Rigidity re. principle bundle
Reduction → 1st theorem

Two topics in particle accelerator beams: Vlasov-Maxwell treatment of coherent synchrotron radiation and topological treatment of spin polarization

by

Klaus Heinemann

Diplom Physiker, University of Hamburg, 1986

Committee Chair
James A. Ellison

DISSERTATION

Submitted in Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy
Mathematics

The University of New Mexico

Albuquerque, New Mexico

December, 2009

Acknowledgments

Two topics in particle accelerator beams: Vlasov-Maxwell treatment of coherent synchrotron radiation and topological treatment of spin polarization

by

Klaus Heinemann

ABSTRACT OF DISSERTATION

Submitted in Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy
Mathematics

The University of New Mexico

Albuquerque, New Mexico

December, 2009

Two topics in particle accelerator beams: Vlasov-Maxwell treatment of coherent synchrotron radiation and topological treatment of spin polarization

by

Klaus Heinemann

Diplom Physiker, University of Hamburg, 1986

Ph.D., Mathematics, University of New Mexico, 2009

Abstract

My old Questions on how far the MAP paper
goes beyond BEH 2004 (LL9) ;
Flexibility by relaxing smoothness ...?
On countability?
Spectrum

Contents

Chapter 1

Introduction to spin-orbit tori

In this chapter we make some general remarks for the orientation of the reader.

1.1 Physical context

SOME MORE PHYSICS STUFF WILL BE ADDED TO THIS SECTION

This work studies the combined system of discrete time equations of motion (2.1),(2.2) which play a central role in the benchmark study of polarized beams in storage rings and in the present section we say a bit about this context.

Dynamical variables describing the classical motion of a spin 1/2 particle (e.g., the proton, electron, muon) are the orbital phase space variable $u \in \mathbb{R}^{2d}$ and the spin variable $\tilde{S} \in \mathbb{R}^3$, where $d = 1, 2, 3$. In the situation of polarized beams in a storage ring u and $\tilde{S}$ are, in the flow formalism, functions of the angular position around the storage ring, the azimuth $\theta = 2\pi s/L$, where s is the distance around the storage ring and L is the circumference. Thus $u = u(\theta), \tilde{S} = \tilde{S}(\theta)$.

Assuming that the orbital motion is integrable one can choose u to consist of d

pairs of action–angle variables, i.e., $u = \begin{pmatrix} \tilde{\phi} \\ \tilde{J} \end{pmatrix}$, where $\tilde{\phi}, \tilde{J} \in \mathbb{R}^d$. Neglecting the Stern-Gerlach force acting from S onto u, the equations of motion read as

$$\frac{d\tilde{\phi}}{d\theta} = \tilde{\omega}(\tilde{J}) \,, \tag{1.1}$$

$$\frac{d\tilde{J}}{d\theta} = 0 \,, \tag{1.2}$$

$$\frac{d\tilde{S}}{d\theta} = \mathcal{A}(\theta, \phi, \tilde{J})\tilde{S} \,, \tag{1.3}$$

where the d components of $\tilde{\omega}(\tilde{J})$ are called the "orbital tunes" and $\mathcal{A}$ is a real skew-symmetric 3×3 matrix, i.e., $\mathcal{A}_{12} = -\mathcal{A}_{21}, \mathcal{A}_{13} = -\mathcal{A}_{31}$ and $\mathcal{A}_{23} = -\mathcal{A}_{32}$. Furthermore $\mathcal{A}(\theta, \tilde{\phi}, \tilde{J})$ is 2π-periodic in θ and in the d components of $\tilde{\phi}$. The function $\mathcal{A}$ is determined by the accelerator's electromagnetic field (or by some modeling of that field) and by the Thomas–Bargmann–Michel–Telegdi (T–BMT) equation [Ja98]. In fact, eq. (1.3) is an incarnation of the T–BMT equation. Analogously eq. (1.1),(1.2) are an incarnation of the Lorentz force law. We can call the pair $(\tilde{\omega}, \mathcal{A})$ the "spin-orbit system" in the flow formalism.

To proceed to the map formalism we write the formal solution of eq. (1.1),(1.2),(1.3) as

$$\tilde{\phi}(\theta) = \tilde{\phi}(\theta_0) + (\theta - \theta_0)\tilde{\omega}(\tilde{J}(\theta_0)) \,, \tag{1.4}$$

$$\tilde{J}(\theta) = \tilde{J}(\theta_0) \,, \tag{1.5}$$

$$\tilde{S}(\theta) = \tilde{\Psi}\Big(\theta, \theta_0; \tilde{\phi}(\theta_0), \tilde{J}(\theta_0)\Big) S(\theta_0) \,, \tag{1.6}$$

where θ_0 is an arbitrary initial azimuth and $\tilde{\Psi}(\theta, \theta_0; \tilde{\phi}(\theta_0), \tilde{J}(\theta_0)) \in SO(3)$ (for the definition of $SO(3)$, see after eq. (2.2)). It follows from eq. (1.4),(1.5),(1.6) that

$$\tilde{\phi}(\theta_0 + 2\pi) = \tilde{\phi}(\theta_0) + 2\pi\tilde{\omega}(\tilde{J}_0) \,, \tag{1.7}$$

$$\tilde{J}(\theta_0 + 2\pi) = \tilde{J}(\theta_0) \,, \tag{1.8}$$

$$\tilde{S}(\theta_0 + 2\pi) = \tilde{\Psi}\Big(\theta_0 + 2\pi, \theta_0; \tilde{\phi}(\theta_0), \tilde{J}(\theta_0)\Big)\tilde{S}(\theta_0) \,. \tag{1.9}$$

Fixing θ_0 we can call the pair $(\tilde{\omega}, \tilde{\Psi}(\theta_0 + 2\pi\cdot, \theta_0; \cdot, \cdot)$ the "spin-orbit system" in the map formalism. We see that $\tilde{J}$ is a constant of motion and, accordingly, the equations of motion (1.7),(1.8),(1.9) for $\tilde{\phi}, \tilde{J}$ and $\tilde{S}$ can be viewed as the family of equations of motion (1.7),(1.9) for $\tilde{\phi}$ and $\tilde{S}$ parametrized by $\tilde{J}(\theta_0)$. Thus, fixing the parameters $\theta_0, \tilde{J}$, we can call the pair $(\tilde{\omega}(\tilde{J}), \tilde{\Psi}(\theta_0 + 2\pi\cdot, \theta_0; \cdot, \tilde{J})$ the "spin-orbit torus at $\tilde{J}$" in the map formalism (analogously one can define the spin-orbit torus in the flow formalism). This pair is the topic of this work and because of its importance we will use, by fixing the parameters $\theta_0, \tilde{J}$, the following abbreviations:

$$\phi(n) := \tilde{\phi}(\theta_0 + 2\pi n) \ , \tag{1.10}$$

$$S(n) := \tilde{S}(\theta_0 + 2\pi n) \ , \tag{1.11}$$

$$\Psi(n; x) := \tilde{\Psi}(\theta_0 + 2\pi, \theta_0; x, \tilde{J}) \ , \tag{1.12}$$

where $n \in \mathbb{Z}, x \in \mathbb{R}^d$. Note that since $\mathcal{A}(\theta, \tilde{\phi}, \tilde{J})$ is 2π-periodic in the d components of $\tilde{\phi}$, $\tilde{\Psi}(\theta, \theta_0; x, \tilde{J})$ and $\Psi(n; x)$ are 2π-periodic in the d components of x.

1.2 Synopsis

This work is structured as follows.

In Chapter 2 we introduce the most basic concepts. In particular, in Section 2.1 we introduce the spin-orbit torus (ω, Ψ) where ω is the orbital tune vector and $\Psi(n; \cdot)$ is a n-turn spin transfer matrix which is modeled after the situation of Eq. (1.12). We also introduce in Section 2.1 the symbol $\mathcal{SOT}(d, \omega)$ for the set of all spin-orbit tori which have the orbital tune vector $\omega \in \mathbb{R}^d$ and the symbol $\mathcal{SOT}(d)$ for the set of all spin-orbit tori which have an orbital tune vector in $\mathbb{R}^d$. We then establish some basic relations between the $\Psi(n; \cdot)$ for different values of the integer n.

This leads naturally in Section 2.2 to the definition of the $\mathbb{Z}$-action, $L_{\omega,\Psi}$, on $\mathbb{R}^{d+3}$ which is a function associated with every spin-orbit torus $(\omega, \Psi) \in \mathcal{SOT}(d, \omega)$

encoding the information about the spin-orbit torus in a very useful form. Some group theoretical properties of $L_{\omega,\Psi}$ are discussed too. Also the $\mathbb{Z}$-action L_ω on $\mathbb{R}^d$ is introduced which formalizes the orbital translations on $\mathbb{R}^d$ associated with each $(\omega, \Psi) \in \mathcal{SOT}(d, \omega)$. In Section 2.3 we consider a distribution or field of spins constructed by attaching a spin to each $\phi_0 \in \mathbb{R}^d$ at $n = 0$ and thereby introduce the polarization fields (and, as a special subclass, the spin fields) associated with every (ω, Ψ). We also define the $\mathbb{Z}$-action $L^{(PF)}_{\omega,\Psi}$ which governs the time evolution of the polarization fields.

Chapter 2 is closed with Section 2.4 where the impact of Homotopy Theory on the present work is outlined and where some related concepts and facts are mentioned which are needed in this work. In particular we show how to exploit the 2π-periodicity of some functions and we point out how Homotopy Theory is related with the $SO(3)$-index. The $SO(3)$-index is based on the quaternion formalism of $\mathbb{S}^3$ which is employed in this work to deal with continuous $SO(3)$-valued functions.

We are particularly interested in spin-orbit tori for which spin precesses around a fixed axis and perhaps even at a fixed rate. Such a fixed rate leads to the definition of spin tune of first kind. Moreover to fully exploit those spin-orbit tori we need a transformation group which allows us to transform the spin motion from one spin-orbit torus to another. Thus in Chapter 3 we introduce the transformation group (=group action), $R_{d,\omega}$, on $\mathcal{SOT}(d, \omega)$. The group action $R_{d,\omega}$ is motivated by some observations made at the beginning of Section 3.1 of how spin-orbit tori should be transformed into each other in an efficient way. This leads leads to the notion of the $R_{d,\omega}$-orbit. Roughly speaking, an $R_{d,\omega}$-orbit of a spin-orbit torus, (ω, Ψ), is the set of spin-orbit tori which can be reached by (ω, Ψ) via varying the parameters of $R_{d,\omega}$, i.e., via varying over the underlying group, $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Thus with Chapter 3 we begin to consider the set $\mathcal{SOT}(d, \omega)$ as a whole and we will see that spin-orbit tori, which belong to the same $R_{d,\omega}$-orbit, share many of their properties. The way in which spin-orbit trajectories and polarization fields transform with $R_{d,\omega}$ from one spin-orbit torus to another is stated in Theorem 3.3 of Section 3.1. The aim of studying reference frames in which spins precess around a fixed axis, possibly at a fixed rate, prompts the definition in Section 3.2 of trivial, almost trivial and weakly trivial spin-orbit tori to embrace these cases. Section 3.2 also shows how Homotopy Theory impacts on weakly trivial spin-orbit tori via the $SO_3(2)$-index. Then in Section 3.3 we use $R_{d,\omega}$ acting on trivial, almost trivial and weakly trivial spin-orbit tori to classify spin-orbit tori into coboundaries, almost coboundaries, weak coboundaries, and those which are not weak coboundaries. Thus we deal with four major subsets of $SOT(d,\omega)$ (where some of them overlap - see the inclusions (3.18)). The terminology of "weak coboundary" etc. is taken from the fact that, given a spin-orbit torus (ω,Ψ) in $SOT(d,\omega)$, the function Ψ is a continuous $SO(3)$-cocycle over the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$. Section 3.3 displays the close connection between the concepts of weak coboundary and invariant spin field (ISF) and the impact of Homotopy Theory on weak coboundaries. In Section 3.4 we define for every spin-orbit torus a (possibly empty) set of spin tunes of first kind (and the associated spin-orbit resonances) which are reincarnations of the spin tunes introduced by Yokoya [Yok1] and show that this set is nonempty iff the spin-orbit torus is an almost coboundary. Spin tunes of the first kind are always associated with almost coboundaries so they are always associated with invariant spin fields. In Section 3.5 we present the celebrated uniqueness theorem of Yokoya [Yok1], which relates the uniqueness issue of the invariant spin field with the condition of spin-orbit resonance of first kind. In Section 3.6 we put the present work, and weak coboundaries in particular, into the context of polarized beam physics. In particular we relate the present work with other work. In Section 3.7 we address the question of whether two weakly trivial spin-orbit tori belong to the same $R_{d,\omega}$-orbit. In particular the relevance of the small divisor problem and Diophantine sets of orbital tunes is pointed out.

In Chapter 4 we widen and deepen the study of spin-orbit tori by using the tool

of quasiperiodic functions. In particular we show that, off orbital resonance, the existence of just one quasiperiodic spin trajectory ensures the existence of an ISF. Then in Section 4.2 we consider reference frames, called simple precession frames, in which spins precess around an axis which can be any spin trajectory and we define a phase advance for spin motion in such a frame. In Section 4.3 we introduce special simple precession frames, called uniform precession frames, for which the phase advance is the same from turn to turn and, armed with this concept, we define for every spin-orbit torus a (possibly empty) set of spin tunes of second kind (and the associated spin-orbit resonances) and show that the spin tunes of second kind are identical with the spin tunes of first kind in most situations. In this work the spin tunes of second kind mainly serve the purpose to analyze the spin tunes of first kind. In Section 4.5 we resume the theme of Section 3.7 and, on the basis of Corollary 4.12, we are able to outline an algorithm employed in the code SPRINT for computing spin tunes of first and second kind. In Section 4.6 we show how Homotopy Theory has an impact on the individual values of the spin tunes of first kind, i.e., how it affects the structure of the sets $\Xi_1(\omega,\Psi)$. Section 4.7 returns to the question, already addressed in Section 3.3, of whether the existence of an ISF implies that a spin-orbit torus can be transformed to become a weakly trivial one.

Chapter 5 reconsiders the basic $\mathbb{Z}$-actions $L_{\omega,\Psi}$ and $L^{(PF)}_{\omega,\Psi}$ used in Chapters 2,3,4 and introduces further associated $\mathbb{Z}$-actions. In particular, in Section 5.1 it is shown how the peculiar structure of eq. (2.5) follows from the fact that $L_{\omega,\Psi}$ is a skew-product of the orbital $\mathbb{Z}$-action L_ω. In Section 5.2 we show that the $\mathbb{Z}$-action $L_{\omega,\Psi}$ is an extension of the $\mathbb{Z}$-action $L^{(T)}_{\omega,\Psi}$. We thereby relate the orbital translations on $\mathbb{R}^d$ to the corresponding orbital translations on the d-torus $\mathbb{T}^d$. Thus Section 5.2 gives a brief glimpse into the $\mathbb{T}^d$-treatment of spin-orbit tori. In Section 5.3 we widen the perspective by showing how a single principal $SO(3)$-bundle, $\lambda_{SOT(d)}$, underlies $SOT(d)$. It leads us in Section 5.3.5 to Theorem 5.5a, which is a special case of Zimmer's celebrated reduction theorem. As an application of this we obtain

Theorem 5.5b which shows the concept of the invariant spin field in a new light.

The appendices, A-E, provide material needed in Chapters 2-5. While most of the material of Appendices A-D is standard, these appendices intend to provide sufficient precision and to make this work essentially self contained. At the end of this work is a guide which is supposed to help the reader with some subjects appearing in the text.

1.3 Scope and limitations

We now mention the possible merits and shortcomings of this work.

The intention and flavor of this work is to present a piece of mathematical physics. In fact an abundance of mathematical definitions is introduced, which transfigure the topic of spin-orbit tori into a mathematical theory. Accordingly, an abundance of lemmas, propositions, theorems, corollaries is stated and the proofs are, without exception, intended to be rigorous.

Three important issues related with this work, but not covered by it at all, are the spinor formalism, the synthesis of families of spin-orbit tori into spin-orbit systems and the use of Borel algebras. Note that the spinor formalism deals with spinor valued functions which are associated with the spin trajectories and spinor valued functions which are associated with the polarization fields (in contrast, the present work uses the 3D formalism where the spin lives in $\mathbb{R}^3$). Note also that both associations can be performed via liftings w.r.t. the so-called complex Hopf bundle whose projection map has domain $\mathbb{S}^3$ and range $\mathbb{S}^2$. It turns out that that the spinor formalism can be pursued along similar lines as the quaternion formalism in Sections B.2,B.3 (the latter is based on the Hurewicz fibration $(\mathbb{S}^3, p_2, SO(3))$). In fact if in the quaternion formalism one replaces the Hurewicz fibration $(\mathbb{S}^3, p_2, SO(3))$ by the complex Hopf

bundle (the latter is a Hurewicz fibration, too) then one obtains the spinor formalism [Hei]. In contrast, the issue of the synthesis of families of spin-orbit tori into spin-orbit systems seems to have a less geometrical and more analytical flavor. While in this work the emphasis is on continuous functions, large parts of spin-orbit theory can be formulated by using Borel measurable functions [Hei]. Such an approach is feasible for the statistical description of spin-orbit tori and it allows to apply more tools from Ergodic Theory, e.g., Birkhoff's ergodic theorem [EH].

This work puts some effort into the taxonomy of spin-orbit tori, in particular, due to their importance, some effort into the taxonomy of weak coboundaries. A minor shortcoming is that many results focus on the generic case where $(1, \omega)$ is nonresonant. However since the nongeneric case can be reduced to the generic case, it would be easy to modify and prove many of our results for the nongeneric case [Hei]. The following conjecture, which I call the "ISF-conjecture", plays a fruitful role in polarized beam physics. The ISF-conjecture, which, at least to my knowledge (see also Section 3.6), is unsettled, goes as follows: "If a spin-orbit torus (ω, Ψ) is off orbital resonance, then it has an invariant spin field". Albeit no attempt is made in this work to settle the ISF-conjecture, the present work presents some conditions which transform the ISF-conjecture into equivalent conjectures. For example, by Theorems 3.9,3.10, a $(\omega, \Psi) \in SOT(d, \omega)$ with $d = 1$ is a weak coboundary iff it has an ISF. Note finally that numerical procedures exist which 'solve' the ISF problem numerically (see Section 3.6).

Chapter 2

The spin-orbit tori

In this section we introduce the most basic concepts and facts needed for this work.

2.1 Introducing the spin-orbit tori (ω, Ψ)

The main purpose of this section is to state Definition 2.1 which introduces the basic entity of this work, the "spin-orbit torus". The orbital motion underlying the definition of (ω, Ψ) is a translational motion in $\mathbb{R}^d$, where d is the number of degrees of freedom (whenever we write $\mathbb{R}^k$, this implies that k is a positive integer).

As pointed out in Chapter 1, the orbital motion in the present work is assumed to be integrable, so its simplest formulation is by choosing the orbital variables as an angles $\phi_1, ..., \phi_d$ which are the components of $\phi \in \mathbb{R}^d$. Accordingly the orbital motion is a constant translation of ϕ per turn. In contrast, the spin motion is modeled after the situation of Eq. (1.12), i.e., after the T—BMT equation so our spin variable S is $\mathbb{R}^3$-valued and its motion is a rotation which is affected by the orbital motion and can therefore be very complicated. For more details on the T—BMT aspect see the remarks after Definition 2.1.

In this work the time variable n is integer valued, i.e., the time axis is the set $\mathbb{Z}$ of integers. Thus the spin-orbit trajectories $\begin{pmatrix} \phi \\ S \end{pmatrix} : \mathbb{Z} \to \mathbb{R}^{d+3}$ have to satisfy the following discrete time equations of motion

$$\phi(n+1) = \phi(n) + 2\pi\omega \,, \qquad (2.1)$$

$$S(n+1) = \Psi(1; \phi(n))S(n) \,, \qquad (2.2)$$

where $\omega \in \mathbb{R}^d$ and $\Psi(1;\cdot) \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$.

Here $\mathbb{Z}$ denotes the set of integers and $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ denotes the set of 2π–periodic and continuous functions from $\mathbb{R}^d$ into $SO(3)$ (for the definition of $\mathcal{C}_{per}(\mathbb{R}^d, X)$ with topological space X, see Section B.1). Note that a function on $\mathbb{R}^d$ is called 2π–periodic if it is 2π–periodic in each of its d arguments. The set $SO(3)$ consists of those real 3×3–matrices R with $\det(R) = 1$ for which $R^T R = I_{3\times3}$ where R^T denotes the transpose of R and $I_{3\times3}$ denotes the 3×3 unit matrix. As is common, the topology of $SO(3)$ is defined as the relative topology from $\mathbb{R}^{3\times3}$ whence the nine components of $\Psi(1;\cdot)$ are continuous functions from $\mathbb{R}^d$ into $\mathbb{R}$. Thus these components are functions in $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ where $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^k)$ denotes the set of 2π–periodic and continuous functions from $\mathbb{R}^d$ into $\mathbb{R}^k$. That the 2π–periodicity of $\Psi(1;\cdot)$ has to be imposed follows from (1.12). Loosely speaking, $\Psi(1;\cdot)$ is 2π–periodic since $\phi_1, ..., \phi_d$ are angular variables.

The terminology 'orbital motion' is common in polarized beam physics and it should not be confused with the mathematical meaning of 'orbital' in the context of group actions where one deals with orbit spaces (see Appendix A). For the present work $\mathbb{R}^d$ is the appropriate carrier of the orbital motion but if one would go deeper into the matter of spin-orbit tori then the d-torus $\mathbb{T}^d$ is an important alternative. To give a brief glimpse into this matter see Section 5.2 where we employ the orbital motion on $\mathbb{T}^d$. While for the most part of this work $\mathbb{R}^d$ is the arena of the orbital motion, the d-torus $\mathbb{T}^d$ plays an ubiquitous role in this work in the study of the sets

$\mathcal{C}_{per}(\mathbb{R}^d, X)$ as is outlined in Section 2.4.

The system (2.1),(2.2) is autonomous because its r.h.s. does not *explicitly* depend on n (it depends on n only via $\phi(n)$ and $S(n)$!). We summarize the three basic facts about the system (2.1),(2.2): it is autonomous and nonlinear, it is uniquely determined by ω and $\Psi(1;\cdot)$, and the "orbital trajectories" $\phi(\cdot)$ are unaffected by the "spin trajectories" $S(\cdot)$.

By induction in n one obtains from eq. (2.1),(2.2) that every spin-orbit trajectory $\begin{pmatrix}\phi\\ S\end{pmatrix}$ satisfies, for $n \in \mathbb{Z}$,

$$\begin{pmatrix}\phi(n)\\ S(n)\end{pmatrix} = \begin{pmatrix}\phi(0)+2\pi n\omega\\ \Psi(n;\phi(0))S(0)\end{pmatrix}, \tag{2.3}$$

where, for $\phi \in \mathbb{R}^d$,

$$\begin{aligned}
&\Psi(0;\phi) := I_{3\times 3}\,,\\
&\Psi(n;\phi) := \Psi(1;\phi+2\pi(n-1)\omega)\cdots\Psi(1;\phi+2\pi\omega)\Psi(1;\phi)\,, \qquad (n=1,2,...)\\
&\Psi(n;\phi) = \Psi^T(1;\phi+2\pi n\omega)\cdots\Psi^T(1;\phi-4\pi\omega)\Psi^T(1;\phi-2\pi\omega)\,. \;\; (n=-1,-2,...)
\end{aligned} \tag{2.4}$$

The function $\Psi : \mathbb{Z}\times\mathbb{R}^d \to SO(3)$ defined by eq. (2.4) is uniquely determined by ω and $\Psi(1;\cdot)$. Note also that $\Psi(n;\cdot) \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ and that, by eq. (2.4), for $m,n \in \mathbb{Z}, \phi \in \mathbb{R}^d$,

$$\Psi(n+m;\phi) = \Psi(n;\phi+2\pi m\omega)\Psi(m;\phi)\,. \tag{2.5}$$

Furthermore eq. (2.5) implies eq. (2.4).

Clearly eq. (2.3) is the solution of the $n=0$ Cauchy problem of the system of eq. (2.1),(2.2). We call S in eq. (2.3) the "spin trajectory over $\phi(0)$". It follows from

eq. (2.3) that, for $m,n \in \mathbb{Z}$,

$$\begin{pmatrix}\phi(n)\\ S(n)\end{pmatrix} = \begin{pmatrix}\phi(m)+2\pi(n-m)\omega\\ \Psi(n;\phi(m)-2\pi m\omega)\Psi^T(m;\phi(m)-2\pi m\omega)S(m)\end{pmatrix}. \tag{2.6}$$

We are led to:

Definition 2.1 *(Spin-orbit torus) Given a $\omega \in \mathbb{R}^d$, a pair (ω,Ψ) is called a d-dimensional* spin-orbit torus *if Ψ is a function $\Psi : \mathbb{Z}\times\mathbb{R}^d \to SO(3)$ satisfying eq. (2.5) and, for every n in $\mathbb{Z}$, $\Psi(n;\cdot) \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. We call ω the* orbital tune vector *of the spin-orbit torus. I denote, for $\omega \in \mathbb{R}^d$, the set of those spin-orbit tori, whose orbital tune vector is ω, by $SOT(d,\omega)$. The set of all d-dimensional spin-orbit tori I denote by $SOT(d)$ and the set of all spin-orbit tori by SOT. A function $\begin{pmatrix}\phi\\ S\end{pmatrix} : \mathbb{Z} \to \mathbb{R}^{d+3}$ is called a* spin-orbit trajectory of (ω,Ψ) *if it satisfies eq. (2.1),(2.2). Accordingly ϕ is called an* orbital trajectory of (ω,Ψ) *and S is called a* spin trajectory of (ω,Ψ) over $\phi(0)$. □

In the remaining parts of this section we give some comments on Definition 2.1.

Clearly, for a given $\omega \in \mathbb{R}^d$, there are as many elements in the set $SOT(d,\omega)$ and as many equations of moton (2.1),(2.2) as there are functions $\Psi(1;\cdot)$) in $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. To put this into perspective one has to recall that the spin-orbit tori are modeled after the situation of Eq. (1.12), i.e., after the T—BMT equation. Therefore the spin-orbit tori obtained from (1.12) constitute only a small subset of SOT. Thus in effect the present work demonstrates that important features of the spin-orbit tori (1.12) can be studied without using (1.12), i.e., without using the T—BMT equation at all. For example, while the uniqueness theorem of Yokoya (see Section 3.5) holds for a vast set of spin-orbit tori, only a small (but, of course very important) subset of those spin-orbit tori is connected with (1.12) and the T—BMT equation.

Since $\Psi(n;\phi) \in SO(3)$, the angle between two spin trajectories over the same $\phi(0)$ is a constant of motion. Of course the Euclidean norm $|S(n)|$ of $S(n)$ is a constant of motion, too.

It follows from eq. (2.5) that, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$, we have the useful formula

$$\Psi^T(n;\phi) = \Psi(-n;\phi + 2\pi n\omega) . \tag{2.7}$$

Picking, for $(\omega, \Psi) \in SOT(d,\omega)$, a $\phi_0 \in \mathbb{R}^d$, then the spin equation of motion (2.2) for the corresponding orbital trajectory $\phi(n) = \phi_0 + 2\pi n\omega$ reads as

$$S(n+1) = \Psi(1;\phi_0 + 2\pi n\omega)S(n) . \tag{2.8}$$

Of course, every function $S : \mathbb{Z} \to \mathbb{R}^3$, which satisfies eq. (2.8), is a spin trajectory over ϕ_0 of (ω, Ψ) (and vice versa). Moreover if $S : \mathbb{Z} \to \mathbb{R}^3$ satisfies eq. (2.8), then the function $\begin{pmatrix} \phi \\ S \end{pmatrix}$, with $\phi(n) = \phi_0 + 2\pi n\omega$, is a spin-orbit trajectory of (ω, Ψ).

While the system of equations of motion (2.1),(2.2) for $\begin{pmatrix} \phi \\ S \end{pmatrix}$ is autonomous and nonlinear, the equation of motion (2.8) for S is linear and non-autonomous.

