Nonlinear Motion of a Point Charge in the 3D Space Charge Field of a Gaussian Bunch

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Abstract

The nonlinear motion of a point charge in the three dimensional space charge field of a Gaussian bunch is analyzed. The 3D space charge field of a Gaussian bunch is derived and the results are compared to the well known Bassetti-Erskine formula for the transverse (2D) space charge fields. The results are applied to the nonlinear motion of an electron in the space charge field of a positron bunch and to the motion of an ion in the space charge field of a electron bunch.

1 Introduction

The beam in a storage ring can interact with the charged particles which can be present in the vacuum chamber. In a positron storage ring an electron cloud can build up in the vacuum chamber due to photoemission or secondary emission while in an electron storage ring ions are created by ionization of the residual gas. In both cases the interaction of the circulating positron or electron beam with a cloud of electrons or ions may result in a degeneration of the beam emittance or in beam instabilities. Simulation codes often use the Bassetti-Erskine formula [1] to calculate in a weak-strong approach the interaction between the bunch and the charged particle [2, 3]. Only the transverse kick is calculated in this approach.

In this report the three dimensional space charge field of a Gaussian bunch is calculated and the results are compared with the Bassetti-Erskine formula.
results are applied to the nonlinear motion of an electron in the space charge field of a positron bunch and to an ion in the space charge field of an electron bunch.

The following situation is considered (see Fig. 1): a Gaussian bunch with total charge $Q_b$ is moving with the velocity $v_b$ along the $z$-axis of the laboratory frame $K$. The electric field of the bunch is calculated in the comoving frame $\bar{K}$ and transformed to the laboratory frame $K$ where the Lorentz-Force on a point charge $Q_0$ is calculated.

![Diagram](image)

Figure 1: Laboratory frame $K$ and rest frame $\bar{K}$ of a Gaussian bunch

From the velocity of the beam $v_b$ the normalized velocity $\beta_b$ and the relativistic factor $\gamma_b$ are calculated as

$$\beta_b = \frac{v_b}{c}, \quad \gamma_b = \frac{1}{\sqrt{1 - \beta_b^2}}. \quad (1)$$

The charge density in the comoving (rest) frame is:

$$\rho(\bar{x}, \bar{y}, \bar{z}) = \frac{Q_b}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} \exp\left(-\frac{1}{2}\frac{\bar{x}^2}{\sigma_x^2}\right) \exp\left(-\frac{1}{2}\frac{\bar{y}^2}{\sigma_y^2}\right) \exp\left(-\frac{1}{2}\frac{\bar{z}^2}{\sigma_z^2}\right), \quad (2)$$

where $\bar{x}, \bar{y}, \bar{z}$ are the coordinates in the comoving frame $\bar{K}$, which are related to the coordinates in the laboratory frame via a Lorentz-Transformation (see Appendix A) in the following way:

$$\bar{x} = x, \quad \bar{y} = y, \quad \bar{z} = \gamma_b (z - \beta_b c t). \quad (3)$$

The beam dimensions in the comoving frame are:

$$\sigma_{\bar{x}} = \sigma_x, \quad \sigma_{\bar{y}} = \sigma_y, \quad \sigma_{\bar{z}} = \gamma_b \sigma_z. \quad (4)$$

The bunch is much longer in the comoving frame $\bar{K}$ than in the laboratory frame where the bunch length appears to be Lorentz contracted to the length $\sigma_z = \sigma_{\bar{z}}/\gamma_b$. 

2
The electric space charge fields are calculated from the solution of the Poisson equation in the comoving frame \( \bar{K} \):

\[
\nabla^2 \Phi(\bar{x}, \bar{y}, \bar{z}) = -\frac{1}{\epsilon_0} \rho(\bar{x}, \bar{y}, \bar{z}).
\]

The electric field in the comoving frame \( \bar{E} = -\nabla \Phi \) is transformed to an electric and magnetic field in the laboratory frame (see Appendix A):

\[
E = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} \gamma_b \bar{E}_x \\ \gamma_b \bar{E}_y \\ \gamma_b \bar{E}_z \end{pmatrix}, \quad B = \begin{pmatrix} -\beta_b \bar{E}_y/c \\ \beta_b \bar{E}_x/c \\ 0 \end{pmatrix}.
\]

The Lorentz force on a point charge \( Q_0 \) with velocity \( u \) is:

\[
F = Q_0 (\mathbf{E} + u \times \mathbf{B}) = Q_0 \begin{pmatrix} (1 - \beta_b u_z/c)E_x \\ (1 - \beta_b u_z/c)E_y \\ E_z + \beta_b (E_x u_x/c + E_y u_y/c) \end{pmatrix}.
\]

The Lorentz force on a comoving point charge, i.e. \( u_z/c = \beta_b \), is rather small, since \( 1 - \beta_b u_z/c = 1/\gamma_b^2 \), while the force on a colliding point charge \( (u_z/c = -\beta_b) \) is about a factor two larger than on a point charge at rest. In the next section the fields in the comoving frame \( \bar{K} \) are calculated.

## 2 Electric fields in the rest frame of the bunch

### 2.1 Solution of the Poisson’s equation

Solution of the Poisson’s equation (5) for a Gaussian charge distribution (2) is [11] (see Appendix B):

\[
\Phi(\bar{x}, \bar{y}, \bar{z}) = \frac{Q_b}{4 \pi \epsilon_0} \frac{1}{\sqrt{\pi}} \int_0^\infty dq \frac{\exp(-q_x^2 \bar{q}_x^2 - q_y^2 \bar{q}_y^2 - q_z^2 \bar{q}_z^2)}{\sqrt{q_x q_y q_z}},
\]

with

\[
q_x = q + 2 \sigma_x^2, \quad q_y = q + 2 \sigma_y^2, \quad q_z = q + 2 \sigma_z^2.
\]

The potential \( \Phi(\bar{x}, \bar{y}, \bar{z}) \) is now rewritten in a form which was used in [1] for the two dimensional potential.

The variable \( q \) is transformed into the new integration variable \( \xi \) according to:

\[
\xi^2 = \frac{q_y}{q_x} = \frac{q + 2 \sigma_y^2}{q + 2 \sigma_x^2}, \quad \text{or} \quad q = \frac{2 (\sigma_x^2 - \sigma_y^2)}{1 - \xi^2} - 2 \sigma_x^2.
\]

Furthermore it is assumed that \( \sigma_x > \sigma_y \) and the following quantities \( \kappa \) and \( \sigma_0 \) are defined:

\[
\kappa = \frac{\sigma_y}{\sigma_x}, \quad \sigma_0 = \sqrt{2 (\sigma_x^2 - \sigma_y^2)}.
\]
Since \( dq/d\xi = 2 q^3/\sqrt{q_y}/\sigma_0^2 \) one obtains the following transformed solution of the Poisson’s equation:

\[
\Phi(\bar{x}, \bar{y}, \bar{z}) = \frac{Q_b}{4 \pi \epsilon_0} \sqrt{\frac{2}{\pi}} \frac{1}{\sigma_0^2} \int_1^1 d\xi \frac{q_{\bar{x}}}{\sqrt{q_z}} \exp\left(\frac{-\bar{x}^2 - \bar{y}^2 - \bar{z}^2}{q_{\bar{x}} q_{\bar{y}} q_{\bar{z}}}\right) \tag{12}
\]

with

\[
q_{\bar{x}} = \sigma_0^2 \frac{1}{1 - \xi^2}, \quad q_{\bar{y}} = \sigma_0^2 \frac{\xi^2}{1 - \xi^2}, \quad q_{\bar{z}} = 2 (\sigma_z^2 - \sigma_x^2) + \sigma_0^2 \frac{1}{1 - \xi^2}. \tag{13}
\]

To make the comparison with the Bassetti-Erskine formula (see Appendix C) easier a constant \( \Phi_0 \) is defined as

\[
\Phi_0 = \frac{Q_b}{2 \epsilon_0 \sqrt{\pi} \sigma_0}. \tag{14}
\]

