See Summary + Outlook

Chapter 5

Introduction to spin-orbit tori

I now come to the second part of this thesis which consists of Chapters 5-10 and Appendices B-G. It presents the topic of spin-orbit tori as a mathematical theory and it is based on the map formalism equations of motion (6.1),(6.2).

5.1 Physical context and mathematical approach

I begin with some brief general remarks on the physical context for the orientation of the reader. More details can be found in [BEH04, Hof, MSY, Vo].

Spin is of central importance for the understanding of the behavior of fundamental particles and their interactions. This is made clear, for example, in [SPIN09] where up-to-date accounts of experimental and theoretical work are given. In particular, the differential cross sections for particle-particle interactions depend on the spin states of the particles. These interactions are typically studied by colliding a beam of spin-1/2 particles (e.g. electrons or protons) either with another beam of spin-1/2 particles or with nuclei located at a fixed 'target'. Various considerations, such as the need for high energies, often dictate that the particles circulate in a beam consisting of a train of separate bunches in a so-called storage ring. Typically the motion of a bunch for 10^9 turns around the ring is of interest. The particle interactions to be studied in such a storage ring take place at the centers of detectors mounted at specially configured interaction points. The task of Accelerator Physics is to provide and describe the transport of the bunches through the interaction points and it requires mathematical tools which are different from those needed to describe the collision processes in the interaction points (the latter tools are from Quantum Field Theory). This thesis deals exclusively with the Accelerator Physics aspects and its tools are from Dynamical Systems Theory. Descriptions of storage rings can be found in standard text books. See for example [CT, Wi]. However, to summarize, the common feature of a storage ring is that the electrically charged particles are confined by combinations of electric and magnetic fields to move in bunches on approximately circular orbits in a vacuum tube. The dimensions of a bunch are millimeters whence they are very small compared to the average radius of the ring which can be kilometers. A bunch typically contains around $N = 10^{11}$ particles. Accelerator Physics involves various levels of description depending on how accurately one wants to study the bunches. So I now have to characterize the level needed for this thesis. At this level a phase-space variable $\tilde{u}$ and a spin variable $\tilde{S}$ provide a classical description of a particle located at $\tilde{u} \in \mathbb{R}^6$ with spin value $\tilde{S} \in \mathbb{R}^3$. Experiments aimed at exploiting the influence of spin on particle-particle collisions usually require that the bunches be spin polarized. This means that the polarization $\tilde{P} := (1/N) \sum_{i=1}^{N} \tilde{S}_i$, namely the average over the spin vectors $\tilde{S}_1, ..., \tilde{S}_N$ of the bunch be non-zero. Thus the task of Polarized Beam Physics is to provide and describe the transport of bunches through the interaction points such that $|\tilde{P}|$ is 'sufficiently' large. Note that in the definition of $\tilde{P}$ the spin vectors have to be normalized, i.e., $|\tilde{S}_i| = 1$. Nevertheless for the purposes of this work there is no need to assume that the spin vectors are normalized. For the purposes of this thesis I ignore all interactions between the particles, the emission of electromagnetic

Definition 6.1 *(Spin-orbit torus) Given a $\omega \in \mathbb{R}^d$, a pair (ω, A) is called a 'd-dimensional spin-orbit torus' if $A \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. I call ω the 'orbital tune vector' of the spin-orbit torus. The function $\Psi_{\omega,A} : \mathbb{Z} \times \mathbb{R}^d \to SO(3)$ is defined by (6.4) and $\Psi_{\omega,A}(n;\cdot)$ is called the 'n-turn spin transfer matrix of (ω, A)'. I denote, for $\omega \in \mathbb{R}^d$, the set of those spin-orbit tori, whose orbital tune vector is ω, by $\mathcal{SOT}(d,\omega)$. The set of all d-dimensional spin-orbit tori I denote by $\mathcal{SOT}(d)$ and the set of all spin-orbit tori by $\mathcal{SOT}$. A function $\begin{pmatrix} \phi \\ S \end{pmatrix} : \mathbb{Z} \to \mathbb{R}^{d+3}$ is called a 'spin-orbit trajectory of (ω, A)' if it satisfies (6.1),(6.2). Accordingly ϕ is called an 'orbital trajectory of (ω, A)' and S is called a 'spin trajectory of (ω, A) over $\phi(0)$'.* □

In the remaining parts of this section I give some comments on Definition 6.1.

