The application of the amplitude dependent spin tune for the study of high order spin–orbit resonances in storage rings

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D.P. Barber and G.H. Hoffstätter

Deutsches Elektronen–Synchrotron (DESY), Hamburg, Germany

M. Vogt

University of New Mexico, Albuquerque, USA.

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Philosophy

A proper understanding of spin–orbit resonance structure at high energy in storage rings can only be obtained with a correct definition of the "spin tune". This in turn requires establishing a proper coordinate system for "measuring" spin precession and that leads to the notion of the "invariant spin field",

which in turn facilitates discussion of:

- Stationary polarisation states.
- Maximum attainable polarization.
- Perturbation theory — if needed, e.g. noise, non-linear fields, beam-beam.....

A vector field $\hat{f}$ of unit length in real 3–D space covering the 6–D phase space at each s and 1–turn periodic:

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$$\hat{f}(\vec{u};s) \text{ with } \vec{u} \equiv (x, p_x, y, p_y, z, p_z)$$

$$\hat{f}(\vec{u}; s+C) = \hat{f}(\vec{u}; s)$$

$$\frac{d\hat{f}}{ds} = \frac{\partial \hat{f}}{\partial s} + \sum_{k=I,II,III} \frac{dx_k}{ds}\frac{\partial \hat{f}}{\partial x_k} + \frac{dp_k}{ds}\frac{\partial \hat{f}}{\partial p_k} = \vec{G}_{\hat{f}}(\vec{u};s) = \underbrace{\vec{F}(\vec{u};s) \times \hat{f}}_{\text{fixed length} \rightarrow \text{precession}}$$

$$=====> \frac{\partial \hat{f}}{\partial s} = \{H_{orb}, \hat{f}\} + \vec{F}(\vec{u};s) \times \hat{f}$$

Now insist that this is the T–BMT equation!

$$\frac{d\hat{f}}{ds} = \vec{\Omega}(\vec{u};s) \times \hat{f}$$

Rename: $\hat{f} \longrightarrow \hat{n}$

Have now set up the:

The Invariant Spin Field, $\hat{n}$

- **So far just phase space: particles come later!**
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$$\vec{n}(M(\vec{u};s);s) = R_{3\times 3}(\vec{u};s)\vec{n}(\vec{u};s)$$

This is NOT the eigenproblem $\vec{N}(\vec{u};s) = R_{3\times 3}(\vec{u};s)\vec{N}(\vec{u};s)$

- On the closed orbit $\hat{n}(\vec{u};s) \longrightarrow \hat{n}(\vec{0};s) \equiv \hat{n}_0(s)$.
- ===> $\hat{n}$ and $\hat{n}_0(s)$ should not be confused!!!
- $\hat{n}$ is called the INVARIANT SPIN FIELD.
- The invariant spin field for 1 plane of orbit motion is a smooth closed vector curve. But $\hat{n}$ is NOT a "closed spin solution"!!!!
- For 3 planes of orbit motion $\hat{n}$ is on a smooth surface but is not closed.

- Again: this is NOT the eigenproblem $\vec{N}(\vec{u};s) = R_{3\times 3}(\vec{u};s)\vec{N}(\vec{u};s)$

- $\vec{N}(\vec{u};s)$: with 1 plane of motion it gives a smooth closed vector curve which repeats after 1 turn.

- But beyond the first turn the FIELD $\vec{N}(\vec{u};s)$ does not repeat: IT IS NOT AN INVARIANT FIELD.

- $\vec{N}$ is NOT EVEN A SOLUTION OF THE T–BMT EQUATION everywhere!

More:

- Existence of $\hat{n}$: for general 6–D orbits, heavy mathematics, non–trivial, lots of work at DESY.
- But the DESY algorithms so far work: pragmatism.
- $$P_{lim}(s) \equiv \int d^6u \; \rho_{\rm eq}(\vec{u};s)\hat{n}(\vec{u};s)$$
  is the maximum stationary polarization that can be achieved: all points in phase space fully polarized.
- ===> estimate this before simulating acceleration!!!
- At very high energy the resonance phenomena are very dense even with snakes ===> models, folk lore, received wisdom, popular prejudice are all useless. Need tracking simulations.

The single resonance model (SRM).

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Easy to show that

$$\vec{n}(\psi + 2\pi Q) = R_{3\times 3}(\psi)\vec{n}(\psi)$$

Note that as $\delta = \nu - Q$ ranges over $\pm\infty$, N passes through horizontal an infinite number of times!

But $\hat{n}$ passes through the horizontal once ===>

The N from $\vec{N}(\psi) = R_{3\times 3}(\psi)\vec{N}(\psi)$ is totally useless!