The definition of \( \Phi_0 \) may look arbitrary at this point but it will turn out that a simple connection between the two dimensional Bassetti-Erskine formula and the three dimensional results can be established. The potential is now:

\[
\Phi(\bar{x}, \bar{y}, \bar{z}) = \Phi_0 \frac{1}{\sigma_0^2} \int_1^1 d\xi \frac{q_{\bar{x}}}{\sqrt{q_z}} \exp\left(\frac{-\bar{x}^2 - \bar{y}^2 - \bar{z}^2}{q_{\bar{x}} q_{\bar{y}} q_{\bar{z}}}\right) \tag{15}
\]

### 2.2 Electric fields

The electric fields in the comoving frame are now derived from the potential \( \Phi(\bar{x}, \bar{y}, \bar{z}) \) as:

\[
\vec{E}_x = -\frac{\partial}{\partial x} \Phi(\bar{x}, \bar{y}, \bar{z}) = \Phi_0 \frac{\bar{x}}{\sigma_0^2} \int_1^1 d\xi \frac{1}{\sqrt{q_z}} \exp\left(\frac{-\bar{x}^2 - \bar{y}^2 - \bar{z}^2}{q_{\bar{x}} q_{\bar{y}} q_{\bar{z}}}\right) \tag{16}
\]

\[
\vec{E}_y = -\frac{\partial}{\partial y} \Phi(\bar{x}, \bar{y}, \bar{z}) = \Phi_0 \frac{\bar{y}}{\sigma_0^2} \int_1^1 d\xi \frac{1}{\xi^2} \frac{1}{\sqrt{q_z}} \exp\left(\frac{-\bar{x}^2 - \bar{y}^2 - \bar{z}^2}{q_{\bar{x}} q_{\bar{y}} q_{\bar{z}}}\right) \tag{17}
\]

\[
\vec{E}_z = -\frac{\partial}{\partial z} \Phi(\bar{x}, \bar{y}, \bar{z}) = \Phi_0 \frac{\bar{z}}{\sigma_0^2} \int_1^1 d\xi \frac{q_{\bar{x}}}{\sqrt{q_z}} \exp\left(\frac{-\bar{x}^2 - \bar{y}^2 - \bar{z}^2}{q_{\bar{x}} q_{\bar{y}} q_{\bar{z}}}\right) \tag{18}
\]

with

\[
q_{\bar{x}} = \sigma_0^2 \frac{1}{1 - \xi^2}, \quad q_{\bar{y}} = \sigma_0^2 \frac{\xi^2}{1 - \xi^2}, \quad q_{\bar{z}} = 2 (\sigma_z^2 - \sigma_x^2) + \sigma_0^2 \frac{1}{1 - \xi^2}. \tag{19}
\]
and

\[
\exp \left( -\frac{\bar{x}^2}{q_x} - \frac{\bar{y}^2}{q_y} - \frac{\bar{z}^2}{q_z} \right) = \exp \left( -(1 - \xi^2) \left( \frac{\bar{x}^2}{\sigma_0^2 \xi^2} + \frac{\bar{y}^2}{\sigma_0^2 \xi^2} + \frac{\bar{z}^2}{2 (\sigma_z^2 - \sigma_x^2) (1 - \xi^2) + \sigma_0^2} \right) \right). \tag{20}
\]

It has been assumed that the transverse beam size does not depend on the longitudinal position. This is a good approximation if one considers only short regions in an accelerator where changes of the beam size due to external magnetic fields can be neglected.

### 2.3 Approximation of the Transverse Electric fields

For an electron beam energy of (say) 5 GeV the Lorentz factor is large: \( \gamma_b \approx 9.8 \cdot 10^3 \). A bunch length of \( \sigma_z = 10 \text{ mm} \) in the laboratory frame corresponds to a bunch length of \( \sigma_z = \gamma_b \sigma_z = 98 \text{ m} \) in the comoving frame of the bunch. Therefore the following relation:

\[
\sigma_z \gg \sigma_x > \sigma_y \tag{21}
\]

is valid for all ultra relativistic bunches and one obtains

\[
\frac{1}{\sqrt{q_z}} \approx \frac{1}{\sigma_z \sqrt{2}}, \quad \text{and} \quad \exp \left( -\frac{\bar{z}^2}{q_z} \right) \approx \exp \left( -\frac{\bar{z}^2}{2 \sigma_z^2} \right). \tag{22}
\]

In this approximation the transverse electric fields can be written as:

\[
\bar{E}_x = \frac{\Phi_0}{\sigma_z \sqrt{2 \pi}} \exp \left( -\frac{\bar{z}^2}{2 \sigma_z^2} \right) f_x(\bar{x}, \bar{y}), \tag{23}
\]

\[
\bar{E}_y = \frac{\Phi_0}{\sigma_z \sqrt{2 \pi}} \exp \left( -\frac{\bar{z}^2}{2 \sigma_z^2} \right) f_y(\bar{x}, \bar{y}), \tag{24}
\]

with the functions

\[
f_x(\bar{x}, \bar{y}) = \frac{\bar{x}}{\sigma_0 \sqrt{\pi}} \int_\kappa^1 \! d\xi \exp \left( -(1 - \xi^2) \left( \frac{\bar{x}^2}{\sigma_0^2 \xi^2} + \frac{\bar{y}^2}{\sigma_0^2 \xi^2} \right) \right) \tag{25}
\]

\[
f_y(\bar{x}, \bar{y}) = \frac{\bar{y}}{\sigma_0 \sqrt{\pi}} \int_\kappa^1 \! d\xi \frac{1}{\xi^2} \exp \left( -(1 - \xi^2) \left( \frac{\bar{x}^2}{\sigma_0^2 \xi^2} + \frac{\bar{y}^2}{\sigma_0^2 \xi^2} \right) \right). \tag{26}
\]

The functions \( f_x \) and \( f_y \) are related to the complex error function in the following way (see Appendix C):

\[
f_x(\bar{x}, \bar{y}) - i f_y(\bar{x}, \bar{y}) = -i \left( w(\zeta_2) - \exp(\zeta_1^2 - \zeta_2^2) w(\zeta_1) \right) \tag{27}
\]

with

\[
\zeta_1 = \frac{x}{\sigma_0} \kappa + i \frac{y}{\sigma_0} \frac{1}{\kappa} \quad \text{and} \quad \zeta_2 = \frac{x}{\sigma_0} + i \frac{y}{\sigma_0}. \tag{28}
\]
This establishes the connection between the fields of a 3D Gaussian bunch with the Bassetti-Erskine formula.

The linear part of the function \( f_x \) and \( f_y \) are:

\[
f_{x,Lin}(\bar{x}, \bar{y}) = \frac{\bar{x}}{\sigma_0} \frac{2}{\sqrt{\pi}} \int_{-\infty}^{1} d\xi \frac{2}{\sigma_0} \sqrt{1 - \xi^2} \approx \frac{\bar{x}}{\sigma_0^{1/2}} \sqrt{\frac{2}{\pi}} \left( 1 - \frac{\bar{x} - \bar{y}}{\sqrt{\sigma_x^2 - \sigma_y^2}} \right)
\]

(29)

\[
f_{y,Lin}(\bar{x}, \bar{y}) = \frac{\bar{y}}{\sigma_0} \frac{2}{\sqrt{\pi}} \int_{-\infty}^{1} d\xi \frac{2}{\sigma_0} \sqrt{1 - \xi^2} \approx \frac{\bar{y}}{\sigma_0^{1/2}} \sqrt{\frac{2}{\pi}} \left( 1 - \frac{\bar{y} - \bar{x}}{\sqrt{\sigma_y^2 - \sigma_y^2}} \right)
\]

(30)

As an example the functions \( f_x \) and \( f_y \) are plotted in Fig. 2 for a Gaussian bunch with \( \sigma_x = 1 \) mm and \( \sigma_y = 0.1 \) mm. Furthermore the linear approximations of the functions are also plotted.

