Using the amplitude dependent spin tune to study high order spin–orbit resonances in storage rings [1]

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Abstract. We define the amplitude dependent spin tune and illustrate its use for identifying spin–orbit resonances and evaluating their strength.

INTRODUCTION

A proper understanding of the spin–orbit resonance structure at high energy in storage rings can only be obtained with a correct definition of the "spin tune". This requires establishing a proper coordinate system for "measuring" spin precession and that, in turn, requires the notion of the "invariant spin field". This paper illustrates that programme. More details can be found in [1–6]. Our calculations were made with the spin–orbit tracking code `SPRINT` [5,6]. The algorithms in `SPRINT` are *non–perturbative*. `SPRINT` has extensive facilities for the *dynamical* variation of the reference energy and tunes so that it has been possible to carry out realistic tracking simulations of acceleration.

THE INVARIANT SPIN FIELD

The transverse and longitudinal motion of particles in storage rings is described in terms of three pairs of canonical coordinates $\vec{u} = (q_1, p_1, q_2, p_2, q_3, p_3)$. The independent variable is the distance along the ring l. There is a corresponding classical Hamiltonian $h_{\rm orb}(\vec{u}; l)$. In distorted rings $\vec{u}$ describes motion with respect to the resulting closed orbit. In the absence of spin flip, spin motion for electrons and protons moving in electric and magnetic fields is described by the T–BMT equation [1] $d\vec{S}/dl = \vec{\Omega} \times \vec{S}$ where $\vec{S}$ is the rest frame spin expectation value of

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the particle ("the spin") and $\vec{\Omega}$ depends on the electric and magnetic fields, the velocity and the energy so that it depends on $\vec{u}$ and l.

As a first step in setting up a coordinate system for spin we attach a laboratory space 3–vector $\hat{f}(\vec{u};l)$ of fixed unit length, to every point $(\vec{u};l)$. At this stage $\hat{f}$ is a definite but freely chosen smooth vector function of $\vec{u}$ and l. The rate of change of $\hat{f}$ along some path in $(\vec{u},\, l)$ space is $\frac{d\hat{f}}{dl} = \frac{\partial \hat{f}}{\partial l} + \sum_{k=1}^{3} \frac{dq_k}{dl}\frac{\partial \hat{f}}{\partial q_k} + \frac{dp_k}{dl}\frac{\partial \hat{f}}{\partial p_k}$. Then along a particle trajectory, and in terms of a Poisson bracket, the equation of motion takes the form $\partial \hat{f}/\partial l \; + \; \{\hat{f}, h_{\rm orb}\} = \vec{F}_{\hat{f}}(\vec{u};l)$. Since, by choice, $||\hat{f}||$ is invariant, the motion of $\hat{f}$ must be a rotation so that $\vec{F}_{\hat{f}}$ must have the form $\vec{G}(\vec{u};l) \times \hat{f}$. We now choose $\hat{f}$ so that it obeys the T–BMT equation: $d\hat{f}/dl = \vec{\Omega} \times \hat{f}$ along particle orbits. Moreover we require that it reflects the periodicity of the magnet structure by being 1–turn periodic in l, i.e. $\hat{f}(\vec{u};l+C) = \hat{f}(\vec{u};l)$ where C is the ring circumference. We denote this special choice by $\hat{n}(\vec{u};l)$. Except at the spin–orbit resonances to be discussed later, $\hat{n}(\vec{u};l)$ is unique.

Thus $\hat{n}(\vec{u};l)$ is a pre–established 1–turn periodic *vector field* on $(\vec{u},l)$ obeying the T–BMT equation. For one turn $\hat{n}(\vec{M}(\vec{u};l);l+C) = \hat{n}(\vec{M}(\vec{u};l);l) = R_{3\times 3}(\vec{u};l)\hat{n}(\vec{u};l)$ where $\vec{M}(\vec{u};l)$ is the new phase space vector after one turn starting at $\vec{u}$ and l and $R_{3\times 3}(\vec{u};l)$ is the corresponding spin transfer matrix. If a spin $\vec{S}$ is followed along an orbit, the scalar product $\vec{S}\cdot\hat{n}$ of $\vec{S}$ and the local $\hat{n}$ is invariant since both vectors obey the T–BMT precession equation. Thus with respect to the *local unique pre-established* $\hat{n}$ the motion of $\vec{S}$ is very simple, namely a precession around $\hat{n}$. On the closed orbit $\hat{n}(\vec{u};l)$ becomes $\hat{n}(\vec{0};l)$ which we denote by $\hat{n}_0(l)$. Obviously $\hat{n}_0(l+C) = \hat{n}_0(l)$. It is given by the real unit eigenvector of the 1–turn 3×3 spin transport matrix on the closed orbit.

