Chapter 5

Introduction to spin-orbit tori

I now come to the second part of this thesis which consists of Chapters 5-10 and Appendices B-G. It presents the topic of spin-orbit tori as a mathematical theory and it is based on the map formalism equations of motion (6.1),(6.2).

5.1 Physical context and mathematical approach

I begin with some brief general remarks on the physical context for the orientation of the reader. More details can be found in [BEH04, Hof, MSY, Vo].

Spin is of central importance for the understanding of the behavior of fundamental particles and their interactions. This is made clear, for example, in [SPIN09] where up-to-date accounts of experimental and theoretical work are given. In particular, the differential cross sections for particle-particle interactions depend on the spin states of the particles. These interactions are typically studied by colliding a beam of spin-1/2 particles (e.g. electrons or protons) either with another beam of spin-1/2 particles or with nuclei located at a fixed 'target'. Various considerations, such as the need for high energies, often dictate that the particles circulate in a beam consisting of a train of separate bunches in a so-called storage ring. Typically the motion of a bunch for 10^9 turns around the ring is of interest. The particle interactions to be studied in such a storage ring take place at the centers of detectors mounted at specially configured interaction points. The task of Accelerator Physics is to provide and describe the transport of the bunches through the interaction points and it requires mathematical tools which are different from those needed to describe the collision processes in the interaction points (the latter tools are from Quantum Field Theory). This thesis deals exclusively with the Accelerator Physics aspects and its tools are from Dynamical Systems Theory. Descriptions of storage rings can be found in standard text books. See for example [CT, Wi]. However, to summarize, the common feature of a storage ring is that the electrically charged particles are confined by combinations of electric and magnetic fields to move in bunches on approximately circular orbits in a vacuum tube. The dimensions of a bunch are millimeters whence they are very small compared to the average radius of the ring which can be kilometers. A bunch typically contains around $N = 10^{11}$ particles. Accelerator Physics involves various levels of description depending on how accurately one wants to study the bunches. So I now have to characterize the level needed for this thesis. At this level a phase-space variable $\tilde{u}$ and a spin variable $\tilde{S}$ provide a classical description of a particle located at $\tilde{u} \in \mathbb{R}^6$ with spin value $\tilde{S} \in \mathbb{R}^3$. Experiments aimed at exploiting the influence of spin on particle-particle collisions usually require that the bunches be spin polarized. This means that the polarization $\tilde{P} := (1/N)\sum_{i=1}^{N} \tilde{S}_i$, namely the average over the spin vectors $\tilde{S}_1, ..., \tilde{S}_N$ of the bunch be non-zero. Thus the task of Polarized Beam Physics is to provide and describe the transport of bunches through the interaction points such that $|\tilde{P}|$ is 'sufficiently' large. Note that in the definition of $\tilde{P}$ the spin vectors have to be normalized, i.e., $|\tilde{S}_i| = 1$. Nevertheless for the purposes of this work there is no need to assume that the spin vectors are normalized. For the purposes of this thesis I ignore all interactions between the particles, the emission of electromagnetic radiation by the particles and the effects of the electric and magnetic fields set up in the vacuum pipe by the particles themselves. This leads to a classical Hamiltonian description (for a derivation of the Hamiltonian from Quantum Physics, see [BH98]). Furthermore I shall neglect the extremely small Stern-Gerlach force acting from $\tilde{S}$ onto $\tilde{u}$ [BEH04] (for details on the relativistic Stern-Gerlach force in Accelerator Physics, see e.g. [He96]). Then the particle motion is described by the equation for the Lorentz force and the spin motion by the Thomas-Bargmann-Michel-Telegdi (T-BMT) equation [Ja]. Thus the equations of motion for the combined $\tilde{u}$,$\tilde{S}$ system are no longer Hamiltonian (albeit the equations of motion for $\tilde{u}$ are still Hamiltonian).