![Figure 2: The functions \( f_x(x,0) \) and \( f_y(0,y) \) and their linear approximations are plotted for a Gaussian beam with \( \sigma_x = 1 \) mm and \( \sigma_y = 0.1 \) mm versus the horizontal/vertical position. The left graph shows \( f_x(x,0) \) versus the horizontal position in units of \( \sigma_x \) while the right graph shows \( f_y(0,y) \) versus the vertical positron in units of \( \sigma_y \).](image)

2.4 Approximation of the Longitudinal Electric field

Using the same approximation as before \( (\sigma_z \gg \sigma_x > \sigma_y) \) one obtains from Eqn. 18:

\[
\frac{q_x}{q_z} \frac{1}{\sqrt{\gamma_z}} \sim \frac{\sigma_0^2}{2(1 - \xi^2)} \frac{\bar{z}}{\sigma_{\bar{z}}^3} \sim \frac{1}{\gamma_0^2}.
\]

(33)
The longitudinal electric field scales as $1/\gamma^2$ and may be neglected for many applications ($E_z \approx 0$).

Using the same approximation as in the previous section (see Eqn. (22)) one obtains:

$$E_z = \frac{\Phi_0}{\sigma_z \sqrt{2\pi}} \exp\left(-\frac{z^2}{2\sigma_z^2}\right) f_z(\bar{x}, \bar{y}, \bar{z}), \quad (34)$$

with

$$f_z(\bar{x}, \bar{y}, \bar{z}) = \frac{\bar{z}}{\sigma_0} \frac{2}{\sqrt{\pi}} \int_\kappa^1 d\xi h_\bar{z}(\xi) \exp\left(-(1-\xi^2)\left(\frac{\bar{x}^2}{\sigma_0^2} + \frac{\bar{y}^2}{\sigma_0^2} \frac{1}{\xi^2}\right)\right) \quad (35)$$

$$h_\bar{z}(\xi) = \frac{\sigma_0^2}{2(\sigma_z^2 - \sigma_x^2)(1-\xi^2) + \sigma_0^2}. \quad (35')$$

In the special case $\bar{x} = 0$ and $\bar{y} = 0$ the following analytic expression$^1$ is a good approximation of the integral:

$$\int_\kappa^1 d\xi h_\bar{z}(\xi) \approx \frac{\sigma_0^2}{2\sigma_z^2} \ln\left(\frac{2\sqrt{2}\sigma_z}{\sigma_0}\right), \quad (36)$$

and therefore

$$f_z(0, 0, \bar{z}) \approx \frac{\bar{z}}{\sigma_z} \frac{2}{\sqrt{\pi}} \frac{\sigma_0}{2\sigma_z} \ln\left(\frac{2\sqrt{2}\sigma_z}{\sigma_0}\right). \quad (37)$$

Instead of the transformation of the variable $q$ according to Eqn. (10) one can also use the following transformation:

$$\chi^2 = \frac{q\bar{y}}{q\bar{z}} \quad \text{or} \quad q = \frac{\sigma_1^2}{1-\chi^2} - 2\sigma_z^2, \quad (38)$$

with $\sigma_1 = \sqrt{2(\sigma_z^2 - \sigma_y^2)}$. With this transformation one obtains for the longitudinal electric field an alternate representation:

$$E_z = \frac{\Phi_0}{\sigma_z \sqrt{2\pi}} \frac{\bar{z}}{\sigma_1^3} \frac{2}{\pi} \int_{\sigma_y/\sigma_z}^1 d\chi \frac{1}{\sqrt{q\bar{z}}} \exp\left(-\frac{\bar{x}^2}{q\bar{x}} - \frac{\bar{y}^2}{q\bar{y}} - \frac{\bar{z}^2}{q\bar{z}}\right) \quad (39)$$

$$= \frac{\Phi_0}{\sigma_z \sqrt{2\pi}} \frac{\bar{z}}{\sigma_1^3} \frac{2}{\pi} \int_{\sigma_y/\sigma_z}^1 d\chi \sqrt{\frac{\sigma_1^2 (1-\chi^2)}{\sigma_z^2 - 2(\sigma_z^2 - \sigma_x^2)(1-\chi^2)}} \exp\left(-\frac{\bar{x}^2}{q\bar{x}} - \frac{\bar{y}^2}{q\bar{y}} - \frac{\bar{z}^2}{q\bar{z}}\right),$$

$^1$The result is based on the integral [4]

$$\int_0^1 d\xi \frac{b^2}{a^2 (1-\xi^2) + b^2} = \frac{b^2}{a \sqrt{a^2 + b^2}} \arctanh\left(\frac{a}{\sqrt{a^2 + b^2}}\right).$$
with

\[ \exp \left( -\frac{x^2}{\sigma_1^2} - \frac{y^2}{\sigma_2^2} - \frac{z^2}{\sigma_3^2} \right) \]

\[ = \exp \left( -(1 - \chi^2) \left( \frac{x^2}{\sigma_1^2 - 2\sigma_2^2} (1 - \chi^2) + \frac{y^2}{\sigma_1^2} \frac{1}{\chi^2} + \frac{z^2}{\sigma_1^2} \right) \right). \tag{40} \]

Figure 3: The longitudinal field $\vec{E}_z(0, 0, \bar{z})/\Phi_0$ multiplied by $\gamma_b^2 \approx 9.6 \times 10^7$ plotted for a Gaussian beam with $\sigma_x = 1$ mm, $\sigma_y = 0.1$ mm and $\sigma_z = \gamma_b \sigma_z = 98$ m versus the longitudinal position.

3 Electric and Magnetic fields in the Laboratory frame

3.1 Lorentz Transformation of the electric field

The electric and magnetic fields in the laboratory frame can be obtained from the electric field in the rest frame of the bunch via a Lorentz Transformation, see Eqn. (6) and Appendix A.

The magnetic field is related to the electric field in the following way:

\[ B_x(x, y, z, t) = -\frac{\beta_b}{c} E_y(x, y, z, t) \]
\[ B_y(x, y, z, t) = \frac{\beta_b}{c} E_z(x, y, z, t) \]
\[ B_z(x, y, z, t) = 0. \tag{41} \]

For the electric field one obtains:

\[ E_x(x, y, z, t) = E_0 \frac{x}{\sigma_0} \frac{2}{\sqrt{\pi}} \int_\kappa d\xi \ h_0(\xi) \ \exp \left( h_\epsilon(\xi, x, y, z, t) \right) \]
\[ E_y(x,y,z,t) = E_0 \frac{y}{\sigma_0} \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \frac{1}{\xi^2} h_0(\xi) \exp\left(h_e(\xi,x,y,z,t)\right) \] (42)

\[ E_z(x,y,z,t) = E_0 \frac{z - \beta_b c t}{\sigma_0} \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi h_z(\xi) h_0(\xi) \exp\left(h_e(\xi,x,y,z,t)\right) \]

with

\[ \sigma_0 = \sqrt{2(\sigma_x^2 - \sigma_y^2)} \]

\[ \kappa = \frac{\sigma_y}{\sigma_x} \]

\[ E_0 = \frac{1}{\sigma_z \sqrt{2 \pi} \frac{Q_b}{\sigma_0}} \]

\[ h_0(\xi) = \sqrt{\frac{2 \gamma_b^2 \sigma_z^2 (1 - \xi^2)}{2 (\gamma_b^2 \sigma_z^2 - \sigma_x^2) (1 - \xi^2) + \sigma_0^2}} \] (43)

\[ h_z(\xi) = \sqrt{\frac{2 (\gamma_b^2 \sigma_z^2 - \sigma_x^2) (1 - \xi^2) + \sigma_0^2}{(\gamma_b^2 \sigma_z^2 - \sigma_x^2) (1 - \xi^2) + \sigma_0^2}} \]

\[ h_e(\xi,x,y,z,t) = -(1 - \xi^2) \left(\frac{x^2}{\sigma_0} + \frac{1}{\xi^2} \frac{y^2}{\sigma_0}\right) + \frac{\gamma_b^2 (z - \beta_b c t)^2}{2 (\gamma_b^2 \sigma_z^2 - \sigma_x^2) (1 - \xi^2) + \sigma_0^2} \]