Furthermore, two spin-orbit tori $(\omega, \Psi), (\omega', \Psi')$ are equal iff $\omega' = \omega$ and $\Psi(1;\cdot) = \Psi'(1;\cdot)$. However if $(\omega, \Psi) \in SOT(d,\omega)$ and if $\omega, \omega' \in \mathbb{R}^d$ differ only by an element of $\mathbb{Z}^d$ then, due to the 2π-periodicity of $\Psi(n;\cdot)$, the spin-orbit tori $(\omega, \Psi), (\omega', \Psi)$ are essentially the same since the associated equation of motion (2.8) is the same for both.

To interpret Definition 2.1 along the lines of Section 1.1 in the context of the map formalism for polarized beams in storage rings, the reader should view $\phi(n)$ as the value of the orbital angle variable and $S(n)$ as the value of the spin variable after n "turns" around the storage ring. In this context, $\Psi(n;\cdot)$ is called the "n-turn spin transfer matrix". This means that n can be as large as 10^9 whence the

present section is definitely not the last word to be said about spin-orbit trajectories. In particular the numerical calculation of $\Psi(n;\cdot)$ for large n is a challenging task. Furthermore this calculation can be hampered by the circumstance that $\Psi(1;\cdot)$ is only approximately known. These circumstances grant a more involved discussion of spin-orbit tori in this work.

2.2 Introducing the $\mathbb{Z}$-action $L_{\omega,\Psi}$ associated with every spin-orbit torus (ω, Ψ)

Since the equations of motion (2.1),(2.2) are autonomous, each spin-orbit torus (ω, Ψ) is associated with a $\mathbb{Z}$-action $L_{\omega,\Psi}$ which determines the time evolution of the spin-orbit trajectories as follows (for details on group actions in general and $\mathbb{Z}$-actions in particular, see Appendix A). Defining the function $L_{\omega,\Psi} : \mathbb{Z} \times \mathbb{R}^{d+3} \to \mathbb{R}^{d+3}$, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$, by

$$L_{\omega,\Psi}(n;\phi,S) := \begin{pmatrix} \phi + 2\pi n\omega \\ \Psi(n;\phi)S \end{pmatrix} , \tag{2.9}$$

we obtain from eq. (2.3) that, for every spin-orbit trajectory $\begin{pmatrix} \phi \\ S \end{pmatrix}$ of (ω, Ψ) and every $n \in \mathbb{Z}$,

$$\begin{pmatrix} \phi(n) \\ S(n) \end{pmatrix} = L_{\omega,\Psi}(n;\phi(0),S(0)) . \tag{2.10}$$

Clearly, by eq. (2.4),(2.5),(2.9), we have, for $m, n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$L_{\omega,\Psi}(0;\phi,S) = \begin{pmatrix} \phi \\ S \end{pmatrix} , \tag{2.11}$$

$$L_{\omega,\Psi}(m+n;\phi,S) = L_{\omega,\Psi}(m;L_{\omega,\Psi}(n;\phi,S)) , \tag{2.12}$$

$$L_{\omega,\Psi}(m+n;\phi,S) = L_{\omega,\Psi}(n;L_{\omega,\Psi}(m;\phi,S)) . \tag{2.13}$$

We conclude from eq. (2.11),(2.12) that $L_{\omega,\Psi}$ is a left $\mathbb{Z}$-action on $\mathbb{R}^{d+3}$ and from eq. (2.11),(2.13) that $L_{\omega,\Psi}$ is a right $\mathbb{Z}$-action on $\mathbb{R}^{d+3}$. Left actions are commonly called actions. Since $L_{\omega,\Psi}$ is a $\mathbb{Z}$-action on $\mathbb{R}^{d+3}$, one calls $(\mathbb{R}^{d+3}, L_{\omega,\Psi})$ a $\mathbb{Z}$-space. In a more loose sense, $L_{\omega,\Psi}$ would be called the general solution map of (2.1),(2.2). Note that, since the group $\mathbb{Z}$ is Abelian, the property (2.13) follows from eq. (2.12) confirming the simple fact that left $\mathbb{Z}$-actions are always right $\mathbb{Z}$-actions and vice versa. Note also that $L_{\omega,\Psi}(n;\cdot)$ is continuous whence $(\mathbb{R}^{d+3}, L_{\omega,\Psi})$ is a topological $\mathbb{Z}$-space. Note also that, because $L_{\omega,\Psi}$ is a $\mathbb{Z}$-action, we have, for $n = 1,2,...$, that $L_{\omega,\Psi}(n;\cdot)$ is the n-fold composition of $L_{\omega,\Psi}(1;\cdot)$ and, for $n = -1,-2,...$, that $L_{\omega,\Psi}(n;\cdot)$ is the $|n|$-fold composition of $L_{\omega,\Psi}(-1;\cdot)$. While all these details on $L_{\omega,\Psi}$ are trivial, they intend to set the stage for later chapters where we have to study more group actions.

If $\omega \in \mathbb{R}^d$ then we define the function $L_\omega : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$, by

$$L_\omega(n;\phi) := \phi + 2\pi n\omega \, . \tag{2.14}$$

Clearly L_ω is a $\mathbb{Z}$-action on $\mathbb{R}^d$ and moreover $(\mathbb{R}^d, L_\omega)$ is a topological $\mathbb{Z}$-space.

In Section 5.1 it will be shown how the peculiar structure of eq. (2.5) follows from the fact that $L_{\omega,\Psi}$ is a skew-product of the orbital $\mathbb{Z}$-action L_ω.

Given a spin-orbit torus (ω,Ψ) in $SOT(d,\omega)$, it follows from (2.5) and Appendix A that Ψ is a continuous $SO(3)$-cocycle over the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$ whence $(L_\omega, \Psi) \in COC(\mathbb{R}^d, \mathbb{Z}, SO(3))$. We thus have a natural injection $\rho_{SOT(d)}$: $SOT(d) \to COC(\mathbb{R}^d, \mathbb{Z}, SO(3))$, defined for $(\omega,\Psi) \in SOT(d)$ by

$$\rho_{SOT(d)}(\omega,\Psi) := (L_\omega, \Psi) \, . \tag{2.15}$$

2.3 Introducing the polarization fields of every spin-orbit torus (ω,Ψ) and the associated $\mathbb{Z}$-action $L^{(PF)}_{\omega,\Psi}$

Each spin-orbit torus is associated with a set of functions, called "polarization fields", which are introduced in this section. The time evolution of the polarization fields of a spin-orbit torus (ω,Ψ) is determined by the $\mathbb{Z}$-action $L^{(PF)}_{\omega,\Psi}$ introduced below.

As displayed by Theorem 3.9, in this work the main purpose of polarization fields is that invariant spin fields (which are special polarization fields) are building blocks of the group action $R_{d;\omega}$ on $SOT(d,\omega)$. This group action, to be introduced in Section 3.1, allows to study $SOT(d,\omega)$ as a whole and exploits some fundamental symmetry properties of $SOT(d,\omega)$ leading in particular to a definition of spin tune (see Definition 3.11). Not pursued in this work (and only briefly mentioned in Section 3.6) is a second purpose of polarization fields being an important tool in the statistical treatment of spin-orbit motion [EH]. The statistical treatment is needed for coping with the fact that a storage ring bunch contains many particles (typically 10^{10}).

To motivate the concept of polarization field, consider an initial assignment of spins $G : \mathbb{R}^d \to \mathbb{R}^3$, i.e., a spin attached to every point $\phi_0 \in \mathbb{R}^d$. Under the $\mathbb{Z}$-action $L_{\omega,\Psi}$ the point $\begin{pmatrix} \phi_0 \\ G(\phi_0) \end{pmatrix}$ evolves to $\begin{pmatrix} \phi_0 + 2\pi n\omega \\ \Psi(n;\phi_0)G(\phi_0) \end{pmatrix}$ at time n. Denoting $\phi_0 + 2\pi n\omega$ by ϕ and $\Psi(n;\phi_0)G(\phi_0)$ by $\mathcal{S}_G(n,\phi)$ we have

$$\mathcal{S}_G(n,\phi) = \Psi(n;\phi - 2\pi n\omega)G(\phi - 2\pi n\omega) \, . \tag{2.16}$$

The 2π–periodicity of G has to be imposed for the same reason as mentioned in Section 2.1, namely because the components of ϕ are angle variables. We are thus led to:

'G' is also used to denote a group.

Definition 2.2 *(Polarization field, spin field) Let (ω,Ψ) be a spin-orbit torus. We call a function $\mathcal{S}_G : \mathbb{Z}\times\mathbb{R}^d \to \mathbb{R}^3$ a "polarization field" of (ω,Ψ), if it satisfies (2.16) for all ϕ, n and if $G \in \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$. The function G will be called the 'generator' of $\mathcal{S}_G$.*

We call a polarization field $\mathcal{S}_G$ "invariant" if $\mathcal{S}_G(n,\cdot)$ is independent of n. A polarization field $\mathcal{S}_G$ with $|\mathcal{S}_G(n,\phi)| = 1$ is called a "spin field". An invariant polarization field which is a spin field is called an "invariant spin field (ISF)". □

Remark:

(1) It follows from Definition 2.1 and eq. (2.16) that if $\mathcal{S}_G$ is an invariant polarization field then, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$,

$$G(\phi) = \Psi(n;\phi - 2\pi n\omega)G(\phi - 2\pi n\omega)\ . \tag{2.17}$$

This has an interesting implication in the case when the components of ω are rational since then we can choose n in (2.17) sufficiently large such that the components of $n\omega$ are integers. Then (2.17) becomes, due to the 2π-periodicity of $\Psi(n;\cdot)$ and G, an eigenvalue value problem for $G(\phi)$:

$$G(\phi) = \Psi(n;\phi)G(\phi)\ . \tag{2.18}$$

It also follows that if the components of ω are not rational then, by rational approximation of ω, one obtains an approximation of an invariant polarization field by solutions of eigenvalue problems. □

By eq. (2.5),(2.16) we get the following equation of motion for a polarization field $\mathcal{S}_G$

$$\mathcal{S}_G(n+1,\phi) = \Psi(1;\phi - 2\pi\omega)\mathcal{S}_G(n,\phi - 2\pi\omega)\ . \tag{2.19}$$

If $\mathcal{S}_G$ is a polarization field then $\mathcal{S}_G(0,\cdot) = G(\cdot) \in \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$ and $\mathcal{S}_G(n,\cdot) \in \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$. Clearly, the equation of motion (2.19) for $\mathcal{S}_G$ is linear and autonomous. Defining the function $L^{(PF)} \equiv L^{(PF)}_{\omega,\Psi}$ by $L^{(PF)}_{\omega,\Psi} : \mathbb{Z}\times\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3) \to \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$ by

$$L^{(PF)}_{\omega,\Psi}(n;G) := \mathcal{S}_G(n,\cdot) = \Psi(n;\cdot - 2\pi n\omega)G(\cdot - 2\pi n\omega)\ , \tag{2.20}$$

it follows easily from eq. (2.5),(2.16) that $L^{(PF)}_{\omega,\Psi}$ is a $\mathbb{Z}$-action on $\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$, i.e., that
$(\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3), L^{(PF)}_{\omega,\Psi})$ is a $\mathbb{Z}$-space. Thus by eq. (2.16)

$$\mathcal{S}_G(n,\cdot) = L^{(PF)}_{\omega,\Psi}(n-m;\mathcal{S}_G(m,\cdot))\ . \tag{2.21}$$

Loosely speaking, $L^{(PF)}$ is the flow map associated with eq. (2.19). Clearly, every $G \in \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$ gives a unique polarization field $\mathcal{S}_G$ for a given spin-orbit torus. In particular, each d-dimensional spin-orbit torus has as many polarization fields as the set $\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$ has elements. We see that the role which the $\mathbb{Z}$-action $L^{(PF)}$ plays for polarization fields, is analogous to the role which the $\mathbb{Z}$-action $L_{\omega,\Psi}$ plays for spin-orbit trajectories. Note also that G is a fixpoint of $L^{(PF)}$ iff the polarization field $\mathcal{S}_G$ is invariant. Since $L^{(PF)}$ is a group action of the group $\mathbb{Z}$ we easily conclude:

Proposition 2.3 *Let (ω,Ψ) be a spin-orbit torus. A polarization field $\mathcal{S}_G$ of (ω,Ψ) is invariant iff*

$$L^{(PF)}_{\omega,\Psi}(1;G) = G\ . \tag{2.22}$$

In other words, $\mathcal{S}_G$ is invariant, iff for all ϕ,

$$G(\phi) = \Psi(1;\phi - 2\pi\omega)G(\phi - 2\pi\omega)\ . \tag{2.23}$$

□

A polarization field $\mathcal{S}_G$ is a spin field iff $|G(\phi)| = 1$ for all ϕ. Defining the 2-sphere $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ and equipping it with the relative topology from $\mathbb{R}^3$

* The spin part of an SOT is not a field? Surely a 1-turn map is also a field.

FAC: B17?

we see that the set $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ of 2π–periodic and continuous functions from $\mathbb{R}^d$ into $\mathbb{S}^2$ is equal to the set of 2π–periodic, normalized (w.r.t. the Euclidean norm), and continuous functions from $\mathbb{R}^d$ into $\mathbb{R}^3$. Thus for every spin field $\mathcal{S}_G$ we have $\mathcal{S}_G(n, \cdot) \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$. Clearly each ISF is a polarization field.

Due to Definition 2.2, every polarization field $\mathcal{S}_G$ fulfills three different conditions: the "dynamical" condition (2.16), the "kinematical" condition that G is 2π–periodic, and the "regularity" condition that G is continuous. In contrast to the dynamical and kinematical conditions, the regularity condition is a matter of choice. The regularity of G can basically vary between the extremes "no regularity condition" and "G being analytic". In this work we choose G to be continuous since the spin-orbit tori are built on continuity, i.e., the $\Psi(n; \cdot)$ are continuous functions.

Since the equation of motion (2.19) for $\mathcal{S}_G$ is linear, $L^{(PF)}_{\omega,\Psi}(n; \cdot)$ is a homomorphism of the additive group $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$, i.e., for $n \in \mathbb{Z}, G, G' \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$,

$$L^{(PF)}_{\omega,\Psi}(n; G + G') = L^{(PF)}_{\omega,\Psi}(n; G) + L^{(PF)}_{\omega,\Psi}(n; G') \,. \tag{2.24}$$

Eq. (2.24) allows, by the technique of twisted cocycles [HK1, HK2, Zim1], to define cohomology groups for any spin-orbit torus, which give further insight into $SOT(d, \omega)$ in general and into the ISF conjecture in particular [Hei]. However this is beyond the scope of the present work.

2.4 Homotopy Theory relevant for spin-orbit tori

Throughout this work we will see some impact of Homotopy Theory on the theory of spin-orbit tori and in this section we introduce some basic features (the details are worked out in Appendix B).

Let X be a path-connected topological space. In the context of spin-orbit tori, we are especially interested in $X = SO(3)$ and $X = \mathbb{S}^2$ (recall that spin transfer matrices

are $SO(3)$-valued functions and that spin fields are $\mathbb{S}^2$-valued functions). The use of Homotopy Theory for $\mathcal{C}_{per}(\mathbb{R}^d, X)$ is twofold. Firstly, we use it by applying the Homotopy Lifting theorem (see Lemma B.6 in Section B.1) which in turn is used in many of those proofs of this work which involve the sets $\mathcal{C}_{per}(\mathbb{R}^d, X)$. Secondly, Homotopy Theory gives us the useful equivalence relation $\simeq^{2\pi}_X$ on $\mathcal{C}_{per}(\mathbb{R}^d, X)$, as follows. To explain this equivalence relation we first note, by Proposition B.4, that any two functions in $\mathcal{C}_{per}(\mathbb{R}^d, X)$ are homotopic w.r.t. X, i.e., $[\mathbb{R}^d, X]$ is a singleton. In other words, the equivalence relation $\simeq_X$ on $\mathcal{C}_{per}(\mathbb{R}^d, X)$ is useless. However, since the functions in $\mathcal{C}_{per}(\mathbb{R}^d, X)$ are 2π-periodic, we can associate, as detailed in Section B.3, every function $g \in \mathcal{C}_{per}(\mathbb{R}^d, X)$ with a function $G := FAC_d(g; X) \in \mathcal{C}(\mathbb{T}^d, X)$ which is uniquely determined by g via the relation $G \circ p_{4,d} = g$. Thus we call two functions $g_0, g_1 \in \mathcal{C}_{per}(\mathbb{R}^d, X)$ "2π-homotopic w.r.t. X", written $g_0 \simeq^{2\pi}_X g_1$, if $FAC_d(g_0; X), FAC_d(g_1; X)$ are homotopic w.r.t. X, i.e., if $FAC_d(g_0; X) \simeq_X FAC_d(g_1; X)$. Clearly $\simeq^{2\pi}_X$ is an equivalence relation on $\mathcal{C}_{per}(\mathbb{R}^d, X)$ and we denote the set of equivalence classes by $[\mathbb{R}^d, X]_{2\pi}$. Obviously the function which maps the $\simeq_X$-equivalence class of a $F \in \mathcal{C}(\mathbb{T}^d, X)$ to the $\simeq^{2\pi}_X$-equivalence class of $F \circ p_{4,d}$, is a bijection from $[\mathbb{T}^d, X]$ onto $[\mathbb{R}^d, X]_{2\pi}$. Thus every statement about $[\mathbb{R}^d, X]_{2\pi}$ corresponds to a statement about $[\mathbb{T}^d, X]$.

The point to be made here is that for the topological spaces X of interest, in general two functions in $\mathcal{C}(\mathbb{T}^d, X)$ are not homotopic w.r.t. X whence, in general, two functions in $\mathcal{C}_{per}(\mathbb{R}^d, X)$ are not 2π-homotopic w.r.t. X, i.e., $[\mathbb{R}^d, X]_{2\pi}$ is not a singleton. In particular we will see below that, for no positive integer d, $[\mathbb{R}^d, SO(3)]_{2\pi}$ is a singleton and that, by Proposition B.18c and Theorem B.24, $[\mathbb{R}^d, \mathbb{S}^2]_{2\pi}$ is not a singleton for any $d \geq 2$. The meaning of this is, loosely speaking, that, among the functions in $\mathcal{C}_{per}(\mathbb{R}^d, X)$, the ones which are especially simple are the g which are "2π-nullhomotopic w.r.t. X", i.e., for which $FAC_d(g; X)$ is nullhomotopic w.r.t. X (the latter condition means that $FAC_d(g; X)$ is homotopic w.r.t. X to a constant function). Note that, by Proposition B.18c, all 2π-nullhomotopic functions in

SO(3) index: something about double valuedness of SO(2)

$C_{per}(\mathbb{R}^d, X)$ are 2π-homotopic w.r.t. X, i.e., belong to the same element of $[\mathbb{R}^d, X]_{2\pi}$. Thus if $[\mathbb{R}^d, X]_{2\pi}$ is not a singleton then $C_{per}(\mathbb{R}^d, X)$ contains functions which are not 2π-nullhomotopic w.r.t. X. As we will see in this work, the fact that $[\mathbb{R}^d, SO(3)]_{2\pi}$ and, for $d \geq 2$, $[\mathbb{R}^d, \mathbb{S}^2]_{2\pi}$ are not singletons, contributes to the structural richness of the sets $SOT(d, \omega)$. Note that, in the context of polarized beams in storage rings, the case $d = 3$ is the most important one whereas the cases $d = 1, 2$ come next in terms of importance.

We wrap up this brief section by mentioning several important facts and concepts valid for the case $X = SO(3)$ and it first of all has to be pointed out that in our study of $SO(3)$-valued functions in Appendix B the "quaternion formalism" is employed which consists in representing $SO(3)$-valued functions by $\mathbb{S}^3$-valued functions. For every positive integer d there is a function $Ind_{3,d} : C_{per}(\mathbb{R}^d, SO(3)) \to \{1, -1\}^d$, defined by Definition B.14 and called the "$SO(3)$-index", which, due to Proposition B.18c, has the property that, if $g_0, g_1 \in C_{per}(\mathbb{R}^d, SO(3))$ and $g_0 \simeq^{2\pi}_{SO(3)} g_1$, then $Ind_{3,d}(g_0) = Ind_{3,d}(g_1)$. Since, by Theorem B.15a, the function $Ind_{3,d}$ is onto $\{1, -1\}^d$ we observe that $[\mathbb{R}^d, SO(3)]_{2\pi}$ is not a singleton. Moreover, for $d = 1, 2$, the function $Ind_{3,d}$ completely determines $[\mathbb{R}^d, SO(3)]_{2\pi}$ since, by Theorem B.22c, we have, for $g_0, g_1 \in C_{per}(\mathbb{R}^d, SO(3))$, that $g_0 \simeq^{2\pi}_{SO(3)} g_1$ iff $Ind_{3,d}(g_0) = Ind_{3,d}(g_1)$. For the most important case, $d = 3$, the structure of $[\mathbb{R}^d, SO(3)]_{2\pi}$ is even richer. In fact, Definition B.21 gives us a function $DEG : C_{per}(\mathbb{R}^3, SO(3)) \to \mathbb{Z}$, which is onto $\mathbb{Z}$ and, due to Theorem B.22f, has the property that, for $g_0, g_1 \in C_{per}(\mathbb{R}^3, SO(3))$, we have $g_0 \simeq^{2\pi}_{SO(3)} g_1$ iff $DEG(g_0) = DEG(g_1)$ and $Ind_{3,3}(g_0) = Ind_{3,3}(g_1)$. Thus, for $d = 3$, $[\mathbb{R}^d, SO(3)]_{2\pi}$ has infinitely many elements. We also conclude that, for $d = 1, 2, 3$, the $SO(3)$-index and the function DEG are sufficient to determine the equivalence class of every $g \in C_{per}(\mathbb{R}^d, SO(3))$ w.r.t. the equivalence relation $\simeq^{2\pi}_{SO(3)}$ whence to determine the equivalence class of every $F \in C(\mathbb{T}^d, SO(3))$ w.r.t. the equivalence relation $\simeq_{SO(3)}$.

$\{1,-1\}^d$ is a set?

Before we state the following proposition, note that we consider $\{1, -1\}$ as a multiplicative group with identity 1 and $\{1, -1\}^d$ as the d-fold direct product of the group $\{1, -1\}$. The following proposition is the most basic result of how Homotopy Theory impacts spin-orbit tori via the $SO(3)$-index.

Proposition 2.4 *If $(\omega, \Psi) \in SOT(d, \omega)$ then, for an arbitrary integer n, we have*

$$Ind_{3,d}(\Psi(n; \cdot)) = (Ind_{3,d}(\Psi(1; \cdot)))^n , \tag{2.25}$$

where $(Ind_{3,d}(\Psi(1; \cdot)))^n$ denotes the n-th power of $Ind_{3,d}(\Psi(1; \cdot))$ w.r.t. the group multiplication in $\{1, -1\}^d$.

Proof of Proposition 2.4: See Section E.1. □

Chapter 3

Transforming spin-orbit tori

In this chapter we study the right group action $R_{d,\omega}$ on $SOT(d,\omega)$ for the group $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ and the associated equivalence relation $\sim_{d,\omega}$ by which two spin-orbit tori $SOT(d,\omega)$ are equivalent iff they belong to the same $R_{d,\omega}$-orbit. The group action $R_{d,\omega}$ is an outgrowth of the observation (see Section 3.1) that spin-orbit tori can be transformed into each other in a natural way. In fact in each $SOT(d,\omega)$ we have a large family of pairs of spin-orbit tori whose topological $\mathbb{Z}$-spaces $(\mathbb{R}^{d+3}, L_{\omega,\Psi})$ are conjugate by conjugating homeomorphisms which form a family L_T labelled by the $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. In particular we obtain in Section 3.1 a transformation law for spin-orbit tori and polarization fields.

The group action $R_{d,\omega}$ allows to define the spin tune (spin tune of first kind) in an elegant way. We will see that two spin-orbit tori which belong to the same $R_{d,\omega}$-orbit, share important properties, e.g., they have the same spin tunes of first kind (see Proposition 3.12) and either both of them have an ISF or both of them have no ISF (see Theorem 3.3d). In other words, spin-orbit tori, whose topological $\mathbb{Z}$-spaces $(\mathbb{R}^{d+3}, L_{\omega,\Psi})$ are conjugate by a homeomorphism L_T, resemble each other. Thus to a large extent the study of $SOT(d,\omega)$ reduces to the study of just one spin-orbit torus

per $R_{d,\omega}$-orbit.

This, of course, raises the question if an $R_{d,\omega}$-orbit contains spin-orbit tori which are more 'simply structured' than others. Indeed (see also Section 3.6) it is widely believed and based on numerical evidence that, generically, the spin-orbit tori of practical relevance are "weak coboundaries" (see Definition 3.6) which means that each of them lies on the same $R_{d,\omega}$-orbit as a "weakly trivial" spin-orbit torus (see Definition 3.4). Thus, generically, the main features of spin-orbit tori can be studied on weakly trivial spin-orbit tori, which indeed are simply structured. Note also that the $SO(3)$-indices and the $SO_3(2)$-indices associated with a weakly trivial spin-orbit torus carry important topological information (see Proposition 3.5). It is even believed that, generically, the spin-orbit tori of practical relevance are not only weak coboundaries but also "almost coboundaries" (see Definition 3.6). As their name suggests, almost coboundaries lie on the same $R_{d,\omega}$-orbit as "almost trivial" spin-orbit tori. Most importantly, almost coboundaries are those spin-orbit tori which carry spin tunes (in fact, spin tunes of first kind - see Definition 3.11). "Coboundaries" (see Definition 3.6) are those almost coboundaries which are on spin-orbit resonance of first kind. Coboundaries, by definition, lie on the same $R_{d,\omega}$-orbit as "trivial" spin-orbit tori, which indeed are the simplest spin-orbit tori of all (see Definition 3.4).

3.1 Introducing the transformations of spin-orbit tori and the group action $R_{d,\omega}$ on $SOT(d,\omega)$

In this section we introduce the group action $R_{d,\omega}$ and the associated equivalence relation $\sim_{d,\omega}$.

The motivation for $R_{d,\omega}$ comes from the practical need to transform spin trajecto-

Defn (3.2): I guess that R is a SOT just summarises (3.4)

ries in order to simplify (analytical and numerical) computations. The archetypical way to transform a spin trajectory goes, in the context of spin-orbit tori, as follows. Let a spin-orbit torus (ω, Ψ) be given with a spin trajectory $S(\cdot)$ over some ϕ_0. Then a function $t : \mathbb{Z} \to SO(3)$ transforms $S(\cdot)$ into the function $S' : \mathbb{Z} \to \mathbb{R}^3$ via $S'(n) := t^T(n)S(n)$ (using t^T instead of t is just a convention). Of course, since $S(\cdot)$ satisfies the equation of motion (2.8), one observes that $S'(\cdot)$ satisfies the equation of motion

$$S'(n+1) = t^T(n+1)\Psi(1; \phi_0 + 2\pi n\omega)t(n)S'(n) , \tag{3.1}$$

where $n \in \mathbb{Z}$. Clearly $S'(\cdot)$ has many features of a spin motion, e.g., $|S'(n)| = |S(n)|$ is independent of n and $S'(n)$ is uniquely determined by $S'(0)$ and n. Perhaps surprisingly however, in general $S'(\cdot)$ is not a spin trajectory of any spin-orbit torus! This follows from the fact that $\Psi(1; \phi_0 + 2\pi n\omega)$ is always an ω-quasiperiodic function of n whereas $t^T(n+1)\Psi(1; \phi_0 + 2\pi n\omega)t(n)$ in general is not a quasiperiodic function of n at all, since t may not be quasiperiodic. Note that quasiperiodic functions play a major role in Chapter 4 and are defined in Section C.1.

So spin traj's must be Q-P?

Part d) of the following proposition now comes as a relief.

