

































Where N is the total number of quads, R and T are 1^{st} and 2^{nd} order matrices of the total beamline, and we also took into account nonzero position and angle of the injected beam at the entrance.

Predicting orbit motion and chromatic dilution ... random case

Let's assume now that the beam is injected along the reference line, then:

$$x_*(t) = \sum_{i=1}^{N} b_i x_i(t)$$
 $\eta_x(t) = \sum_{i=1}^{N} d_i x_i(t)$

Assume that quads misalignments, averaged over many cases, is zero. Let's find the nonzero variance

$$\langle x_*^2(t) \rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} b_i b_j \langle x_i(t) \cdot x_j(t) \rangle \qquad \langle \eta_x^2(t) \rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} d_i d_j \langle x_i(t) \cdot x_j(t) \rangle$$

Let's first consider a very simple case. In case of **random** uncorrelated misalignment we have $\langle x_j(t) \cdot x_j(t) \rangle = \sigma_x^2 \, \delta_{ij}$ (σ_x is rms misalignment, not the beam size)

So that, for example $\langle x_*^2(t) \rangle = \sigma_x^2 \sum_{i=1}^N b_i^2$

And similar for dispersion

Now we would like to know what are these **b** and **d** coefficients.

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Predicting orbit motion escaping the complete randomness					
Now you have everything to calculate b and d coefficients and find, for example, the rms of the orbit motion at the exit for the simplest case – completely random uncorrelated misalignments.					
Completely random and uncorrelated means that misalignments of two neighboring points, even infinitesimally close to each other, would be completely independent.					
If we would assume that such random and uncorrelated behavior occur in time also, I.e. for any infinitesimally small Dt the misalignments will be random (no "memory" in the system) then it would be obvious that such situation is physically impossible. Simply because its spectrum correspond to white noise, I.e. goes to infinite frequencies, thus having infinite energy.					
We have to assume that things do not get changed infinitely fast, nor in space, neither in time. I.e., there is some correlation with previous moments of time, or with neighboring points in space.					
Let's consider the random walk (drunk sailor). In this case, together with randomness, there is certain memory in this process: the sailor makes the next step relative to the position he is at the present point.					
Extension of random walk model to multiple points in space and time is described by the famous ATL [B.Baklakov, et al, 1991].					
 N.B. Nonzero correlation (often called auto-correlation, when talking about correlation in time) would necessarily mean that spectrum decrease with frequency, saving the energy conservation law. More on this later in the lecture. 					



















Predicting orbit motion for
arbitrary misalignments
So, we would like to calculate, for example,
$$\langle x_{*}^{2}(t) \rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} d_{i} d_{j} \langle x_{i}(t) \cdot x_{j}(t) \rangle$$
 in case of
arbitrary properties of misalignments
One can introduce the spatial harmonics $\mathbf{x}(t,\mathbf{k})$ of wave number
 $\mathbf{k}=2\pi/\lambda$, with λ being he spatial period of displacements:
The displacement $\mathbf{x}(t,s)$ can
be written using the back
 $\mathbf{x}(t,s) = \int_{-\infty}^{\infty} x(t,k)(e^{iks}-1) \frac{dk}{2\pi}$ which ensures that at the
entrance $\mathbf{x}(t,\mathbf{s}=\mathbf{0})=\mathbf{0}$.
Then the variance of dispersion is
 $\langle \eta_{x}^{2}(t) \rangle = \sum_{i} \sum_{j} d_{i}d_{j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \mathbf{x}(t,k_{1})\mathbf{x}^{*}(t,k_{2}) \rangle (e^{ik_{1}s_{i}}-1)(e^{-ik_{2}s_{j}}-1) \frac{dk_{1}}{2\pi} \frac{dk_{2}}{2\pi}$
We can rewrite it as
 $\langle \eta_{x}^{2}(t) \rangle = \sum_{i} \sum_{j} d_{i}d_{j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(t,k)(e^{iks_{i}}-1)(e^{-iks_{j}}-1) \frac{dk}{2\pi}$
Where we defined the spatial power spectrum of displacements $\mathbf{x}(t,\mathbf{s})$ as
 $P(t,k) = \lim_{\mathcal{L}\to\infty} \frac{1}{\mathcal{L}}\mathbf{x}(t,k)\mathbf{x}^{*}(t,k) = \lim_{\mathcal{L}\to\infty} \frac{1}{\mathcal{L}} \left| \int_{-\mathcal{L}/2}^{\mathcal{L}/2} \mathbf{x}(t,s)e^{-iks}ds \right|^{2}$