\subsection{3.2 Approximation of the electric field}

In the same approximation as in the previous section \( \gamma_b \gg 1 \) or \( \sigma_z \gg \sigma_x > \sigma_y \) one obtains:

\[ E_x(x,y,z,t) = g_z(z - \beta_b c t) \frac{Q_b}{2 \epsilon_0 \sqrt{2 \pi} \sqrt{\sigma_x^2 - \sigma_y^2}} f_x(x,y) \] (44)

\[ E_y(x,y,z,t) = g_z(z - \beta_b c t) \frac{Q_b}{2 \epsilon_0 \sqrt{2 \pi} \sqrt{\sigma_x^2 - \sigma_y^2}} f_y(x,y) \]

\[ E_z(x,y,z,t) = g_z(z - \beta_b c t) \frac{Q_b}{2 \epsilon_0 \sqrt{2 \pi} \sqrt{\sigma_x^2 - \sigma_y^2}} f_z(x,y,z,t) \]

\approx 0,

where \( g_z(s) \) is the Gaussian longitudinal charge density

\[ g_z(s) = \frac{1}{\sigma_z \sqrt{2 \pi}} \exp\left(-\frac{s^2}{2 \sigma_z^2}\right), \] (45)

and \( f_x(x,y), f_y(x,y) \) are the functions, which are defined in section 2.3. The function \( f_z(x,y,z,t) \) is the function \( f_z \) of Eqn. (35) rewritten in terms of the
variables of the laboratory frame ($\sigma_z = \gamma_b \sigma_z$):

$$f_z(x, y, z, t) = \frac{z - ct}{\sigma_0} \frac{2}{\sqrt{\pi}} \int_0^1 d\xi \, h_z(\xi) \exp \left( - (1 - \xi^2) \left( \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} \frac{1}{\xi^2} \right) \right)$$  \hfill (46)

(note that $h_z(\xi) \sim 1/\gamma_b^2$).

The time average of the electric fields are:

$$\int_{-\infty}^{\infty} dt \, E_{x,y}(x, y, z, t) = \frac{1}{\beta_b c} \frac{Q_b}{2 \epsilon_0 \sqrt{2 \pi} \sqrt{\sigma_x^2 - \sigma_y^2}} f_{x,y}(x, y)$$  \hfill (47)

$$\int_{-\infty}^{\infty} dt \, E_z(x, y, z, t) = 0.$$  \hfill (48)

If the linear part of the functions $f_x(x, y)$ and $f_y(x, y)$ is used (see Eqn. (30) and (32)) one obtains:

$$\int_{-\infty}^{\infty} dt \, E_x(x, y, z, t) = \frac{1}{\beta_b c} \frac{Q_b}{2 \pi \epsilon_0} \frac{x}{\sigma_x (\sigma_x + \sigma_y)}$$  \hfill (49)

$$\int_{-\infty}^{\infty} dt \, E_y(x, y, z, t) = \frac{1}{\beta_b c} \frac{Q_b}{2 \pi \epsilon_0} \frac{y}{\sigma_y (\sigma_x + \sigma_y)}.$$  \hfill (50)

4 Relativistic equation of motion

4.1 General considerations

To study the motion of an electron or ion in the electric field of the positron or electron bunch one has to solve the (in general) relativistic equation of motion:

$$\frac{d}{dt} \mathbf{p} = \frac{d}{dt} (m_0 \gamma \mathbf{v}) = \mathbf{F},$$  \hfill (51)

where $\mathbf{v}$ is the velocity, $m_0$ the rest mass, and

$$\gamma = \left( 1 - \mathbf{v} \cdot \mathbf{v}/c^2 \right)^{-1/2}$$  \hfill (52)

is the relativistic $\gamma$-factor of the point charge $Q_0$ (electron or ion) and $\mathbf{F}$ is the Lorentz-Force (see Eqn. 7). The time derivative of the momentum $\mathbf{p}$ is:

$$\frac{d}{dt} \mathbf{p} = m_0 \gamma \frac{d}{dt} \mathbf{v} + m_0 \gamma^3 \frac{1}{c^2} \left( \mathbf{v} \cdot \frac{d}{dt} \mathbf{v} \right)$$  \hfill (53)

$$= m_0 \gamma \frac{d}{dt} \mathbf{v} + \mathbf{v} \frac{1}{c^2} \left( \mathbf{v} \cdot \frac{d}{dt} \mathbf{p} \right),$$  \hfill (54)

where in the second equation it has been used that

$$\mathbf{v} \cdot \frac{d}{dt} \mathbf{p} = m_0 \gamma \left( 1 + \gamma^2 \frac{\mathbf{v} \cdot \mathbf{v}}{c^2} \right) (\mathbf{v} \cdot \frac{d}{dt} \mathbf{v}) = m_0 \gamma^3 \left( \mathbf{v} \cdot \frac{d}{dt} \mathbf{v} \right).$$  \hfill (55)
Note that \( \mathbf{v} \cdot \frac{d}{dt} \mathbf{p} \) is also equal to the time derivative of the energy \( E = m_0 \gamma c^2 \) of the point charge:

\[
\frac{d}{dt} E = \frac{d}{dt} \left( m_0 c^2 \gamma \right) = m_0 \gamma^3 (\mathbf{v} \cdot \frac{d}{dt} \mathbf{v}) = \mathbf{v} \cdot \frac{d}{dt} \mathbf{p}
\]  

(56)

Now one can solve the Eqn. 54 for \( \frac{dv}{dt} \) and substitute the force \( \mathbf{F} \) for \( \frac{dp}{dt} \):

\[
\frac{d}{dt} \mathbf{v} = \frac{1}{m_0 \gamma} \left( \mathbf{F} - \frac{1}{c^2} (\mathbf{v} \cdot \mathbf{F}) \mathbf{v} \right)
\]  

(57)

\[
= \frac{1}{m_0 \gamma} \sqrt{1 - \frac{\mathbf{v} \cdot \mathbf{v}}{c^2}} \left( \mathbf{F} - \frac{1}{c^2} (\mathbf{v} \cdot \mathbf{F}) \mathbf{v} \right)
\]  

(58)

For the Lorentz force \( \mathbf{F} = Q_0 (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \) one obtains

\[
\frac{d}{dt} \mathbf{v} = \frac{Q_0}{m_0 \gamma} \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{1}{c^2} (\mathbf{E} \cdot \mathbf{v}) \mathbf{v} \right).
\]  

(59)

The vector product \( \mathbf{v} \times \mathbf{B} \) can be also written as

\[
\mathbf{v} \times \mathbf{B} = \begin{pmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{pmatrix} \mathbf{v}
\]  

(60)

**4.2 Motion in the space charge field of a Gaussian bunch**

Now it is assumed that the point charge \( Q_0 \) is moving in the space charge field of a Gaussian bunch. The magnetic field in the laboratory is related to the electric field as (see section 3)

\[
\mathbf{B} = \beta_b \frac{1}{c} \begin{pmatrix} -E_y \\ E_x \\ 0 \end{pmatrix},
\]  

(61)

where \( \beta_b c \) is the velocity of the beam along the z-axis. The equation of motion for the trajectory \( \mathbf{r} \) of the point charge \( Q_0 \) is

\[
\frac{d}{dt} \begin{pmatrix} \mathbf{r} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ G(\mathbf{r}, \mathbf{v}, t) \end{pmatrix},
\]  

(62)

with

\[
G(\mathbf{r}, \mathbf{v}, t) = \frac{Q_0}{m_0} \sqrt{1 - \frac{\mathbf{v} \cdot \mathbf{v}}{c^2}} \begin{pmatrix} -E_y \\ E_x \\ -E_y \end{pmatrix} \mathbf{v} - \frac{1}{c^2} (\mathbf{E} \cdot \mathbf{v}) \mathbf{v}
\]  

(63)

Often it is convenient to measure the position of the point charge \( Q_0 \) in units of the dimensions of the Gaussian bunch and to use the variable \( \tau = c t / \sigma_z \) instead.
of the time \( t \). Let \( D \) be the diagonal matrix with the beam dimension on the diagonal and \( D^{-1} \) the inverse matrix of \( D \)

\[
D = \begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}, \quad D^{-1} = \begin{pmatrix} \sigma_x^{-1} & 0 & 0 \\ 0 & \sigma_y^{-1} & 0 \\ 0 & 0 & \sigma_z^{-1} \end{pmatrix}.
\]

(64)

Then the normalized position is

\[
\varrho = D^{-1} r.
\]