Proposition 3.1 *a) Let $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Then the map $L_T : \mathbb{R}^{d+3} \to \mathbb{R}^{d+3}$, defined by*

$$L_T(\phi, S) := (\phi, T^T(\phi)S) , \tag{3.2}$$

is a homeomorphism onto $\mathbb{R}^{d+3}$ and its inverse L_T^{-1} is defined by $L_T^{-1}(\phi, S) := (\phi, T(\phi)S)$, i.e., $L_T^{-1} = L_{T^T}$.

b) Let $(\omega, \Psi) \in \mathcal{SOT}(d, \omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Then, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in$

$\mathbb{R}^3$,

Defining L_T or proposition?

$$\left(L_T \circ L_{\omega,\Psi}(n; \cdot) \circ L_T^{-1}\right)\begin{pmatrix}\phi \\ S\end{pmatrix} = \begin{pmatrix}\phi + 2\pi n\omega \\ T^T(\phi + 2\pi n\omega)\Psi(n;\phi)T(\phi)S\end{pmatrix} . \tag{3.3}$$

Moreover $(\omega, \Psi') \in \mathcal{SOT}(d, \omega)$ where

$$\Psi'(n; \phi) := T^T(\phi + 2\pi n\omega)\Psi(n; \phi)T(\phi) . \tag{3.4}$$

Furthermore, L_T is a continuous $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L_{\omega,\Psi})$ to the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L_{\omega,\Psi'})$, i.e., for $n \in \mathbb{Z}$,

$$L_{\omega,\Psi'}(n; \cdot) = L_T \circ L_{\omega,\Psi}(n; \cdot) \circ L_T^{-1} . \tag{3.5}$$

Thus the topological $\mathbb{Z}$-spaces $(\mathbb{R}^{d+3}, L_{\omega,\Psi})$ and $(\mathbb{R}^{d+3}, L_{\omega,\Psi'})$ are conjugate.

c) (Transformation rule of spin-orbit trajectories) Let $(\omega, \Psi) \in \mathcal{SOT}(d, \omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. If $\begin{pmatrix}\phi(\cdot) \\ S(\cdot)\end{pmatrix}$ is a spin-orbit trajectory of the spin-orbit torus (ω, Ψ), then $\begin{pmatrix}\phi(\cdot) \\ S'(\cdot)\end{pmatrix}$ is a spin-orbit trajectory of the spin-orbit torus (ω, Ψ') where Ψ' is given by eq. (3.4) and where, for $n \in \mathbb{Z}$,

$$\begin{pmatrix}\phi(n) \\ S'(n)\end{pmatrix} := L_T(\phi(n), S(n)) = \begin{pmatrix}\phi(n) \\ T^T(\phi(n))S(n)\end{pmatrix} . \tag{3.6}$$

Backwards?

d) (Transformation rule of spin trajectories) Let $(\omega, \Psi) \in \mathcal{SOT}(d, \omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Let also $\phi_0 \in \mathbb{R}^d$ and let $t : \mathbb{Z} \to SO(3)$ be defined by $t(n) := T(\phi_0 + 2\pi n\omega)$. If $S(\cdot)$ is a spin trajectory, over ϕ_0, of the spin-orbit torus (ω, Ψ) then $S'(\cdot)$, defined by $S'(n) := t^T(n)S(n)$, is a spin trajectory, over ϕ_0, of the spin-orbit torus (ω, Ψ') where Ψ' is given by (3.4).

Proof of Proposition 3.1: See Section E.2. □

(3.4) is at fixed n. Just changing coord spin for spin

With Proposition 3.1b we see, for every $\omega \in \mathbb{R}^d$, that every $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ associates any $(\omega, \Psi) \in \mathcal{SOT}(d,\omega)$ with onother $(\omega, \Psi') \in \mathcal{SOT}(d,\omega)$. This we coin into the following definition:

Definition 3.2 *Let $\omega \in \mathbb{R}^d$. We define the function $R_{d,\omega} : \mathcal{C}_{per}(\mathbb{R}^d, SO(3)) \times \mathcal{SOT}(d,\omega) \to \mathcal{SOT}(d,\omega)$ by $R_{d,\omega}(T;\omega,\Psi) := (\omega,\Psi')$ where $(\omega,\Psi) \in \mathcal{SOT}(d,\omega), T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$, and where the function $\Psi' : \mathbb{Z} \times \mathbb{R}^d \to SO(3)$ is given by eq. (3.4). Note that, by Proposition 3.1b, $R_{d,\omega}$ is indeed a function from $\mathcal{C}_{per}(\mathbb{R}^d, SO(3)) \times \mathcal{SOT}(d,\omega)$ into $\mathcal{SOT}(d,\omega)$. If $R_{d,\omega}(T;\omega,\Psi) = (\omega,\Psi')$ then we call T a "transfer field from (ω,Ψ) to (ω,Ψ')".* □

The following theorem states the basic properties of $R_{d,\omega}$.

Theorem 3.3 *a) Let $\omega \in \mathbb{R}^d$. Then, for $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$, $(\omega,\Psi) \in \mathcal{SOT}(d,\omega)$,*

$$L_{R_{d,\omega}(T;\omega,\Psi)} = L_T \circ L_{\omega,\Psi}(n;\cdot) \circ L_T^{-1} . \tag{3.7}$$

Furthermore $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ is a group under pointwise multiplication of $SO(3)$-valued functions and $R_{d,\omega}$ is a right $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$-action on $\mathcal{SOT}(d,\omega)$.

b) (Transformation rule of spin-orbit trajectories) Let $(\omega,\Psi) \in \mathcal{SOT}(d,\omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. If $\begin{pmatrix}\phi(\cdot)\\ S(\cdot)\end{pmatrix}$ is a spin-orbit trajectory of the spin-orbit torus (ω,Ψ), then $\begin{pmatrix}\phi(\cdot)\\ S'(\cdot)\end{pmatrix}$, defined by (3.6), is a spin-orbit trajectory of the spin-orbit torus $R_{d,\omega}(T;\omega,\Psi)$.

c) (Transformation rule of polarization fields) Let $(\omega,\Psi) \in \mathcal{SOT}(d,\omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Let also $\mathcal{S}_G$ be a polarization field of the spin-orbit torus (ω,Ψ).

Then $\mathcal{S}'$, defined by

$$\mathcal{S}'(n,\phi) := T^T(\phi)\mathcal{S}_G(n,\phi) , \tag{3.8}$$

is a polarization field of the spin-orbit torus $R_{d,\omega}(T;\omega,\Psi)$ and the generator of $\mathcal{S}'$ is $T^T G$. Thus for every $n \in \mathbb{Z}, G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$

$$L^{(PF)}_{\omega,\Psi'}(n;G) = T^T L^{(PF)}_{\omega,\Psi}(n;TG) . \tag{3.9}$$

If the polarization field $\mathcal{S}_G$ is invariant, then so is $\mathcal{S}'$. If the polarization field $\mathcal{S}_G$ is a spin field, then so is $\mathcal{S}'$.

d) Let $(\omega,\Psi),(\omega,\Psi') \in \mathcal{SOT}(d,\omega)$ belong to the same $R_{d,\omega}$-orbit. Then either both spin-orbit tori have an ISF or none of them.

e) Let $(\omega,\Psi),(\omega,\Psi') \in \mathcal{SOT}(d,\omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Then $R_{d,\omega}(T;\omega,\Psi) = (\omega,\Psi')$ iff

$$\Psi'(1;\phi) := T^T(\phi + 2\pi\omega)\Psi(1;\phi)T(\phi) . \tag{3.10}$$

f) Let $(\omega,\Psi),(\omega,\Psi') \in \mathcal{SOT}(d,\omega)$ belong to the same $R_{d,\omega}$-orbit. Then, for every integer n, $\Psi(n;\cdot),\Psi'(n;\cdot)$ have the same $SO(3)$-index, i.e., $Ind_{3,d}(\Psi(n;\cdot)) = Ind_{3,d}(\Psi'(n;\cdot))$. If $d = 1,2$ then, for every integer n, $\Psi(n;\cdot) \simeq^{2\pi}_{SO(3)} \Psi'(n;\cdot)$.

Proof of Theorem 3.3: See Section E.3. □

If $(\omega,\Psi),(\omega,\Psi') \in \mathcal{SOT}(d,\omega)$ lie on the same $R_{d,\omega}$-orbit then we write $(\omega,\Psi) \sim_{d,\omega} (\omega,\Psi')$. It follows from Theorem 3.3a that $\sim_{d,\omega}$ is an equivalence relation on $\mathcal{SOT}(d,\omega)$. It also follows from Theorem 3.3a that, for each $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$, the function $R_{d,\omega}(T;\cdot)$ is a bijection from $\mathcal{SOT}(d,\omega)$ onto $\mathcal{SOT}(d,\omega)$. Clearly each $R_{d,\omega}(T;\cdot)$ transforms spin-orbit tori into spin-orbit tori and the associated transformation of spin-orbit trajectories and polarization fields is given by parts b),c) of Theorem 3.3 respectively.

Since $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ is a group under pointwise multiplication of $SO(3)$-valued functions, the constant function in $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ whose constant value is $I_{3\times3}$, is the unit element of the group. If there is no danger of confusion, we denote the unit element by $I_{3\times3}$. Furthermore the inverse of $f \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ is the transpose f^T since $(f^T f)(\phi) = f^T f(\phi) = I_{3\times3}$. Since the group $SO(3)$ is not Abelian, so is the group $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$.

As announced at the beginning of Chapter 3, spin-orbit tori on the same $R_{d,\omega}$-orbit share some important properties and with Theorems 3.3d,f we have got a first glimpse on that and more in that vein will follow. This raises the following issue. While, by Proposition 3.1b, spin-orbit tori on the same $R_{d,\omega}$-orbit have conjugate topological $\mathbb{Z}$-spaces $(\mathbb{R}^{d+3}, L_{\omega,\Psi})$ this does not exclude more general conjugacy relations in $\mathcal{SOT}(d,\omega)$. Although we here cannot pursue more general conjugacy relations, it is in fact conceivable that there are pairs of spin-orbit tori in $\mathcal{SOT}(d,\omega)$ whose topological $\mathbb{Z}$-spaces $(\mathbb{R}^{d+3}, L_{\omega,\Psi})$ are conjugate but which do not lie on the same $R_{d,\omega}$-orbit. Nevertheless it is questionable if those pairs of spin-orbit tori would share properties like the one in Theorem 3.3d.

Since the group $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ is not Abelian, it is easy to see that $R_{d,\omega}$ is *not* a left $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$-action on $\mathcal{SOT}(d,\omega)$. However, as every right action has its 'dual' left action, we could use the left $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$-action $L_{d,\omega}$ on $\mathcal{SOT}(d,\omega)$ defined by $L_{d,\omega}(T;\omega,\Psi) := R_{d,\omega}(T^T;\omega,\Psi)$ and the subsequent theory would be just 'dual' to the theory based on $R_{d,\omega}$. Nevertheless we stick, for convenience, with $R_{d,\omega}$.

Remark:

(1) That $R_{d,\omega}$ is so useful in this work is granted to the fact that the equations of motion (2.1), (2.2) are autonomous. In a more general situation where the

accelerator is not a storage ring, eq. (2.1), (2.2) maybe generalized to a non-autonomous system of the form

$$\phi(n+1) = \phi(n) + 2\pi\omega\,, \qquad S(n+1) = \Psi(1,n;\phi(n))S(n)\,. \quad (3.11)$$

Accordingly the definition of $\mathcal{SOT}(d,\omega)$ would be modified and the group action $R_{d,\omega}$ would be modified to a right G-action where G consists of functions $T : \mathbb{Z} \times \mathbb{R}^d \to SO(3)$ where $T(n,\cdot) \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. □

3.2 Introducing weakly trivial spin-orbit tori

As mentioned at the beginning of Chapter 3, simply structured spin-orbit tori will play an important role in this work and the following definition specifies what a "simply structured" spin-orbit torus is.

Definition 3.4 *(Trivial, almost trivial, weakly trivial spin-orbit torus) A spin-orbit torus (ω,Ψ) is called "trivial" if $\Psi(n;\phi) = I_{3\times3}$. The set of trivial spin-orbit tori in $\mathcal{SOT}(d,\omega)$ is denoted by $\mathcal{T}(d,\omega)$. A spin-orbit torus (ω,Ψ) is called "almost trivial" if Ψ is $SO_3(2)$-valued and if, for every integer n, $\Psi(n;\phi)$ is independent of ϕ where $SO_3(2) \subset SO(3)$ is defined by Definition B.2. We denote the set of almost trivial spin-orbit tori in $\mathcal{SOT}(d,\omega)$ by $\mathcal{AT}(d,\omega)$. A spin-orbit torus (ω,Ψ) is called "weakly trivial" if Ψ is $SO_3(2)$-valued and the set of weakly trivial spin-orbit tori in $\mathcal{SOT}(d,\omega)$ is denoted by $\mathcal{WT}(d,\omega)$.* □

The fact, that $(\omega, I_{3\times3})$ is a spin-orbit torus, is obvious since, for $\Psi = I_{3\times3}$, eq. (2.5) is just the identity $I_{3\times3} = I_{3\times3}$.

We now draw some simple consequences. Firstly, for each $\omega \in \mathbb{R}^d$, there exists exactly one trivial spin-orbit torus (ω,Ψ), i.e., $\mathcal{T}(d,\omega) = \{(\omega, I_{3\times3})\}$. Secondly

$$\mathcal{T}(d,\omega) \subset \mathcal{AT}(d,\omega) \subset \mathcal{WT}(d,\omega) \subset \mathcal{SOT}(d,\omega)\,. \qquad (3.12)$$

Thirdly every weakly trivial spin-orbit torus has the constant ISF's $\mathcal{S}_G = e^3$ and $\mathcal{S}_G = -e^3$ where e^i denotes the i-th unit vector (see Definition B.2).

For the following proposition, we note that the topology of $SO_3(2)$ is defined as the relative topology from $\mathbb{R}^{3\times 3}$ (see also Definition B.2). Thus if $(\omega, \Psi) \in \mathcal{WT}(d,\omega)$ then, for every $n \in \mathbb{Z}$, the function $\Psi(n;\cdot)$ belongs to $\mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ whence has a unique phase function (which is an element of $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$) and has a unique $SO_3(2)$-index (which is an element of $\mathbb{Z}^d$). Note that the $SO_3(2)$-index is defined by Definition B.12. Note also that, for $\mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$, each of the d components of $Ind_{2,d}(g)$ can be interpreted, in an obvious way, as a winding number in the plane $\mathbb{R}^2$. However this aspect of the $SO_3(2)$-index plays no role in this work. Denoting the fractional part of a real number x by $\lfloor x \rfloor$, we obtain

Proposition 3.5 *a) (Structure of weakly trivial spin-orbit tori) Let $(\omega, \Psi) \in \mathcal{WT}(d,\omega)$. Then, for every positive integer n,*

$$\Psi(n;\phi) = \exp\Big(\mathcal{J}[nN^T\phi + \pi n(n-1)N^T\omega + 2\pi \sum_{j=0}^{n-1} g(\phi + 2\pi j\omega)]\Big)\,, \tag{3.13}$$

where $N := Ind_{2,d}(\Psi(1;\cdot)), g := PHF(\Psi(1;\cdot))$ and $\mathcal{J}$ is defined by eq. (B.1). Also, for every $n \in \mathbb{Z}$,

$$Ind_{2,d}(\Psi(n;\cdot)) = nInd_{2,d}(\Psi(1;\cdot))\,. \tag{3.14}$$

Thus defining $f : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}$ by $f(n,\cdot) := PHF(\Psi(n;\cdot))$, we have $f(1,\cdot) = g(\cdot)$ and, for every $n \in \mathbb{Z}$,

$$\Psi(n;\phi) = \exp(\mathcal{J}[nN^T\phi + 2\pi f(n,\phi)])\,. \tag{3.15}$$

Moreover $\Psi(n;\cdot)$ is 2π-nullhomotopic w.r.t. $SO(3)$ iff $Ind_{3,d}(\Psi(n;\cdot)) = (1,...,1)^T$. Furthermore the $SO(3)$-index of $\Psi(n;\cdot)$ reads as $Ind_{3,d}(\Psi(n;\cdot)) = ((-1)^{nN_1}, ..., (-1)^{nN_d})^T$.

b) (Structure of almost trivial spin-orbit tori) If $(\omega, \Psi) \in \mathcal{AT}(d,\omega)$, then, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$,

$$\Psi(n;\phi) = \Psi(n;0) = \exp(\mathcal{J}2\pi n\nu)\,, \tag{3.16}$$

where $\nu := PH(\Psi(1;0))$ (recall Definition B.2). Moreover if $(\omega, \Psi) \in \mathcal{AT}(d,\omega)$ then, for every $n \in \mathbb{Z}$, $Ind_2(\Psi(n;\cdot)) = 0$ and $PHF(\Psi(n;\cdot))$ is the constant function in $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ whose value is $\lfloor n\nu \rfloor$ where $\nu := PH(\Psi(1;0))$. Furthermore, a $(\omega, \Psi) \in \mathcal{AT}(d,\omega)$ is trivial iff $PH(\Psi(1;0)) = 0$.

c) (The one-turn criterion) Let $(\omega, \Psi) \in \mathcal{SOT}(d,\omega)$. Then $(\omega, \Psi) \in \mathcal{WT}(d,\omega)$ iff $\Psi(1;\cdot)$ is $SO_3(2)$-valued. Moreover $(\omega, \Psi) \in \mathcal{AT}(d,\omega)$ iff $\Psi(1;\cdot)$ is $SO_3(2)$-valued and constant.

d) Let $(\omega, \Psi), (\omega, \Psi') \in \mathcal{WT}(d,\omega)$. If n is an even integer then $\Psi(n;\cdot) \simeq^{2\pi}_{SO(3)} \Psi'(n;\cdot)$. If n is an odd integer then $\Psi(n;\cdot) \simeq^{2\pi}_{SO(3)} \Psi'(n;\cdot)$ iff $Ind_{3,d}(\Psi(1;\cdot)) = Ind_{3,d}(\Psi'(1;\cdot))$. For every integer n, $(\omega, \Psi) \sim_{d,\omega} (\omega, \Psi')$ implies $\Psi(n;\cdot) \simeq^{2\pi}_{SO(3)} \Psi'(n;\cdot)$.

Proof of Proposition 3.5: See Section E.4. □

Note that the last claim in Proposition 3.5a confirms Proposition 2.4. Note also that, by Proposition 3.5c and eq. (2.4), there are as many weakly trivial spin-orbit tori in every $\mathcal{SOT}(d,\omega)$ as there are elements in $\mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ and that there are as many almost trivial spin-orbit tori in every $\mathcal{SOT}(d,\omega)$ as there are elements in $[0, 1)$. Clearly, the cardinalities of $\mathcal{WT}(d,\omega)$ and $\mathcal{AT}(d,\omega)$ are the same.

3.3 Introducing weak coboundaries as special cocycles

Recalling Section 2.2, given a spin-orbit torus (ω, Ψ) in $SOT(d,\omega)$, the function Ψ is a continuous $SO(3)$-cocycle over the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$. This terminology comes from Nonabelian Group Cohomology and Dynamical Systems Theory and, in fact, from this terminology we also borrow the term "coboundary" which will be introduced now (the weaker notions of "almost coboundary" and "weak coboundary" is my terminology).

Definition 3.6 *(Coboundary, almost coboundary, weak coboundary)*

A spin-orbit torus $(\omega, \Psi) \in SOT(d,\omega)$ is called a "coboundary" if it belongs to the $R_{d,\omega}$-orbit of the trivial spin-orbit torus $(\omega, I_{3\times3})$. We denote the set of coboundaries in $SOT(d,\omega)$ by $CB(d,\omega)$. A spin-orbit torus $(\omega, \Psi) \in SOT(d,\omega)$ is called an "almost coboundary" if it belongs to the $R_{d,\omega}$-orbit of a spin-orbit torus in $AT(d,\omega)$. We denote the set of almost coboundaries in $SOT(d,\omega)$ by $ACB(d,\omega)$. A spin-orbit torus $(\omega, \Psi) \in SOT(d,\omega)$ is called a "weak coboundary" if it belongs to the $R_{d,\omega}$-orbit of a spin-orbit torus in $WT(d,\omega)$. We denote the set of weak coboundaries in $SOT(d,\omega)$ by $WCB(d,\omega)$. □

Recalling Section 3.1, $\sim_{d,\omega}$ is an equivalence relation on $SOT(d,\omega)$ whence, by Definitions 3.4,3.6,

$$T(d,\omega) \subset CB(d,\omega)\,, \quad AT(d,\omega) \subset ACB(d,\omega)\,, \quad WT(d,\omega) \subset WCB(d,\omega)\,, \tag{3.17}$$

$$CB(d,\omega) \subset ACB(d,\omega) \subset WCB(d,\omega) \subset SOT(d,\omega)\,. \tag{3.18}$$

For the relevance of coboundaries, almost coboundaries, and weak coboundaries, see Section 3.6.

Physical picture of coboundaries?

Proposition 3.7 *a) Let $(\omega, \Psi) \in WCB(d,\omega)$ and $T \in C_{per}(\mathbb{R}^d, SO(3))$ with $(\omega, \Psi') := R_{d,\omega}(T; \omega, \Psi) \in WT(d,\omega)$. If $N := Ind_{2,d}(\Psi'(1;\cdot))$ then $Ind_{3,d}(\Psi(n;\cdot)) = ((-1)^{nN_1}, ..., (-1)^{nN_d})^T$ for arbitrary integer n.*

b) Let $(\omega, \Psi) \in ACB(d,\omega)$. Then, for every $n \in \mathbb{Z}$, $\Psi(n;\cdot)$ is 2π-nullhomotopic w.r.t. $SO(3)$ and $Ind_{3,d}(\Psi(n;\cdot)) = (1, ..., 1)^T$.

Proof of Proposition 3.7: See Section E.5. □

Lemma 3.8 *a) Let R be in $SO(3)$ and $Re^3 = e^3$. Then $R \in SO_3(2)$.*

b) A spin-orbit torus (ω, Ψ) is weakly trivial iff $\Psi(1;\phi)e^3 = e^3$.

Proof of Lemma 3.8: See Section E.6. □

The following theorem expresses the most important property of weak coboundaries.

Theorem 3.9 *Let $(\omega, \Psi) \in SOT(d,\omega)$. Then, for every $T \in C_{per}(\mathbb{R}^d, SO(3))$, we have $R_{d,\omega}(T; \omega, \Psi) \in WT(d,\omega)$ iff the third column, Te^3, of T is the generator of an ISF of (ω, Ψ). Moreover $(\omega, \Psi) \in WCB(d,\omega)$ iff there exists a $T \in C_{per}(\mathbb{R}^d, SO(3))$ such that Te^3 is the generator of an ISF of (ω, Ψ).*

Proof of Theorem 3.9: See Section E.7. □

Theorem 3.9 shows that the existence of an ISF is a necessary condition for a spin-orbit torus to be a weak coboundary, however Theorem 3.10, below, shows that it is not always a sufficient condition. Note also that, since, as pointed out after Proposition 3.5, there are (uncountably) many weakly trivial spin-orbit tori in every

$\mathcal{SOT}(d,\omega)$, it follows from Theorem 3.9 that invariant spin fields are important building blocks of $R_{d,\omega}$ for every (d,ω).

As we just learned from Theorem 3.9, every weak coboundary has an ISF. We now address the converse question: is a spin-orbit torus a weak coboundary, if it has an ISF? A partial answer is given by the following theorem which uses some concepts introduced in Section 2.4 and which are borrowed from Homotopy Theory.

Theorem 3.10 *Let $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ and let $(\omega,\Psi) \in \mathcal{SOT}(d,\omega)$ such that G is the generator of an ISF S_G of (ω,Ψ). Then the following hold.*

a) If G is 2π-nullhomotopic w.r.t. $\mathbb{S}^2$ then $(\omega,\Psi) \in \mathcal{WCB}(d,\omega)$ and a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists such that $R_{d,\omega}(T;\omega,\Psi) \in \mathcal{WT}(d,\omega)$ and $G = Te^3$.

b) If $d = 1$ then $(\omega,\Psi) \in \mathcal{WCB}(1,\omega)$ and a $T \in \mathcal{C}_{per}(\mathbb{R}, SO(3))$ exists such that $R_{1,\omega}(T;\omega,\Psi) \in \mathcal{WT}(1,\omega)$ and $G = Te^3$.

c) If $d = 2$ then a $T \in \mathcal{C}_{per}(\mathbb{R}^2, SO(3))$ exists such that $R_{2,\omega}(T;\omega,\Psi) \in \mathcal{WT}(2,\omega)$ and $G = Te^3$ iff G is 2π-nullhomotopic w.r.t. $\mathbb{S}^2$.

Proof of Theorem 3.10: See Section E.8. □

Let $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ and let $(\omega,\Psi) \in \mathcal{SOT}(d,\omega)$ such that G is the generator of an ISF of (ω,Ψ). It is clear by Theorem 3.10a that if (ω,Ψ) is not a weak coboundary, then G is not 2π-nullhomotopic w.r.t. $\mathbb{S}^2$. That this situation does occur, is the content of Theorem 4.19 (of course, due to Theorem 3.10b, this situation only occurs if $d \geq 2$).

3.4 Introducing spin tune and spin-orbit resonance of first kind

Definition 3.11 *(Spin tune of first kind, spin-orbit resonance of first kind) Let $(\omega,\Psi) \in \mathcal{SOT}(d,\omega)$. Then the subset $\Xi_1(\omega,\Psi)$ of $[0,1)$ is defined by*

$$\Xi_1(\omega,\Psi) := \{PH(\Psi'(1;0)) : (\omega,\Psi') \in \mathcal{AT}(d,\omega) \,\&\, (\omega,\Psi') \sim_{d,\omega} (\omega,\Psi)\} . \quad (3.19)$$

We call ν a "spin tune of first kind of (ω,Ψ)" if $\nu \in \Xi_1(\omega,\Psi)$.

We say that (ω,Ψ) is "on spin-orbit resonance of first kind" iff $0 \in \Xi_1(\omega,\Psi)$. We say that (ω,Ψ) is "off spin-orbit resonance of first kind" iff $\Xi_1(\omega,\Psi)$ is nonempty and $0 \notin \Xi_1(\omega,\Psi)$. □

Definition 3.11 will be discussed, in the physics context, in Section 3.6.

It is clear that if $(\omega,\Psi) \in \mathcal{AT}(d,\omega)$ then, since $(\omega,\Psi) \sim_{d,\omega} (\omega,\Psi)$, $PH(\Psi(1;0)) \in \Xi_1(\omega,\Psi)$. Of course, $\Xi_1(\omega,\Psi)$ is nonempty iff (ω,Ψ) is an almost coboundary. Thus (ω,Ψ) has no spin tune of first kind iff (ω,Ψ) is not an almost coboundary.

By Proposition 3.5 it is clear that there is a vast supply of spin-orbit tori which have spin tunes of first kind. On the other hand in Section 3.7 we will find a vast supply of spin-orbit tori which have no spin tune of first kind (see Remark 6 in Section 4.5).

In Section 4.4 (see Proposition 4.9a) we will observe that the sets $\Xi_1(\omega,\Psi)$ have a simple structure. This result, as several others, goes beyond Chapter 3 since it relies on the machinery of quasiperiodic functions worked out in Chapter 4.

Proposition 3.12 *a) Let $(\omega,\Psi), (\omega,\Psi') \in \mathcal{SOT}(d,\omega)$. If $(\omega,\Psi) \sim_{d,\omega} (\omega,\Psi')$, then $\Xi_1(\omega,\Psi) = \Xi_1(\omega,\Psi')$. If $(\omega,\Psi) \in \mathcal{ACB}(d,\omega)$ then $(\omega,\Psi) \sim_{d,\omega} (\omega,\Psi')$ iff $\Xi_1(\omega,\Psi) =$*

$\Xi_1(\omega, \Psi')$.

b) A $(\omega, \Psi) \in SOT(d, \omega)$ is on spin-orbit resonance of first kind iff $(\omega, \Psi) \in CB(d, \omega)$. A $(\omega, \Psi) \in SOT(d, \omega)$ is off spin-orbit resonance of first kind iff $(\omega, \Psi) \in (ACB(d, \omega) \setminus CB(d, \omega))$.

c) Let $(\omega, \Psi), (\omega, \Psi') \in SOT(d, \omega)$ with $(\omega, \Psi) \sim_{d,\omega} (\omega, \Psi')$. Then either both spin-orbit tori are coboundaries or none of them, and either both are almost coboundaries or none of them, and either both are weak coboundaries or none of them.

d) Let $(\omega, \Psi) \in SOT(d, \omega)$. Then $(\omega, \Psi) \in ACB(d, \omega)$ iff there exists a $(\omega, \Psi') \in SOT(d, \omega)$ such that $\Psi(n; \phi)$ is independent of ϕ and $(\omega, \Psi) \sim_{d,\omega} (\omega, \Psi')$.