Predicting orbit motion for arbitrary misalignments

So, we see that we can write the variance of dispersion (and very similar for the offset) in such a way, that the lattice properties and displacement properties are separated:

$$\langle \eta_x^2(t) \rangle = \int_{-\infty}^{\infty} P(t,k) G(k) \frac{dk}{2\pi}$$

Here G(k) is the so-called spectral response function of the considered transport line (in terms of dispersion):

$$G(k) = g_c^2(k) + g_s^2(k)$$

where

$$g_c(k) = \sum_{i=1}^{N} d_i [\cos(ks_i) - 1]$$
 and $g_s(k) = \sum_{i=1}^{N} d_i \sin(ks_i)$

The spectral function for the offset will be the same, but \mathbf{d}_i substituted by \mathbf{b}_i

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Arbitrary ground motion can be fully described, for a linear collider, by a 2-D power spectrum $P(\omega,k)$

If a 2-D spectrum of ground motion is given, the spatial power spectrum **P**(**t**,**k**) can be found as

$$P(t,k) = \int_{-\infty}^{\infty} P(\omega,k) 2[1 - \cos(\omega t)] \frac{d\omega}{2\pi}$$

Example of 2-D spectrum for ATL motion:

The 2-D spectrum can be used to find variance of misalignment. Again, assume that there is an inertial reference frame, where coordinates of our linac are $x_{abs}(t,s)$. And assume that at t=0 the linac was perfectly aligned, and that misalignment with respect to this original positions is $x(t,s) = x_{abs}(t,s) - x_{abs}(t=0,s)$, its variance is given by

$$\langle \left(\mathbf{x}(t,s+L) - \mathbf{x}(t,s) \right)^2 \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{P}(\omega,k) 2 \cdot \left[1 - \cos(\omega t) \right] 2 \cdot \left[1 - \cos(kL) \right] \frac{d\omega}{2\pi} \frac{dk}{2\pi}$$

You can easily verify, for example, that for ATL spectrum it gives the ATL formula

The (directly measurable !) spectrum of relative motion is given by

$$\rho(\omega,L) = \int_{-\infty}^{\infty} P(\omega,k) 2[1 - \cos(kL)] dk/2\pi$$

 $P(\omega,k) = \frac{A}{\omega^2 k^2}$ And for P(t,k): $P(t,k) = \frac{A \cdot t}{k^2}$

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Behavior of spectral functionsRemember that before assuming that beams injected without offset we wrote that $x_*(t) = R_{11} x_{inj}(t) + R_{12} x_{inj}(t) + \sum_{i=1}^{N} b_i x_i(t)$ $\eta_x(t) = T_{116} x_{inj}(t) + T_{126} x_{inj}'(t) + \sum_{i=1}^{N} d_i x_i(t)$ It is easy to show that the coefficients b (and d) follow certain rules, which can be found in the nextway. By considering a rigid displacement of the whole beam line, it is easy to find the identity $\sum_{i=1}^{N} b_i = 1 - R_{11}$ and $\sum_{i=1}^{N} d_i = -T_{116}$ On the other hand, one can show by tilting the whole beamline by a constant angle that the
coefficients satisfy for thin lenses the following identity: $\sum_{i=1}^{N} b_i s_i + R_{12} = s_{exit}$ and $\sum_{i=1}^{N} d_i s_i + T_{126} = 0$ These rules allow to find behavior of the spectral
functions at small k: $g_c(k \to 0) \approx O(k^2)$ $g_s(k \to 0) \approx -k \cdot R_{12} + O(k^3)$ You see that if R_{12} is zero, effect of long wavelength is suppressed as k^2





Slow but short λ ground motion						
• Diffusive or ATL motion: $\Delta X^2 \sim A_D TL$ (minutes-month) (T - elapsed time, L - separation between two points)						
	Place	Α μm ² /(m·s)		~20µm displacement		
	HERA	~ 10-5 -		over 20m in one month		
	FNAL surface	~ 1-few*10 ⁻⁶				
	SLAC*	~ 5*10 ⁻⁷				
	Aurora mine*	_₹ 2*10-7				
	Sazare mine	~ 10 ⁻⁸				
* 39	* Further measurements in Aurora mine, SLAC & FNAL are ongoing					



































