(65)

With this definition one obtains:

\[
r = D \varrho, \quad v = \frac{c}{\sigma_z} D \frac{d}{d\tau} \varrho, \quad \frac{d}{dt} v = \left( \frac{c}{\sigma_z} \right)^2 D \frac{d^2}{d\tau^2} \varrho.
\]

(66)

The equation of motion can now be rewritten using the variables \( \tau \) and \( \varrho \):

\[
\frac{d}{d\tau} \left( \begin{pmatrix} \varrho \\ \frac{d}{d\tau} \varrho \end{pmatrix} \right) = \left( \begin{pmatrix} \frac{d}{d\tau} \varrho \\ \frac{d}{d\tau} \frac{d}{d\tau} \varrho \end{pmatrix} \right),
\]

(67)

with

\[
G(D \varrho, \frac{c}{\sigma_z} D \frac{d}{d\tau} \varrho, \frac{\sigma_z}{c} \tau) = C_0 \left( \sigma_z D^{-1} \frac{E}{E_0} + \beta_b \left( \begin{pmatrix} 0 & 0 & -\frac{\sigma_x E_x}{E_0} \\ 0 & 0 & -\frac{\sigma_y E_y}{E_0} \\ \frac{\sigma_x}{\sigma_z} E_x & \frac{\sigma_y}{\sigma_z} E_y & 0 \end{pmatrix} \right) \right)
\]

(68)

\[
-\frac{1}{c^2} \left( \frac{E}{E_0} \cdot \frac{D}{\sigma_z} \frac{d}{d\tau} \varrho \right) \left( \frac{d}{d\tau} \varrho \right),
\]

\[
\frac{1}{\gamma} = \sqrt{1 - \left( \frac{\sigma_z d}{\sigma_z d\tau \varrho_x} \right)^2 - \left( \frac{\sigma_y d}{\sigma_z d\tau \varrho_y} \right)^2 - \left( \frac{d}{d\tau \varrho_z} \right)^2},
\]

(69)

and a constant \( C_0 \)

\[
C_0 = \frac{Q_b}{m_0 c^2} \sigma_z E_0.
\]

(70)

\( E_0 \) is the amplitude of the electric field, which is used to normalize the electric field strength. If the approximations from section 3 of the electric fields are used one obtains

\[
E_0 = \frac{1}{\sigma_z \sqrt{2\pi}} \frac{Q_b}{2\epsilon_0 \sqrt{\pi} \sigma_0},
\]

(71)
and

\[
\frac{E_x}{E_0} = \exp \left( -\frac{(\varrho_z - \beta_b \tau)^2}{2} \right) f_x(\sigma_x, \sigma_y, \varrho_y)
\]

\[
\frac{E_y}{E_0} = \exp \left( -\frac{(\varrho_z - \beta_b \tau)^2}{2} \right) f_y(\sigma_x, \sigma_y, \varrho_y).
\]

(72)

5 Applications

5.1 Motion of an electron in the space charge field of the bunch

In positron storage rings electrons produced by photoemission, ionization and secondary emission accumulate in the vacuum chamber forming an "electron cloud". The previously obtained formulae for the motion of a point charge in the space charge field of a Gaussian bunch are now applied to the motion of one electron of an electron cloud. It is assumed that the electron is initially at rest in the laboratory frame. Usually, the transverse kick on the electron is calculated from the ratio of the change of the transverse momentum and the longitudinal momentum: \( \Delta p_\perp / p_\parallel \). Since the electron from the electron cloud may have a very small longitudinal momentum \((p_\parallel \approx 0)\) the following approach is adopted for the kick on the electron:

\[
\Delta r'_\perp = \frac{1}{c} \Delta v_\perp \approx \frac{1}{m_0 c} \Delta p_\perp = \frac{-e}{m_0 c} \int_{-\infty}^{\infty} dt E_\perp(x, y, z, t).
\]

(73)

Using the linear approximation for the electric field (Eqn. (50)) one obtains for the vertical kick:

\[
\Delta y' = -\frac{N_b e^2}{m_0 c^2} \frac{1}{2\pi \epsilon_0 \sigma_y (\sigma_x + \sigma_y)} \frac{y}{\sigma_x + \sigma_y} = -\frac{2 N_b r_e}{\sigma_x + \sigma_y} \frac{y}{\sigma_y},
\]

(74)

for a total bunch charge of \(Q_b = N_b e\) positrons. \(r_e\) is the classical electron radius

\[
r_e = \frac{e^2}{4 \pi \epsilon_0 m_0 c^2} = 2.818 \times 10^{-15} \text{ m}.
\]

(75)

It has been assumed that the bunch is ultra relativistic, i.e. \(\beta_b = 1\). The vertical angular oscillation frequency \(\omega_e\) of the electron is

\[
\frac{\omega_{e,y}}{c} = \sqrt{\frac{N_b}{2 \sigma_x \sigma_y (\sigma_x + \sigma_y)}}.
\]

(76)
assuming an effective bunch length of $2\sigma_z$. The bunch acts as a thin lens with focal length $1/(2\sigma_z (\omega_{e,y}/c)^2)$:

$$
\begin{pmatrix}
y_f \\
y'_f
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2\sigma_z (\omega_{e,y}/c)^2 & 1 \end{pmatrix} \begin{pmatrix} y_i \\
y'_i
\end{pmatrix},
$$

(77)

where $(y_i, y'_i)$, and $(y_f, y'_f)$ are the initial and final phase space coordinates of the electron before and after the interaction with the bunch. In the horizontal plane one obtains in analogy to the vertical plane:

$$
\Delta x' = -\frac{2N_br_e x}{\sigma_x + \sigma_y} \frac{\omega_{e,x}}{c} = \sqrt{\frac{N_b}{2\sigma_z}} \frac{2r_e}{(\sigma_x + \sigma_y)}.
$$

(78)

The beam parameters of KEKB-LER \[2\] are used to calculate the kicks and the oscillation frequency of an electron in the linear approximation. The results are summarized in table 1. The kick of the positron bunch on the electron from the cloud is rather strong and the period of one (vertical) oscillation $2\pi c/\omega_{e,y}$, as calculated from the linear approximation, is only a factor 1.7 larger than the RMS bunch length $\sigma_z$. Therefore it is expected that the (nonlinear) motion of the electron will significantly differ from the result obtained from the linear kick approximation.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positron energy</td>
<td>$E_b$</td>
<td>3.5</td>
<td>GeV</td>
</tr>
<tr>
<td>RMS beam size</td>
<td>$\sigma_x$</td>
<td>420</td>
<td>$\mu$m</td>
</tr>
<tr>
<td></td>
<td>$\sigma_y$</td>
<td>60</td>
<td>$\mu$m</td>
</tr>
<tr>
<td>RMS bunch length</td>
<td>$\sigma_z$</td>
<td>4</td>
<td>mm</td>
</tr>
<tr>
<td>Charge of one bunch</td>
<td>$Q_b$</td>
<td>5.3</td>
<td>nC</td>
</tr>
<tr>
<td>Number of positrons per bunch</td>
<td>$N_b$</td>
<td>$3.3\cdot10^{10}$</td>
<td></td>
</tr>
<tr>
<td>oscillation frequency of an electron</td>
<td>$\omega_{e,x}/c$</td>
<td>$2\pi \cdot 16.2$</td>
<td>GHz</td>
</tr>
<tr>
<td></td>
<td>$\omega_{e,y}/c$</td>
<td>$2\pi \cdot 42.9$</td>
<td>GHz</td>
</tr>
<tr>
<td>kick for $1 \times \sigma_{x,y}$ offset</td>
<td>$2N_br_e/(\sigma_x + \sigma_y)$</td>
<td>0.387</td>
<td>rad</td>
</tr>
</tbody>
</table>

Table 1: Main parameters of KEK-LER \[2\] and the kicks and the oscillation frequency of an electron in the linear approximation.