Proof of Proposition 3.12: See Section E.9. □

Propositions 3.12a, 3.12c give us again properties shared by spin-orbit tori which belong to the same $R_{d,\omega}$-orbit.

Concerning Proposition 3.12d, we note that, by eq. (2.4), $\Psi(n; \phi)$ is independent of ϕ for all integers n iff $\Psi(1; \phi)$ is independent of ϕ. Moreover it is easy to see that if $\Psi(1; \phi)$ is independent of ϕ, then the function $\Psi(n)$ of n is a group homomorphism from the additive group $\mathbb{Z}$ into the multiplicative group $SO(3)$, i.e., $\Psi(n + m) = \Psi(n)\Psi(m)$. In particular this is the case for almost trivial (ω, Ψ).

3.5 Yokoya's uniqueness theorem

If a spin-orbit torus has an ISF S_G then also $-S_G$ is an ISF. Thus for spin-orbit tori which have an ISF, the question arises if they have two or more ISF's. The following

celebrated theorem gives a partial answer (its importance is pointed out in Section 3.6).

Theorem 3.13 *(Yokoya's uniqueness theorem) Let $(\omega, \Psi) \in SOT(d, \omega)$ and let $(1, \omega)$ be nonresonant. Let (ω, Ψ) have an ISF S_G and an ISF which is different from S_G and $-S_G$. Then (ω, Ψ) is on spin-orbit resonance of first kind.*

Proof of Theorem 3.13: See Section E.10. □

3.6 Putting weak coboundaries into perspective

We now can begin to put things into perspective. Based on numerical and experimental evidence from storage rings, it is widely believed that the practically relevant spin-orbit tori are almost coboundaries (whence weak coboundaries) which is a strong motivation for many of the concepts introduced in Chapter 3. Much of the numerical evidence comes from the code SPRINT which, among other things, contains a numerical procedure which transforms a given almost coboundary into a weakly trivial spin-orbit torus and then transforms this weakly trivial spin-orbit torus into an almost trivial spin-orbit torus which then yields a spin tune of first kind (for more details on this code, see Section 4.5).

Nevertheless one knows of counterexamples, since one has discovered [BV], by numerical means, spin-orbit tori which do not have an ISF, i.e., which, by Theorem 3.9, are not weak coboundaries (and these results were subsequently confirmed by analytical means). However, I am not aware of a spin-orbit torus off orbital resonance which does not have an ISF. It is therefore useful here to state the following conjecture, which I call the "ISF-conjecture": "If a spin-orbit torus (ω, Ψ) is off orbital resonance, then it has an ISF". While, at least to my knowledge, the ISF-conjecture is unsettled, it is definitely true that spin-orbit tori exist off orbital resonance, which are not weak coboundaries (see Theorem 4.19).

Spin tunes of first kind are important tools in the simulation and analysis of polarized beams in storage rings since spin-orbit resonances of first kind impose serious limitations on the polarization in a storage ring. On the other hand, by Theorem 3.13, we see that, off orbital resonance and off spin-orbit resonance of first kind, the invariant spin field is unique up to a sign, i.e., only two ISF's exist in that situation. Thus in this case one can expect that the invariant spin field is an important characteristic of (ω, Ψ) and so it perhaps comes as no surprise that, off orbital resonance and off spin-orbit resonance of first kind, the invariant spin field allows to compute the maximal possible polarization in a storage ring [BEH, Hof]. This makes the invariant spin field an important tool in the statistical treatment of spin-orbit motion.

It is here the right place to make also some remarks on the relation of the concept of spin tune of first kind with other works. Let $(\omega, \Psi) \in \mathcal{WCB}(d, \omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Then, in the context of the flow formalism, T is called, in the terminology of [BEH], an "invariant frame field" of (ω, Ψ) if $R_{d,\omega}(T; \omega, \Psi) \in \mathcal{WT}(d, \omega)$ and T is called a "uniform invariant frame field" of (ω, Ψ) if $R_{d,\omega}(T; \omega, \Psi) \in \mathcal{AT}(d, \omega)$. The point to be made here is that in Yokoya's fundamental paper [Yok1], uniform invariant frame fields are used (in the context of the flow formalism) to define spin tunes so that indeed spin tunes of first kind are reincarnations of Yokoya's spin tunes. In constrast the spin tunes defined for the flow formalism in [BEH] and their counterparts in the map formalism (introduced in Section 4.4 of the present work), are the spin tunes of second kind which are based on the tool of quasiperiodic functions and are nonetheless essentially equal to the spin tunes of first kind. In fact, by Proposition 4.9a, the spin tunes of first and second kind are identical for almost coboundaries. In this work the main purpose of the spin tunes of second kind is to

Skipping notes on old page 23 (Aug 19, 2009)

enhance our knowledge of the spin tunes of first kind. Note also that [Yok1] builds on earlier work by Derbenev and Kondratenko [DK72, DK73] and that [BEH] can be roughly characterized as refining [Yok1] by employing quasiperiodic functions. In turn, the present work refines [BEH] by employing group actions allowing thus to systematically build up a transformation theory of spin-orbit tori.

3.7 Transformations between weakly trivial spin-orbit tori

Clearly each $\mathcal{SOT}(d, \omega)$ is the disjoint union of the $R_{d,\omega}$-orbits. Thus of obvious interest is the issue, how this foliation looks like, e.g., how it depends on d and ω. Since (recall Section 3.6) we are mainly interested in almost coboundaries (or, slightly more generally, weak coboundaries), we will only study the subset of $\mathcal{SOT}(d, \omega)$ which consists of the $R_{d,\omega}$-orbits of weak coboundaries. Thus we have to deal with the following question: when do two weakly trivial spin-orbit tori in $\mathcal{SOT}(d, \omega)$ belong to the same $R_{d,\omega}$-orbit? Perhaps surprisingly, this question can be pursued rather easily. As a matter of fact we only treat the generic case where spin-orbit tori are off orbital resonance (the case on orbital resonance can be tackled by the same techniques). Therefore in this section we state and prove Theorem 3.14 which gives sufficient and necessary conditions for two weakly trivial spin-orbit tori to be on the same $R_{d,\omega}$-orbit. We also point out (see Remark 2) how these conditions are related to small-divisor problems and Diophantine sets of orbital tunes. Corollary 3.15 then shows how things further simplify if one of the spin-orbit tori is almost trivial. In Sections 4.4, 4.5 we will, by using the machinery of quasiperiodic functions, obtain results related with, and going beyond, Theorem 3.14 and Corollary 3.15. In particular in Section 4.5 we will see the practical importance of the material from the present section.

Point out that on s-o res. there are no spin tunes

Defining

$$\mathcal{J}' := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{3.20}$$

and using

$$\mathcal{J}'\mathcal{J}\mathcal{J}' = -\mathcal{J}\ , \tag{3.21}$$

we obtain:

Theorem 3.14 *Let $(1,\omega)$ be nonresonant and $(\omega,\Psi_i) \in \mathcal{WT}(d,\omega)$ where $i=1,2$. Thus, by Proposition 3.5a, we have, for $\phi \in \mathbb{R}^d$, $i=1,2$,*

$$\Psi_i(1;\phi) = \exp(\mathcal{J}[M_i^T\phi + 2\pi f_i(\phi)])\ , \tag{3.22}$$

where $M_i := Ind_2(\Psi_i(1;\cdot))$, $f_i := PHF(\Psi_i(1;\cdot))$. Then, abbreviating the zeroth Fourier coefficient by $f_{i,0} := (1/2\pi)^d \int_0^{2\pi}\cdots\int_0^{2\pi} f_i(\phi)d\phi_1\cdots d\phi_d$ and defining $\tilde f_i := f_i - f_{i,0} \in \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R})$, the following hold:

a) If $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ such that $R_{d,\omega}(T;\omega,\Psi_1) = (\omega,\Psi_2)$ then, after abbreviating $N := Ind_2(T)$, $g := PHF(T)$, we get

$$M_1 = M_2\ , \tag{3.23}$$

$$(f_{1,0} - f_{2,0} - N^T\omega) \in \mathbb{Z}\ , \tag{3.24}$$

and, for all $\phi \in \mathbb{R}^d$,

$$g(\phi + 2\pi\omega) - g(\phi) = \tilde f_1(\phi) - \tilde f_2(\phi)\ . \tag{3.25}$$

If $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ such that $R_{d,\omega}(T\mathcal{J}';\omega,\Psi_1) = (\omega,\Psi_2)$ then, after abbreviating $N := Ind_2(T)$, $g := PHF(T)$, we get

$$M_1 = -M_2\ , \tag{3.26}$$

$$(f_{1,0} + f_{2,0} - N^T\omega) \in \mathbb{Z}\ , \tag{3.27}$$

and, for all $\phi \in \mathbb{R}^d$,

$$g(\phi + 2\pi\omega) - g(\phi) = \tilde f_1(\phi) + \tilde f_2(\phi)\ . \tag{3.28}$$

b) If $(\omega,\Psi_1) \sim_{d,\omega} (\omega,\Psi_2)$ then a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ exists such that either $R_{d,\omega}(T;\omega,\Psi_1) = (\omega,\Psi_2)$ or $R_{d,\omega}(T\mathcal{J}';\omega,\Psi_1) = (\omega,\Psi_2)$.

c) $(\omega,\Psi_1) \sim_{d,\omega} (\omega,\Psi_2)$ iff the following criterion holds:

Either

$M_1 = M_2$ and $g \in \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R})$, $N \in \mathbb{Z}^d$ exist such that (3.24), (3.25) hold,

or

$M_1 = -M_2$ and $g \in \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R})$, $N \in \mathbb{Z}^d$ exist such that (3.27), (3.28) hold.

In the former case $R_{d,\omega}(T;\omega,\Psi_1) = (\omega,\Psi_2)$ where

$$T(\phi) := \exp(\mathcal{J}[N^T\phi + 2\pi g(\phi)])\ , \tag{3.29}$$

and in the latter case $R_{d,\omega}(T\mathcal{J}';\omega,\Psi_1) = (\omega,\Psi_2)$ where T is given by eq. (3.29).

Proof of Theorem 3.14: See Section E.11. □

Note that the nontrivial part of the proof of Theorem 3.14 is part b).

Remarks:

(2) Perhaps the most important conclusion from Theorem 3.14 is that generically the spin-orbit tori (ω,Ψ_1), (ω,Ψ_2) do not belong to the same $R_{d,\omega}$-orbit. To make this point clear, let $(1,\omega)$ be nonresonant and let us adopt the notation of Theorem 3.14.

If $M_1^2 - M_2^2 \neq 0$, $f_{1,0} - f_{2,0} \notin Y_\omega$, and $f_{1,0} + f_{2,0} \notin Y_\omega$ then, by Theorem 3.14c, one has $(\omega, \Psi_1) \not\sim_{d,\omega} (\omega, \Psi_2)$ (recall the definition (C.1) of Y_ω). In addition, a small divisor problem enhances this effect as follows. Even if $M_1 - M_2 = 0$ and $f_{1,0} - f_{2,0} \in Y_\omega$ hold, in general one cannot solve eq. (3.25) for g since the Fourier coefficients of a provisional g are in general hampered by a small divisor problem preventing them to decay sufficiently fast to make g an element of $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ (note also that these Fourier coefficients are, except for the zeroth Fourier coefficient, uniquely determined by $\tilde{f}_1, \tilde{f}_2$). Analogously, even if $M_1 + M_2 = 0$ and $f_{1,0} + f_{2,0} \in Y_\omega$ hold, in general one cannot solve eq. (3.28) for g due to an analogous small divisor problem. Note however if one restricts ω to some appropriate Diophantine sets, then one can solve eq. (3.25),(3.28) (whence, in that case, $(\omega, \Psi_1) \sim_{d,\omega} (\omega, \Psi_2)$). For further details on Diophantine sets and related references, see [DEV].

We conclude, for nonresonant $(1, \omega)$, that the group action $R_{d,\omega}$ is not transitive (recall the definition of 'transitive' in Appendix A). This comes as a relief since $\sim_{d,\omega}$ would be pretty useless if all spin-orbit tori in $\mathcal{SOT}(d, \omega)$ would lie on the same $R_{d,\omega}$-orbit. Note also that, even without Theorem 3.14, it is obvious that the $R_{d,\omega}$-orbits of (ω, Ψ_1) and (ω, Ψ_2) contain many spin-orbit tori.

Of course, by the definition of weak coboundaries, we also conclude for nonresonant $(1, \omega)$ that, generically, weak coboundaries in $\mathcal{SOT}(d, \omega)$ do not belong to the same $R_{d,\omega}$-orbit.

(3) Let us again adopt the notation of Theorem 3.14 and let $(1, \omega)$ be nonresonant and $(\omega, \Psi_1) \sim_{d,\omega} (\omega, \Psi_2)$. Theorem 3.14b does *not* claim that *every* $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ with $R_{d,\omega}(T; \omega, \Psi_1) = (\omega, \Psi_2)$ is either in $\mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ or of the form $T = T'\mathcal{J}'$ with $T' \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$. However the proof of Theorem 3.14b implies that, if $(\omega, \Psi_1), (\omega, \Psi_2)$ are not coboundaries, then *every* $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ with $R_{d,\omega}(T; \omega, \Psi_1) = (\omega, \Psi_2)$ is of this simple form,

i.e., either $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ or $T = T'\mathcal{J}'$ with $T' \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$. □

Note also that Theorem 3.14c confirms Proposition 3.5d.

The following corollary reconsiders the situation of Theorem 3.14 in the special case when the spin-orbit torus (ω, Ψ_2) is almost trivial.

Corollary 3.15 *Let $(1, \omega)$ be nonresonant and $(\omega, \Psi_1) \in \mathcal{WT}(d, \omega)$, $(\omega, \Psi_2) \in \mathcal{AT}(d, \omega)$. Thus, by Proposition 3.5, we have, for $\phi \in \mathbb{R}^d$,*

$$\Psi_1(1; \phi) = \exp(\mathcal{J}[M_1^T \phi + 2\pi f_1(\phi)]) , \tag{3.30}$$

$$\Psi_2(1; \phi) = \exp(\mathcal{J} 2\pi\nu) , \tag{3.31}$$

where $M_1 := Ind_2(\Psi_1(1; \cdot))$, $f_1 := PHF(\Psi_1(1; \cdot))$, $\nu := PH(\Psi_2(1; 0)) \in [0, 1)$. Then, abbreviating the zeroth Fourier coefficient of f_1 by $f_{1,0} := (1/2\pi)^d \int_0^{2\pi} \cdots \int_0^{2\pi} f_1(\phi) d\phi_1 \cdots d\phi_d$ and defining $\tilde{f}_1 := f_1 - f_{1,0} \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$, the following hold:

a) If $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ such that $R_{d,\omega}(T; \omega, \Psi_1) = (\omega, \Psi_2)$ then, after abbreviating $N := Ind_2(T)$, $g := PHF(T)$, we get

$$M_1 = 0 , \tag{3.32}$$

$$(f_{1,0} - \nu - N^T \omega) \in \mathbb{Z} , \tag{3.33}$$

and, for all $\phi \in \mathbb{R}^d$,

$$g(\phi + 2\pi\omega) - g(\phi) = \tilde{f}_1(\phi) . \tag{3.34}$$

If $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ such that $R_{d,\omega}(T\mathcal{J}'; \omega, \Psi_1) = (\omega, \Psi_2)$ then we have eq. (3.32) and, after abbreviating $N := Ind_2(T)$, $g := PHF(T)$, we get

$$(f_{1,0} + \nu - N^T \omega) \in \mathbb{Z} , \tag{3.35}$$

Corollary 3.15 etc.
Tie back to Cor 3.4
Tie forward to Sect 4.5, Remark 6
See e-mail of 8/10/09: Why is the stuff in that e-mail not explicit here?

and, for all $\phi \in \mathbb{R}^d$, we get eq. (3.34).

b) $(\omega, \Psi_1) \sim_{d,\omega} (\omega, \Psi_2)$ *iff the following criterion holds:*

Either

$M_1 = 0$ *and* $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$, $N \in \mathbb{Z}^d$ *exist such that (3.33), (3.34) hold,*

or

$M_1 = 0$ *and* $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$, $N \in \mathbb{Z}^d$ *exist such that (3.34), (3.35) hold.*

In the former case $R_{d,\omega}(T;\omega,\Psi_1) = (\omega,\Psi_2)$ *where* T *is given by eq. (3.29) and in the latter case* $R_{d,\omega}(T\mathcal{J}';\omega,\Psi_1) = (\omega,\Psi_2)$ *where* T *is given by eq. (3.29).*

Proof of Corollary 3.15: See Section E.12. □

Chapter 4

Quasiperiodic functions as tools for studying spin-orbit tori

Quasiperiodic functions on $\mathbb{Z}$ come up naturally for spin-orbit tori since, as already pointed out at the beginning of Section 3.1, the expression $\Psi(1;\phi_0+2\pi n\omega)$, occurring in the spin equation of motion (2.8), is an ω-quasiperiodic function of n (note that quasiperiodic functions are defined in Section C.1). In Sections 4.1-4.4 we develop the basic machinery of quasiperiodic functions needed for spin-orbit tori. While some of the results of Sections 4.1-4.4 are interesting per se (notably Theorems 4.1,4.3,4.5), their main purpose is to improve, in Sections 4.5-4.7, on the themes which we started in Chapter 3. Thus the tranformation theory of spin-orbit tori, developed in Chapter 3, stays in the foreground also in the present section. In particular we stick with our credo mentioned in Section 3.6, that the emphasis is on weak coboundaries.

(note that) → . Note that

4.1 Relations between polarization fields and spin trajectories

The following theorem is about the characteristic curves of polarization fields.

Theorem 4.1 *a) Let $(\omega, \Psi) \in SOT(d,\omega)$. Let $\mathcal{S}_G$ be a polarization field for this spin-orbit torus and let $\phi_0 \in \mathbb{R}^d$. Then the "characteristic curve" $S : \mathbb{Z} \to \mathbb{R}^3$, defined by $S(n) := \mathcal{S}_G(n, \phi_0 + 2\pi n\omega)$, is a spin trajectory over ϕ_0 for (ω, Ψ). If the polarization field $\mathcal{S}_G$ is invariant, then $S(n) = G(\phi_0 + 2\pi n\omega)$ and the spin trajectory S is ω-quasiperiodic.*

b) Let $(\omega, \Psi) \in SOT(d,\omega)$ and let $(1,\omega)$ be nonresonant (for the definition of 'nonresonant', see Section C.1). Let (ω, Ψ) have, for some $\phi_0 \in \mathbb{R}^d$, an ω–quasiperiodic spin trajectory S over ϕ_0. Then (ω, Ψ) has a unique invariant polarization field $\mathcal{S}_G$ such that, for all integers n,

$$S(n) = G(\phi_0 + 2\pi n\omega) \, . \tag{4.1}$$

If in addition S is normalized to 1, i.e., $|S(n)| = 1$ then $\mathcal{S}_G$ is an ISF of (ω, Ψ).

Proof of Theorem 4.1: See Section E.13. □

Note that by Theorem 4.1, and off orbital resonance, a nonzero ω–quasiperiodic spin trajectory over ϕ_0 exists for *every* ϕ_0, if a nonzero ω-quasiperiodic spin trajectory exists over *some* ϕ_0.

Since for every spin trajectory S we have that $|S|$ is constant, it follows from Theorem 4.1b that if, off orbital resonance, at least one nonzero ω-quasiperiodic spin trajectory exists, then (ω, Ψ) has an ISF.

In spite of Theorem 4.1b, every spin trajectory S over a ϕ_0 is the characteristic curve of infinitely many polarization fields. In fact, every polarization field $\mathcal{S}_G$ which satisfies $G(\phi_0) = S(0)$ also satisfies, for every integer n, $S(n) = \mathcal{S}_G(n, \phi_0 + 2\pi n\omega) = \Psi(n;\phi_0)G(\phi_0)$. However it follows from Theorem 4.1b that, in the special case when $(1,\omega)$ is nonresonant and S is ω–quasiperiodic, there is among those infinitely many polarization fields $\mathcal{S}_G$, which satisfy $S(n) = \mathcal{S}_G(n, \phi_0 + 2\pi n\omega)$, only one that is invariant.

Recalling Section 3.6, we do not pursue to solve the ISF-conjecture. Thus by Theorem 4.1b we leave open the question if nonzero ω–quasiperiodic spin trajectories exist off orbital resonance.

However, as mentioned in Section 3.6, 'most' relevant spin-orbit tori are almost coboundaries whence, by Theorem 3.9 and Theorem 4.1a, have nonzero ω–quasiperiodic spin trajectories.

Since for every spin trajectory S we have that $|S|$ is constant, it follows from Theorem 4.1b that if, off orbital resonance, at least one nonzero ω–quasiperiodic spin trajectory exists, then (ω, Ψ) has an ISF.

Moreover, it follows from the proof of Theorem 4.1b that the invariant polarization field $\mathcal{S}_G$ is uniquely determined by $S(0)$. One takes advantage of this fact if one computes the ISF by the technique of stroboscopic averaging (for remarks on stroboscopic averaging, see Section 4.5).

4.2 Simple precession frames

With the group action $R_{d\omega}$ introduced in Chapter 3, we kind of automatically arrive, in the present section, at the concept of the simple precession frame. We recall from Definition 3.2 that if $(\omega, \Psi), (\omega, \Psi') \in SOT(d,\omega)$ and $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ such that

$R_{d,\omega}(T;\omega,\Psi) = (\omega,\Psi')$ then eq. (3.4) holds. Thus picking a $\phi_0 \in \mathbb{R}^d$, the function $t : \mathbb{Z} \to SO(3)$, defined by $t(n) := T(\phi_0 + 2\pi n\omega)$, satisfies

$$t^T(n+1)\Psi(1;\phi_0 + 2\pi n\omega)t(n) = \Psi'(1;\phi_0 + 2\pi n\omega)\ . \qquad (4.2)$$

Let in addition $(\omega,\Psi') \in \mathcal{WT}(d,\omega)$. Then by Lemma 3.8b, eq. (4.2) implies

$$t(n+1)e^3 = \Psi(1;\phi_0 + 2\pi n\omega)t(n)e^3\ . \qquad (4.3)$$

Comparing eq. (2.8),(4.3), we find that the third column of t is a spin trajectory of (ω,Ψ) over ϕ_0. This leads us to the following definition.

Definition 4.2 *(Simple precession frame)*

Let $(\omega,\Psi) \in \mathcal{SOT}(d,\omega)$ and let $\phi_0 \in \mathbb{R}^d$. A function $t : \mathbb{Z} \to SO(3)$ is called a "simple precession frame (SPF) of (ω,Ψ) over ϕ_0" if its third column is a spin trajectory over ϕ_0, i.e., if eq. (4.3) holds for all integers n. □

By the remarks before Definition 4.2 it is clear that if $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ and $R_{d,\omega}(T;\omega,\Psi) \in \mathcal{WT}(d,\omega)$ then $T(\phi_0 + 2\pi n\omega)$, as a function of n, is an SPF over ϕ_0. Thus the "characteristic curves" of T are SPF's (for more details on this, see Theorem 4.3 below).

If t is an SPF over ϕ_0 then, by eq. (4.3), $e^3 = t^T(n+1)\Psi(1;\phi_0 + 2\pi n\omega)t(n)e^3$, hence, by Lemma 3.8a, a function $\lambda : \mathbb{Z} \to [0,1)$ exists such that for all n

$$t^T(n+1)\Psi(1;\phi_0 + 2\pi n\omega)t(n) = \exp(2\pi\lambda(n)\mathcal{J})\ . \qquad (4.4)$$

Clearly λ is unique. We call λ the "differential phase function" of t. We see that t "transforms" $\Psi(1;\phi_0 + 2\pi n\omega)$ via eq. (4.4) into the matrix $\exp(2\pi\lambda(n)\mathcal{J})$ which has a simple block diagonal form and this is the origin of the term "simple". Defining

the function $\mu : \mathbb{Z} \to [0,1)$ by

$$\mu(n) := \begin{cases} 0 & \text{if } n = 0 \\ \lfloor \lambda(0) + \ldots + \lambda(n-1) \rfloor & \text{if } n > 0 \\ \lfloor -\lambda(-1) - \ldots - \lambda(n) \rfloor & \text{if } n < 0 \end{cases}\ , \qquad (4.5)$$

we obtain, by eq. (2.4),(4.4), that, for $n \in \mathbb{Z}$,

$$\Psi(n;\phi_0) = t(n)\exp(2\pi\mu(n)\mathcal{J})t^T(0)\ . \qquad (4.6)$$

Note that μ is uniquely determined by Ψ, ϕ_0, t via eq. (4.6) and satisfies $\lfloor \mu(n+1) - \mu(n) \rfloor = \lambda(n)$ so that we call μ the "integral phase function" of t. Clearly a function $t : \mathbb{Z} \to SO(3)$ is an SPF over ϕ_0 iff a function $\mu : \mathbb{Z} \to \mathbb{R}$ exists such that eq. (4.6) holds for all integers n.

Remarks:

(1) Let $(\omega,\Psi) \in \mathcal{SOT}(d,\omega)$ and let $\phi_0 \in \mathbb{R}^d$. If f is an arbitrary function $f : \mathbb{Z} \to \mathbb{R}$ and if R is a constant $SO(3)$-matrix then, by using eq. (2.5),(4.4), the function t, defined by $t(n) := \Psi(n;\phi_0)R\exp(-\mathcal{J}2\pi f(n))$, is an SPF over ϕ_0 with the differential phase function $\lambda(n) = \lfloor f(n+1) - f(n) \rfloor$. We see by this construction that, for every ϕ_0, a large abundance of SPF's, over ϕ_0, exists.

(2) We here discuss a sometimes useful property of SPF's. Let $(\omega,\Psi) \in \mathcal{SOT}(d,\omega)$ and let t be an SPF of (ω,Ψ) over some ϕ_0 with differential phase function λ. Let j be an integer and let the function $t' : \mathbb{Z} \to SO(3)$ be defined by $t'(n) := t(n+j)$. It follows from eq. (4.3) that for all integers n

$$t'(n+1)e^3 = t(n+1+j)e^3 = \Psi(1;\phi_0 + 2\pi(n+j)\omega)t(n+j)e^3$$
$$= \Psi(1;\phi_0 + 2\pi(n+j)\omega)t'(n)e^3\ ,$$

whence, by eq. (2.8), the third column of t' is a spin trajectory over $\phi_0 + 2\pi j\omega$. Thus t' is an SPF over $\phi_0 + 2\pi j\omega$. We also obtain from eq. (4.4) that for all n

$t'^T(n+1)\Psi(1;\phi_0+2\pi(n+j)\omega)t'(n) = t^T(n+1+j)\Psi(1;\phi_0+2\pi(n+j)\omega)t(n+j) = \exp(2\pi\lambda(n+j)\mathcal{J})$, hence the differential phase function λ' of t' is given by $\lambda'(n) := \lambda(n+j)$. If t is ω–quasiperiodic and $\tilde{t}$ is an ω-generator of t then $\tilde{t}(\cdot + 2\pi j\omega)$ is an ω-generator of t' whence t' is ω–quasiperiodic. □

Since an ω–quasiperiodic SPF t is $SO(3)$-valued, it follows from Definition C.1 that t has an ω-generator $\tilde{t}$ which is $\mathbb{R}^{3\times3}$-valued, albeit in general not $SO(3)$-valued. Nevertheless, the situation simplifies when $(1,\omega)$ is nonresonant, as Part b) of the following theorem shows.