Spin tune and the definition of resonance

- $\hat{n}(\vec{u}; s)$ is a unique unit vector field on phase space obeying T–BMT.
- Attach 2 other unit vectors $\hat{n}_1(\vec{u}; s), \hat{n}_2(\vec{u}; s)$ to each $(\vec{u}; s)$ ===>
- $(\hat{n}_1, \hat{n}_2, \hat{n})$ form a right handed orthonormal coordinate system at each $(\vec{u}; s)$ ===>
- have a **local** coordinate system at each point $\vec{u}$ and s ===> defines parallel transport c.f. gauge theory, differential geometry.

Still no particles in the story!

Now add particles flying through phase space.

- Away from orbital resonance $\hat{n}_1(\vec{u};s), \hat{n}_2(\vec{u};s)$ can be chosen so that spins flying through phase space precess around $\hat{n}$ **uniformly** w.r.t. the $\hat{n}_1(\vec{u};s), \hat{n}_2(\vec{u};s)$ plane:

- $\vec{S} \cdot \hat{n}$ is constant along an orbit, in fact an **integral of motion**.

- The precession rate $\nu(J_x, J_y, J_z)$ is **independent** of $\psi_x, \psi_y, \psi_z, s$.

- That's the **spin tune**! ===> action–angle variables for spin.

- The resonance condition is

$$\nu(J_x, J_y, J_z) \;=\; k_0 + k_x Q_x + k_y Q_y + k_z Q_z$$

- $\nu(J_x, J_y, J_z) \neq \nu_0$ off the closed orbit.

- $\nu(J_x, J_y, J_z)$ is NOT extracted from the eigenvalues of the eigenproblem for $\vec{N}$: that gives some number depending on the ψ_i!!

Odds and ends on real Spin Tune

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- With the correct definition one can identify high order resonances properly and look for consistency.

- Use the correct definition to search for windows in the orbital tune diagram ===> dramatic increase in "spin aperture" (M. Vogt Thesis 2000).

- SRM with 2 snakes: $\nu(J_y) = 1/2$!! independently of J_y or Q_y except at orbital resonances.

- Froisart–Stora formula applies for high order resonances out in phase space !!!

The vector $\hat{n}$ was introduced in the context of action–angle variables for spin by Derbenev and Kondratenko (1973) to get a semiclassical spin quantization axis for calculating electron radiative spin flip rates,

— I come from another direction.

The Froissart–Stora formula

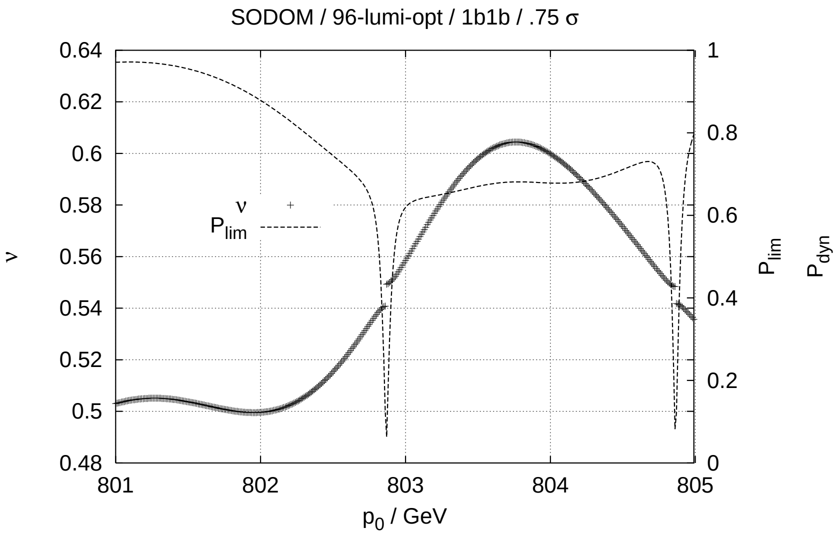
$$\frac{P_{\text{final}}}{P_{\text{initial}}} = 2\ e^{-\frac{\pi|\epsilon|^2}{2\alpha}} - 1$$

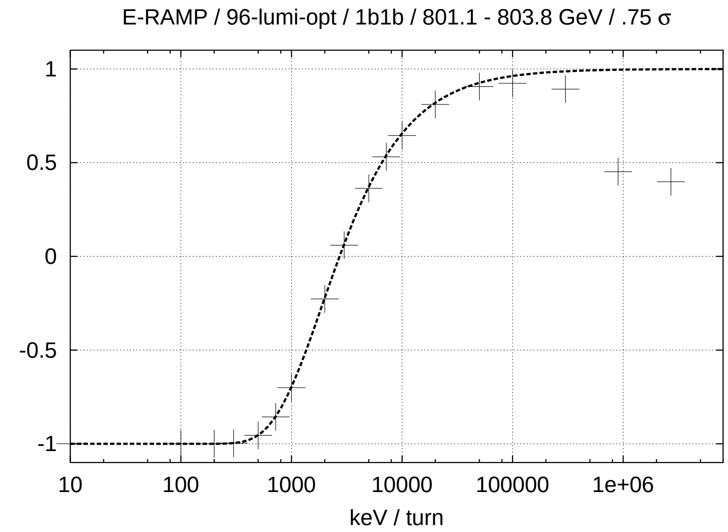
- ϵ is the "resonance strength", a measure of the dominant spin perturbation at resonance,
- α expresses the rate of resonance crossing.