The (nonlinear) equation of motion according to Eqn. (67) is numerically solved with a Runge-Kutta \[5\] solver using a commercial code \[6\]. In total 1500 integration steps are used to solve the equation of motion in the interval from 0 to 15 for the variable $\tau = ct/\sigma_z$. The positron bunch is at the origin of the laboratory frame at time $t = 0$, while the electron is at rest at $z_0 = 5\sigma_z$ (again time $t = 0$). The bunch is moving with the velocity $v_b = c$ toward the electron. The situation is shown in Fig. 4. The numerical solutions of the vertical motion

\[2\] low energy ring of the B-factory at KEK, Tsukuba, Japan
are shown in Fig. 5 and 6 and for the horizontal motion in Fig. 7 and 8. The equation of motion is solved in the normalized coordinates $\mathbf{q} = D^{-1} \mathbf{r}$. The vertical kick on the electron is related to the derivative of $\mathbf{q}$ with respect to $\tau$ in the following way:

$$\frac{d}{d\tau} q_y = \frac{\sigma_z}{c} \frac{1}{\sigma_y} \frac{d}{dt} q_y = \frac{\sigma_z}{c} \frac{1}{\sigma_y} \frac{d}{dt} y = \frac{\sigma_z}{c} \frac{1}{\sigma_y} cy'. \quad (79)$$

Therefore one obtains for the vertical and in the same way for the horizontal kick:

$$y' = \frac{\sigma_y}{\sigma_z} \frac{d}{d\tau} q_y, \quad x' = \frac{\sigma_x}{\sigma_z} \frac{d}{d\tau} q_x. \quad (80)$$

The linear model for the interaction of the electron with the bunch is a very poor approximation of the electron motion since the oscillation period of the electron in the bunch potential is of the same order as the rms bunch length. There is almost no longitudinal motion of the electron. The change of the longitudinal position $(z - z_0)/\sigma_z$ is shown versus $\tau = ct/\sigma_z$ in Fig. 9.
Figure 5: Vertical position of the electron versus the normalized time $\tau$ for different initial offsets of the electron (0.5, 1.0, 2.0, and $0.3 \times \sigma_y$). The initial horizontal offset is zero in all cases.

Figure 6: Vertical kick of the electron versus the normalized time $\tau$ for different initial offsets of the electron (0.5, 1.0, 2.0, and $0.3 \times \sigma_y$). The initial horizontal offset is zero in all cases.
Figure 7: Horizontal position of the electron versus the normalized time $\tau$ for different initial offsets of the electron (0.5, 1.0, 2.0, and 3.0 $\times \sigma_x$). The initial vertical offset is zero in all cases.

Figure 8: Horizontal kick of the electron versus the normalized time $\tau$ for different initial offsets of the electron (0.5, 1.0, 2.0, and 3.0 $\times \sigma_x$). The initial vertical offset is zero in all cases.
Figure 9: Longitudinal motion of the electron versus the normalized time $\tau$ for an initial offset of $1.0 \times \sigma_y$. The initial horizontal offset is zero and the initial longitudinal position $z_0$ is $5.0 \times \sigma_z$.

5.2 Motion of an ion in the space charge field of an electron bunch

In the vacuum chamber of an accelerator the electron bunches can ionize the residual gas molecules. Subsequently the ions may interact with the electron bunches, which may degenerate the performance of accelerator when ions are trapped in the beam potential. The kick on the ion due to the electron bunch is:

$$\Delta r'_\perp = \frac{e}{A_{ion} m_p} c \int_{-\infty}^{\infty} dt \mathbf{E}_\perp(x, y, z, t)$$

$$= -2 N_b r_p \frac{\pi}{A_{ion}} \sqrt{\frac{1}{2}} \frac{1}{\sqrt{\sigma_x^2 + \sigma_y^2}} \mathbf{f}_\perp(x, y),$$

where $N_b$ is the electron bunch population, $r_p$ is the classical proton radius

$$r_p = \frac{e^2}{4 \pi \epsilon_0 m_p c^2} = 1.538 \times 10^{-18} \text{ m},$$

$m_p$ the proton mass and $A_{ion}$ the atomic mass number of the ion. The function $\mathbf{f}_\perp(x, y) = (f_x(x, y), f_y(x, y))^t$ is the Bassetti-Erskine presentation of the transverse electric field, see section 2.3.

Using again the linear approximation for the electric field one obtains for the
vertical and horizontal kick on a positively charged ion:

\[
\Delta y' = -\frac{2 N_b r_p}{(\sigma_x + \sigma_y) A_{ion}} \frac{y}{\sigma_y}, \quad \Delta x' = -\frac{2 N_b r_p}{(\sigma_x + \sigma_y) A_{ion}} \frac{x}{\sigma_x}.
\]  

(84)

for a total bunch charge of \(Q_b = -N_b e\) electrons. The angular oscillation frequency \(\omega_{ion}\) of the ion is

\[
\omega_{ion,y} = \sqrt{\frac{N_b}{2 \sigma_z A_{ion}}} \frac{2 r_p}{\sigma_y (\sigma_x + \sigma_y)}, \quad \omega_{ion,x} = \sqrt{\frac{N_b}{2 \sigma_z A_{ion}}} \frac{2 r_p}{\sigma_x (\sigma_x + \sigma_y)}. \]

(85)

The motion of a carbon oxide (CO) ion \((A_{ion} = 28)\) is considered for the beam parameters of the synchrotron light source PETRA III, which started beam operation with positrons in 2009. But in the future PETRA III may be likely operated with electrons. The horizontal emittance is \(\epsilon_x = 1\) nm and for the simulations it is assumed that the vertical emittance is about 1 % of the horizontal emittance. The single bunch intensity will be \(0.5 \cdot 10^{10}\) electrons in the 960 bunch mode and a total current of 100 mA. The important parameters for the interaction of an ion with a single bunch are summarized in table 2. The beam size has been calculated for the design emittance and the optic functions in a FODO cell. The numerical solutions of the vertical motion are shown in Fig. 10 and 11 for an initial offset of the CO-ion of one \(\sigma_y\). Results for the normalized horizontal and vertical position versus the time variable \(\tau = ct/\sigma_z\) are shown in Fig. 12 and 13. Again 1500 integration steps are used to solve the equation of motion in the interval from \(\tau = 0\) to \(\tau = 15\) with a Runge-Kutta solver. The situation shown in Fig. 13 may be compared with the linear approximation. The electron bunch acts as a thin lens on the ion and subsequently the ion drifts over a distance of \(10 \sigma_z\). The corresponding transport matrix is:

\[
\begin{pmatrix}
y_1 \\
y_0
\end{pmatrix} =
\begin{pmatrix}
1 & 10 \sigma_z \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-k_0 & 1
\end{pmatrix}
\begin{pmatrix}
y_0 \\
y_0'
\end{pmatrix},
\]

(86)
where
\[ k_0 = \frac{2 N_b r_p}{A_{ion} (\sigma_x + \sigma_y)} = 0.543 \mu \text{rad}. \] (87)

For \( y'_0 = 0 \) one obtains for the final position \( y_1 \) normalized to the initial offset \( y_0 \):
\[ \frac{y_1}{y_0} = 1 - k_0 \frac{10 \sigma_z}{\sigma_y} = 0.993, \] (88)
and under the same assumptions in the horizontal plane:
\[ \frac{x_1}{x_0} = 1 - k_0 \frac{10 \sigma_z}{\sigma_x} = 0.999935, \] (89)

The results in Fig. 12 and 13 clearly show the non linear deviations from the linear result from Eqn. (89) and (88). The relative deviation of the linear approximation from the nonlinear Bassetti-Erskine function, i.e. the quantities (see section 2.3)
\[ \frac{f_{x,Lin}(x,0) - f_x(x,0)}{f_x(x,0)} \quad \text{and} \quad \frac{f_{y,Lin}(0,y) - f_y(0,y)}{f_y(0,y)}, \] (90)
are plotted in Fig. 14.

Figure 10: Vertical position of an ion versus the normalized time \( \tau \) for an initial offset of one \( \sigma_y \).
Figure 11: Vertical kick of an ion versus the normalized time $\tau$ for an initial offset of one $\sigma_y$.

Figure 12: Normalized horizontal position of an ion versus the normalized time $\tau$ for different initial offsets $x_0$ (0.5, 1.0, and $2.0 \times \sigma_x$).
Figure 13: Normalized vertical position of an ion versus the normalized time $\tau$ for different initial offsets $y_0$ (0.5, 1.0, and $2.0 \times \sigma_x$).