Theorem 4.3 *a) Let $(\omega,\Psi) \in \mathcal{WCB}(d,\omega)$ and $(\omega,\Psi') := R_{d,\omega}(T;\omega,\Psi) \in \mathcal{WT}(d,\omega)$ with $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Then, for an arbitrary $\phi_0 \in \mathbb{R}^d$ the function $t : \mathbb{Z} \to SO(3)$, defined by $t(n) := T(\phi_0 + 2\pi n\omega)$, is an ω–quasiperiodic SPF of (ω,Ψ) over ϕ_0. Furthermore the differential phase function λ of t satisfies, for $n \in \mathbb{Z}$,*

$$\lambda(n) = \lfloor \frac{N_1^T\phi_0}{2\pi} + N_n^T\omega + f(1,\phi_0+2\pi n\omega)\rfloor$$
$$= \lfloor \frac{N_1^T\phi_0}{2\pi} + nN_1^T\omega + f(1,\phi_0+2\pi n\omega)\rfloor \,, \tag{4.7}$$

and the integral phase function μ of t satisfies, for $n \in \mathbb{Z}$,

$$\mu(n) = \lfloor \frac{N_n^T\phi_0}{2\pi} + f(n,\phi_0)\rfloor = \lfloor \frac{nN_1^T\phi_0}{2\pi} + f(n,\phi_0)\rfloor \,, \tag{4.8}$$

where $N_n := Ind_2(\Psi'(n;\cdot)), f(n,\cdot) := PHF(\Psi'(n;\cdot))$.

b) Let $(\omega,\Psi) \in \mathcal{SOT}(d,\omega)$ and let $(1,\omega)$ be nonresonant. Let also (ω,Ψ) have an ω-quasiperiodic SPF t over some ϕ_0. Then a unique $T \in \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^{3\times3})$ exists such that, for all integers n, $t(n) = T(\phi_0 + 2\pi n\omega)$. Moreover $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Furthermore, $(\omega,\Psi) \in \mathcal{WCB}(d,\omega)$ and $R_{d,\omega}(T;\omega,\Psi) \in \mathcal{WT}(d,\omega)$.

Proof of Theorem 4.3: See Section E.14. □

As mentioned in Section 3.6, 'most' relevant spin-orbit tori are weak coboundaries whence, by Theorem 4.3a, have ω-quasiperiodic SPF's. However as Theorem 4.19 shows there are spin-orbit tori off orbital resonance which are not weak coboundaries whence, by Theorem 4.3b, have no ω–quasiperiodic SPF.

4.3 Uniform precession frames

In this section we introduce "uniform precession frames" which are special SPF's. As we shall see in the next section, uniform precession frames lead to the definition of the "spin tune of second kind".

Definition 4.4 *(Uniform precession frame)*

Let $(\omega,\Psi) \in \mathcal{SOT}(d,\omega)$ and let $\phi_0 \in \mathbb{R}^d$. Let also T be a simple precession frame of (ω,Ψ) over ϕ_0 and let its differential phase function be denoted by λ. Then t is called a "uniform precession frame (UPF) over ϕ_0" if $\lambda(n)$ is independent of n. The constant value, say ν, of λ is then called the "uniform precession rate (UPR)" of t. Thus by eq. (4.4)

$$t^T(n+1)\Psi(1;\phi_0+2\pi n\omega)t(n) = \exp(2\pi\nu\mathcal{J}) \,, \tag{4.9}$$

and, by eq. (4.5), the integral phase function μ of t reads as $\mu(n) = \lfloor n\nu\rfloor$ and whence by eq. (4.6)

$$\Psi(n;\phi_0) = t(n)\exp(\mathcal{J}2\pi n\nu)t^T(0) \,. \tag{4.10}$$

We denote by $\Xi_2(\omega,\Psi,\phi_0)$ the set of those UPR's which correspond to an ω–quasiperiodic UPF over ϕ_0 and we define $\Xi_2(\omega,\Psi) := \bigcup_{\phi_0\in\mathbb{R}^d}\Xi_2(\omega,\Psi,\phi_0)$. □

It follows from Definition 4.4 that a function $t : \mathbb{Z} \to SO(3)$ is a UPF over ϕ_0 iff a $\nu \in [0,1)$ exists such that either eq. (4.9) or eq. (4.10) holds for all $n \in \mathbb{Z}$.

Of course any UPR is uniquely determined by the corresponding UPF but the converse is not true, i.e., different UPF's can have the same UPR.

Remarks:

(3) Let $(\omega, \Psi) \in SOT(d,\omega)$ and let t be a UPF of (ω, Ψ) over some $\phi_0 \in \mathbb{R}^d$. Let ν denote the UPR of t and let j be an integer. From Remark 2 we know that the function $t' : \mathbb{Z} \to SO(3)$, defined by $t'(n) := t(n+j)$, is an SPF over $\phi_0 + 2\pi j\omega$ and that its differential phase function λ' is given by $\lambda'(n) := \lambda(n+j) = \nu$, where λ is the differential phase function of t. Thus λ' has the constant value ν whence t' is a UPF over $\phi_0 + 2\pi j\omega$ with UPR ν. It also follows from Remark 2 that t' is ω-quasiperiodic if t is ω–quasiperiodic. Thus, for every integer j, $\Xi_2(\omega, \Psi, \phi_0 + 2\pi j\omega) = \Xi_2(\omega, \Psi, \phi_0)$.

(4) Let $(\omega, \Psi) \in SOT(d,\omega)$ and $\phi_0 \in \mathbb{R}^d$. By Remark 1 we know that $\Psi(\cdot;\phi_0)$ is an SPF over ϕ_0 with the differential phase function $\lambda(n) = 0$. Thus $\Psi(\cdot;\phi_0)$ is an UPF over ϕ_0 with UPR 0. □

Theorem 4.5 *a) Let $(\omega, \Psi) \in SOT(d,\omega)$. If $\nu \in \Xi_2(\omega, \Psi, \phi_0)$ for some $\phi_0 \in \mathbb{R}^d$ then every spin trajectory of (ω, Ψ) over ϕ_0 is (ω, ν)–quasiperiodic.*

b) Let $(\omega, \Psi) \in ACB(d,\omega)$ and $(\omega, \Psi') := R_{d,\omega}(T;\omega,\Psi) \in AT(d,\omega)$ with $T \in C_{per}(\mathbb{R}^d, SO(3))$. Then for an arbitrary $\phi_0 \in \mathbb{R}^d$ the function $t : \mathbb{Z} \to SO(3)$, defined by $t(n) := T(\phi_0 + 2\pi n\omega)$, is an ω-quasiperiodic UPF over ϕ_0 with UPR $\nu = PH(\Psi'(1;0))$.

c) Let $(\omega, \Psi) \in SOT(d,\omega)$ and let $(1,\omega)$ be nonresonant. Let (ω, Ψ) have an ω-quasiperiodic UPF t over some $\phi_0 \in \mathbb{R}^d$ with UPR ν. Then a unique $T \in C_{per}(\mathbb{R}^d, \mathbb{R}^{3\times3})$ exists such that, for all integers n, $t(n) = T(\phi_0 + 2\pi n\omega)$. Moreover $T \in C_{per}(\mathbb{R}^d, SO(3))$. Furthermore, $(\omega, \Psi) \in ACB(d,\omega)$ and $(\omega, \Psi') := R_{d,\omega}(T;\omega,\Psi) \in AT(d,\omega)$ with $PH(\Psi'(1;0)) = \nu \in \Xi_1(\omega, \Psi)$.

Proof of Theorem 4.5: See Section E.15. □

As mentioned in Section 3.6, 'most' relevant spin-orbit tori are almost coboundaries whence, by Theorem 4.5b, have ω–quasiperiodic UPF's. However, as mentioned after Theorem 4.3, there are spin-orbit tori off orbital resonance which have no ω–quasiperiodic SPF whence have no ω–quasiperiodic UPF.

Theorem 4.5a allows to do spectral analysis of spin trajectories as follows. In fact if $\nu \in \Xi_2(\omega, \Psi, \phi_0)$ and S is a spin trajectory of (ω, Ψ) over ϕ_0 then, by Lemma C.4d and Remark 1 in Section C.3, the spectrum of each component S_i of S is a subset of $Y_{(\omega,\nu)}$ (the spectrum of a complex valued function on $\mathbb{Z}$ is defined in Section C.3).

The following theorem (Theorem 4.6) reveals the structure of the sets $\Xi_2(\omega, \Psi, \phi_0)$ (and this in turn will reveal, in the next section, the structure of the sets $\Xi_1(\omega, \Psi)$). To prepare for the following theorem let $(\omega, \Psi) \in SOT(d,\omega)$ and let $\phi_0 \in \mathbb{R}^d$.

We first recall from Definition C.1 that, for $\omega \in \mathbb{R}^d$, Y_ω is defined by $Y_\omega := \{m^T\omega + n : m \in \mathbb{Z}^d, n \in \mathbb{Z}\}$. For the following theorem we need the equivalence relation $\sim_\omega$ on $[0,1)$ by which elements $\nu_1, \nu_2 \in [0,1)$ are equivalent iff there exist $(\varepsilon, y) \in \{1,-1\} \times Y_\omega$ such that $\nu_2 = \varepsilon\nu_1 + y$. The equivalence class of a $\nu \in [0,1)$ is denoted by $[\nu]_\omega$. Clearly

$$[\nu]_\omega = \{(\varepsilon\nu + y) \in [0,1) : \varepsilon \in \{1,-1\}, y \in Y_\omega\}$$
$$= \{[\varepsilon\nu + y] : \varepsilon \in \{1,-1\}, y \in Y_\omega\} = \{[\varepsilon\nu + j^T\omega] : \varepsilon \in \{1,-1\}, j \in \mathbb{Z}^d\}\,. \quad (4.11)$$

To get a feel for the equivalence relation $\sim_\omega$ we now show that if ν is in $\Xi_2(\omega, \Psi, \phi_0)$ then

$$[\nu]_\omega \subset \Xi_2(\omega, \Psi, \phi_0)\,. \quad (4.12)$$

Which bits connect SPFs to WT and AT? Need a Commentary Also UPF, ACB, AT

In fact if $\nu \in \Xi_2(\omega, \Psi, \phi_0)$ then by Definition 4.4 an ω–quasiperiodic UPF t exists over ϕ_0 which has UPR ν. We pick a $y \in Y_\omega$ and define the function $t' : \mathbb{Z} \to SO(3)$ by $t'(n) := t(n)\exp(-\mathcal{J}2\pi n y)$. Clearly t' is an ω–quasiperiodic function. Furthermore for $n \in \mathbb{Z}$ we have, by eq. (4.9),

$$
\begin{aligned}
&t'^T(n+1)\Psi(1;\phi_0 + 2\pi n\omega)t'(n) \\
&= \exp(\mathcal{J}2\pi(n+1)y)t^T(n+1)\Psi(1;\phi_0 + 2\pi n\omega)t(n)\exp(-\mathcal{J}2\pi n y) \\
&= \exp(\mathcal{J}2\pi(n+1)y)\exp(2\pi\nu\mathcal{J})\exp(-\mathcal{J}2\pi n y) = \exp(\mathcal{J}2\pi(\nu+y))\,.
\end{aligned}
$$

Thus t' is an ω–quasiperiodic UPF over ϕ_0 with UPR $\lfloor \nu + y \rfloor$. We define the function $t'' : \mathbb{Z} \to SO(3)$ by $t''(n) := t(n)\exp(\mathcal{J}2\pi n y)\mathcal{J}'$, where $\mathcal{J}'$ is given by eq. (3.20). Clearly t'' is an ω–quasiperiodic function. Furthermore for $n \in \mathbb{Z}$ we have by eq. (4.9)

$$
\begin{aligned}
&t''^T(n+1)\Psi(1;\phi_0 + 2\pi n\omega)t''(n) \\
&= \mathcal{J}'\exp(-\mathcal{J}2\pi(n+1)y)t^T(n+1)\Psi(1;\phi_0 + 2\pi n\omega)t(n)\exp(\mathcal{J}2\pi n y)\mathcal{J}' \\
&= \mathcal{J}'\exp(-\mathcal{J}2\pi(n+1)y)\exp(2\pi\nu\mathcal{J})\exp(\mathcal{J}2\pi n y)\mathcal{J}' = \mathcal{J}'\exp(\mathcal{J}2\pi(\nu-y))\mathcal{J}' \\
&= \exp(\mathcal{J}'\mathcal{J}\mathcal{J}'2\pi(\nu-y)) = \exp(-\mathcal{J}2\pi(\nu-y)) = \exp(\mathcal{J}2\pi(-\nu+y))\,,
\end{aligned}
$$

where in the fifth equality we used eq. (3.21). Thus t'' is an ω–quasiperiodic UPF over ϕ_0 with UPR $\lfloor -\nu + y \rfloor$. We have therefore shown that if $\nu \in \Xi_2(\omega, \Psi, \phi_0)$ and $\varepsilon \in \{1, -1\}, y \in Y_\omega$ then $\lfloor \varepsilon\nu + y \rfloor \in \Xi_2(\omega, \Psi, \phi_0)$ so that, by eq. (4.11), the inclusion (4.12) holds, as was to be proven. While obtaining (4.12) was elementary, the following theorem strengthens this inclusion to an equality. Since the proof of Theorem 4.6 involves rather sophisticated properties of quasiperiodic functions, this indicates that eq. (4.13) is a much deeper property than (4.12).

Theorem 4.6 *(Structure of $\Xi_2(\omega, \Psi, \phi_0)$) Let $(\omega, \Psi) \in SOT(d, \omega)$ and let $\phi_0 \in \mathbb{R}^d$. If $\nu \in \Xi_2(\omega, \Psi, \phi_0)$ then*

$$\Xi_2(\omega, \Psi, \phi_0) = \lfloor \nu \rfloor_\omega\,. \tag{4.13}$$

Proof of Theorem 4.6: See Section E.16. □

4.4 Introducing spin tune and spin-orbit resonance of second kind

In this work the main purpose of UPF's and UPR's is to enhance our knowledge of the spin tunes and spin-orbit resonances of first kind. The following theorem gives a first glance at the relation between spin tunes of first kind and UPR's, in particular between $\Xi_1(\omega, \Psi)$ and $\Xi_2(\omega, \Psi, \phi_0)$.

Theorem 4.7 *a) Let (ω, Ψ) be a spin-orbit torus. If $\nu \in \Xi_1(\omega, \Psi)$ then $\lfloor \nu \rfloor_\omega \subset \Xi_1(\omega, \Psi)$. Moreover, if $y \in ([0,1) \cap Y_\omega)$ then $\lfloor y \rfloor_\omega = [0,1) \cap Y_\omega$. Furthermore either $([0,1) \cap Y_\omega) \subset \Xi_1(\omega, \Psi)$ or $\Xi_1(\omega, \Psi) \cap Y_\omega = \emptyset$.*

b) Let $(\omega, \Psi) \in SOT(d, \omega)$. Then for all $\phi_0 \in \mathbb{R}^d$

$$\Xi_1(\omega, \Psi) \subset \Xi_2(\omega, \Psi, \phi_0)\,. \tag{4.14}$$

Moreover, if $\Xi_1(\omega, \Psi)$ is nonempty, then, for all $\phi_0 \in \mathbb{R}^d$, $\Xi_1(\omega, \Psi) = \Xi_2(\omega, \Psi, \phi_0)$.

c) Let $(\omega, \Psi) \in SOT(d, \omega)$ and let $(1, \omega)$ be nonresonant. Then, for all $\phi_0 \in \mathbb{R}^d$, $\Xi_1(\omega, \Psi) = \Xi_2(\omega, \Psi, \phi_0)$.

d) Let $(\omega, \Psi), (\omega, \Psi') \in SOT(d, \omega)$ with $(\omega, \Psi) \sim_{d,\omega} (\omega, \Psi')$ and let $\phi_0 \in \mathbb{R}^d$. Then $\Xi_2(\omega, \Psi', \phi_0) = \Xi_2(\omega, \Psi, \phi_0)$.

Proof of Theorem 4.7: See Section E.17. □

In the case of most practical interest, i.e., when (ω, Ψ) is an almost coboundary, the sets $\Xi_1(\omega, \Psi)$ and $\Xi_2(\omega, \Psi, \phi_0)$ are equal by Theorem 4.7b. The following definition of spin tune of second kind transfers the spin tune definition in [BEH] from the flow formalism to the map formalism.

(Lotte?)

Definition 4.8 *(Spin tune of second kind, spin-orbit resonance of second kind) Let $(\omega, \Psi) \in SOT(d, \omega)$. Then (ω, Ψ) is said to be "well-tuned" if all $\Xi_2(\omega, \Psi, \phi_0)$ are nonempty and equal, where ϕ_0 varies over $\mathbb{R}^d$. Otherwise (ω, Ψ) is said to be "ill-tuned". Of course, if (ω, Ψ) is well tuned, then, due to Definition 4.4 all $\Xi_2(\omega, \Psi, \phi_0)$ are equal to $\Xi_2(\omega, \Psi)$, where again ϕ_0 varies over $\mathbb{R}^d$. For a well-tuned spin-orbit torus we call the elements of $\Xi_2(\omega, \Psi)$ "spin tunes of second kind".*

If the spin-orbit torus is well-tuned then it is said to be "on spin-orbit resonance of second kind" if 0 is a spin tune of second kind and it is said to be "off spin-orbit resonance of second kind" if 0 is not a spin tune of second kind. □

Proposition 4.9 *a) Let $(\omega, \Psi) \in SOT(d, \omega)$. If $(\omega, \Psi) \in ACB(d, \omega)$ then (ω, Ψ) is well-tuned and the spin tunes of first and second kind are the same. If $\nu \in \Xi_1(\omega, \Psi)$ then $\Xi_1(\omega, \Psi) = [\nu]_\omega$. If (ω, Ψ) is well-tuned and if ν is a spin tune of second kind then, for all $\phi_0 \in \mathbb{R}^d$, $\Xi_2(\omega, \Psi) = \Xi_2(\omega, \Psi, \phi_0) = [\nu]_\omega$.*

b) Let $(\omega, \Psi), (\omega, \Psi') \in SOT(d, \omega)$ and $(\omega, \Psi) \in ACB(d, \omega)$. Then either $\Xi_1(\omega, \Psi) \cap \Xi_1(\omega, \Psi') = \emptyset$ or $\Xi_1(\omega, \Psi) = \Xi_1(\omega, \Psi')$. In the former case $(\omega, \Psi) \not\sim_{d,\omega} (\omega, \Psi')$ and in the latter case $(\omega, \Psi) \sim_{d,\omega} (\omega, \Psi')$, $(\omega, \Psi') \in ACB(d, \omega)$.

c) If (ω, Ψ) is a spin-orbit torus and if $(1, \omega)$ is nonresonant then the following hold. The spin-orbit torus (ω, Ψ) is well-tuned iff $(\omega, \Psi) \in ACB(d, \omega)$. If (ω, Ψ) is well-tuned then $\Xi_1(\omega, \Psi) = \Xi_2(\omega, \Psi)$.

Explain importance of Prop 4.9: are all properties neatly bundled?

A summary of meaning?

d) For every spin-orbit torus (ω, Ψ) the following hold. If ν is a spin tune of second kind of (ω, Ψ) then each spin trajectory of (ω, Ψ) is (ω, ν)-quasiperiodic. If ν is a spin tune of first kind of (ω, Ψ) then each spin trajectory of (ω, Ψ) is (ω, ν)-quasiperiodic.

e) A $(\omega, \Psi) \in SOT(d, \omega)$ is well-tuned iff the $\Xi_2(\omega, \Psi, \phi_0)$ have a common element when ϕ_0 varies over $\mathbb{R}^d$.

f) If $(\omega, \Psi) \in SOT(d, \omega)$ then the following hold. The set $\Xi_1(\omega, \Psi)$ and the sets $\Xi_2(\omega, \Psi, \phi_0)$, where ϕ_0 varies over $\mathbb{R}^d$, have countably many elements. The spin-orbit torus is ill-tuned if $\Xi_2(\omega, \Psi)$ has uncountably many elements.

Map paper?

g) If $(\omega, \Psi), (\omega, \Psi') \in SOT(d, \omega)$ with $(\omega, \Psi) \sim_{d,\omega} (\omega, \Psi')$ then the following hold. Either both spin-orbit tori $(\omega, \Psi), (\omega, \Psi')$ are well-tuned or both of them are ill-tuned. Moreover if the spin-orbit tori $(\omega, \Psi), (\omega, \Psi')$ are well-tuned then they have the same spin tunes of second kind.

Proof of Proposition 4.9: See Section E.18. □

Remark:

Assumes that this choice is possible. Circles

(5) An important conclusion from Proposition 4.9a is that generically two almost coboundaries $(\omega, \Psi), (\omega, \Psi') \in ACB(d, \omega)$ do not (need) belong to the same $R_{d,\omega}$-orbit, as follows. In fact, picking $\nu \in \Xi_1(\omega, \Psi), \nu' \in \Xi_1(\omega, \Psi')$ such that $[\nu]_\omega \neq [\nu']_\omega$, we have, by Proposition 4.9a, that $\Xi_1(\omega, \Psi') = [\nu']_\omega \neq [\nu]_\omega = \Xi_1(\omega, \Psi)$ whence, by Proposition 3.12a, $(\omega, \Psi) \not\sim_{d,\omega} (\omega, \Psi')$.

We now address the topic of spin-orbit resonances of first and second kind.

* But ψ, ψ' on same orbit & share spin tunes — see Propn 3.12a

Obvious that ψ, ψ' need not be on the same orbit? Translate?

Proposition 4.10 *a) If a spin-orbit torus is on spin-orbit resonance of first kind then it is on spin-orbit resonance of second kind. If a spin-orbit torus is off spin-orbit resonance of first kind then it is off spin-orbit resonance of second kind.*

b) Let (ω, Ψ) be a spin-orbit torus. Then (ω, Ψ) is on spin-orbit resonance of second kind iff all of its spin trajectories are ω-quasiperiodic.

c) A $(\omega, \Psi) \in \mathcal{SOT}(d, \omega)$ is on spin-orbit resonance of first kind iff $\Xi_1(\omega, \Psi) = [0, 1) \cap Y_\omega$. Furthermore a $(\omega, \Psi) \in \mathcal{SOT}(d, \omega)$ is on spin-orbit resonance of first kind iff (ω, Ψ) has a spin tune ν of first kind such that $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ exist with

$$\nu = m^T \omega + n \, . \tag{4.15}$$

d) A $(\omega, \Psi) \in \mathcal{SOT}(d, \omega)$ is on spin-orbit resonance of second kind iff, for all $\phi_0 \in \mathbb{R}^d$, $\Xi_2(\omega, \Psi, \phi_0) = [0, 1) \cap Y_\omega$. Furthermore a $(\omega, \Psi) \in \mathcal{SOT}(d, \omega)$ is on spin-orbit resonance of second kind iff it has a spin tune ν of second kind such that $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ which satisfy eq. (4.15).

e) If $(\omega, \Psi), (\omega, \Psi') \in \mathcal{SOT}(d, \omega)$ are on spin-orbit resonance of first kind, then $(\omega, \Psi) \sim_{d,\omega} (\omega, \Psi')$.

f) If $(\omega, \Psi), (\omega, \Psi') \in \mathcal{SOT}(d, \omega)$ with $(\omega, \Psi) \sim_{d,\omega} (\omega, \Psi')$ then the following hold. Either both of $(\omega, \Psi), (\omega, \Psi')$ are on spin-orbit resonance of second kind or none of them. Furthermore either both of them are off spin-orbit resonance of second kind or none of them.

g) (Yokoya's uniqueness theorem revisited) Let $(\omega, \Psi) \in \mathcal{SOT}(d, \omega)$ and let $(1, \omega)$ be

nonresonant. Let (ω, Ψ) have an ISF S_G and an ISF which is different from S_G and $-S_G$. Then (ω, Ψ) is on spin-orbit resonance of second kind.

Proof of Proposition 4.10: See Section E.19. □

4.5 The SPRINT Theorem and a corresponding spin tune algorithm

We now resume the theme of Section 3.7 and pose the question under which circumstances a weakly trivial spin-orbit torus is an almost coboundary. As a matter of fact, as in Section 3.7, we confine to the case off orbital resonance for which Theorem 4.11 answers the question. Based on Theorem 4.11 we then prove the "SPRINT Theorem" (Corollary 4.12) and demonstrate its practical importance by outlining, after Corollary 4.12, an algorithm, used in the code SPRINT, to compute spin tunes of first and second kind.

Theorem 4.11 *Let $(1, \omega)$ be nonresonant and $(\omega, \Psi_1) \in \mathcal{WT}(d, \omega)$. Thus, by Proposition 3.5a, eq. (3.30) holds for $\phi \in \mathbb{R}^d$, where $M_1 := Ind_2(\Psi_1(1; \cdot))$, $f_1 := PHF(\Psi_1(1; \cdot))$. Then, abbreviating the zeroth Fourier coefficient of f_1 by $f_{1,0} := (1/2\pi)^d \int_0^{2\pi} \cdots \int_0^{2\pi} f_1(\phi) d\phi_1 \cdots d\phi_d$ and defining $\tilde{f}_1 := f_1 - f_{1,0} \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$, the following hold:*

a) $(\omega, \Psi_1) \in \mathcal{ACB}(d, \omega)$ iff the following conditions are satisfied: $M_1 = 0$ and a $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ exists such that eq. (3.34) is true for all $\phi \in \mathbb{R}^d$.

b) Let $M_1 = 0$ and let $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ exist such that eq. (3.34) holds for all

Date: Thu, 8 Oct 2009 11:22:25 -0600 (MDT)
From: Klaus Heinemann <heineman@math.unm.edu>
To: mpybar@mail.desy.de
Subject: two answers

Hi Desmond,
since my previous email was a little imprecise,
here a slight update on my answers to your two questions:

(i) I here sharpen the claim of the previous email. about the claim in Sect 3.6 which points to Sect 3.7.

Claim: Infinitely many orbits of R(d,w) exist which have no spin tune of first kind.

Proof: Let (1,w) be nonresonant. Choose Psi_1 in Corollary 3.15b such that M_1 is
nonzero. Then Corollary 3.15b implies that (Psi_1,w) is not on the same R(d,w)-orbit as (Psi_2,w). However since (Psi_2,w) is an arbitrary almost trivial SOT we thus obtained that (Psi_1,w) cannot lie on the R(d,w)-orbit of any almost trivial SOT. Thus, by Definition 3.6, (Psi_1,w) is not an almost coboundary. It follows by a remark after Definition 3.11 that (Psi_1,w) has no spin tunes of first kind!
Of course all SOT's which belong to the R(d,w)-orbit of (Psi_1,w) have no spin tunes of first kind (this follows for example from Proposition B3.12a) which implies that there are infinitely many SOT's which have no spin tune of first kind.
To ~~even~~ show that there are even infinitely many R(d,w)-orbits of SOT's which have no spin tune of first kind, we now modify (Psi_1,w) by replacing M_1 by M_1' such that M_1' is neither zero nor equal to
M_1 or -M_1. Clearly the resulting SOT, (Psi_1',w), has no spin tunes of first kind. Moreover, by Theorem 3.14b, (Psi_1,w) and (Psi_1',w) do not belong to the same R(d,w)-orbit.
END OF PROOF

(ii) Your question concerning page 28 can be answered more precisely as follows.

Let S(n) be a spin trajectory over phi0\in R^d and let G be in C_per(R^d,R^3) such that S(n) is a characteristic curve of the polarization field S_G. Note that G(phi0)=S(0).
? Now choose a G'\in C_per(R^d,R^3) such that G'(phi0)=G(phi0)
(obviously there are infinitely many of those G'!).
Applying Definition 2.2 we observe that S(n) is a characteristic curve of the polarization field S_G'. Thus there indeed exists infinitely many polarization fields whose characteristic curve is S(n).
Cheers, Klaus

Date: Wed, 21 Oct 2009 22:16:04 +0200 (CEST)
From: Desmond Barber <mpybar@mail.desy.de>
To: Klaus Heinemann <heineman@math.unm.edu>
Cc: Jim Ellison <ellison@math.unm.edu>
Subject: The synopsis

From Desmond Barber at DESY, Germany.
http://www.desy.de/~mpybar. REPLY TO: mpybar@mail.desy.de

Hi Klaus,

so here is the extended synopsis. I've also tuned my suggestions for the Introduction. I wanted to get this off last week but we had a major problem with the Unix/Linux data bases so that everyone was thrown out of gear and couldn't concentrate on real work. The last many weeks have been crazy and that has led to my lack of focus.