Although dynamical systems are usually analyzed by taking time as the independent variable, this is usually not convenient for storage rings since there, the vacuum tube and the electric and magnetic guide fields have a fixed, 1-turn periodic, approximately circular spatial layout. It is then common practice to define the angular distance, $\theta = 2\pi s/L$, around the ring where s is the distance around the ring and L is the circumference. The equations of motion for $\tilde{u}$ and $\tilde{S}$ are then transformed into forms in which θ is the independent variable. The one-turn periodicity of the positions of the electric and magnetic guide fields then becomes a 2π–periodicity in θ of the equations of motion for $\tilde{u}$ and $\tilde{S}$. As a next step one constructs the curvilinear closed orbit, i.e., the orbit along which the particle motion is one-turn periodic and one defines coordinates with respect to this orbit. Then $\tilde{u}$ consists of three pairs of canonical variables. For example, two of the pairs can describe transverse motion and one pair can describe longitudinal (synchrotron) motion within a bunch. One of this latter pair quantifies the deviation of the particle energy from the energy of a 'reference particle' fixed at the center of a bunch and the other describes the time delay w.r.t. the reference particle [BHR]. With respect to the average radius of the closed orbit and the nominal particle energy, the canonical position variable and the energy variable are very small.

Spin and particle motion in storage rings is usually described using either the 'flow formalism' or the 'map formalism'. In the flow formalism $\tilde{u}$ and $\tilde{S}$ are functions of θ: $\tilde{u} = \tilde{u}(\theta), \tilde{S} = \tilde{S}(\theta)$ and in the map formalism one samples $\tilde{S}$ and $\tilde{u}$ at a fixed θ turn by turn.

In this thesis I focus on the map formalism which I now derive from the flow formalism. The magnetic and electric fields in storage rings are usually set up so that the motion of the particles is close to integrable. In the following I shall assume that it is exactly integrable. Once the spin motion has been classified on this basis, the effect of non-integrability can be included as a perturbation. I therefore choose $\tilde{u}$ to consist of d pairs of action–angle variables, i.e., $\tilde{u} = \begin{pmatrix} \tilde{\phi} \\ \tilde{J} \end{pmatrix}$, where $\tilde{\phi}, \tilde{J} \in \mathbb{R}^d$ and where $d = 3$ is the case of main interest. Then in the flow formalism one writes

$$\frac{d\tilde{\phi}}{d\theta} = \tilde{\omega}(\tilde{J}) , \qquad \tilde{\phi}(\theta_0) = \phi_0 , \tag{5.1}$$

$$\frac{d\tilde{J}}{d\theta} = 0 , \qquad \tilde{J}(\theta_0) = J_0 , \tag{5.2}$$

$$\frac{d\tilde{S}}{d\theta} = \mathcal{A}(\theta, \tilde{\phi}, \tilde{J})\tilde{S} , \qquad \tilde{S}(\theta_0) = S_0 , \tag{5.3}$$

where the d components of $\tilde{\omega}(\tilde{J})$ are called the 'orbital tunes' and $\mathcal{A}$ is a real skew–symmetric 3×3 matrix, i.e., $\mathcal{A}_{12} = -\mathcal{A}_{21}, \mathcal{A}_{13} = -\mathcal{A}_{31}$ and $\mathcal{A}_{23} = -\mathcal{A}_{32}$. The function $\mathcal{A}$ is derived from the rotation rate vector of the T-BMT equation [BEH04] and it is 2π-periodic in θ and in the d components of $\tilde{\phi}$. Of course, (5.3) is an incarnation of the T–BMT equation. Analogously (5.1),(5.2) are an incarnation of the Lorentz force law. One can call the pair $(\tilde{\omega}, \mathcal{A})$ the 'spin-orbit system' in the flow formalism and it was studied in [BEH04].

To proceed from the flow formalism to the map formalism I write the solution of

(5.1),(5.2),(5.3) as

$$\tilde{\phi}(\theta) = \phi_0 + (\theta - \theta_0)\tilde{\omega}(J_0) \,, \tag{5.4}$$

$$\tilde{J}(\theta) = J_0 \,, \tag{5.5}$$

$$\tilde{S}(\theta) = \tilde{\Psi}(\theta, \theta_0; \phi_0, J_0) S_0 \,, \tag{5.6}$$

where $\tilde{\Psi}$ is the principal solution matrix for $d\tilde{S}/d\theta = \mathcal{A}(\theta, \phi_0 + (\theta - \theta_0)\tilde{\omega}(J_0), J_0)\tilde{S}$ and where $\tilde{\Psi}(\theta, \theta_0; \phi_0, J_0)$ is 2π-periodic in the d components of ϕ_0 and $\tilde{\Psi}$ is $SO(3)$-valued. For the definition of $SO(3)$, see after (6.2). It follows from (5.4),(5.5),(5.6) that