Figure 14: Deviation of the linear approximation from the nonlinear Bassetti-Erskine formula versus the normalized horizontal and vertical position ($x/\sigma_x$ and $y/\sigma_y$).
Acknowledgment

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References


A Lorentz Transformation

A frame $\bar{K}$ is moving with the velocity $v_r$ along the $z$-axis of the laboratory frame $K$. The coordinates and the electric and magnetic fields in both frames are related to each other via a Lorentz transformation [8].

\[
\mathbf{v}_r = \begin{pmatrix} 0 \\ 0 \\ v_r \end{pmatrix}, \quad \beta_r = \frac{v_r}{c}, \quad \gamma_r = \frac{1}{\sqrt{1 - \beta_r^2}} \quad (91)
\]

Coordinate transformation:
\[
\begin{align*}
ct &= \gamma_r (ct - \beta_r z) \\
x &= \bar{x} \\
y &= \bar{y} \\
z &= \gamma_r (z - \beta_r ct)
\end{align*} \quad (92)
\]

transformation of electric and magnetic fields:
\[
\begin{align*}
\bar{E}_z &= E_z \\
\bar{E}_\perp &= \gamma_r \left( \mathbf{E} + v_r \times \mathbf{B} \right)_\perp \\
E_z &= \bar{E}_z \\
E_\perp &= \gamma_r \left( \bar{E} - v_r \times \bar{B} \right)_\perp \\
\bar{B}_z &= B_z \\
\bar{B}_\perp &= \gamma_r \left( \mathbf{B} - \frac{1}{c^2} v_r \times \mathbf{E} \right)_\perp \\
B_z &= \bar{B}_z \\
B_\perp &= \gamma_r \left( \bar{B} + \frac{1}{c^2} v_r \times \bar{E} \right)_\perp
\end{align*} \quad (93)
\]

Lorentz force on charge $q$ and velocity $\mathbf{v}$ in the frame $K$:
\[
\mathbf{F} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (94)
\]
A.1 Components

The electric field $\vec{E}$ and magnetic field $\vec{B}$ in the frame $\bar{K}$ are transformed to the fields $\vec{E}$ and $\vec{B}$ in the laboratory frame $K$:

$$ E = \begin{pmatrix} \gamma_\tau (\bar{E}_x + v_r \bar{B}_y) \\ \gamma_\tau (\bar{E}_y - v_r \bar{B}_x) \\ \bar{E}_z \end{pmatrix}, $$  \hspace{1cm} (95)

$$ B = \begin{pmatrix} \gamma_\tau (\bar{B}_x - v_r \bar{E}_y / c^2) \\ \gamma_\tau (\bar{B}_y + v_r \bar{E}_x / c^2) \\ \bar{B}_z \end{pmatrix}. $$  \hspace{1cm} (96)

In the special case of a pure electric field in the reference frame $\bar{K}$ ($\vec{B} = 0$) one obtains:

$$ E = \begin{pmatrix} \gamma_\tau \bar{E}_x \\ \gamma_\tau \bar{E}_y \\ \bar{E}_z \end{pmatrix}, $$  \hspace{1cm} (97)

and

$$ B = \begin{pmatrix} -\gamma_\tau v_r \bar{E}_y / c^2 \\ \gamma_\tau v_r \bar{E}_x / c^2 \\ 0 \end{pmatrix}. $$  \hspace{1cm} (98)

The Lorentz-Force on a charge $q$ with a velocity $\vec{u}$ in the reference frame $K$ is:

$$ F = q \begin{pmatrix} \gamma_\tau \bar{E}_x (1 - u_z v_r / c^2) \\ \gamma_\tau \bar{E}_y (1 - u_z v_r / c^2) \\ \bar{E}_z + \gamma_\tau v_r (\bar{E}_x u_x + \bar{E}_y u_y) / c^2 \end{pmatrix}. $$  \hspace{1cm} (99)
B Solution of the Poisson Equation

B.1 Integral representation of the Green’s function

The solution of the Poisson equation
\[ \nabla^2 \Phi(r) = -\frac{1}{\epsilon_0}\rho(r) \] (100)
for an arbitrary charge distribution \( \rho(r) \) can be written as [10, 9]
\[ \Phi(r) = \frac{1}{\epsilon_0} \int d^3\hat{r} G(r, \hat{r}) \rho(\hat{r}), \] (101)
where
\[ G(r, \hat{r}) = \frac{1}{4\pi} \frac{1}{|r - \hat{r}|} \] (102)
is the Greens function of the Poisson equation
\[ \nabla^2 G(r, \hat{r}) = -\delta^3(r - \hat{r}). \] (103)

Often it is useful to rewrite the Green’s function as an integral representation [11, 12, 13]:
\[ G(r, \hat{r}) = \frac{1}{2\pi^{3/2}} \int_0^\infty d\xi \exp(-|r - \hat{r}|^2 \xi^2), \] (104)
using the well known integral formula \( \int_0^\infty d\xi \exp(-x^2 \xi^2) = \sqrt{\pi}/(2x) \). With the substitution \( \xi = 1/\sqrt{q} \) one obtains for the Green’s function [12]:
\[ G(r, \hat{r}) = \frac{1}{4\pi^{3/2}} \int_0^\infty dq \frac{1}{q^{3/2}} \exp(-|r - \hat{r}|^2/q), \] (105)
and finally for the potential \( \Phi(r) \) of the charge distribution \( \rho(r) \):
\[ \Phi(r) = \frac{1}{4\pi \epsilon_0} \int_0^\infty dq \frac{1}{q^{3/2} \sqrt{\pi}} \int d^3\hat{r} \rho(\hat{r}) \exp(-|r - \hat{r}|^2/q). \] (106)

B.2 Potential of a Gaussian charge distribution

The general result of Eqn. (106) is now applied to a Gaussian charge distribution:
\[ \rho(x, y, z) = \frac{Q}{(2\pi)^{3/2} \sigma_x \sigma_y \sigma_z} \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2}\right). \] (107)

One obtains:
\[ \int_{-\infty}^{\infty} d\hat{x} \frac{1}{\sqrt{2\pi \sigma_x}} \exp\left(-\frac{\hat{x}^2}{2\sigma_x^2}\right) \frac{1}{\sqrt{q}} \exp\left(-\frac{|x - \hat{x}|^2}{q}\right) = \frac{\exp\left(\frac{x^2}{q + 2\sigma_x^2}\right)}{\sqrt{q + 2\sigma_x^2}} \] (108)
for the integration with respect to the variable \( \hat{x} \). The potential of a Gaussian charge distribution is therefore:

\[
\Phi(x, y, z) = \frac{Q}{4 \pi \epsilon_0} \frac{1}{\sqrt{\pi}} \int_0^\infty dq \frac{\exp \left( -\frac{x^2}{q_x} - \frac{y^2}{q_y} - \frac{z^2}{q_z} \right)}{\sqrt{q_x q_y q_z}}
\]  

(109)

with

\[
q_x = q + 2 \sigma_x^2, \quad q_y = q + 2 \sigma_y^2, \quad q_z = q + 2 \sigma_z^2
\]  

(110)
C The Bassetti Erskine Formula and the Complex Error Function

C.1 A two dimensional potential

In appendix B the potential of a three dimensional Gaussian charge distribution was obtained. A two dimensional form of the potential, which is defined as:

\[
\phi(x, y) = \frac{Q'}{4 \pi \varepsilon_0} \int_0^\infty dq \frac{\exp \left( -\frac{x^2}{q + 2\sigma_x^2} - \frac{y^2}{q + 2\sigma_y^2} \right)}{\sqrt{q + 2\sigma_x^2} \sqrt{q + 2\sigma_y^2}} - 1.
\]  

(111)

was used in ref. [14, 15] to study space charge forces and beam-beam effects. \(Q'\) is the longitudinal line charge density.