Anyway, for the Synopsis I have added my points in capitals. I have also slotted in questions in a couple of places.

I might suggest a couple more points in the next days.

You will also want to rewrite my comments when you can do a better job and you will want to correct my comments where they are wrong. In any case feel free to ignore part or all of my stuff.

Now some additional points (not exhaustive, a few more are probably coming). I'll check some again myself:

1)
For calculating the ADST: there is also SODOM (KY), MILES and LIMES (Mane)
Look with Google for refs. I can help too.

2)
Somehow point out more clearly in the main text (did I miss it?) that spin tunes only exist if the Diophantuine conditions are OK. Mention BEH and Q-periodicity.

3)
Do you allow the electric and magnetic fields to be discontinuities in theta (as in the map paper)?

4)
Try to explain in words the main difference (physics) between spin tunes of first and second kind.

5)
Explain (if it's possible) the physics in Remark 2 on page 25.
Of course, this is tied up with the use of arbitrarily many SOT's

6)
Section 6. Expand a bit.

Explain heuristically the extent to which your work allows (if it does) pathological cases off orbital resonance to by identified. Perhaps mention again the need for Diophantine conditions for allowing spin tunes.

Perhaps mention further topics needing your kind of rigorous approach:

Extending to orbital resonance mentioning discontinuities in the ''ISF'' [BV]

Limit of the rational approx to get spin tunes and ISF.

Gelfand: take a look at
http://www.timesonline.co.uk/tol/comment/obituaries/article6879978.ece

Des.

```
[ Part 2, ""  Application/X-TEX  28KB. ]
[ Unable to print this part. ]
```

Date: Wed, 7 Apr 2010 19:58:04 +0200 (CEST)
From: Desmond Barber <mpybar@mail.desy.de>
To: Jim Ellison <ellison@math.unm.edu>
Subject: Remark 6

From Desmond Barber at DESY, Germany.
http://www.desy.de/~mpybar. REPLY TO: mpybar@mail.desy.de

Hi Jim.

So here are steeping stones to Remark 6 in Section 8.5.

Section 3.4 mentions Section 7.7 with regard to the number of s-o tori with no spin tune of first kind and it refers to Remark 6.

Section 7.7 then deals with spin tunes. It also has Corollary 7.15 at the end but as far as I can see there is no clear statement about what that's for.

Anyway, although the sets in Remark 6 are identical, they are empty.

So, for my taste this is not a very strong example of a converse of Corollary 7.12a. On the other hand it might still be worth stating that converse of Corollary 7.12a is not guarenteed.

Des.

$\phi \in \mathbb{R}^d$ (thus, by Theorem 4.11a, $(\omega, \Psi_1) \in ACB(d,\omega)$). Then picking a $N \in \mathbb{Z}^d$ and defining $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO_3(2))$ by eq. (3.29), the following hold. The spin-orbit torus $(\omega, \Psi_2) := R_{d,\omega}(T;\omega,\Psi_1)$ is almost trivial and, for $\phi \in \mathbb{R}^d$, we have

$$\Psi_2(1;\phi) = \exp(\mathcal{J}2\pi\nu_2)\,, \tag{4.16}$$

where $\nu_2 := \lfloor f_{1,0} - N^T\omega \rfloor$. Moreover $\nu_2 \in \Xi_1(\omega,\Psi_1)$. The spin-orbit torus $(\omega,\Psi_3) := R_{d,\omega}(T\mathcal{J}';\omega,\Psi_1)$ is almost trivial and, for $\phi \in \mathbb{R}^d$, we have

$$\Psi_3(1;\phi) = \exp(\mathcal{J}2\pi\nu_3)\,, \tag{4.17}$$

where $\nu_3 := \lfloor -f_{1,0} + N^T\omega \rfloor$. Moreover $\nu_3 \in \Xi_1(\omega,\Psi_1)$.

c) Let $(\omega,\Psi_1) \in ACB(d,\omega)$. Then (ω,Ψ_1) is well-tuned and

$$[\lfloor f_{1,0} \rfloor]_\omega = \Xi_1(\omega,\Psi_1) = \Xi_2(\omega,\Psi_1)\,. \tag{4.18}$$

Proof of Theorem 4.11: See Section E.20. □

Remark:

(6) Clearly, those spin-orbit tori in Theorem 4.11a, with $M_1 \neq 0$, are not almost coboundaries. Another consequence of Theorem 4.11a is the following. Let $(1,\omega)$ be nonresonant and let $(\omega,\Psi_1),(\omega,\Psi_2) \in \mathcal{WT}(d,\omega)$ such that $M_1, M_2 \neq 0$ and $M_1^2 - M_2^2 \neq 0$ where $M_i := Ind_2(\Psi_i(1;\cdot))$ $(i = 1,2)$. Thus, by Theorem 3.14c, we observe that $(\omega,\Psi_1) \not\sim_{d,\omega} (\omega,\Psi_2)$. Moreover, by Theorem 4.11a, $(\omega,\Psi_1),(\omega,\Psi_2)$ are not almost coboundaries whence $\Xi_1(\omega,\Psi_1) = \Xi_1(\omega,\Psi_2) = \emptyset$. Therefore $(\omega,\Psi_1),(\omega,\Psi_2)$ provide an example of two spin-orbit tori in the same $SOT(d,\omega)$ and with identical Ξ_1 but which are not on the same $R_{d,\omega}$-orbit. Thus this example shows that, in general, the converse of the first claim in Proposition 3.12a is not true.

The following corollary to Theorem 4.11 we call the "SPRINT Theorem" since it presents the facts used by the code SPRINT for the numerical calculation of spin tunes (of first and second kind) via stroboscopic averaging (for details on this code, see the remarks after Corollary 4.12). Note that the notation $\Psi_1, M_1, f_1, f_{1,0}$ used in Corollary 4.12 serves to facilitate the comparison with Theorem 4.11.

Corollary 4.12 *(The SPRINT Theorem) Let $(\omega,\Psi) \in ACB(d,\omega)$ and let $(1,\omega)$ be nonresonant. Let us choose a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ such that $(\omega,\Psi_1) := R_{d,\omega}(T;\omega,\Psi) \in \mathcal{WT}(d,\omega)$. Thus, by Proposition 3.5a, eq. (3.30) holds for $\phi \in \mathbb{R}^d$, where $M_1 := Ind_2(\Psi_1(1;\cdot))$, $f_1 := PHF(\Psi_1(1;\cdot))$. Abbreviating the zeroth Fourier coefficient of f_1 by $f_{1,0} := (1/2\pi)^d \int_0^{2\pi} \cdots \int_0^{2\pi} f_1(\phi)d\phi_1 \cdots d\phi_d$, the following hold:*

a) The spin-orbit tori (ω,Ψ) and (ω,Ψ_1) are well-tuned and their spin tunes of first and second kind satisfy

$$[\lfloor f_{1,0} \rfloor]_\omega \;=\; \Xi_1(\omega,\Psi) = \Xi_2(\omega,\Psi) = \Xi_1(\omega,\Psi_1) = \Xi_2(\omega,\Psi_1)\,. \tag{4.19}$$

b) We have $M_1 = 0$ and, for $\phi \in \mathbb{R}^d, n = 1,2,...$,

$$\Psi_1(n;\phi) = \exp\Big(\mathcal{J}2\pi \sum_{j=0}^{n-1} f_1(\phi + 2\pi j\omega)\Big)\,. \tag{4.20}$$

Moreover, the zeroth Fourier coefficient of f_1 reads as

$$f_{1,0} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f_1(2\pi j\omega)\,. \tag{4.21}$$

c) The function $t : \mathbb{Z} \to SO(3)$, defined by $t(n) := T(2\pi n\omega)$, is an ω-quasiperiodic SPF of (ω,Ψ) over $0 \in \mathbb{R}^d$ and for $n = 1,2,...$ we have

$$\Psi(n;0) = t(n)\exp\Big(\mathcal{J}2\pi \sum_{j=0}^{n-1} f_1(2\pi j\omega)\Big)t^T(0)\,. \tag{4.22}$$

The function $S : \mathbb{Z} \to \mathbb{S}^2$, defined by $S(n) := \Psi(n;0)t(0)e^1$ is a spin trajectory of (ω, Ψ) over $0 \in \mathbb{R}^d$. Moreover for $n = 1, 2, ...,$

$$\exp\Big(i2\pi \sum_{j=0}^{n-1} f_1(2\pi j\omega) \Big) = (e^1 + ie^2)^T t^T(n) S(n) . \qquad (4.23)$$

where, as common, i denotes the complex root of -1 lying in the upper complex plane.

Proof of Corollary 4.12: See Section E.21. □

Corollary 4.12 is of great practical interest for the numerical calculation of spin tunes (of first and second kind) via stroboscopic averaging in the code SPRINT [epac98, spin98, Hof, Vog, spin2000b, spin2000a] (note that SPRINT also employs a second method, which is due to Yokoya [Yok2] and different from stroboscopic averaging, but which is of no relevance for the point we want to make here). Thus in the following paragraph I sketch, by using the notation of Corollary 4.12, that particular algorithm in SPRINT which computes, via stroboscopic averaging, spin tunes of first and second kind. Note that SPRINT performs this algorithm not just for a single spin-orbit torus but for a whole family of spin-orbit tori (which constitute the spin-orbit system to be dealt with in a storage ring). This important circumstance, which is explained in Remark 7 below, is essential for putting the algorithm into perspective.

Now I outline the algorithm as it is used, up to some modifications which do not matter here, by the code SPRINT. Let $(\omega, \Psi) \in SOT(d, \omega)$ be an almost coboundary and let it be off orbital resonance, i.e., let $(1, \omega)$ be nonresonant. On one hand, the algorithm computes, via the technique of stroboscopic averaging, an ISF $\mathcal{S}_G$ of (ω, Ψ). As a matter of fact, the algorithm merely computes $\mathcal{S}_G$ at the points $\phi = 0$ and $\phi = 2\pi N\omega$ for some sufficiently large positive integer N, i.e., computes the points $G(0)$ and $G(2\pi N\omega)$ in $\mathbb{S}^2$. From that, by a simple orthonormalization procedure, the algorithm computes a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ whose third column is

G. As a matter of fact, the algorithm merely computes T at the points $\phi = 0$ and $\phi = 2\pi N\omega$, i.e., computes the points $T(0) = t(0)$ and $T(2\pi N\omega) = t(N)$ in $SO(3)$. Note incidentally that, by Theorem 3.9, one has $R_{d,\omega}(T;\omega,\Psi) \in WT(d,\omega)$. So let us abbreviate $(\omega, \Psi_1) := R_{d,\omega}(T;\omega,\Psi) \in WT(d,\omega)$, i.e., we are in the situation of Corollary 4.12. On the other hand the algorithm computes in a recursive way, via spin tracking, the points $S(1), ..., S(N)$ in $\mathbb{S}^2$ where $S(n) := \Psi(n;0)t(0)e^1$. Now Corollary 4.12 enters the game since the algorithm uses the data $t(N), S(N)$ to compute a spin tune as follows. If N is sufficiently large (order of magnitude $N = 100000$), then, by Corollary 4.12b, we have

$$N f_{1,0} \approx \sum_{j=0}^{N-1} f_1(2\pi j\omega) ,$$

whence by Corollary 4.12c,

$$\exp(i2\pi N \lfloor f_{1,0} \rfloor) = \exp(i2\pi N f_{1,0}) \approx \exp\Big(i2\pi \sum_{j=0}^{N-1} f_1(2\pi j\omega) \Big)$$
$$= (e^1 + ie^2)^T t^T(N) S(N) .$$

Thus for large N we have a (unique) $\nu \in [0,1)$ such that

$$\exp(i2\pi N\nu) = (e^1 + ie^2)^T t^T(N) S(N) , \qquad (4.24)$$
$$\lfloor f_{1,0} \rfloor \approx \nu , \qquad (4.25)$$

whence ν is an approximation of $\lfloor f_{1,0} \rfloor$. Solving therefore eq. (4.24) for $\nu \in [0,1)$ the algorithm obtains an approximation of $\lfloor f_{1,0} \rfloor$. However, by Corollary 4.12a, $\lfloor f_{1,0} \rfloor$ is a spin tune of first and second kind of (ω, Ψ). Thus ν is an approximation of a spin tune of first and second kind of (ω, Ψ) which completes our outline of the algorithm.

In retrospect we see that the algorithm, being a blend of concepts and facts established in Chapters 3 and 4, computes $t(N), S(N)$ and applies (4.24). The computation of $t(N), S(N)$ is done by tracking, i.e., by solving the equations of motion (2.1),(2.2) in a recursive way. Note that stroboscopic averaging is a certain way of summing up spin tracking data.

Remark:

(7) We recall from the Introduction (see Section 1.1) that, in the situation of a storage ring, one is not only faced with a single spin-orbit torus but with a continuous family of spin-orbit tori labelled by an action-parameter J, i.e., with a spin-orbit system. Then the spin tune $\lfloor f_{1,0} \rfloor$ unfolds into a family of spin tunes paramterized by J. This function $\lfloor f_{1,0} \rfloor$ of J is called the amplitude dependent spin tune (ADST) and experience shows that it is piecewise continuous in J. The piecewise continuity in J is owed to the continuity of ω in J and to the fact that T is constructed in a way such that it depends piecewise continuously on the parameter J. The latter is achieved, thanks to the stroboscopic averaging technique, by constructing the above mentioned ISF $\mathcal{S}_G$ (whose generator G is the third column of T) such that G is a piecewise continuous function of the parameter J and by performing the orthonormalization procedure, which leads from G to T, in a piecewise continuous way.

Of course, the code SPRINT has to discretize the continuous J-values into a grid, and, once having choosen this grid sufficiently dense, SPRINT nicely exhibits the piecewise continuous dependence of $\lfloor f_{1,0} \rfloor$ on J.

4.6 The impact of Homotopy Theory on spin tunes of first kind

In this section we state and prove Theorem 4.17. Parts c) and d) of this theorem display how Homotopy Theory has an impact on the individual values of the spin tunes of first kind. In fact, in the situation of Theorems 4.17c,d, $\Xi_1(\omega, \Psi)$ partitions

into sets in a way, such each of these sets is associated with a certain subset of $[\mathbb{R}^d, SO(3)]_{2\pi}$. For more details and the practical implications of this, see the remarks after Theorem 4.17. Recall that $[\mathbb{R}^d, SO(3)]_{2\pi}$ is defined in Definition B.19.

Definition 4.15 *Let $(\omega, \Psi) \in SOT(d, \omega)$ and $s \in \{1, -1\}^d$. Then $\Xi_1^s(\omega, \Psi)$ is defined by*

$$\Xi_1^s(\omega, \Psi) := \{PH(\Psi'(1;0)) : (\omega, \Psi') = R_{d,\omega}(T; \omega, \Psi) \in AT(d, \omega), \\ T \in C_{per}(\mathbb{R}^d, SO(3)), Ind_{3,d}(T) = s\} .$$

Clearly for every $(\omega, \Psi) \in SOT(d, \omega)$ we have

$$\Xi_1(\omega, \Psi) = \bigcup_{s \in \{1,-1\}^d} \Xi_1^s(\omega, \Psi) . \tag{4.26}$$

With $\chi \in \mathbb{R}^k, s \in \{1, -1\}^k$ we define

$$Y_\chi^s := \{m^T\chi + n : m \in \mathbb{Z}^k, n \in \mathbb{Z}, s = ((-1)^{m_1}, ..., (-1)^{m_k})^T\} \subset Y_\chi ,$$
$$Y_\chi^{half} := \{\frac{m^T\chi + n}{2} : n \in \mathbb{Z}, m \in \mathbb{Z}^k, ((-1)^{m_1}, ..., (-1)^{m_k}) \neq (1, ..., 1)\} ,$$

where Y_χ is given by Definition C.1. □

Proposition 4.16 *If $(\omega, \Psi) \in WCB(d, \omega)$ and $s \in \{1, -1\}^d$ then there exists $T \in C_{per}(\mathbb{R}^d, SO(3))$ with $SO(3)$-index s such that $R_{d,\omega}(T; \omega, \Psi) \in WT(d, \omega)$. If $(\omega, \Psi) \in ACB(d, \omega)$ then, for every $t \in \{1, -1\}^d$, $\Xi_1^t(\omega, \Psi)$ is nonempty.*

Proof of Proposition 4.16: See Section E.22. □

If $\Xi_1(\omega, \Psi)$ is nonempty then, by Proposition 4.16, each $\Xi_1^s(\omega, \Psi)$ is nonempty which raises the option to see some structure in $\Xi_1^s(\omega, \Psi)$ leading us to the question if the $\Xi_1^s(\omega, \Psi)$ overlap or don't, i.e., the question if the union on the rhs of eq. (4.26) is disjoint or not. Theorem 4.17 gives us conditions under which the $\Xi_1^s(\omega, \Psi)$ indeed don't overlap. For the implications of this, see the remarks after Theorem 4.17.

Theorem 4.17 *Let $(\omega,\Psi) \in SOT(d,\omega)$ and let $(1,\omega)$ be nonresonant. Then the following hold.*

a) Let $(\omega,\Psi) \in \mathcal{ACB}(d,\omega)$ and let $T_1, T_2 \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ such that $(\omega,\Psi_i) := R_{d,\omega}(T_i;\omega,\Psi) \in \mathcal{AT}(d,\omega)$ where $i=1,2$. Abbreviating $\nu_i := PH(\Psi_i(1;0))$, where $i=1,2$, and $s := Ind_{3,d}(T_1^T T_2)$ then either $(\nu_1-\nu_2) \in Y^s_\omega$ or $(\nu_1+\nu_2) \in Y^s_\omega$.

b) Let $(\omega,\Psi) \in \mathcal{ACB}(d,\omega)$. If one picks, by using Proposition 4.16, a ν in $\Xi_1^{(1,\dots,1)}(\omega,\Psi)$ then one obtains, for every $s \in \{1,-1\}^d$,

$$\Xi_1^s(\omega,\Psi) \subset \{\varepsilon\nu + y : y \in Y^s_\omega, \varepsilon \in \{1,-1\}\}\,. \tag{4.27}$$

c) If $\Xi_1(\omega,\Psi) \cap Y^{half}_\omega = \emptyset$ and $s,t \in \{1,-1\}^d$ with $s \neq t$ then $\Xi_1^s(\omega,\Psi) \cap \Xi_1^t(\omega,\Psi) = \emptyset$.

d) Let (ω,Ψ) have an ISF $\mathcal{S}_G$ and let it also have an ISF which is different from $\mathcal{S}_G$ and $-\mathcal{S}_G$. Then $\Xi_1(\omega,\Psi) \neq \emptyset$ and, for $s \neq t$, $\Xi_1^s(\omega,\Psi) \cap \Xi_1^t(\omega,\Psi) = \emptyset$.

e) Either $\Xi_1(\omega,\Psi) \subset Y^{half}_\omega$ or $\Xi_1(\omega,\Psi) \cap Y^{half}_\omega = \emptyset$.

Remark: The burden of the proof of Theorem 4.17 is on the proof of Theorem 4.17a.

Proof of Theorem 4.17: See Section E.23. □

Since Theorems 4.17c,d give us conditions under which the $\Xi_1^s(\omega,\Psi)$ don't overlap they display at the same time how Homotopy Theory impacts the spin tunes of first kind, as follows. Let $(\omega,\Psi) \in \mathcal{ACB}(d,\omega)$ and $s^1 \neq s^2$ such that $\Xi_1^{s^1}(\omega,\Psi) \cap \Xi_1^{s^2}(\omega,\Psi) = \emptyset$. If $\nu_i \in \Xi_1^{s^i}(\omega,\Psi)$ then, by Definition 4.15, a $T_i \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists with $Ind_{3,d}(T_i) = s^i$ and such that $(\omega,\Psi_i) := R_{d,\omega}(T_i;\omega,\Psi) \in \mathcal{AT}(d,\omega)$ and $\nu_i = PH(\Psi_i(1;0))$ where $i=1,2$. Since $s^1 \neq s^2$ we have $Ind_{3,d}(T_1) \neq Ind_{3,d}(T_2)$

✱: SU(2) quaternions are 4π-periodic
Ask in ABQ.

whence, by Proposition B.18e, $T_1 \not\simeq^{2\pi}_{SO(3)} T_2$, i.e., T_1, T_2 are not 2π-homotopic w.r.t. $SO(3)$.

We now discuss some aspects of the situation, in which the $\Xi_1^s(\omega,\Psi)$ don't overlap, that are not only of theoretical but also of practical interest. Let $(\omega,\Psi) \in \mathcal{ACB}(d,\omega)$ such that the $\Xi_1^s(\omega,\Psi)$ don't overlap. Then the elements of $\Xi_1^{(1,\dots,1)}(\omega,\Psi)$ are rather exceptional as follows. We recall from Definition 4.15 that for each element ν of $\Xi_1^{(1,\dots,1)}(\omega,\Psi)$ a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists with $Ind_{3,d}(T) = (1,\dots,1)^T$ and such that $(\omega,\Psi') := R_{d,\omega}(T;\omega,\Psi) \in \mathcal{AT}(d,\omega)$ and $\nu = PH(\Psi'(1;0))$. Note that, by Definitions B.12,B.14, every lifting of T w.r.t. $(\mathbb{S}^3, p_2, SO(3))$ is a function $\tilde{T} \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^3)$, i.e., is 2π-periodic. Thus in computer codes which compute T in the quaternion formalism, i.e., which deal with $\tilde{T}$, the elements of $\Xi_1^{(1,\dots,1)}(\omega,\Psi)$ require a 2π-periodic $\tilde{T}$ whereas each element of $\Xi_1(\omega,\Psi) \setminus \Xi_1^{(1,\dots,1)^T}(\omega,\Psi)$ requires a $\tilde{T}$ which is not 2π-periodic. In other words, the spin tunes of first kind which are associated with 2π-periodic $\tilde{T}$'s, are rather exceptional. This phenomenon, which occurs in a similar way also in the spinor formalism (the latter formalism is mentioned in Section 1.3), was observed in [Hof, Section 4.1],[Yok2] and accordingly the present section is inspired by these two works.

? ✱ Interpret ✱.

4.7 Further properties of invariant spin fields

Lemma 4.18 *Let $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ be of class C^1 and let $\omega \in \mathbb{R}^d$. Then a $(\omega,\Psi) \in SOT(d,\omega)$ exists which has an ISF $\mathcal{S}_G$ generated by G.*

Proof of Lemma 4.18: See Section E.24. □

We now resume the theme of Theorem 3.10.

Already done? Physics?

Theorem 4.19 *Let ω be in $\mathbb{R}^d$ such that $(1,\omega)$ is nonresonant and $d \geq 2$. Then* *there exists a $(\omega, \Psi) \in (\mathcal{SOT}(d,\omega) \setminus \mathcal{WCB}(d,\omega))$ which has an ISF $\mathcal{S}_G$. For every such spin-orbit torus, $\mathcal{S}_G$ and $-\mathcal{S}_G$ are the only ISF's.*

Proof of Theorem 4.19: See Section E.25. □

Chapter 5

Reconsidering the $\mathbb{Z}$-actions $L_{\omega,\Psi}$ and $L^{(PF)}_{\omega,\Psi}$

In this section we reconsider the $\mathbb{Z}$-actions $L_{\omega,\Psi}$ and $L^{(PF)}_{\omega,\Psi}$ introduced in Section 2.

5.1 Carving out the topological $\mathbb{Z}$-spaces $(\mathbb{R}^{d+3}, L_{\omega,\Psi})$ as skew products of the topological $\mathbb{Z}$-spaces $(\mathbb{R}^d, L_\omega)$

Means?

Proposition 5.1 *Let (ω, Ψ) be a d-dimensional spin-orbit torus. Then the function $h : \mathbb{R}^{d+3} \to \mathbb{R}^d$, defined, for $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3$, by $h(\phi_1, ..., \phi_d, S) := (\phi_1, ..., \phi_d)^T$, is a continuous $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L_{\omega,\Psi})$ to the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$. Moreover, the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3}, L_{\omega,\Psi})$ is a skew product of the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$.*

Proof of Proposition 5.1: See Section E.26. □

With Proposition 5.1 we can now put eq. (2.5) into perspective. In fact, while in Section 2.1 we derived eq. (2.5) from eq. (2.1),(2.2) we now derive eq. (2.5) in a different way. Since, by Proposition 5.1, $(\mathbb{R}^{d+3}, L_{\omega,\Psi})$ is a skew product of $(\mathbb{R}^d, L_\omega)$ we can apply Remark 1 in Section A. According to Remark 1 in Section A we get, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$L_{\omega,\Psi}(n;\phi,S) = \begin{pmatrix} L_\omega(n;\phi) \\ L''(n;\phi,S) \end{pmatrix} , \tag{5.1}$$

where the function $L'' : \mathbb{Z} \times \mathbb{R}^{d+3} \to \mathbb{R}^3$ satisfies, for $m, n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$L''(n+m;\phi,S) = L''(n;\phi + 2\pi m\omega, L''(m;\phi,S)) , \tag{5.2}$$

where we also have used eq. (2.14). Imposing the condition that $L''(n;\phi,S)$ is linear in S we get, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$L''(n;\phi,S) = L'''(n;\phi)S , \tag{5.3}$$

where L''' is a function from $\mathbb{Z} \times \mathbb{R}^d$ into the set of real 3×3 matrices. It follows from eq. (5.2),(5.3) that, for $m, n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$,

$$L'''(n+m;\phi) = L'''(n;\phi + 2\pi m\omega)L'''(m;\phi) , \tag{5.4}$$

which is indeed eq. (2.5) expressed in terms of L'''. We conclude that eq. (2.5) follows from the facts that $(\mathbb{R}^{d+3}, L_{\omega,\Psi})$ is a skew product of $(\mathbb{R}^d, L_\omega)$ and that $L_{\omega,\Psi}(n;\phi,S)$ is linear in S.

5.2 Carving out the topological $\mathbb{Z}$-spaces $(\mathbb{R}^{d+3}, L_{\omega,\Psi})$ as extensions of the topological $\mathbb{Z}$-spaces $(\mathbb{T}^d \times \mathbb{R}^3, L^{(T)}_{\omega,\Psi})$

As mentioned in Section 2.1, the spin-orbit motion in $\mathbb{R}^{d+3}$ is closely related to an associated spin-orbit motion in $\mathbb{T}^d \times \mathbb{R}^3$ which is characterized by the $\mathbb{Z}$-action $L^{(T)}_{\omega,\Psi}$

on $\mathbb{T}^d \times \mathbb{R}^3$ that is defined in Proposition 5.2b. In fact while the emphasis in the present work is on orbital motion in $\mathbb{R}^d$, a deeper study of spin-orbit tori will stronger focus on orbital motion in $\mathbb{T}^d$ and therefore the present section give a brief glimpse into this.

The d-torus $\mathbb{T}^d$ is defined by Definition B.2. Proposition 5.2, stated below, exhibits the relation between $L^{(T)}_{\omega,\Psi}$ and $L_{\omega,\Psi}$. But before we come to that we define the map $p_{5,d} : \mathbb{R}^{d+3} \to \mathbb{T}^d \times \mathbb{R}^3$, for $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3$, by

$$p_{5,d}(\phi,S) := \begin{pmatrix} p_{4,d}(\phi) \\ S \end{pmatrix} = \begin{pmatrix} \exp(i\phi) \\ S \end{pmatrix} , \tag{5.5}$$

will turn out to be a $\mathbb{Z}$-map from $(\mathbb{R}^{d+3}, L_{\omega,\Psi})$ to $(\mathbb{T}^d \times \mathbb{R}^3, L^{(T)}_{\omega,\Psi})$. Note that, choosing the product topology on $\mathbb{T}^d \times \mathbb{R}^3$, we see that $p_{5,d}$ is a continuous. Moreover, $p_{5,d}$ is onto $\mathbb{T}^d \times \mathbb{R}^3$.

If (ω, Ψ) is a d-dimensional spin-orbit torus then $\Psi(n;\cdot)$ is 2π-periodic whence it has a unique factor $\Psi'(n;\cdot)$ w.r.t. $(\mathbb{R}^d, p_{4,d}, \mathbb{T}^d)$, i.e., $\Psi' : \mathbb{Z} \times \mathbb{T}^d \to SO(3)$ is the unique map such that for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$,

$$\Psi(n;\phi) = \Psi'(n;p_{4,d}(\phi)) . \tag{5.6}$$

Note that $\Psi'(n;\cdot)$ is continuous. We can now state the proposition.