Examples of the field $\hat{n}$ at 800 GeV for a HERA proton optic with a suitable arrangement of Siberian Snakes are shown in figure 1. In these particular simulations the protons only execute stable linear vertical betatron motion of fixed amplitude. Each picture shows the locus, on the surface of a sphere, of the tip of the $\hat{n}$ vector as the betatron phase varies at a point on the ring where $\hat{n}_0$ is vertical. The parameters are shown in the captions. For each picture in figure 1, the phase space coordinates of a particle are not 1–turn periodic but at a fixed position on the ring ("azimuth") they lie on a closed elliptical curve at positions depending on its vertical betatron phase. Likewise a spin at some $\vec{u}$ set parallel to $\hat{n}$ and tracked, is not 1–turn periodic but on tracking it turn to turn, it lies on the closed curve, parametrised by the orbital phase, of the field $\hat{n}$. Thus, like the invariant orbital ellipses, the curves of figure 1 are invariant when tracked from turn to turn and we therefore call $\hat{n}(\vec{u};l)$ the *invariant spin field*. As the amplitude is increased, the invariant spin field becomes convoluted, especially near the spin–orbit resonances to be discussed below. For motion with one degree of freedom, the loci on the sphere are closed as, for example, in figure 1. For more than one degree of freedom, the phase space coordinates lie on invariant tori and the loci of $\hat{n}$ do not close in general although the field $\hat{n}$ is still an invariant of the 1–turn spin–orbit

map. If the spins for an ensemble of particles distributed uniformly around the phase space ellipses for figure 1, are all set initially parallel to $\hat{n}_0$ and then tracked, the beam polarisation at that azimuth oscillates. If they are set parallel to $\hat{n}$, the beam polarisation is stationary. The maximum *stationary* beam polarisation that can be reached is $P_{\rm lim}(l) = ||\int d^6u\ w_{\rm st}(\vec{u};l)\hat{n}(\vec{u};l)||$ where $w_{\rm st}(\vec{u};l)$ is the normalised stationary phase space density. For motion on a vertical betatron ellipse $P_{\rm lim}$ is just given by the average of $\hat{n}$ over the betatron phase [3]. On the 64π mm mrad ellipse $P_{\rm lim}$ is much smaller than for the 4π mm mrad ellipse — it pays to devise ways to keep the spread of $\hat{n}$ small. $P_{\rm lim}$ should be calculated before carrying out simulations of acceleration. If $P_{\rm lim}$ is small such a simulation is not worthwhile. Note that for $\vec{u} \neq \vec{0}$, the constraint $\hat{n}(\vec{u};l+C) = \hat{n}(\vec{u};l)$ for the invariant spin field

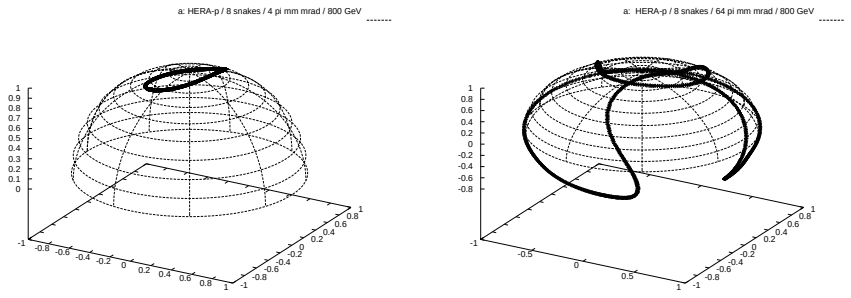

FIGURE 1. The field $\hat{n}$ in HERA–p calculated with `SPRINT` *on* the 4π mm mrad (left) and the 64π mm mrad (right) ellipses at 800 GeV. A normalised emittance of 4π mm mrad $\equiv$ "$1-\sigma$".

is obviously *not* equivalent to the closure condition $\vec{N}(\vec{u};l) = R_{3\times3}(\vec{u};l)\vec{N}(\vec{u};l)$. In fact, the calculation of the real $\hat{n}(\vec{u};l)$ is computationally nontrivial and requires either "stroboscopic averaging" [2], Fourier analysis as in `SODOM-II` [7] or "adiabatic anti–damping". All three algorithms are non–perturbative and are implemented in the code `SPRINT` [5,6].

Although we have concentrated on protons and have introduced the invariant spin field as an essential geometrical object it was first motivated by Derbenev and Kondratenko [8,9], for providing semiclassical spin quantisation axes when calculating radiative spin flip for electrons.

THE AMPLITUDE DEPENDENT SPIN TUNE

To complete the construction of our coordinates for describing spin motion, two other unit vectors $\hat{n}_1(\vec{u};l)$ and $\hat{n}_2(\vec{u};l)$ are attached to all $(\vec{u},l)$ such that the sets $(\hat{n}_1,\hat{n}_2,\hat{n})$ form local orthonormal coordinate systems at all points in phase space at each l. Like $\hat{n}$, $\hat{n}_1$ and $\hat{n}_2$ are 1–turn periodic in l: $\hat{n}_i(\vec{u};l+C) = \hat{n}_i(\vec{u};l)$ for $i\in\{1,2\}$. But unlike $\hat{n}$ they do not obey the T–BMT equation. As pointed out above the motion of $\vec{S}$ is a precession around $\hat{n}$. Now, with the basis vectors $\hat{n}_1$ and

$\hat{n}_2$ we have a way to quantify the rate of spin precession around $\hat{n}$: it is the rate of rotation of the projection of $\vec{S}$ onto the $\hat{n}_1, \hat{n}_2$ plane. Except for the uninteresting case of running on orbital resonance, the fields $\hat{n}_1(\vec{u}; l)$ and $\hat{n}_2(\vec{u}; l)$ can be chosen so that the rate of precession is constant and independent of the starting orbital phases [1–6]. The number of precessions per turn "measured" in this way is called the spin tune ν. The spin tune depends only on the orbital amplitudes — a tune depending in some way on phases would hardly be a useful quantity since it would have to change as the phases advance.