$$\tilde{\Psi}(\theta_2, \theta_0; \phi_0, J_0) = \tilde{\Psi}(\theta_2, \theta_1; \phi_0 + (\theta_1 - \theta_0)\tilde{\omega}(J_0), J_0)\tilde{\Psi}(\theta_1, \theta_0; \phi_0, J_0) \,,$$

whence, for integers m, n,

$$\begin{aligned} &\tilde{\Psi}(\theta_0 + 2\pi(n+m), \theta_0; \phi_0, J_0) \\ &= \tilde{\Psi}(\theta_0 + 2\pi n, \theta_0; \phi_0 + 2\pi m\tilde{\omega}(J_0), J_0)\tilde{\Psi}(\theta_0 + 2\pi m, \theta_0; \phi_0, J_0) \,, \end{aligned} \tag{5.7}$$

where I used the fact that, due to the 2π-periodicity of $\mathcal{A}(\theta, \cdot, \cdot)$ in θ,

$$\tilde{\Psi}(\theta + 2\pi m, \theta_0 + 2\pi m; \phi_0, J_0) = \tilde{\Psi}(\theta, \theta_0; \phi_0, J_0) \,. \tag{5.8}$$

Without loss of generality one can take $\theta_0 = 0$ and so, by letting

$$\phi(n) := \tilde{\phi}(2\pi n) \,, \tag{5.9}$$

$$J(n) := \tilde{J}(2\pi n) \,, \tag{5.10}$$

$$S(n) := \tilde{S}(2\pi n) \,, \tag{5.11}$$

I obtain from (5.4),(5.5),(5.6)

$$\phi(n+1) = \phi(n) + 2\pi\tilde{\omega}(J(n)) \,, \qquad \phi(0) = \phi_0 \,, \tag{5.12}$$

$$J(n+1) = J(n) \,, \qquad J(0) = J_0 \,, \tag{5.13}$$

$$S(n+1) = \tilde{\Psi}(2\pi, 0; \phi(n), J(n))S(n) \,, \qquad S(0) = S_0 \,. \tag{5.14}$$

The initial value problem (5.12),(5.13), (5.14) characterizes the 'spin-orbit system' $(\tilde{\omega}, \tilde{\Psi}(2\pi, 0; \cdot, \cdot))$ taken in the map formalism. Letting

$$\omega := \tilde{\omega}(J_0) , \tag{5.15}$$

$$\Psi(n; x) := \tilde{\Psi}(2\pi n, 0; x, J_0) , \tag{5.16}$$

I obtain from (5.12),(5.13), (5.14)

$$\phi(n+1) = \phi(n) + 2\pi\omega , \qquad \phi(0) = \phi_0 , \tag{5.17}$$

$$S(n+1) = A(\phi(n))S(n) , \qquad S(0) = S_0 , \tag{5.18}$$

where

$$A(\cdot) := \tilde{\Psi}(2\pi, 0; \cdot, J_0) , \tag{5.19}$$

and from (5.7) the 'cocycle condition'

$$\Psi(n+m; \phi) = \Psi(n; \phi + 2\pi m\omega)\Psi(m; \phi) . \tag{5.20}$$

Note that $A(\cdot) = \Psi(1; \cdot)$. The initial value problem (5.17),(5.18) characterizes the 'spin-orbit torus' (ω, A) taken in the map formalism. Thus (5.17),(5.18) are the basic equations for this second part of the thesis. We will see in Section 6.1 that Ψ is uniquely determined by ω and A, whence I will use for Ψ the notation $\Psi_{\omega,A}$. In this work I will assume that A is continuous and accordingly continuity is assumed in many other definitions as well. For example, the generators of the invariant spin fields (see Definition 6.2) and the transfer fields (see Definition 7.2) between spin-orbit tori are continuous functions. In contrast, in [BEH04] A is of class C^1 since $\tilde{\Psi}(\cdot, J_0)$ is of class C^1 (as well as the invariant spin fields and the transfer fields). Note that assuming mere continuity in the present work is fruitful since I here deal with the map formalism (in contrast, in the flow formlism of [BEH04] it is natural to impose the C^1-property since one has to deal with differential equations).