Clearly, for a given $\omega \in \mathbb{R}^d$, there are as many elements in the set $\mathcal{SOT}(d,\omega)$ and as many equations of moton (6.1),(6.2) as there are elements in $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. To put this into perspective one has to recall that the spin-orbit tori are modeled after the situation of (5.19), i.e., after the T—BMT equation. Therefore the spin-orbit tori obtained from (5.19) constitute only a small subset of $\mathcal{SOT}$. Thus in effect the present work demonstrates that important features of the spin-orbit tori can be studied without using (5.19), i.e., without referring to the actual T—BMT equation at all. For example, while the uniqueness theorem of Yokoya (see Section 7.5) holds for a vast set of spin-orbit tori, only a small (but, of course very important) subset of those spin-orbit tori is connected with (5.19) and the T—BMT equation.

Since $\Psi_{\omega,A}(n;\phi) \in SO(3)$, the angle between two spin trajectories over the same $\phi(0)$ is a constant of motion. Of course the Euclidean norm $|S(n)|$ of $S(n)$ is a constant of motion, too.

It follows from (6.6) that, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$, we have the useful formula

$$\Psi^T_{\omega,A}(n;\phi) = \Psi_{\omega,A}(-n;\phi + 2\pi n\omega) . \tag{6.7}$$

Picking, for $(\omega, A) \in \mathcal{SOT}(d,\omega)$, a $\phi_0 \in \mathbb{R}^d$, then the equation of spin motion (6.2) for the corresponding orbital trajectory $\phi(n) = \phi_0 + 2\pi n\omega$ reads as

$$S(n+1) = A(\phi_0 + 2\pi n\omega)S(n) \,. \tag{6.8}$$

Of course, every function $S : \mathbb{Z} \to \mathbb{R}^3$, which satisfies (6.8), is a spin trajectory over ϕ_0 of (ω, A) (and vice versa). Moreover if $S : \mathbb{Z} \to \mathbb{R}^3$ satisfies (6.8), then the function $\begin{pmatrix} \phi \\ S \end{pmatrix}$, with $\phi(n) = \phi_0 + 2\pi n\omega$, is a spin-orbit trajectory of (ω, A).

While the system of equations of motion (6.1),(6.2) for $\begin{pmatrix} \phi \\ S \end{pmatrix}$ is autonomous and nonlinear, the equation of motion (6.8) for S is linear and non-autonomous.

Furthermore, if $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and if $\omega, \omega' \in \mathbb{R}^d$ differ only by an element of $\mathbb{Z}^d$ then, due to the 2π–periodicity of A, the spin-orbit tori $(\omega, A), (\omega', A)$ are essentially the same since the associated equation of motion (6.8) is the same for both.

To interpret Definition 6.1 along the lines of Section 5.1 in the context of the map formalism for polarized beams in storage rings, the reader should view $\phi(n)$ as the value of the orbital angle variable and $S(n)$ as the value of the spin variable after n 'turns' around the storage ring. This means that n can be as large as 10^9 whence the present section is definitely not the last word to be said about spin-orbit trajectories. In particular the numerical calculation of $\Psi_{\omega,A}(n;\cdot)$ for large n is a challenging task. Furthermore this calculation can be hampered by the circumstance that A is only approximately known. These circumstances warrant the more involved discussion of spin-orbit tori in this work.

particular, each d-dimensional spin-orbit torus has as many polarization fields as the set $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ has elements. We see that the role which the $\mathbb{Z}$-action $L^{(PF)}_{\omega,A}$ plays for polarization fields, is analogous to the role which the $\mathbb{Z}$-action $L_{\omega,A}$ plays for spin-orbit trajectories. Note also that G is a fixpoint of $L^{(PF)}_{\omega,A}$ iff the polarization field $\mathcal{S}_G$ is invariant. Since $L^{(PF)}_{\omega,A}$ is a group action of the group $\mathbb{Z}$ one easily concludes:

Proposition 6.3 *Let (ω, A) be a spin-orbit torus. A polarization field $\mathcal{S}_G$ of (ω, A) is invariant iff*

$$L^{(PF)}_{\omega,A}(1; G) = G \,. \tag{6.22}$$

In other words, $\mathcal{S}_G$ is invariant, iff for all ϕ,

$$G(\phi) = A(\phi - 2\pi\omega)G(\phi - 2\pi\omega) \,. \tag{6.23}$$

□

Proof of Theorem 9.5: See Section F.30. □

Note that (6.23) will be interpreted by Theorem 9.5b as a symmetry property of (ω, A) along the lines of reduction theory.

A polarization field $\mathcal{S}_G$ is a spin field iff $|G(\phi)| = 1$ for all ϕ. Defining the 2-sphere $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ and equipping it with the relative topology from $\mathbb{R}^3$ we see that the set $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ of 2π–periodic and continuous functions from $\mathbb{R}^d$ into $\mathbb{S}^2$ is equal to the set of 2π–periodic, normalized (w.r.t. the Euclidean norm), and continuous functions from $\mathbb{R}^d$ into $\mathbb{R}^3$. Thus for every spin field $\mathcal{S}_G$ we have $\mathcal{S}_G(n, \cdot) \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$. Clearly each ISF is a polarization field.

Due to Definition 6.2, every polarization field $\mathcal{S}_G$ fulfills three different conditions: the 'dynamical' condition (6.16), the 'kinematical' condition that G is 2π–periodic, and the 'regularity' condition that G is continuous. In contrast to the dynamical and kinematical conditions, the regularity condition is a matter of choice. The regularity of G can basically vary between the extremes 'no regularity condition' and 'G being real analytic'. In this work I choose G to be continuous since the spin-orbit tori are built on continuity, i.e., the $\Psi_{\omega,A}(n;\cdot)$ are continuous functions.

Since the equation of motion (6.19) for $\mathcal{S}_G$ is linear, $L^{(PF)}_{\omega,A}(n;\cdot)$ is a homomorphism of the additive group $\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$, i.e., for $n\in\mathbb{Z}, G, G'\in\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$,

$$L^{(PF)}_{\omega,A}(n;G+G') = L^{(PF)}_{\omega,A}(n;G) + L^{(PF)}_{\omega,A}(n;G')\ . \tag{6.24}$$

Eq. (6.24) allows, by the technique of twisted cocycles [HK1, HK2, Zi1], to define cohomology groups for any spin-orbit torus, which give further insight into $SOT(d,\omega)$ in general and into the ISF conjecture in particular [He]. However this is beyond the scope of the present work.

6.4 Homotopy Theory relevant for spin-orbit tori

Throughout this work I will see some impact of Homotopy Theory on the theory of spin-orbit tori and in this section I introduce some basic features (the details are worked out in Appendix C).

Let X be a path-connected topological space. In the context of spin-orbit tori, one is especially interested in $X = SO(3)$ and $X = \mathbb{S}^2$ (recall that spin transfer matrices are $SO(3)$-valued functions and that spin fields are $\mathbb{S}^2$-valued functions). The use of Homotopy Theory for $\mathcal{C}_{per}(\mathbb{R}^d,X)$ is twofold. Firstly, I use it by applying the Homotopy Lifting Theorem (see Lemma C.6 in Section C.1) which in turn is used in many of those proofs of this work which involve the sets $\mathcal{C}_{per}(\mathbb{R}^d,X)$. Secondly, Homotopy Theory gives us the useful equivalence relation $\simeq^{2\pi}_X$ on $\mathcal{C}_{per}(\mathbb{R}^d,X)$, as follows. To explain this equivalence relation I first note, by Proposition C.4, that any two functions in $\mathcal{C}_{per}(\mathbb{R}^d,X)$ are homotopic w.r.t. X, i.e., $[\mathbb{R}^d,X]$ is a singleton.