Spins are particularly strongly perturbed, and the locus of $\hat{n}$ is then expected to be very convoluted, when the spin tune is near resonance with the orbital tunes: $\nu(J_1, J_2, J_3) = k_0 + k_1 Q_1 + k_2 Q_2 + k_3 Q_3$ where the Q's are the amplitude dependent tunes of the orbital modes, the k's are integers and the J's are orbital amplitudes. The spin tune on the closed orbit, $\nu(0, 0, 0)$, is the number of precessions per turn of an arbitrary spin around $\hat{n}_0(l)$. We denote it by ν_0. Note that contrary to common practice our expression for resonance does not contain ν_0. Indeed, that is the whole point of having a clean definition of spin tune as we now illustrate. Figure 2 (left) shows the dependence of the spin tune on orbital amplitude (=

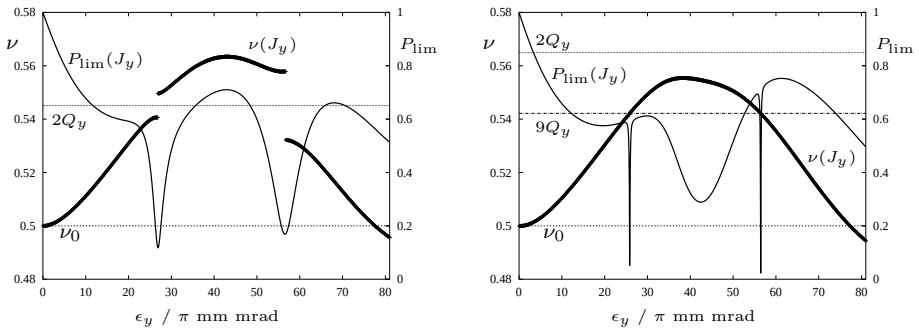

FIGURE 2. The amplitude dependent spin tune ν and $P_{\rm lim}$ *on* phase space ellipses with normalised vertical emittance ϵ_y as calculated with `SPRINT` for HERA–p at 805 GeV. Left: vertical tune $Q_y = 32.2725$, right: $Q_y = 32.2825$.

enclosed normalised emittance) for purely vertical betatron motion in HERA–p at 805 GeV with a suitable arrangement of snakes [3]. On the design orbit, i.e. at zero amplitude, ν is $1/2$ as expected. But it deviates from $1/2$ as the amplitude increases and at 27π mm mrad it jumps symmetrically across the resonant value $2Q_y$. After increasing further, ν then decreases and at a normalised emittance of 56π mm mrad it jumps back across the resonant value $2Q_y$. So ν never actually hits the resonant value but as one can see $P_{\rm lim}$ becomes small around the resonant amplitudes as the locus of $\hat{n}$ becomes convoluted and extends over the whole unit sphere. Thus the behaviours of ν and $P_{\rm lim}$ are mutually consistent. Figure 2 (right) shows the behaviour of ν when Q_y is increased. The second order resonance can no longer be crossed but 9th order resonant behaviour occurs instead. These curves illustrate just how complicated spin motion can be at very high energy. Such phenomena

could obviously not be seen without a properly defined spin tune. For example, a "fake spin tune" erroneously extracted from the complex eigenvalues of $R_{3\times3}$ shows no correlation with dips in $P_{\rm lim}$. That is no surprise since that "fake tune" depends on the orbital phase and is therefore unsuitable for describing long term spin–orbit coherence. With the properly defined ν, the proximity to spin–orbit resonances can be properly judged and the changes in orbital tunes needed to avoid resonances can be properly estimated.

An especially satisfying aspect of these concepts is that they have provided a way to generalise the application of the Froissart–Stora formula [11] for the polarisation loss when passing through resonances. In particular, the size of a resonant jump in ν, $\Delta\nu$, for a high order resonance, is a measure of the strength of the resonance and using the facilities in `SPRINT` it has been possible to parametrise polarisation loss *with respect to* $\hat{n}$, when varying various machine parameters *dynamically* through such high order resonances, in terms of a generalised Froissart–Stora formula [5,6], containing $\Delta\nu$. More details on these calculations will be published elsewhere.

SUMMARY AND CONCLUSION

The use of a coordinate system based on the invariant spin field and of the properly defined spin tune are indispensable for a clear understanding of spin–orbit resonant behaviour in storage rings. Their use allows high order resonances to be cleanly identified, their strengths to be determined and misconceptions based on false definitions of spin tune to be avoided.

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