Although accelerator physicists tend to concentrate on studying spin motion in real storage rings, many of the issues surrounding the so-called invariant spin field (introduced in Section 6.3) and the spin-orbit resonance (introduced in Sections 7.4 and 8.4) depend just on the structure of the initial value problem (5.17),(5.18) and can be treated in isolation from the original physical system. This is the strategy to be adopted here and it clears the way for the focus on purely mathematical matters, in particular for the exploitation of theorems from Topology and Fourier Analysis. For example, the Homotopy Lifting Theorem (see also Section 6.4) facilitates the study of continuous functions (in particular it allows to apply the so-called quaternion formalism to functions like $\Psi(n;\cdot)$ in (5.16)). Another example is Fejér's multivariate theorem which facilitates the study of so-called quasiperiodic functions (in particular it allows, via Theorem 8.6, to characterize the set of the so-called spin tunes of second kind).

Now that the background to this work has been presented as well as an introduction to the map formalism, I finish this chapter with an outline of the structure of the following chapters. For thorough overviews of the importance of the invariant spin field and the so-called amplitude-dependent spin tune for classifying spin motion in storage rings see [BEH04, Hof, Vo]. Note that the spin tunes of first kind introduced in Section 7.4 are the amplitude-dependent spin tunes at a fixed, but arbitrary value of the 'amplitude' J_0.

5.2 Synopsis

Chapters 5-10 and Appendices B-G are structured as follows.

In Chapter 6 I introduce the most basic concepts. In particular, in Section 6.1 I introduce the spin-orbit torus (ω, A) where ω is the orbital tune vector and A is a 1-turn spin transfer matrix which is modeled after the situation of (5.19). I also introduce in Section 6.1 the symbol $\mathcal{SOT}(d,\omega)$ for the set of all spin-orbit tori which have the orbital tune vector $\omega \in \mathbb{R}^d$ and the symbol $\mathcal{SOT}(d)$ for the set of all spin-orbit tori which have an orbital tune vector in $\mathbb{R}^d$. I then derive the n-turn spin transfer matrix $\Psi_{\omega,A}$ from ω and A and establish some basic relations between the $\Psi_{\omega,A}(n;\cdot)$ for different values of the integer n. This leads naturally in Section 6.2 to the definition of the $\mathbb{Z}$-action, $L_{\omega,A}$, on $\mathbb{R}^{d+3}$ which is a function associated with every spin-orbit torus $(\omega, A) \in \mathcal{SOT}(d,\omega)$ encoding the information about the spin-orbit torus in a very useful form. Some group theoretical properties of $L_{\omega,A}$ are discussed too. Also the $\mathbb{Z}$-action L_ω on $\mathbb{R}^d$ is introduced which formalizes the orbital translations on $\mathbb{R}^d$ associated with each $(\omega, A) \in \mathcal{SOT}(d,\omega)$. In Section 6.3 I consider a distribution or field of spins constructed by attaching a spin to each $\phi_0 \in \mathbb{R}^d$ at $n = 0$ and thereby introduce the polarization fields (and, as a special subclass, the spin fields) associated with every (ω, A). I also define the $\mathbb{Z}$-action $L^{(PF)}_{\omega,A}$ which governs the evolution of the polarization fields. Polarization fields are important tools to study the polarization of a bunch (see also Section 5.1), however this aspect of polarization fields plays no role in this work. Chapter 6 is closed with Section 6.4 where the impact of Homotopy Theory on the present work is outlined and where some related concepts and facts are mentioned which are needed in this work. In particular I show how to exploit the 2π-periodicity of some functions and I point out how Homotopy Theory is related with the $SO(3)$-index. The $SO(3)$-index is based on the quaternion formalism of $\mathbb{S}^3$ which is employed in this work to deal with continuous $SO(3)$-valued functions.