Lemma 7.8 *a) Let R be in $SO(3)$ and $Re^3 = e^3$. Then $R \in SO_3(2)$.*

b) A spin-orbit torus (ω, A) is weakly trivial iff $A(\phi)e^3 = e^3$.

Proof of Lemma 7.8: See Section F.6. □

The following theorem expresses the most important property of weak coboundaries.

Theorem 7.9 *Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$. Then, for every $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$, we have $R_{d,\omega}(T; \omega, A) \in \mathcal{WT}(d, \omega)$ iff the third column, Te^3, of T is the generator of an ISF of (ω, A). Moreover $(\omega, A) \in \mathcal{WCB}(d, \omega)$ iff there exists a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ such that Te^3 is the generator of an ISF of (ω, A).*

Proof of Theorem 7.9: See Section F.7. □

Theorem 7.9 shows that the existence of an ISF is a necessary condition for a spin-orbit torus to be a weak coboundary. However Theorem 7.10, below, shows that this is not a sufficient condition.

As we just learned from Theorem 7.9, every weak coboundary has an ISF. I now address the converse question: is a spin-orbit torus a weak coboundary, if it has an ISF? A partial answer is given by the following theorem which uses some concepts introduced in Section 6.4 and which are borrowed from Homotopy Theory.

Theorem 7.10 *Let $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ and let $(\omega, A) \in \mathcal{SOT}(d, \omega)$ such that G is the generator of an ISF $\mathcal{S}_G$ of (ω, A). Then the following hold.*

a) If G is 2π-nullhomotopic w.r.t. $\mathbb{S}^2$ then $(\omega, A) \in \mathcal{WCB}(d, \omega)$ and a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists such that $R_{d,\omega}(T; \omega, A) \in \mathcal{WT}(d, \omega)$ and $G = Te^3$.

b) If $d = 1$ then $(\omega, A) \in \mathcal{WCB}(1, \omega)$ and a $T \in \mathcal{C}_{per}(\mathbb{R}, SO(3))$ exists such that $R_{1,\omega}(T; \omega, A) \in \mathcal{WT}(1, \omega)$ and $G = Te^3$.

c) If $d = 2$ then a $T \in \mathcal{C}_{per}(\mathbb{R}^2, SO(3))$ exists such that $R_{2,\omega}(T; \omega, A) \in \mathcal{WT}(2,\omega)$ and $G = Te^3$ iff G is 2π-nullhomotopic w.r.t. $\mathbb{S}^2$.

Proof of Theorem 7.10: See Section F.8. □

Let $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ and let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ such that G is the generator of an ISF of (ω, A). It is clear by Theorem 7.10a that if (ω, A) is not a weak coboundary, then G is not 2π-nullhomotopic w.r.t. $\mathbb{S}^2$. That this situation does occur, is the content of Theorem 8.17 (of course, due to Theorem 7.10b, this situation only occurs if $d \geq 2$).

Let $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ and let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ such that G is the generator of an ISF of (ω, A). If $S_0 \in \mathbb{S}^2$ exists such that neither S_0 nor $-S_0$ belong to the image G then it follows easily from Theorem 7.9 that $(\omega, A) \in \mathcal{WCB}(d,\omega)$ (and thus, by Theorem 7.10c, that, for $d = 2$, G is 2π-nullhomotopic w.r.t. $\mathbb{S}^2$). This also implies that if $(\omega, A) \in \mathcal{SOT}(d,\omega)$ has an ISF $\mathcal{S}_G$ then the question, whether $(\omega, A) \in \mathcal{WCB}(d,\omega)$, is connected with the issue of 'how complete' the image of G covers the sphere $\mathbb{S}^2$.

