLC MDR
ν_x	76.32	27.26
ν_y	41.19	11.14

7 Collective Effects

The high charge density associated with the large bunch current and low emittance, makes the beams in linear collider damping rings susceptible to a wide variety of collective phenomena that threaten operational performance. Effects that need to be considered include the following.

- The (broad band) impedance of the vacuum chamber leads to the growth of high frequency modes within individual bunches, that decohere in the presence of nonlinearities, leading to an increase in emittance. This is the microwave instability.
- Higher order modes in the RF cavities and the resistive wall impedance of the vacuum chamber couple the oscillations of different bunches. This may lead to emittance growth, or a limit on the stored beam current.
- Touschek scattering limits the lifetime of the beam, since some of the high transverse momentum of the particles may be transferred into the longitudinal plane through their interaction within a bunch. Particles given a large energy deviation in this way may be outside the momentum acceptance of the ring, and are quickly lost. Although the limited lifetime itself is not significant for operation (the beams are only stored for milliseconds, whereas even a short Touschek lifetime is measured in minutes) it is a consideration for commissioning and tuning of the ring.
- Intra-beam scattering (IBS) is similar to the Touschek effect, in that it involves scattering of particles within the bunch. The difference is that the Touschek effect consider "large-angle" scattering, in which there is a significant exchange of momentum from transverse degrees of freedom to the longitudinal, whereas IBS considers "small-angle" scattering that does not lead to particles being kicked outside the momentum acceptance. The momentum exchange that does occur in IBS leads to emittance growth in much the same way as quantum excitation from synchrotron radiation.
- The space-charge of the bunch leads to a variation of the betatron tune with betatron amplitude and longitudinal position in the bunch. With a large tune spread, the oscillations of numbers of particles within the bunch may become unstable. Particles may be lost, or the emittance of the entire bunch may be increased to a point where the space-charge tune spread is reduced sufficiently for the oscillations to be stabilized.
- In the positron rings, energy can be transferred to electrons produced by synchrotron radiation striking the vacuum chamber walls, or by gas ionization. High-energy electrons hitting the vacuum chamber can lead to a shower of electrons, resulting in a high electron charge density in the chamber, that destabilizes the beam. This is the electron cloud effect.
- In the electron rings, positive ions produced by gas ionization can be trapped in the beam, leading to instability of the beam. Ions can build up over several turns (the well-known "ion trapping" phenomenon) or within the passage of a single bunch train (the "fast ion" instability).
- Variation in beam loading in the RF cavities from gaps between bunch trains can lead to "phase transients" along the bunch train. Phase transients have

implications for the operation of systems downstream of the damping rings, particularly the bunch compressors and main linac.

- Coherent synchrotron radiation can, in certain regimes, produce significantly higher radiated power than the incoherent radiation considered in the discussion of radiation damping. The radiation produced can interact with the beam, driving a longitudinal instability.

Detailed analysis of these effects and discussion of preventive measures is outside the scope of these notes. In some cases, the phenomena are not fully understood, or there is limited experimental data with which to verify the theoretical models. Some discussion may be found in the references given in the Introduction. Estimates of the possible impact of some of the above effects have been made for the NLC Main Damping Ring ("Estimates of Collective Effects in the NLC Main Damping Rings", A. Wolski and S. de Santis, LCC-0080, May 2002).

8 Problems for the Student

1. Show that (in the context of quantum excitation of synchrotron oscillations):

$$\int \frac{P_\gamma}{\rho} \mathrm{d}s = \frac{I_3}{I_2} \int P_\gamma \mathrm{d}s$$

2. Show that the betatron action induced by a particle emitting a photon of energy u is given by:

$$\Delta J = \frac{1}{2}\left(\frac{u}{E_0}\right)^2 \mathcal{H}$$

3. Refer to equation (19), for the rate of change of betatron action in the presence of damping and quantum excitation. Show that:

$$\frac{1}{2E_0^2 C_0} \oint N \langle u^2 \rangle \mathcal{H} \mathrm{d}s = 2C_q \frac{\gamma^2}{\mathcal{J}\tau} \frac{I_5}{I_2}$$

4. Show that for $1/\rho = 0$, $\mathcal{H}$ is a constant (i.e. independent of s).

5. Suppose that the NLC and TESLA damping rings were composed entirely of arc cells, with parameters as given in Table 1. Assume the lattices are circular, with the circumferences given in Table 1. For each lattice, calculate:
   a) the synchrotron radiation integrals;
   b) the damping times;
   c) the equilibrium energy spread;
   d) the natural emittance.

Compare the natural emittance that you find for each lattice, with the minimum theoretically possible for a lattice with the given energy and dipole bending angle. Explain why the lattice might be designed with a much larger natural emittance than the minimum.

6. Using the wiggler parameters given in Table 2, calculate the contributions from the wigglers to the synchrotron radiation integrals in the NLC and TESLA damping rings. Hence estimate for the full rings (including the wiggler):
   a) the damping times;
   b) the equilibrium energy spread;
   c) the natural emittance.

   Using your result for the damping times, and assuming an injected normalized transverse emittance of 150 µm for the NLC and 0.01 m for TESLA, estimate the equilibrium vertical emittance necessary to achieve the extracted vertical emittance of 0.02 µm for both machines.

7. a) Using values given in Table 3, calculate the chromaticities of the arc cells in the NLC and TESLA damping rings.
   b) Assuming that the sextupoles are superposed onto the quadrupoles, calculate the sextupole strengths required to correct the chromaticity to zero in the cell.
   c) Explain why stronger sextupoles than you calculated in (b) will be needed to correct the chromaticity in the full lattice.

8. Show that the RF acceptance of a storage ring is given by:

$$\delta_{RF}^2 = -\frac{4eV_{RF}}{E_0 T_0 \omega_{RF} \alpha_p}\left[\cos(\phi_s) + \left(\phi_s - \frac{\pi}{2}\right)\sin(\phi_s)\right]$$

9. a) Calculate the RF voltage required to give an RF acceptance of ±1.5% in the NLC and TESLA damping rings.
   b) Using your values from (a) calculate:
      i) the synchrotron frequency;
      ii) the equilibrium bunch length.

10. Show that the fixed points of the dynamical system with Hamiltonian given by (25) occur at the values for the action given by (26).

11. Show that for a beam in a coupled lattice, the tunes observed in the control room are given by *both* values of equation (27).

12. Assuming an otherwise perfect lattice, estimate the uncorrelated sextupole vertical alignment error in the NLC and TESLA damping rings that will lead to a betatron coupling at the respective operational limit of each machine.

13. Estimate the rms vertical dispersion in the NLC and TESLA damping rings that will lead to an equilibrium vertical emittance at the respective operational limit of each

machine. Hence, estimate the quadrupole vertical misalignment, quadrupole rotation and sextupole vertical misalignment that will (separately) give the vertical emittance limit.

14. It has been proposed to raise the repetition rate of the NLC from 120 Hz to 180 Hz. This would allow higher luminosity to be delivered to a single interaction point, or simultaneous luminosity to be delivered to two separate interaction points. Produce an outline design for a damping ring suitable for a version of the NLC to be operated at 180 Hz. Assume that the injected normalized transverse emittances are 150 µm, to be damped down to 3 µm and 0.02 µm horizontally and vertically respectively, at extraction. The injected energy spread is 1%, to be damped down to 0.1% or lower, at extraction. The bunch train consists of 192 bunches, with a spacing of 1.4 ns. You should consider all the principle parameters, including:
    - energy;
    - circumference;
    - number of cells;
    - dipole field;
    - wiggler length;
    - RF voltage.