One is particularly interested in spin-orbit tori for which spin precesses around a fixed axis and perhaps even at a fixed rate. Such a fixed rate leads to the definition of spin tune of first kind. Moreover to fully exploit those spin-orbit tori one needs a transformation group which allows to transform the spin motion from one spin-orbit torus to another. Thus in Chapter 7 I introduce the transformation group (=group action), $R_{d,\omega}$, on $\mathcal{SOT}(d,\omega)$. The group action $R_{d,\omega}$ is motivated by some observations made at the beginning of Section 7.1 of how spin-orbit tori should be transformed into each other in an efficient way. This leads to the notion of the $R_{d,\omega}$-

orbit. Roughly speaking, an $R_{d,\omega}$-orbit of a spin-orbit torus, (ω, A), is the set of spin-orbit tori which can be reached from (ω, A) by varying the parameters of $R_{d,\omega}$ over the underlying group, $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Thus with Chapter 7 I begin to consider the set $\mathcal{SOT}(d, \omega)$ as a whole and we will see that spin-orbit tori, which belong to the same $R_{d,\omega}$-orbit, share many of their properties. The way in which spin-orbit trajectories and polarization fields transform with $R_{d,\omega}$ from one spin-orbit torus to another is stated in Theorem 7.3 of Section 7.1. The aim of studying reference frames in which spins precess around a fixed axis, possibly at a fixed rate, prompts the definition in Section 7.2 of trivial, almost trivial and weakly trivial spin-orbit tori to embrace these cases. Section 7.2 also shows how Homotopy Theory impacts on weakly trivial spin-orbit tori via the $SO_3(2)$-index. Then in Section 7.3 I use $R_{d,\omega}$ acting on trivial, almost trivial and weakly trivial spin-orbit tori to classify spin-orbit tori into so-called coboundaries, almost coboundaries, weak coboundaries, and those which are not weak coboundaries. Thus I deal with four major subsets of $\mathcal{SOT}(d, \omega)$ (where some of them overlap - see the inclusions (7.18)). The terminology of 'coboundary' and 'almost coboundary' is borrowed from Dynamical Systems Theory since, given a spin-orbit torus (ω, A) in $\mathcal{SOT}(d, \omega)$, the function $\Psi_{\omega,A}$ is a $SO(3)$-cocycle over the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$. Section 7.3 displays the close connection between the concepts of weak coboundary and invariant spin field (ISF) and the impact of Homotopy Theory on weak coboundaries. In Section 7.4 I define for every spin-orbit torus a (possibly empty) set of spin tunes of first kind (and the associated spin-orbit resonances) which are reincarnations of the spin tunes introduced by Yokoya [Yo1] and show that this set is nonempty iff the spin-orbit torus is an almost coboundary. Spin tunes of the first kind are always associated with almost coboundaries so that they are always associated with invariant spin fields. In Section 7.5 I present the celebrated uniqueness theorem of Yokoya [Yo1], which relates the uniqueness issue of the invariant spin field with the condition of spin-orbit resonance of first kind. In Section 7.6 I put the present work, and weak coboundaries in particular, into the context of Polarized Beam Physics. Thus I relate the present work with other work of Polarized Beam Physics. In Section 7.7 I address the question of whether two weakly trivial spin-orbit tori belong to the same $R_{d,\omega}$-orbit. In particular the relevance of the small divisor problem and Diophantine sets of orbital tunes is pointed out.

In Chapter 8 I widen and deepen the study of spin-orbit tori by using the tool of quasiperiodic functions. In particular I show that, off orbital resonance, the existence of just one quasiperiodic spin trajectory ensures the existence of an ISF. Then in Section 8.2 I consider reference frames, called 'simple precession frames', in which spins precess around an axis which can be any spin trajectory and I define a phase advance for spin motion in such a frame. In Section 8.3 I introduce special simple precession frames, called 'uniform precession frames', for which the phase advance is the same from turn to turn and show their connection with the so-called generalized Floquet Theorem. Armed with the concept of the uniform precession frame I define, in Section 8.4, for every spin-orbit torus a (possibly empty) set of spin tunes of second kind (and the associated spin-orbit resonances) and show that the spin tunes of second kind are identical with the spin tunes of first kind in most situations. In this work the spin tunes of second kind mainly serve to analyze the spin tunes of first kind. In Section 8.5 I resume the theme of Section 7.7 and, on the basis of Corollary 8.12, I am able to outline an algorithm employed in the code SPRINT for computing spin tunes of first and second kind. In Section 8.6 I show how Homotopy Theory has an impact on the individual values of the spin tunes of first kind, i.e., how it affects the structure of the sets $\Xi_1(\omega, A)$. Section 8.7 returns to the question, already addressed in Section 7.3, of whether the existence of an ISF implies that a spin-orbit torus can be transformed to become a weakly trivial one.