7.4 Introducing spin tune and spin-orbit resonance of first kind

Definition 7.11 *(Spin tune of first kind, spin-orbit resonance of first kind) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$. Then the subset $\Xi_1(\omega, A)$ of $[0,1)$ is defined by*

$$\Xi_1(\omega, A) := \{PH(A') : (\omega, A') \in \mathcal{AT}(d,\omega) \,\&\, (\omega, A') \sim_{d,\omega} (\omega, A)\} \,. \qquad (7.19)$$

I call ν a 'spin tune of first kind of (ω, A)' if $\nu \in \Xi_1(\omega, A)$.

I say that (ω, A) is 'on spin-orbit resonance of first kind' iff $0 \in \Xi_1(\omega, A)$. I say *b) Let $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$. Then the $SO_3(2)$-reduction $\widehat{MAIN}_{\lambda_{SOT(d)},SO_3(2)}(F^{-1} \circ G)$ of $\lambda_{SOT(d)}$ is invariant under $\Phi_{\omega,A}(\mathbb{Z})$ iff $\mathcal{S}_G$ is an invariant spin field of (ω, A). In particular (ω, A) has an invariant spin field iff $\lambda_{SOT(d)}$ has a 2π-periodic $SO_3(2)$-reduction which is invariant under $\Phi_{\omega,A}(\mathbb{Z})$.*

Proof of Theorem 9.5: See Section F.30. □

Note by (9.63),(9.67) and Theorem 9.5b that if $(\omega, A) \in \mathcal{SOT}(d)$ and $\mathcal{S}_G$ is an invariant spin field of (ω, A) then the total space of the invariant $SO_3(2)$-reduction $\widehat{MAIN}_{\lambda_{SOT(d)},SO_3(2)}(F^{-1} \circ G)$ of $\lambda_{SOT(d)}$ has the form

$$\check{E}_{F^{-1}\circ G,SO_3(2)} = \{(\phi, R) \in \mathbb{R}^d \times SO(3) : G(\phi) = Re^3\} . \qquad (9.71)$$

Thus (9.71) represents the invariant spin field $\mathcal{S}_G$ by a subset of $\mathbb{R}^d \times SO(3)$, i.e., we have a 'geometrization' of invariant spin fields. Another aspect of Theorem 9.5b is that the existence of an invariant spin field of (ω, A) is a symmetry property of (ω, A).

One more aspect of Theorem 9.5 is the following. While, by Theorem 9.5b, invariant spin fields are linked to 2π-periodic invariant $SO_3(2)$-reductions of $\lambda_{SOT(d)}$, it is easy to show, by Theorem 9.5a, that spin-orbit resonances of first kind are linked to 2π-periodic invariant H-reductions of $\lambda_{SOT(d)}$ where H is the trivial subgroup of $SO(3)$. Thus the existence of spin tunes of first kind of (ω, A) is a symmetry property of (ω, A).

9.3.6 Closing remarks on $\lambda_{SOT(d)}$

I have now completed my coverage of principal bundles since my only objective in this regard was to show how the principal $SO(3)$-bundle $\lambda_{SOT(d)}$ underlies the theory of $\mathcal{SOT}(d)$.

Following the Feres machinery one could extend my study. However this would go beyond the scope of the present work. So I just mention four points. Firstly, by using the linearity of $L^{(3D)}(R;S)$ in S, one can extend the structure group from $SO(3)$ to $GL(3)$ and study, by a 'prolongation' of the principal $SO(3)$-bundle $\lambda_{SOT(d)}$ to a principal $GL(3)$-bundle, the $\mathbb{Z}$-actions $L_{\omega,A}$ and $L_{\omega,A}^{(PF)}$ in terms of vector bundle techniques ($GL(n)$ denotes the group of real nonsingular $n \times n$–matrices). Secondly, one can go beyond Theorem 9.5 to study invariant H-reductions of $\lambda_{SOT(d)}$ in a more general way by asking what closed subgroups H of $SO(3)$ allow for 2π-periodic H-reductions which are invariant under a given spin-orbit torus in $SOT(d)$. For such a study the 'algebraic hull' is an important tool which was introduced by Zimmer in the 1980's. Thirdly one can apply rigidity theorems which allow to discuss properties which are stable (='rigid') under the extension of the group $\mathbb{Z}$ of the evolution variable. Fourthly, the choice of $\lambda_{SOT(d)}$ is not unique. For example an alternative choice is to employ $\mathbb{T}^d$ rather than $\mathbb{R}^d$ in the definition of the total resp. base space of the principal $SO(3)$-bundle. In fact this alternative choice is very convenient when one would go deeper into the matter of spin-orbit tori but for the purposes of the present work the choice of $\lambda_{SOT(d)}$ is sufficient and leads to analogous results as if one would use $\mathbb{T}^d$ instead of $\mathbb{R}^d$.