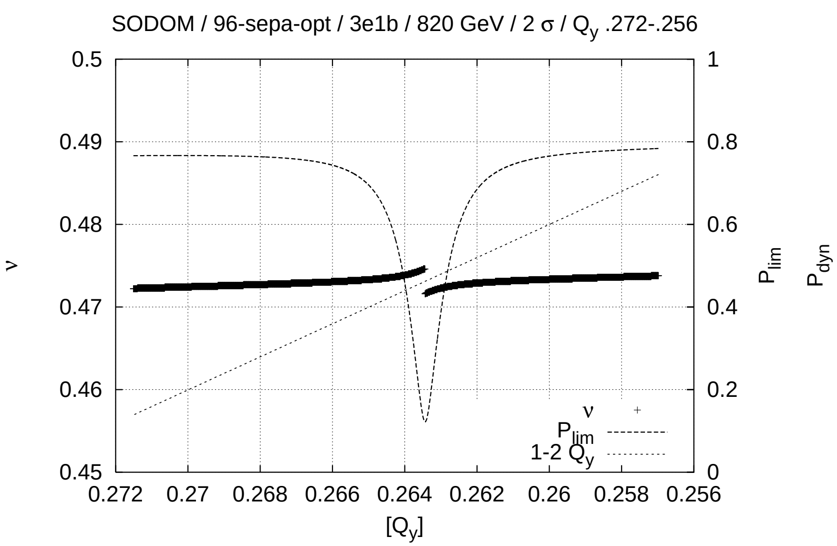
$$P(s) = P_{\mathrm{lim}}(s)|P_{\mathrm{dyn}}|$$

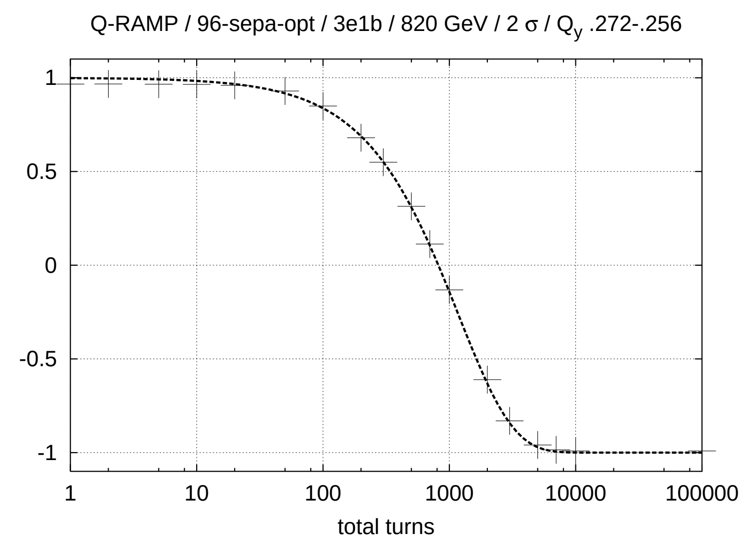
$P_{\mathrm{lim}}(s)$ is a static property of the optic and ring and energy.

P_{dyn} depends on the history: essentially $\left\langle \vec{S} \cdot \hat{n} \right\rangle$.


SODOM / 96-lumi-opt / 1b1b / .75 σ
ν
P_{lim}
p_0 / GeV
P_{lim}
P_{dyn}


E-RAMP / 96-lumi-opt / 1b1b / 801.1 - 803.8 GeV / .75 σ
keV / turn


SODOM / 96-sepa-opt / 3e1b / 820 GeV / 2 σ / Q_y .272-.256
ν
P_{lim}
1-2 Q_y
[Q_y]
P_{lim}
P_{dyn}


Q-RAMP / 96-sepa-opt / 3e1b / 820 GeV / 2 σ / Q_y .272-.256
total turns

- Top left:
  Energy scan of $P_{\rm lim}$ and ν for HERA–p with flatteners and a 4 snake scheme (rad., 45°, rad., 45°) with purely vertical motion at 0.75 σ.

- Top right:
  The dependence of the final $P_{\rm dyn}$ after ramping through the resonance at approximately 802.7 GeV on the energy gain per turn.

- Bottom left:
  Tune scan of $P_{\rm lim}$ and ν for HERA–p with flatteners and a 4 snake scheme (long., $-45°$, rad., 45°) with purely vertical motion at 2 σ.

- Bottom right:
  The dependence of the final $P_{\rm dyn}$ after ramping through the resonance at $[Q_y] \approx 0.2635$ on the total number of turns.