Chapter 9 reconsiders the basic $\mathbb{Z}$-actions $L_{\omega,A}$ and $L^{(PF)}_{\omega,A}$ used in Chapters 6,7,8 and introduces further associated $\mathbb{Z}$-actions. In particular, in Section 9.1 it is shown how the peculiar structure of the cocycle condition (see (5.20) and (6.6)) follows from the fact that $L_{\omega,A}$ is a skew-product of the orbital $\mathbb{Z}$-action L_ω. In Section 9.2 I show that the $\mathbb{Z}$-action $L_{\omega,A}$ is an extension of the $\mathbb{Z}$-action $L^{(T)}_{\omega,A}$. I thereby relate the orbital translations on $\mathbb{R}^d$ to the corresponding orbital translations on the d-torus $\mathbb{T}^d$. Thus Section 9.2 gives a brief glimpse into the $\mathbb{T}^d$-treatment of spin-orbit tori. In Section 9.3 I widen the perspective by showing how a single principal $SO(3)$-bundle, $\lambda_{SOT(d)}$, underlies $SOT(d)$. It leads in Section 9.3.5 to Theorem 9.5a, which is a special case of Zimmer's Reduction Theorem. As an application of this I obtain Theorem 9.5b which shows the concept of the invariant spin field in a new light.

The appendices, B-F, provide material needed in Chapters 6-9. While most of the material of Appendices B-E is standard, these appendices provide sufficient precision and make this part of the thesis essentially self contained. Appendix F contains those proofs which are not given elsewhere. Appendix G contains a guide which will help the reader with some subjects appearing in this part of the thesis.

5.3 Scope and limitations

I now mention the possible merits and shortcomings of this part of the thesis.

The intention and flavor of this work is to present a piece of Mathematical Physics. In fact an abundance of mathematical definitions is introduced, which transfigure the topic of spin-orbit tori into a mathematical theory. Accordingly, an abundance of lemmas, propositions, theorems, corollaries is stated and the proofs are, without exception, intended to be rigorous.

Three important issues related with this work, but not covered by it at all, are the spinor formalism, the synthesis of families of spin-orbit tori into spin-orbit systems and the use of Borel algebras. Note that the spinor formalism deals with spinor valued functions which are associated with the spin trajectories and spinor valued functions which are associated with the polarization fields (in contrast, the present work uses the 3D formalism where the spin lives in $\mathbb{R}^3$). Note also that both associations can be performed via liftings w.r.t. the so-called complex Hopf bundle whose projection has domain $\mathbb{S}^3$ and range $\mathbb{S}^2$. It turns out that that the spinor formalism can be pursued along similar lines as the quaternion formalism in Sections C.2,C.3 (the latter is based on the Hurewicz fibration $(\mathbb{S}^3, p_2, SO(3))$). In fact if in the quaternion formalism one replaces the Hurewicz fibration $(\mathbb{S}^3, p_2, SO(3))$ by the complex Hopf bundle (the latter is a Hurewicz fibration, too) then one obtains the spinor formalism [He] (for Hurewicz fibrations, see Appendix C). In contrast, the issue of the synthesis of families of spin-orbit tori into spin-orbit systems seems to have a less geometrical and more analytical flavor. While in this work the emphasis is on continuous functions, large parts of spin-orbit theory can be formulated by using Borel measurable functions [He]. Such an approach is feasible for the statistical description of spin-orbit tori (e.g., the study of the polarization) and it allows to apply more tools from Ergodic Theory, e.g., Birkhoff's Ergodic Theorem [EH].