Proposition 5.2 *a) Let $\omega \in \mathbb{R}^d$ and let the function $L^{(T)}_\omega : \mathbb{Z} \times \mathbb{T}^d \to \mathbb{T}^d$ be defined, for $n \in \mathbb{Z}, z \in \mathbb{T}^d$, by*

$$L^{(T)}_\omega(n;z) := \Big(\exp(i2\pi n\omega_1)z_1, ..., \exp(i2\pi n\omega_d)z_d\Big)^T . \tag{5.7}$$

Then $L^{(T)}_\omega$ is a $\mathbb{Z}$-action on $\mathbb{T}^d$. Moreover $(\mathbb{T}^d, L^{(T)}_\omega)$ is a topological $\mathbb{Z}$-space and $p_{4,d}$ is a continuous $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$ to the topological $\mathbb{Z}$-space $(\mathbb{T}^d, L^{(T)}_\omega)$. Furthermore the topological $\mathbb{Z}$-space $(\mathbb{R}^d, L_\omega)$ is an extension of the topological $\mathbb{Z}$-space $(\mathbb{T}^d, L^{(T)}_\omega)$.

b) Let (ω,Ψ) be a d-dimensional spin-orbit torus and let the function $L^{(T)}_{\omega,\Psi}:\mathbb{Z}\times\mathbb{T}^d\times\mathbb{R}^3\to\mathbb{T}^d\times\mathbb{R}^3$ be defined, for $n\in\mathbb{Z}, z\in\mathbb{T}^d, S\in\mathbb{R}^3$, by

$$L^{(T)}_{\omega,\Psi}(n;z,S):=\begin{pmatrix} L^{(T)}_{\omega}(n;z)\\ \Psi'(n;z)S\end{pmatrix}, \tag{5.8}$$

where Ψ' is related to Ψ by (5.6), i.e., each $\Psi'(n;\cdot)$ is the factor of $\Psi(n;\cdot)$ w.r.t. $(\mathbb{R}^d,p_{4,d},\mathbb{T}^d)$.

Then $L^{(T)}_{\omega,\Psi}$ is a $\mathbb{Z}$-action on $\mathbb{T}^d\times\mathbb{R}^3$. Moreover $(\mathbb{T}^d\times\mathbb{R}^3,L^{(T)}_{\omega,\Psi})$ is a topological $\mathbb{Z}$-space and $p_{5,d}$ is a continuous $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3},L_{\omega,\Psi})$ to the topological $\mathbb{Z}$-space $(\mathbb{T}^d\times\mathbb{R}^3,L^{(T)}_{\omega,\Psi})$. Furthermore the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3},L_{\omega,\Psi})$ is an extension of the topological $\mathbb{Z}$-space $(\mathbb{T}^d\times\mathbb{R}^3,L^{(T)}_{\omega,\Psi})$.

c) Let (ω,Ψ) be a d-dimensional spin-orbit torus and let $(\mathbb{T}^d\times\mathbb{R}^3,L)$ be a topological $\mathbb{Z}$-space. If the map $p_{5,d}$ is a $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3},L_{\omega,\Psi})$ to the topological $\mathbb{Z}$-space $(\mathbb{T}^d\times\mathbb{R}^3,L)$, then $L=L^{(T)}_{\omega,\Psi}$.

d) Let $(\mathbb{R}^{d+3},L)$ be a topological $\mathbb{Z}$-space, let (ω,Ψ) be a d-dimensional spin-orbit torus and let the map $p_{5,d}$ be a $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3},L)$ to the topological $\mathbb{Z}$-space $(\mathbb{T}^d\times\mathbb{R}^3,L^{(T)}_{\omega,\Psi})$. Then a $N\in\mathbb{Z}^d$ exists such that, for $n\in\mathbb{Z},\phi\in\mathbb{R}^d, S\in\mathbb{R}^3$,

$$L(n;\phi,S)=\begin{pmatrix}\phi+2\pi n\omega+2\pi nN\\ \Psi(n;\phi)S\end{pmatrix}. \tag{5.9}$$

Conversely, if N is an arbitrary element of $\mathbb{Z}^d$ and if a function $L:\mathbb{Z}\times\mathbb{R}^{d+3}\to\mathbb{R}^{d+3}$ is defined, for $n\in\mathbb{Z},\phi\in\mathbb{R}^d,S\in\mathbb{R}^3$, by (5.9), then $(\mathbb{R}^{d+3},L)$ is a topological $\mathbb{Z}$-space and $p_{5,d}$ is a $\mathbb{Z}$-map from the topological $\mathbb{Z}$-space $(\mathbb{R}^{d+3},L)$ to the topological $\mathbb{Z}$-space $(\mathbb{T}^d\times\mathbb{R}^3,L^{(T)}_{\omega,\Psi})$ making the former an extension of the latter.

Proof of Proposition 5.2: See Section E.27. □

Due to Eq. (E.107) in the proof of Proposition 5.2b and due to Section A the function Ψ' in Proposition 5.2b is a continuous $SO(3)$-cocycle over the topological $\mathbb{Z}$-space $(\mathbb{T}^d,L^{(T)}_{\omega})$.

5.3 A principal $SO(3)$-bundle which underlies $\mathcal{SOT}(d)$

The theory of spin-orbit tori developed so far in this work will in the present section be reconsidered in terms of the principal $SO(3)$-bundle $\lambda_{\mathcal{SOT}(d)}$, defined by (5.12). For every $(\omega,\Psi)\in\mathcal{SOT}(d)$ we recall from Section 2.2 that Ψ is a continuous $SO(3)$-cocycle over the topological $\mathbb{Z}$-space $(\mathbb{R}^d,L_\omega)$. In Section 5.3.1 we will show that this allows us to encode (ω,Ψ) into a group homomorphism, $\Phi_{\omega,\Psi}$, from the group $\mathbb{Z}$ into the automorphism group $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{\mathcal{SOT}(d)})$ of $\lambda_{\mathcal{SOT}(d)}$. This technique was apparently introduced, in the context of Dynamical Systems Theory, by Zimmer in the 1980's [Zim2] and further developed by Feres and coworkers in the 1990's [Fer, Section 6]. Thus for brevity I call this technique the 'Feres machinery'. The Feres machinery shows us in Sections 5.3.3 and 5.3.4 how, via $\Phi_{\omega,\Psi}$, the associated bundle $\lambda_{\mathcal{SOT}(d)}[\mathbb{R}^3,L^{(3D)}]$, which is defined by (5.33), carries the two basic $\mathbb{Z}$-actions, $L_{\omega,\Psi}$ and $L^{(PF)}_{\omega,\Psi}$, of spin-orbit theory. We thus fulfill the motto, mentioned at the beginning of Section 5, of reconsidering $L_{\omega,\Psi}$ and $L^{(PF)}_{\omega,\Psi}$. Furthermore we prove in Section 5.3.5 a theorem, Theorem 5.5a, which is a special case of Zimmer's celebrated reduction theorem. In particular our theorem shows the relation between invariant spin fields and invariant $SO_3(2)$-reductions of $\lambda_{\mathcal{SOT}(d)}$. Note that a reader who is interested in Section 5.3.5 can skip Sections 5.3.3 and 5.3.4. Clearly the present section widens the perspective since it demonstrates how the principal $SO(3)$-bundle $\lambda_{\mathcal{SOT}(d)}$ underlies the theory of spin-orbit tori.

The facts and features of principal bundles and their associated bundles which are needed here are presented in Section D where we follow the elegant treatment of Husemoller's book [Hus] which uses Cartan principal bundles (another textbook which uses Cartan principal bundles is [Mac]).

5.3.1 The principal $SO(3)$-bundle $\lambda_{SOT(d)}$

The principal $SO(3)$-bundle $\lambda_{SOT(d)}$ we introduce in this section is a product principal bundle and its underlying bundle is defined by

$$\xi^{(1)}_{SOT(d)} := (\mathbb{R}^d \times SO(3), p^{(1)}_{SOT(d)}, \mathbb{R}^d)\,, \tag{5.10}$$

where the function $p^{(1)}_{SOT(d)} : \mathbb{R}^d \times SO(3) \to \mathbb{R}^d$ is the projection onto the first component, i.e., $p^{(1)}_{SOT(d)}(\phi, R) := \phi$ for $\phi \in \mathbb{R}^d, R \in SO(3)$. Clearly, by Definition B.1, $\xi^{(1)}_{SOT(d)}$ is a bundle and, since $p^{(1)}_{SOT(d)}$ is onto $\mathbb{R}^d$, it is a fiber structure. Of course $\xi^{(1)}_{SOT(d)}$ is a product bundle. To 'unfold' the bundle $\xi^{(1)}_{SOT(d)}$ into a principal bundle we define the right $SO(3)$-action $R^{(1)}_{SOT(d)}$ on $\mathbb{R}^d \times SO(3)$ by

$$R^{(1)}_{SOT(d)}(R', \phi, R) := (\phi, RR')\,, \tag{5.11}$$

where $\phi \in \mathbb{R}^d, R, R' \in SO(3)$. Clearly $(\mathbb{R}^d \times SO(3), R^{(1)}_{SOT(d)})$ is a topological right $SO(3)$-space. We thus arrive at the quadruple

$$\lambda_{SOT(d)} := (\xi^{(1)}_{SOT(d)}, R^{(1)}_{SOT(d)}) = (\mathbb{R}^d \times SO(3), p^{(1)}_{SOT(d)}, \mathbb{R}^d, R^{(1)}_{SOT(d)})\,. \tag{5.12}$$

In Section D.6.1 it is shown that the topological right $SO(3)$-space $(\mathbb{R}^d \times SO(3), R^{(1)}_{SOT(d)})$ is principal and that $\lambda_{SOT(d)}$ is a principal $SO(3)$-bundle. Note that $\lambda^{(1)}_{SOT(d)}$ is called a product principal $SO(3)$-bundle.

Following Section D.6.1, we denote the set of morphisms from $\xi^{(1)}_{SOT(d)}$ to itself in the category Bun of bundles by $\mathfrak{Mor}_{Bun}(\xi^{(1)}_{SOT(d)})$. Note that, by definition, $\mathfrak{Mor}_{Bun}(\xi^{(1)}_{SOT(d)})$ consists of the pairs $(\varphi, \bar{\varphi})$ for which $\varphi \in \mathcal{C}(\mathbb{R}^d \times SO(3), \mathbb{R}^d \times SO(3))$ and $\bar{\varphi} \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$ such that

$$\bar{\varphi} \circ p^{(1)}_{SOT(d)} = p^{(1)}_{SOT(d)} \circ \varphi\,. \tag{5.13}$$

Following Section D.6.1, we denote the set of morphisms from $\lambda_{SOT(d)}$ to itself in the category $Bun(SO(3))$ of principal $SO(3)$-bundles by $\mathfrak{Mor}_{Bun(SO(3))}(\lambda_{SOT(d)})$. Note that, by definition, $\mathfrak{Mor}_{Bun(SO(3))}(\lambda_{SOT(d)})$ consists of the pairs $(\varphi, \bar{\varphi})$ in $\mathfrak{Mor}_{Bun}(\xi^{(1)}_{SOT(d)})$ for which φ is a $SO(3)$-map from the right G-space $(\mathbb{R}^d \times SO(3), R^{(1)}_{SOT(d)})$ to itself. It follows from (D.79) that $\mathfrak{Mor}_{Bun(SO(3))}(\lambda_{SOT(d)})$ has the following simple form:

$$\mathfrak{Mor}_{Bun(SO(3))}(\lambda_{SOT(d)}) = \Big\{(\varphi, \bar{\varphi}) \in \mathcal{C}(\mathbb{R}^d \times SO(3), \mathbb{R}^d \times SO(3)) \times \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d) : (\forall\, \phi \in \mathbb{R}^d, R \in SO(3))\varphi(\phi, R) = \begin{pmatrix} \bar{\varphi}(\phi) \\ f(\phi)R \end{pmatrix}, f \in \mathcal{C}(\mathbb{R}^d, SO(3))\Big\}\,. \tag{5.14}$$

Note that if $(\varphi, \bar{\varphi}) \in \mathfrak{Mor}_{Bun(SO(3))}(\lambda_{SOT(d)})$ then by (5.14) the functions $\bar{\varphi}, f$ are uniquely determined by φ and φ is uniquely determined by $\bar{\varphi}, f$. Given $(\varphi_i, \bar{\varphi}_i) \in \mathfrak{Mor}_{Bun(SO(3))}(\lambda_{SOT(d)})$ for $i = 1, 2$ and writing, by (5.14), $\varphi_i(\phi, R) = (\bar{\varphi}_i(\phi), f_i(\phi)R)$, the composition law of $Bun(SO(3))$ gives the morphism $(\varphi_2, \bar{\varphi}_2)(\varphi_1, \bar{\varphi}_1) = (\varphi_2 \circ \varphi_1, \bar{\varphi}_2 \circ \bar{\varphi}_1) \in \mathfrak{Mor}_{Bun(SO(3))}(\lambda_{SOT(d)})$ where for $\phi \in \mathbb{R}^d, R \in SO(3)$

$$(\varphi_2 \circ \varphi_1)(\phi, R) = \varphi_2\Big(\bar{\varphi}_1(\phi), f_1(\phi)R\Big) = \begin{pmatrix} (\bar{\varphi}_2 \circ \bar{\varphi}_1)(\phi) \\ f_2(\bar{\varphi}_1(\phi))f_1(\phi)R \end{pmatrix}\,. \tag{5.15}$$

Denoting by $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$ the set of isomorphisms in $\mathfrak{Mor}_{Bun(SO(3))}(\lambda_{SOT(d)})$ it follows from (D.82) that

$$\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)}) = \Big\{(\varphi, \bar{\varphi}) \in \mathcal{C}(\mathbb{R}^d \times SO(3), \mathbb{R}^d \times SO(3)) \times HOMEO(\mathbb{R}^d, \mathbb{R}^d) : (\forall\, \phi \in \mathbb{R}^d, R \in SO(3))\varphi(\phi, R) = \begin{pmatrix} \bar{\varphi}(\phi) \\ f(\phi)R \end{pmatrix}, f \in \mathcal{C}(\mathbb{R}^d, SO(3))\Big\}\,, \tag{5.16}$$

where $HOMEO(\mathbb{R}^d, \mathbb{R}^d)$ denotes the set of homeomorphisms from $\mathbb{R}^d$ onto itself. Note that, for every category, isomorphisms from an object to itself are called automorphisms, which explains the notation $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$. Note that $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$ has a canonical group structure where the multiplication is given by the composition law of $Bun(SO(3))$.

Following Section D.6.5 we now encode the spin-orbit tori in $\mathcal{SOT}(d)$ into subgroups of $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$. Recalling Section 2.2, we have the function $\rho_{SOT(d)}$: $\mathcal{SOT}(d) \to COC(\mathbb{R}^d, \mathbb{Z}, SO(3))$, which is defined for $(\omega, \Psi) \in \mathcal{SOT}(d)$ by (2.15). Since $\rho_{SOT(d)}$ is an injection it allows to encode spin-orbit tori into cocycles. Moreover, recalling Section D.4, we denote by $HOM_{\mathbb{Z}}(\lambda_{SOT(d)})$ the set of group homomorphisms from $\mathbb{Z}$ into $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$ so Section D.6.5 gives us an injection $\rho_{\mathbb{R}^d,\mathbb{Z},SO(3)} : COC(\mathbb{R}^d, \mathbb{Z}, SO(3)) \to HOM_{\mathbb{Z}}(\lambda_{SOT(d)})$ which is defined for $(l, f) \in COC(\mathbb{R}^d, \mathbb{Z}, SO(3))$ by

$$\rho_{\mathbb{R}^d,\mathbb{Z},SO(3)}(l, f) := \Phi \,, \tag{5.17}$$

where, for $n \in \mathbb{Z}$,

$$\Phi(n) := (\varphi(n;\cdot), l(n;\cdot)) \,, \tag{5.18}$$

and where, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, R \in SO(3)$,

$$\varphi(n;\phi,R) := \begin{pmatrix} l(n;\phi) \\ f(n;\phi)R \end{pmatrix} . \tag{5.19}$$

Note that the injection $\rho_{\mathbb{R}^d,\mathbb{Z},SO(3)}$ is a special case of a more general construction which is outlined in Remark 1 of Section D.6.5 and which is based on the cross sections of the bundle $\xi^{(1)}_{SOT(d)}$. It follows from (2.15), (5.17),(5.18) (5.19) that for $(\omega, \Psi) \in \mathcal{SOT}(d)$

$$(\rho_{\mathbb{R}^d,\mathbb{Z},SO(3)} \circ \rho_{SOT(d)})(\omega,\Psi) = \rho_{\mathbb{R}^d,\mathbb{Z},SO(3)}(L_\omega, \Psi) = \Phi_{\omega,\Psi} \,, \tag{5.20}$$

where, for $n \in \mathbb{Z}$,

$$\Phi_{\omega,\Psi}(n) := (\varphi_{\omega,\Psi}(n;\cdot), L_\omega(n;\cdot)) \,, \tag{5.21}$$

and where for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, R \in SO(3)$

$$\varphi_{\omega,\Psi}(n;\phi,R) := \begin{pmatrix} L_\omega(n;\phi) \\ \Psi(n;\phi)R \end{pmatrix} = \begin{pmatrix} \phi + 2\pi n\omega \\ \Psi(n;\phi)R \end{pmatrix} . \tag{5.22}$$

Since $\rho_{SOT(d)}$ and $\rho_{\mathbb{R}^d,\mathbb{Z},SO(3)}$ are one-one, it follows from (5.20) that every spin-orbit torus $(\omega, \Psi) \in \mathcal{SOT}(d)$ is uniquely characterized by the group homomorphism $\Phi_{\omega,\Psi}$ whence (ω, Ψ) is encoded in the subgroup $\Phi_{\omega,\Psi}(\mathbb{Z})$ of $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$. We call the group homomorphisms $\Phi_{\omega,\Psi}$ 'tied' to $\mathcal{SOT}(d)$. Equipping $\mathbb{Z}$ with the discrete topology we conclude from Section D.6.5 that $\rho_{\mathbb{R}^d,\mathbb{Z},SO(3)}$ is a bijection onto $HOM_{\mathbb{Z}}(\lambda_{SOT(d)})$. Thus, given a $\Phi \in HOM_{\mathbb{Z}}(\lambda_{SOT(d)})$ and since $\rho_{\mathbb{R}^d,\mathbb{Z},SO(3)}$ is a bijection onto $HOM_{\mathbb{Z}}(\lambda_{SOT(d)})$, eq. (5.17) holds where $(l, f) \in COC(\mathbb{R}^d, \mathbb{Z}, SO(3))$ is defined by $(l, f) := \rho^{-1}_{\mathbb{R}^d,\mathbb{Z},SO(3)}(\Phi)$. It is easy to see by (5.17),(5.18) (5.19) that Φ is tied to $\mathcal{SOT}(d)$ iff $l(1;\cdot)$ is a translation on $\mathbb{R}^d$ and $f(1;\phi)$ is 2π-periodic in ϕ. Thus not every group homomorphism in $HOM_{\mathbb{Z}}(\lambda_{SOT(d)})$ is tied to $\mathcal{SOT}(d)$.

To discuss $R_{d,\omega}$ in the context of $\lambda_{SOT(d)}$, let $(\omega, \Psi), (\omega, \Psi') \in \mathcal{SOT}(d), T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ and $R_{d,\omega}(T; \omega, \Psi) = (\omega, \Psi')$. Thus by (3.4) we have, for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$,

$$\Psi'(n;\phi) = T^T(L_\omega(n;\phi))\Psi(n;\phi)T(\phi) \,. \tag{5.23}$$

It follows from (5.20) that

$$(\rho_{\mathbb{R}^d,\mathbb{Z},SO(3)} \circ \rho_{SOT(d)})(\omega,\Psi') = \rho_{\mathbb{R}^d,\mathbb{Z},SO(3)}(L_\omega, \Psi') = \Phi_{\omega,\Psi'} \,, \tag{5.24}$$

where, for $n \in \mathbb{Z}$,

$$\Phi_{\omega,\Psi'}(n) = (\varphi_{\omega,\Psi'}(n;\cdot), L_\omega(n;\cdot)) \,, \tag{5.25}$$

and where for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, R \in SO(3)$

$$\varphi_{\omega,\Psi'}(n;\phi,R) := \begin{pmatrix} L_\omega(n;\phi) \\ \Psi'(n;\phi)R \end{pmatrix} = \begin{pmatrix} L_\omega(n;\phi) \\ T^T(L_\omega(n;\phi))\Psi(n;\phi)T(\phi)R \end{pmatrix} , \tag{5.26}$$

where in the second equality we used (5.23). We define $\varphi_T \in \mathcal{C}(\mathbb{R}^d \times SO(3), \mathbb{R}^d \times SO(3)$ for $\phi \in \mathbb{R}^d, R \in SO(3)$ by

$$\varphi_T(\phi,R) := \begin{pmatrix} \phi \\ T(\phi)R \end{pmatrix} . \tag{5.27}$$

Using (D.12),(D.141) the gauge group of $\lambda_{SOT(d)}$ reads as

$$\begin{aligned}
&\mathfrak{Gau}_{Bun(SO(3))}(\lambda_{SOT(d)}) = \\
&\quad \{\varphi \in \mathcal{C}(\mathbb{R}^d \times SO(3), \mathbb{R}^d \times SO(3)) : (\varphi, id_{\mathbb{R}^d}) \in \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})\} \\
&\qquad = \{\varphi \in \mathcal{C}(\mathbb{R}^d \times SO(3), \mathbb{R}^d \times SO(3)) : \\
&\qquad\quad (\forall\, \phi \in \mathbb{R}^d, R \in SO(3))\varphi(\phi, R) = \begin{pmatrix} \phi \\ f(\phi)R \end{pmatrix}, f \in \mathcal{C}(\mathbb{R}^d, SO(3))\}\,, \qquad (5.28)
\end{aligned}$$

whence $\varphi_T \in \mathfrak{Gau}_{Bun(SO(3))}(\lambda_{SOT(d)})$ and $\Phi_T := (\varphi_T, id_{\mathbb{R}^d}) \in \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$. We define $\Phi' \in HOM_{\mathbb{Z}}(\lambda_{SOT(d)})$ for $n \in \mathbb{Z}$ by

$$\begin{aligned}
\Phi'(n) &:= \Phi_T^{-1}\Phi(n)\Phi_T = (\varphi_T, id_{\mathbb{R}^d})^{-1}(\varphi_{\omega,\Psi}(n;\cdot), L_\omega(n;\cdot))(\varphi_T, id_{\mathbb{R}^d}) \\
&= (\varphi_T^{-1} \circ \varphi_{\omega,\Psi}(n;\cdot) \circ \varphi_T, L_\omega(n;\cdot))\,, \qquad (5.29)
\end{aligned}$$

where we also used (5.21). We conclude from (5.22),(5.29), (D.146) that for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, R \in SO(3)$

$$\begin{aligned}
(\Phi'(n))(\phi, R) &= \left(\begin{pmatrix} L_\omega(n;\phi) \\ T^T(L_\omega(n;\phi))\Psi(n;\phi)T(\phi)R \end{pmatrix}, L_\omega(n;\phi)\right) \\
&= \left(\begin{pmatrix} L_\omega(n;\phi) \\ \Psi'(n;\phi)R \end{pmatrix}, L_\omega(n;\phi)\right). \qquad (5.30)
\end{aligned}$$

We conclude from (5.25),(5.26),(5.30) that $\Phi_{\omega,\Psi'} = \Phi'$ whence we have shown that the transformation via $R_{d,\omega}(T;\cdot)$ corresponds in $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$ to a conjugation by Φ_T. In other words, on the level of $\lambda_{SOT(d)}$, the gauge group $\mathfrak{Gau}_{Bun(SO(3))}(\lambda_{SOT(d)})$ takes over the job from the group $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$.

5.3.2 The bundle $\lambda_{SOT(d)}[\mathbb{R}^3, L^{(3D)}]$ associated with $\lambda_{SOT(d)}$

In this section we introduce the bundle $\lambda_{SOT(d)}[\mathbb{R}^3, L^{(3D)}]$ which in the ensuing sections will be the substratum by which $\lambda_{SOT(d)}$ carries the $\mathbb{Z}$-actions $L_{\omega,\Psi}$ and

$L^{(PF)}_{\omega,\Psi}$. We define the topological left $SO(3)$-space $(\mathbb{R}^3, L^{(3D)})$ where the function $L^{(3D)} : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$L^{(3D)}(R; S) := RS\,, \qquad (5.31)$$

with $S \in \mathbb{R}^3, R \in SO(3)$ and where RS is the matrix product of R and S. Following the standard technique of constructing associated bundles, which is outlined in Section D.2 and, for the case of product principal bundles, in Section D.6.2, one defines the function $R^{(2)}_{SOT(d)} : SO(3) \times \mathbb{R}^d \times SO(3) \times \mathbb{R}^3 \to \mathbb{R}^d \times SO(3) \times \mathbb{R}^3$ for $\phi \in \mathbb{R}^d, R, R' \in SO(3), S \in \mathbb{R}^3$, by

$$R^{(2)}_{SOT(d)}(R', \phi, R, S) := \begin{pmatrix} R^{(1)}_{SOT(d)}(R', \phi, R) \\ L^{(3D)}(R'^{-1}; S) \end{pmatrix} = \begin{pmatrix} \phi \\ RR' \\ R'^{-1}S \end{pmatrix}, \qquad (5.32)$$

and observes that $(\mathbb{R}^d \times SO(3) \times \mathbb{R}^3, R^{(2)}_{SOT(d)})$ is a topological right $SO(3)$-space. Denoting the orbit space of $(\mathbb{R}^d \times SO(3) \times \mathbb{R}^3, R^{(2)}_{SOT(d)})$ by $E^{(3)}_{SOT(d)}$, i.e., in the notation of Section A, $E^{(3)}_{SOT(d)} := (\mathbb{R}^d \times SO(3) \times \mathbb{R}^3)/R^{(2)}_{SOT(d)}$ and the canonical surjection: $\mathbb{R}^d \times SO(3) \times \mathbb{R}^3 \to E^{(3)}_{SOT(d)}$ by $p^{(2)}_{SOT(d)}$, one obtains the bundle:

$$\lambda_{SOT(d)}[\mathbb{R}^3, L^{(3D)}] =: \xi^{(3)}_{SOT(d)} = (E^{(3)}_{SOT(d)}, p^{(3)}_{SOT(d)}, \mathbb{R}^d)\,, \qquad (5.33)$$

where the continuous function $p^{(3)}_{SOT(d)} : E^{(3)}_{SOT(d)} \to \mathbb{R}^d$ is the unique function: $E^{(3)}_{SOT(d)} \to \mathbb{R}^d$ which satisfies

$$p^{(3)}_{SOT(d)} \circ p^{(2)}_{SOT(d)} = p^{(1)}_{SOT(d)}\,. \qquad (5.34)$$

One calls $\xi^{(3)}_{SOT(d)}$ the "bundle associated with $\lambda_{SOT(d)}$ via the topological left $SO(3)$-space $(\mathbb{R}^3, L^{(3D)})$". Note again that the above properties of the associated bundle follow from Sections D.2 and D.6.2.