The corresponding electric fields are:

\[
E_x = -\frac{Q'}{4 \pi \varepsilon_0} \int_0^\infty dq \frac{\partial}{\partial x} \psi(x, y, q) = \frac{Q'}{4 \pi \varepsilon_0} \int_0^\infty dq \frac{2 x \psi(x, y, q)}{\sqrt{q + 2\sigma_x^2}},
\]

\[
E_y = -\frac{Q'}{4 \pi \varepsilon_0} \int_0^\infty dq \frac{\partial}{\partial y} \psi(x, y, q) = \frac{Q'}{4 \pi \varepsilon_0} \int_0^\infty dq \frac{2 y \psi(x, y, q)}{\sqrt{q + 2\sigma_y^2}},
\]

(112)

where \(\psi(q, x, y)^3\) is a two dimensional version of the integrand of Eqn.(109):

\[
\psi(x, y, q) = \frac{\exp \left( -\frac{x^2}{q + 2\sigma_x^2} - \frac{y^2}{q + 2\sigma_y^2} \right)}{\sqrt{q + 2\sigma_x^2} \sqrt{q + 2\sigma_y^2}}.
\]

(113)

The integrals for the electric fields can be rewritten using a new variable \(\xi\) [1]

\[
\xi = \sqrt{\frac{q + 2\sigma_y^2}{q + 2\sigma_x^2}} \quad \text{or} \quad q = -\frac{\sigma_0^2}{\xi^2 - 1} - 2\sigma_x^2,
\]

(114)

with \(\sigma_0 = \sqrt{2(\sigma_x^2 - \sigma_y^2)}\), assuming that \(\sigma_x > \sigma_y\). With this transformation one obtains for the potential \(\phi(x, y)\):

\[
\phi(x, y) = -\frac{Q'}{2 \pi \varepsilon_0} \int_{\sigma_y/\sigma_0}^1 d\xi \frac{\exp \left( \left( \frac{\xi^2 - 1}{\sigma_0^2} \right) \frac{x^2}{\sigma_0^2} + \left( 1 - \frac{1}{\xi^2} \right) \frac{y^2}{\sigma_0^2} \right) - 1}{\xi^2 - 1},
\]

(115)

---

3Note that the integral \(\int_0^\infty dq \psi(x, y, q)\) is divergent since \(\int_0^{q_{\text{max}}} dq 1/\left(\sqrt{q + 2\sigma_x^2} \sqrt{q + 2\sigma_y^2}\right) \sim \ln(q_{\text{max}})\) is divergent for \(q_{\text{max}} \to \infty\).
and for the electric fields:

\[
E_x = Q' \frac{1}{\pi \varepsilon_0 \sigma_0} \int_{\sigma_y/\sigma_x}^{1} d\xi \frac{x}{\sigma_0} \exp \left( \left( \xi^2 - 1 \right) \frac{x^2}{\sigma_0^2} + (1 - 1/\xi^2) \frac{y^2}{\sigma_0^2} \right)
\]

\[
= Q' \frac{1}{\pi \varepsilon_0 \sigma_0} \frac{x}{\sigma_0} \int_{\sigma_y/\sigma_x}^{1} d\xi \exp(-1 - \xi^2) \exp \left( \frac{x^2}{\sigma_0^2} + \frac{1}{\xi^2} \frac{y^2}{\sigma_0^2} \right)
\]

\[
E_y = Q' \frac{1}{\pi \varepsilon_0 \sigma_0} \int_{\sigma_y/\sigma_x}^{1} d\xi \frac{y}{\sigma_0} \frac{1}{\xi^2} \exp \left( \left( \xi^2 - 1 \right) \frac{x^2}{\sigma_0^2} + (1 - 1/\xi^2) \frac{y^2}{\sigma_0^2} \right)
\]

\[
= Q' \frac{1}{\pi \varepsilon_0 \sigma_0} \frac{y}{\sigma_0} \int_{\sigma_y/\sigma_x}^{1} d\xi \frac{1}{\xi^2} \exp(-1 - \xi^2) \exp \left( \frac{x^2}{\sigma_0^2} + \frac{1}{\xi^2} \frac{y^2}{\sigma_0^2} \right).
\]

C.2 The Complex Error Function

The complex error functions \( \text{erf}(z) \) and \( w(z) \) are defined as [16]:

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} d\zeta \exp(-\zeta^2) \quad (117)
\]

\[
w(z) = \exp(-z^2) \left( 1 + \frac{2i}{\sqrt{\pi}} \int_{0}^{z} d\zeta \exp(\zeta^2) \right). \quad (118)
\]

One can rewrite \( w(z) \) in terms of \( \text{erf}(z) \):

\[
w(z) = \exp(-z^2) \left( 1 - \text{erf}(-iz) \right). \quad (119)
\]

The function \( w(z) \) can be used to rewrite the integral \( \int d\zeta \exp(\zeta^2) \) in the following form:

\[
\int_{z_1}^{z_2} d\zeta \exp(\zeta^2) = -i \frac{\sqrt{\pi}}{2} \exp(z_1^2) \left( w(z_2) - \exp(z_1^2 - z_2^2) w(z_1) \right). \quad (120)
\]

C.3 The Bassetti Erskine Formula

The electric fields from Eqn.(116) can be rewritten with the help of the complex error function \( w(z) \) [1]. One obtains for the complex quantity \( E_x - i E_y \):

\[
E_x(x, y) - i E_y(x, y)
\]

\[
= Q' \frac{1}{\pi \varepsilon_0 \sigma_0} \int_{\sigma_y/\sigma_x}^{1} d\xi \left( \frac{x}{\sigma_0} - i \frac{y}{\sigma_0} \frac{1}{\xi^2} \right) \exp \left( \left( \xi^2 - 1 \right) \frac{x^2}{\sigma_0^2} + (1 - 1/\xi^2) \frac{y^2}{\sigma_0^2} \right)
\]

\[
= Q' \frac{1}{\pi \varepsilon_0 \sigma_0} \exp \left( -\left( \frac{x}{\sigma_0} + i \frac{y}{\sigma_0} \right)^2 \right) \int_{\sigma_y/\sigma_x}^{1} d\xi \left( \frac{x}{\sigma_0} - i \frac{y}{\sigma_0} \frac{1}{\xi^2} \right) \exp \left( \frac{x^2}{\sigma_0^2} + i \frac{y^2}{\sigma_0^2} \right).
\]
The integral can be transformed into an integral in the complex plane along the path
\[ \zeta = \frac{x}{\sigma_0} \xi + i \frac{y}{\sigma_0} \xi, \quad \text{with } \frac{\sigma_y}{\sigma_x} \leq \xi \leq 1 : \quad (121) \]

\[ E_x(x, y) - i E_y(x, y) = \frac{Q'}{\pi \epsilon_0 \sigma_0} \exp \left( -z_2^2 \right) \int_{z_1}^{z_2} d\zeta \exp \left( \zeta^2 \right), \quad (122) \]

with
\[ z_1 = \frac{x \sigma_y}{\sigma_0 \sigma_x} + i \frac{y}{\sigma_0 \sigma_y / \sigma_x} \]
\[ z_2 = \frac{x}{\sigma_0} + i \frac{y}{\sigma_0}. \quad (123) \]

Note that
\[ z_1^2 - z_2^2 = -\frac{x^2}{2 \sigma_x^2} - \frac{y^2}{2 \sigma_y^2}. \quad (124) \]

Equation (122) can be rewritten in terms of the complex error function \( w(z) \) (Eqn. (120)):
\[ E_x(x, y) - i E_y(x, y) = \frac{Q'}{4 \pi \epsilon_0} \frac{-i 2 \sqrt{\pi}}{\sigma_0} \left( w(z_2) - \exp(z_1^2 - z_2^2) w(z_1) \right), \quad \text{or} \]
\[ E_x(x, y) = \frac{Q'}{2 \epsilon_0 \sqrt{\pi} \sigma_0} \text{Im} \left( w(z_2) - \exp(z_1^2 - z_2^2) w(z_1) \right), \quad (125) \]
\[ E_y(x, y) = \frac{Q'}{2 \epsilon_0 \sqrt{\pi} \sigma_0} \text{Re} \left( w(z_2) - \exp(z_1^2 - z_2^2) w(z_1) \right). \quad (126) \]