This work puts some effort into the taxonomy of spin-orbit tori, in particular, due to their importance, some effort into the taxonomy of weak coboundaries. A minor shortcoming is that many results focus on the generic case where $(1, \omega)$ is nonresonant. However since the nongeneric case can be reduced to the generic case, it would be easy to modify and prove many of my results for the nongeneric case [He]. The following conjecture, which I call the 'ISF-conjecture', plays a fruitful role in Polarized Beam Physics. The ISF-conjecture, which, at least to my knowledge (see also Section 7.6), is unsettled, goes as follows: "If a spin-orbit torus (ω, A) is off orbital resonance, then it has an invariant spin field". Albeit no attempt is made in this work to settle the ISF-conjecture, the present work presents some conditions which transform the ISF-conjecture into equivalent conjectures. For example, by Theorems 7.9,7.10, a $(\omega, A) \in \mathcal{SOT}(d, \omega)$ with $d = 1$ is a weak coboundary iff it has an ISF. Note finally that numerical procedures exist which 'solve' the ISF problem numerically (see Section 7.6).

Chapter 10

Summary of spin-orbit tori and outlook

As pointed out in the Introduction, the second part of this thesis studies spin-orbit tori in terms of the map formalism equations of motion (6.1),(6.2) which plays a central role in the mathematical study of polarized beams in storage rings.

From a technical point of view a distinguishing feature of the present work is to formulate all concepts and properties in mathematical terms. Accordingly the mathematical notion of spin-orbit torus is introduced and a number of properties of spin-orbit tori are derived. Most of my definitions that are related to spin-orbit tori are distilled from established concepts in Polarized Beam Physics which are then translated into the language of Mathematics. The subsets $\mathcal{CB}(d,\omega) \subset \mathcal{ACB}(d,\omega) \subset \mathcal{WCB}(d,\omega)$ of the set $\mathcal{SOT}$ of spin-orbit tori have been introduced and discussed in some detail. I noted that spin-orbit tori (ω, A) of interest are almost coboundaries, i.e., are in $\mathcal{ACB}(d,\omega)$ and they have the form $A(\phi) = T^T(\phi + 2\pi\omega)\exp(\mathcal{J}2\pi\nu)T(\phi)$.

To my knowledge the results of the thesis are either new (e.g., Theorem 9.5b about the impact of Principal Bundle Theory on invariant spin fields) or were never formulated in mathematically precise terms whence were never rigorously proved before (e.g., Corollary 8.12 aka the SPRINT Theorem). Note that some results (e.g., Yokoya's uniqueness theorem 7.13) were rigorously proved before for the flow formalism (see [BEH04]).

I have gathered quite a bit of insight into the invariant spin field (as well as into the spin tune) which is central for Polarized Beam Physics, as explained in Section 7.6. From Section 6.3 we know that an invariant spin field is tied with the equation $G(\phi) = A(\phi - 2\pi\omega)G(\phi - 2\pi\omega)$. I formulated the ISF conjecture which states that if (ω, A) is off orbital resonance, i.e., $(1, \omega)$ nonresonant, then an invariant spin field exists. Theorem 7.9 states that if (ω, A) is a weak coboundary, then an invariant spin field exists. Theorem 7.10a states that if $\mathcal{S}_G$ is an invariant spin field and if G is 2π-nullhomotopic then (ω, A) is a weak coboundary. Theorem 8.17 states that there are spin-orbit tori which have an invariant spin field and which are not weak coboundaries. Finally Theorem 9.5b shows that the existence of an invariant spin field of (ω, A) is a symmetry property of (ω, A). In fact Theorem 9.5b ties the existence of an invariant spin field to an $SO_3(2)$-reduction of the principal $SO(3)$-bundle $\lambda_{SOT(d)}$.

It is also worthwile to mention that the machinery of Chapter 9 can be applied to any linear n-dimensional nonautonomous ODE $\dot{y} = Y(t)y$ since the standard procedure of making it autonomous, encodes the ODE into a $GL(n)$-cocycle over the time translations whence encodes it into a principal $GL(n)$-bundle with base space $\mathbb{R}$. This will be addressed in a future publication of the author.

For a detailed outline of this work see Section 5.2. Avenues for further work are of course plentiful. In addition to those mentioned in Section 5.3, one topic of further studies could be the continuation of the work of Section 9.3. In fact, as outlined in Section 9.3.6, there are further applications of the principal $SO(3)$-bundle $\lambda_{SOT(d)}$ in waiting which will shed further light into the matter of spin-orbit tori.