Zimmer Reduction ?

Chapter 10

Summary of spin-orbit tori and outlook

As pointed out in the Introduction, the second part of this thesis studies spin-orbit tori in terms of the map formalism equations of motion (6.1),(6.2) which plays a central role in the mathematical study of polarized beams in storage rings.

From a technical point of view a distinguishing feature of the present work is to formulate all concepts and properties in mathematical terms. Accordingly the mathematical notion of spin-orbit torus is introduced and a number of properties of spin-orbit tori are derived. Most of my definitions that are related to spin-orbit tori are distilled from established concepts in Polarized Beam Physics which are then translated into the language of Mathematics. The subsets $\mathcal{CB}(d,\omega) \subset \mathcal{ACB}(d,\omega) \subset \mathcal{WCB}(d,\omega)$ of the set $\mathcal{SOT}$ of spin-orbit tori have been introduced and discussed in some detail. I noted that spin-orbit tori (ω, A) of interest are almost coboundaries, i.e., are in $\mathcal{ACB}(d,\omega)$ and they have the form $A(\phi) = T^T(\phi + 2\pi\omega)\exp(\mathcal{J}2\pi\nu)T(\phi)$.

To my knowledge the results of the thesis are either new (e.g., Theorem 9.5b about the impact of Principal Bundle Theory on invariant spin fields) or were never P121 formulated in mathematically precise terms whence were never rigorously proved before (e.g., Corollary 8.12 aka the SPRINT Theorem). Note that some results (e.g., Yokoya's uniqueness theorem 7.13) were rigorously proved before for the flow formalism (see [BEH04]).

I have gathered quite a bit of insight into the invariant spin field (as well as into the spin tune) which is central for Polarized Beam Physics, as explained in Section 7.6. From Section 6.3 we know that an invariant spin field is tied with the equation $G(\phi) = A(\phi - 2\pi\omega)G(\phi - 2\pi\omega)$. I formulated the ISF conjecture which states that if (ω, A) is off orbital resonance, i.e., $(1, \omega)$ nonresonant, then an invariant spin field exists. Theorem 7.9 states that if (ω, A) is a weak coboundary, then an invariant spin field exists. Theorem 7.10a states that if $\mathcal{S}_G$ is an invariant spin field and if G is 2π-nullhomotopic then (ω, A) is a weak coboundary. Theorem 8.17 states that there are spin-orbit tori which have an invariant spin field and which are not weak coboundaries. Finally Theorem 9.5b shows that the existence of an invariant spin field of (ω, A) is a symmetry property of (ω, A). In fact Theorem 9.5b ties the existence of an invariant spin field to an $SO_3(2)$-reduction of the principal $SO(3)$-bundle $\lambda_{SOT(d)}$.

It is also worthwile to mention that the machinery of Chapter 9 can be applied to any linear n-dimensional nonautonomous ODE $\dot{y} = Y(t)y$ since the standard procedure of making it autonomous, encodes the ODE into a $GL(n)$-cocycle over the time translations whence encodes it into a principal $GL(n)$-bundle with base space $\mathbb{R}$. This will be addressed in a future publication of the author.

For a detailed outline of this work see Section 5.2. Avenues for further work are of course plentiful. In addition to those mentioned in Section 5.3, one topic of further studies could be the continuation of the work of Section 9.3. In fact, as outlined in Section 9.3.6, there are further applications of the principal $SO(3)$-bundle $\lambda_{SOT(d)}$ in waiting which will shed further light into the matter of spin-orbit tori.