5.3.3 How $\lambda_{SOT(d)}$ carries the $\mathbb{Z}$-action $L_{\omega,\Psi}$

We now have all ingredients at our disposal to apply the Feres machinery. As outlined in Sections D.3 and D.6.4, this machinery provides us with a canonical left $\mathfrak{A}ut_{Bun(SO(3))}(\lambda_{SOT(d)})$-action, $L^{(1)}_{SOT(d)}$, on $E^{(3)}_{SOT(d)}$ and this will allow us in the present section to recover $L_{\omega,\Psi}$. Specializing (D.41) to the present case it is shown in Section D.3.1 that the function $L^{(1)}_{SOT(d)} : \mathfrak{A}ut_{Bun(SO(3))}(\lambda_{SOT(d)}) \times E^{(3)}_{SOT(d)} \to E^{(3)}_{SOT(d)}$ which is defined for $(\varphi, \bar{\varphi}) \in \mathfrak{A}ut_{Bun(SO(3))}(\lambda_{SOT(d)})$ and $\phi \in \mathbb{R}^d, R \in SO(3), S \in \mathbb{R}^3$ by

$$L^{(1)}_{SOT(d)}(\varphi, \bar{\varphi}; p^{(2)}_{SOT(d)}(\phi, R, S)) := p^{(2)}_{SOT(d)}(\varphi(\phi, R), S) , \qquad (5.35)$$

is a left $\mathfrak{A}ut_{Bun(SO(3))}(\lambda_{SOT(d)})$-action on $E^{(3)}_{SOT(d)}$ whence $(E^{(3)}_{SOT(d)}, L^{(1)}_{SOT(d)})$ is a left $\mathfrak{A}ut_{Bun(SO(3))}(\lambda_{SOT(d)})$-space. Note that by Section D.3.1 $L^{(1)}_{SOT(d)}(\varphi, \bar{\varphi}; \cdot)$ is a homeomorphism onto $E^{(3)}_{SOT(d)}$. With now showing that the bundle $\xi^{(3)}_{SOT(d)}$ is trivial we construct a left $\mathfrak{A}ut_{Bun(SO(3))}(\lambda_{SOT(d)})$-space which is conjugate to $(E^{(3)}_{SOT(d)}, L^{(1)}_{SOT(d)})$. Specializing (D.84) to the present case we define the function $r^{(1)}_{SOT(d)} : \mathbb{R}^d \times SO(3) \times \mathbb{R}^3 \to \mathbb{R}^{d+3}$ for $\phi \in \mathbb{R}^d, R \in SO(3), S \in \mathbb{R}^3$ by

$$r^{(1)}_{SOT(d)}(\phi, R, S) := \begin{pmatrix} \phi \\ L^{(3D)}(R; S) \end{pmatrix} = \begin{pmatrix} \phi \\ RS \end{pmatrix} \qquad (5.36)$$

and conclude by Section D.6.2 that there exists a unique function $r^{(2)}_{SOT(d)} : E^{(3)}_{SOT(d)} \to \mathbb{R}^{d+3}$ such that

$$r^{(2)}_{SOT(d)} \circ p^{(2)}_{SOT(d)} = r^{(1)}_{SOT(d)} \qquad (5.37)$$

and that $r^{(2)}_{SOT(d)}$ is a homeomorphism onto $\mathbb{R}^{d+3}$. Defining the bundle

$$\xi^{(4)}_{SOT(d)} = (\mathbb{R}^{d+3}, p^{(4)}_{SOT(d)}, \mathbb{R}^d) , \qquad (5.38)$$

where $p^{(4)}_{SOT(d)}(\phi, S) := \phi$, we know from Section D.6.2 that $(r^{(2)}_{SOT(d)}, id_{\mathbb{R}^d})$ is an isomorphism from $\xi^{(3)}_{SOT(d)}$ to $\xi^{(4)}_{SOT(d)}$ in the category Bun of bundles. Thus the bundle

$\xi^{(3)}_{SOT(d)}$ is trivial. Specializing (D.102) to the present case we define the function $\tilde{L}^{(1)}_{SOT(d)} : \mathfrak{A}ut_{Bun(SO(3))}(\lambda_{SOT(d)}) \times \mathbb{R}^{d+3} \to \mathbb{R}^{d+3}$ for $(\varphi, \bar{\varphi}) \in \mathfrak{A}ut_{Bun(SO(3))}(\lambda_{SOT(d)})$ and $\phi \in \mathbb{R}^d, S \in \mathbb{R}^3$ by

$$\tilde{L}^{(1)}_{SOT(d)}(\varphi, \bar{\varphi}; \phi, S) := r^{(2)}_{SOT(d)}(L^{(1)}_{SOT(d)}(\varphi, \bar{\varphi}; (r^{(2)}_{SOT(d)})^{-1}(\phi, S))) , \qquad (5.39)$$

whence

$$\tilde{L}^{(1)}_{SOT(d)}(\varphi, \bar{\varphi}; \cdot) \circ r^{(2)}_{SOT(d)} = r^{(2)}_{SOT(d)} \circ L^{(1)}_{SOT(d)}(\varphi, \bar{\varphi}; \cdot) . \qquad (5.40)$$

Since $L^{(1)}_{SOT(d)}$ is a left $\mathfrak{A}ut_{Bun(SO(3))}(\lambda_{SOT(d)})$-action on $E^{(3)}_{SOT(d)}$ and $r^{(2)}_{SOT(d)}$ is a bijection onto $\mathbb{R}^{d+3}$ it follows from (5.40) that $\tilde{L}^{(1)}_{SOT(d)}$ is a left $\mathfrak{A}ut_{Bun(SO(3))}(\lambda_{SOT(d)})$-action on $\mathbb{R}^{d+3}$ and that the left $\mathfrak{A}ut_{Bun(SO(3))}(\lambda_{SOT(d)})$-spaces $(E^{(3)}_{SOT(d)}, L^{(1)}_{SOT(d)})$, $(\mathbb{R}^{d+3}, \tilde{L}^{(1)}_{SOT(d)})$ are conjugate. Note also that since $L^{(1)}_{SOT(d)}(\varphi, \bar{\varphi}; \cdot)$ is a homeomorphism onto $E^{(3)}_{SOT(d)}$ and $r^{(2)}_{SOT(d)}$ is a homeomorphism onto $\mathbb{R}^{d+3}$, it follows from (5.40) that $\tilde{L}^{(1)}_{SOT(d)}(\varphi, \bar{\varphi}; \cdot)$ is a homeomorphism onto $\mathbb{R}^{d+3}$. In fact we will now see that $\tilde{L}^{(1)}_{SOT(d)}$ has a very simple structure. Specializing (D.104) to the present case we obtain for $(\varphi, \bar{\varphi}) \in \mathfrak{A}ut_{Bun(SO(3))}(\lambda_{SOT(d)})$ and $\phi \in \mathbb{R}^d, R \in SO(3), S \in \mathbb{R}^3$

$$\tilde{L}^{(1)}_{SOT(d)}(\varphi, \bar{\varphi}; \phi, S) = r^{(1)}_{SOT(d)}(\varphi(\phi, R), L^{(3D)}(R^{-1}; S)) = r^{(1)}_{SOT(d)}(\varphi(\phi, R), R^{-1}S) . \qquad (5.41)$$

Of course if $(\varphi, \bar{\varphi}) \in \mathfrak{A}ut_{Bun(SO(3))}(\lambda_{SOT(d)})$ then by (5.16) we have for $\phi \in \mathbb{R}^d, R \in SO(3)$

$$\varphi(\phi, R) = \begin{pmatrix} \bar{\varphi}(\phi) \\ f(\phi)R \end{pmatrix} , \qquad (5.42)$$

where $f \in \mathcal{C}(\mathbb{R}^d, SO(3))$. Thus by (5.36),(5.41) we obtain for $(\varphi, \bar{\varphi}) \in \mathfrak{A}ut_{Bun(SO(3))}(\lambda_{SOT(d)})$ and $\phi \in \mathbb{R}^d, R \in SO(3), S \in \mathbb{R}^3$ the simple formula

$$\tilde{L}^{(1)}_{SOT(d)}(\varphi, \bar{\varphi}; \phi, S) = r^{(1)}_{SOT(d)}(\bar{\varphi}(\phi), f(\phi)R, R^{-1}S) = \begin{pmatrix} \bar{\varphi}(\phi) \\ f(\phi)S \end{pmatrix} . \qquad (5.43)$$

Note also that (5.43) confirms our assertion that $\tilde{L}^{(1)}_{SOT(d)}(\varphi,\bar{\varphi};\cdot)$ is a homeomorphism onto $\mathbb{R}^{d+3}$. To bring spin-orbit theory into the picture we now pick a spin-orbit torus $(\omega,\Psi)\in\mathcal{SOT}(d)$ and conclude from (5.21),(5.41) that for $n\in\mathbb{Z},\phi\in\mathbb{R}^d,S\in\mathbb{R}^3$

$$\begin{aligned}\tilde{L}^{(1)}_{SOT(d)}(\Phi_{\omega,\Psi}(n);\phi,S) &= \tilde{L}^{(1)}_{SOT(d)}(\varphi_{\omega,\Psi}(n;\cdot),L_{\omega,\Psi}(n;\cdot);\phi,S)\\ &= r^{(1)}_{SOT(d)}(\varphi_{\omega,\Psi}(n;\phi,R),R^{-1}S)\,,\end{aligned}\tag{5.44}$$

where $\varphi_{\omega,\Psi}$ is given by (5.22). It follows from (2.9),(5.22),(5.36), (5.44) the remarkable result that for $n\in\mathbb{Z},\phi\in\mathbb{R}^d,S\in\mathbb{R}^3$

$$\begin{aligned}\tilde{L}^{(1)}_{SOT(d)}(\Phi_{\omega,\Psi}(n);\phi,S) &= r^{(1)}_{SOT(d)}(\varphi_{\omega,\Psi}(n;\phi,R),R^{-1}S)\\ &= r^{(1)}_{SOT(d)}(\phi+2\pi n\omega,\Psi(n;\phi)R,R^{-1}S)=\begin{pmatrix}\phi+2\pi n\omega\\ \Psi(n;\phi)S\end{pmatrix}\\ &= L_{\omega,\Psi}(n;\phi,S)\,.\end{aligned}\tag{5.45}$$

Having thus recovered $L_{\omega,\Psi}$ we put this into perspective by defining the function $\hat{L}_{\omega,\Psi}:\mathbb{Z}\times E^{(3)}_{SOT(d)}\to E^{(3)}_{SOT(d)}$ for $n\in\mathbb{Z},z\in E^{(3)}_{SOT(d)}$ by

$$\hat{L}_{\omega,\Psi}(n;z):=L^{(1)}_{SOT(d)}(\Phi_{\omega,\Psi}(n);z)\,.\tag{5.46}$$

Since $L^{(1)}_{SOT(d)}$ is a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$-action on $E^{(3)}_{SOT(d)}$ and since $\Phi_{\omega,\Psi}$ is a group homomorphism into $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$ it follows from (5.46) that $\hat{L}_{\omega,\Psi}$ is a $\mathbb{Z}$-action on $E^{(3)}_{SOT(d)}$. Since $L^{(1)}_{SOT(d)}(\Phi_{\omega,\Psi}(n);\cdot)$ is continuous, it follows from (5.46) that $\hat{L}_{\omega,\Psi}(n;\cdot)$ is continuous whence $(E^{(3)}_{SOT(d)},\hat{L}_{\omega,\Psi})$ is a topological $\mathbb{Z}$-space. Furthermore we conclude from (5.40),(5.45),(5.46) that for $n\in\mathbb{Z}$

$$\begin{aligned}L_{\omega,\Psi}(n;\cdot)\circ r^{(2)}_{SOT(d)} &= \tilde{L}^{(1)}_{SOT(d)}(\Phi_{\omega,\Psi}(n);\cdot)\circ r^{(2)}_{SOT(d)} = r^{(2)}_{SOT(d)}\circ L^{(1)}_{SOT(d)}(\Phi_{\omega,\Psi}(n);\cdot)\\ &= r^{(2)}_{SOT(d)}\circ\hat{L}_{\omega,\Psi}(n;\cdot)\,.\end{aligned}\tag{5.47}$$

In other words, since $r^{(2)}_{SOT(d)}\in HOMEO(E^{(3)}_{SOT(d)},\mathbb{R}^{d+3})$, (5.47) tells us that the topological $\mathbb{Z}$-spaces $(E^{(3)}_{SOT(d)},\hat{L}_{\omega,\Psi})$ and $(\mathbb{R}^{d+3},L_{\omega,\Psi})$ are conjugate. This fact demonstrates how $\lambda_{SOT(d)}$ carries $L_{\omega,\Psi}$ in a canonical way and it thus establishes $\lambda_{SOT(d)}$

as an appropriate principal bundle. Note also that specializing (D.40) to the present case we observe, for every integer n,

$$(\hat{L}_{\omega,\Psi}(n;\cdot),L_{\omega}(n;\cdot))\in\mathfrak{Mor}_{Bun}(\xi^{(3)}_{SOT(d)})$$

and, by Remark 1 in Section D.3.1, obtain that $(\hat{L}_{\omega,\Psi}(n;\cdot),L_{\omega}(n;\cdot))$ is a fibre morphism on $\xi^{(3)}_{SOT(d)}$ so that (5.47) reveals a close relationship between spin-orbit trajectories and the fibre morphisms on the associated bundle.

5.3.4 How $\lambda_{SOT(d)}$ carries the $\mathbb{Z}$-action $L^{(PF)}_{\omega,\Psi}$

In the previous section we employed the canonical left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$-action $L^{(1)}_{SOT(d)}$ and in the present section we build up on that. In fact, as outlined in detail in Sections D.3.2 and D.6.4, the Feres machinery provides us with a canonical left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$-action, $L^{(2)}_{SOT(d)}$, on the set $\Gamma(\xi^{(3)})$ of cross sections of the associated bundle and it will allow us in the present section to recover $L^{(PF)}_{\omega,\Psi}$. Specializing (D.46) to the present case it is shown in Section D.3.2 that the function $L^{(2)}_{SOT(d)}:\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})\times\Gamma(\xi^{(3)})\to\Gamma(\xi^{(3)})$ defined for $(\varphi,\bar{\varphi})\in\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$ and $\sigma\in\Gamma(\xi^{(3)}),\phi\in\mathbb{R}^d$ by

$$\Big(L^{(2)}_{SOT(d)}(\varphi,\bar{\varphi};\sigma)\Big)(\phi)=L^{(1)}_{SOT(d)}(\varphi,\bar{\varphi};\sigma(\bar{\varphi}^{-1}(\phi)))\,,\tag{5.48}$$

is a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$-action on $\Gamma(\xi^{(3)})$ whence $(\Gamma(\xi^{(3)}),L^{(2)}_{SOT(d)})$ is a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$-space. Clearly $L^{(2)}_{SOT(d)}$ builds up on $L^{(1)}_{SOT(d)}$. Specializing (D.107) to the present case we define the function $r^{(3)}_{SOT(d)}:\Gamma(\xi^{(3)})\to\Gamma(\xi^{(4)})$ for $\sigma\in\Gamma(\xi^{(3)})$ by

$$r^{(3)}_{SOT(d)}(\sigma):=r^{(2)}_{SOT(d)}\circ\sigma\,.\tag{5.49}$$

It is shown in Section D.6.4 that $r^{(3)}_{SOT(d)}$ is a bijection onto $\Gamma(\xi^{(4)})$. Specializing (D.110) to the present case we define the function $\tilde{L}^{(2)}_{SOT(d)}:\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})\times$

$\Gamma(\xi^{(4)}) \to \Gamma(\xi^{(4)})$ for $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$ and $\sigma \in \Gamma(\xi^{(4)})$ by

$$\tilde{L}^{(2)}_{SOT(d)}(\varphi, \bar{\varphi}; \sigma) := r^{(3)}_{SOT(d)}(L^{(2)}_{SOT(d)}(\varphi, \bar{\varphi}; (r^{(3)}_{SOT(d)})^{-1}(\sigma))) , \tag{5.50}$$

whence in analogy with (D.111)

$$\tilde{L}^{(2)}_{SOT(d)}(\varphi, \bar{\varphi}; \cdot) \circ r^{(3)}_{SOT(d)} = r^{(3)}_{SOT(d)} \circ L^{(2)}_{SOT(d)}(\varphi, \bar{\varphi}; \cdot) . \tag{5.51}$$

Since $L^{(2)}_{SOT(d)}$ is a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$-action on $\Gamma(\xi^{(3)})$ and $r^{(3)}_{SOT(d)}$ is a bijection onto $\Gamma(\xi^{(4)})$ it follows from (5.51) that $\tilde{L}^{(2)}_{SOT(d)}$ is a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$-action on $\Gamma(\xi^{(4)})$ and that the left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$-spaces $(\Gamma(\xi^{(3)}), L^{(2)}_{SOT(d)})$, $(\Gamma(\xi^{(4)}), \tilde{L}^{(2)}_{SOT(d)})$ are conjugate. We will now see that $\tilde{L}^{(2)}_{SOT(d)}$ has a very simple structure. In fact specializing (D.113) to the present case we obtain for $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$ and $\sigma \in \Gamma(\xi^{(4)}), \phi \in \mathbb{R}^d$

$$\Big(\tilde{L}^{(2)}_{SOT(d)}(\varphi, \bar{\varphi}; \sigma)\Big)(\phi) = \tilde{L}^{(1)}_{SOT(d)}(\varphi, \bar{\varphi}; \sigma(\bar{\varphi}^{-1}(\phi))) . \tag{5.52}$$

Recalling Definition B.1 we have for $\sigma \in \Gamma(\xi^{(4)})$ that $p^{(4)}_{SOT(d)} \circ \sigma = id_{\mathbb{R}^d}$ whence for $\phi \in \mathbb{R}^d$ we have

$$\sigma(\phi) = \begin{pmatrix} \phi \\ \hat{\sigma}(\phi) \end{pmatrix} , \tag{5.53}$$

where $\hat{\sigma} \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^3)$. We thus obtain by specializing (D.114) to the present case the simple formula

$$\begin{aligned} \Big(\tilde{L}^{(2)}_{SOT(d)}(\varphi, \bar{\varphi}; \sigma)\Big)(\phi) &= \Big(\phi, L^{(3D)}(f(\bar{\varphi}^{-1}(\phi)); \hat{\sigma}(\bar{\varphi}^{-1}(\phi)))\Big) \\ &= \begin{pmatrix} \phi \\ f(\bar{\varphi}^{-1}(\phi))\hat{\sigma}(\bar{\varphi}^{-1}(\phi)) \end{pmatrix} , \end{aligned} \tag{5.54}$$

where $f \in \mathcal{C}(\mathbb{R}^d, SO(3))$ is determined from φ by (5.42). To bring spin-orbit theory into the picture we now pick a spin-orbit torus $(\omega, \Psi) \in \mathcal{SOT}(d)$ and define the function $\hat{L}^{(PF)}_{\omega,\Psi} : \mathbb{Z} \times \Gamma(\xi^{(4)}) \to \Gamma(\xi^{(4)})$ for $n \in \mathbb{Z}, \sigma \in \Gamma(\xi^{(4)})$ by

$$\hat{L}^{(PF)}_{\omega,\Psi}(n; \sigma) := \tilde{L}^{(2)}_{SOT(d)}(\Phi_{\omega,\Psi}(n); \sigma) . \tag{5.55}$$

Since $\tilde{L}^{(2)}_{SOT(d)}$ is a left $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$-action on $\Gamma(\xi^{(4)})$ and since $\Phi_{\omega,\Psi}$ is a group homomorphism into $\mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$ it follows from (5.55) that $\hat{L}^{(PF)}_{\omega,\Psi}$ is a $\mathbb{Z}$-action on $\Gamma(\xi^{(4)})$ whence $(\Gamma(\xi^{(4)}), \hat{L}^{(PF)}_{\omega,\Psi})$ is a $\mathbb{Z}$-space. We conclude from (5.21),(5.22), (5.45),(5.52),(5.53), (5.55) that for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, \sigma \in \Gamma(\xi^{(4)})$

$$\begin{aligned} (\hat{L}^{(PF)}_{\omega,\Psi}(n; \sigma))(\phi) &= (\tilde{L}^{(2)}_{SOT(d)}(\Phi_{\omega,\Psi}(n); \sigma))(\phi) = (\tilde{L}^{(2)}_{SOT(d)}(\varphi_{\omega,\Psi}(n; \cdot), L_{\omega,\Psi}(n; \cdot); \sigma))(\phi) \\ &= \tilde{L}^{(1)}_{SOT(d)}(\varphi_{\omega,\Psi}(n; \cdot), L_{\omega,\Psi}(n; \cdot); \sigma(L_{\omega,\Psi}(-n; \phi))) \\ &= \tilde{L}^{(1)}_{SOT(d)}(\varphi_{\omega,\Psi}(n; \cdot), L_{\omega,\Psi}(n; \cdot); L_{\omega,\Psi}(-n; \phi), \hat{\sigma}(L_{\omega,\Psi}(-n; \phi))) \\ &= \tilde{L}^{(1)}_{SOT(d)}(\varphi_{\omega,\Psi}(n; \cdot), L_{\omega,\Psi}(n; \cdot); \phi - 2\pi n\omega, \hat{\sigma}(\phi - 2\pi n\omega)) \\ &= \tilde{L}^{(1)}_{SOT(d)}(\Phi_{\omega,\Psi}(n); \phi - 2\pi n\omega, \hat{\sigma}(\phi - 2\pi n\omega)) \\ &= \begin{pmatrix} \phi \\ \Psi(n; \phi - 2\pi n\omega)\hat{\sigma}(\phi - 2\pi n\omega) \end{pmatrix} . \end{aligned} \tag{5.56}$$

Since by (5.53) the first component of no $\sigma \in \Gamma(\xi^{(4)})$ carries any information about σ it is not a surprise that the $\mathbb{Z}$-space $(\Gamma(\xi^{(4)}), \hat{L}^{(PF)}_{\omega,\Psi})$ is conjugate to a $\mathbb{Z}$-space which does not carry the redundant first component of (5.53). In fact we define the function $r^{(4)}_{SOT(d)} : \mathcal{C}(\mathbb{R}^d, \mathbb{R}^3) \to \Gamma(\xi^{(4)}_{SOT(d)})$ for $G \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^3)$ and $\phi \in \mathbb{R}^d$ by

$$(r^{(4)}_{SOT(d)}(G))(\phi) := \begin{pmatrix} \phi \\ G(\phi) \end{pmatrix} . \tag{5.57}$$

Note that $r^{(4)}_{SOT(d)}$ is a bijection onto $\Gamma(\xi^{(4)}_{SOT(d)})$. For $\sigma = r^{(4)}_{SOT(d)}(G)$ we have by (5.53), (5.57) that $G = \hat{\sigma}$ whence we conclude from (5.56),(5.57) that for $G \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^3)$ and $n \in \mathbb{Z}, \phi \in \mathbb{R}^d$

$$\begin{aligned} \Big(\hat{L}^{(PF)}_{\omega,\Psi}(n; r^{(4)}_{SOT(d)}(G))\Big)(\phi) &= \begin{pmatrix} \phi \\ \Psi(n; \phi - 2\pi n\omega)G(\phi - 2\pi n\omega) \end{pmatrix} \\ &= \Big(r^{(4)}_{SOT(d)}\Big(\Psi(n; \cdot - 2\pi n\omega)G(\cdot - 2\pi n\omega)\Big)\Big)(\phi) , \end{aligned}$$

so that by (5.50),(5.55) for $G \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^3)$ and $n \in \mathbb{Z}$

$$\begin{aligned} r^{(4)}_{SOT(d)}\Big(\Psi(n;\cdot - 2\pi n\omega)G(\cdot - 2\pi n\omega)\Big) &= \hat{L}^{(PF)}_{\omega,\Psi}(n; r^{(4)}_{SOT(d)}(G)) \\ &= \tilde{L}^{(2)}_{SOT(d)}(\Phi_{\omega,\Psi}(n); r^{(4)}_{SOT(d)}(G)) \\ &= r^{(3)}_{SOT(d)}\Big(L^{(2)}_{SOT(d)}(\Phi_{\omega,\Psi}(n); (r^{(3)}_{SOT(d)})^{-1}(r^{(4)}_{SOT(d)}(G)))\Big) . \end{aligned} \tag{5.58}$$

Defining the function $r^{(5)}_{SOT(d)} : \mathcal{C}(\mathbb{R}^d, \mathbb{R}^3) \to \Gamma(\xi^{(3)}_{SOT(d)})$ by $r^{(5)}_{SOT(d)} := (r^{(3)}_{SOT(d)})^{-1} \circ r^{(4)}_{SOT(d)}$ we observe that $r^{(5)}_{SOT(d)}$ is a bijection onto $\Gamma(\xi^{(3)}_{SOT(d)})$ and that by (5.58) for $G \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^3)$ and $n \in \mathbb{Z}$

$$\begin{aligned} &\Psi(n;\cdot - 2\pi n\omega)G(\cdot - 2\pi n\omega) \\ &= (r^{(4)}_{SOT(d)})^{-1}\bigg(r^{(3)}_{SOT(d)}\Big(L^{(2)}_{SOT(d)}(\Phi_{\omega,\Psi}(n); (r^{(3)}_{SOT(d)})^{-1}(r^{(4)}_{SOT(d)}(G)))\Big)\bigg) \\ &= (r^{(5)}_{SOT(d)})^{-1}\Big(L^{(2)}_{SOT(d)}(\Phi_{\omega,\Psi}(n); r^{(5)}_{SOT(d)}(G))\Big) . \end{aligned} \tag{5.59}$$

By (2.20) we have for $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ and $n \in \mathbb{Z}$ that $\Psi(n;\cdot - 2\pi n\omega)G(\cdot - 2\pi n\omega) = L^{(PF)}_{\omega,\Psi}(n; G)$ whence by (5.59) we obtain the remarkable result that for $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ and $n \in \mathbb{Z}$

$$L^{(PF)}_{\omega,\Psi}(n; G) = (r^{(5)}_{SOT(d)})^{-1}\Big(L^{(2)}_{SOT(d)}(\Phi_{\omega,\Psi}(n); r^{(5)}_{SOT(d)}(G))\Big) , \tag{5.60}$$

which tells us how $\lambda_{SOT(d)}$ carries $L^{(PF)}_{\omega,\Psi}$ in a canonical way. In particular since $L^{(2)}_{SOT(d)}$ acts on $\Gamma(\xi^{(3)}_{SOT(d)})$ we see in (5.60) a close relationship between polarization fields and cross sections of the associated bundle.

5.3.5 Reducing the structure group $SO(3)$

The most important objectives of the Feres machinery are the reduction theorems and the rigidity theorems [Fer] and in this section we will be concerned with the

former (the latter are beyond the scope of this work). The reduction theorems deal, in our context, with the reduction of the structure group $SO(3)$ of $\lambda_{SOT(d)}$ to a closed subgroup of $SO(3)$ and its impact on the dynamics, i.e., on $SOT(d)$. This leads us to Theorem 5.5.

Let H be a closed topological subgroup of $SO(3)$. Recalling Section D.5, a principal H-bundle, $\hat{\lambda}$, is called a H-reduction of $\lambda_{SOT(d)}$ if the total space of $\hat{\lambda}$ is a closed topological subspace $\hat{E}$ of the total space $\mathbb{R}^d \times SO(3)$ of $\lambda_{SOT(d)}$ and if $\hat{\lambda}$ has the form

$$\hat{\lambda} = (\hat{E}, p^{(1)}_{SOT(d)}\Big|\hat{E}, \mathbb{R}^d, R^{(1)}_{SOT(d)}\Big|(H \times \hat{E})) . \tag{5.61}$$

Note that two H-reductions of $\lambda_{SOT(d)}$ are different iff their total spaces are different. The set of all H-reductions of $\lambda_{SOT(d)}$ is denoted by $RED_H(\lambda_{SOT(d)})$. The condition that $\hat{\lambda}$ is a principal H-bundle is a strong restriction on the possible forms of $\hat{E}$ and the following proposition gives an account of this.

Proposition 5.3 *Let H be a closed topological subgroup of $SO(3)$.*
If $f \in \mathcal{C}(\mathbb{R}^d, SO(3)/H)$ then $\check{E}_{f,H}$, defined by

$$\check{E}_{f,H} := \{(\phi, R) \in \mathbb{R}^d \times SO(3) : f(\phi) = RH\} , \tag{5.62}$$

is a closed subspace of $\mathbb{R}^d \times SO(3)$ where $RH := \{RR' : R' \in H\}$ and where the space $SO(3)/H$ is defined in Section D.5. Moreover, if $f \in \mathcal{C}(\mathbb{R}^d, SO(3)/H)$ then the quadruple:

$$\widehat{MAIN}_{\lambda_{SOT(d)},H}(f) := (\check{E}_{f,H}, p^{(1)}_{SOT(d)}\Big|\check{E}_{f,H}, \mathbb{R}^d, R^{(1)}_{SOT(d)}\Big|(H \times \check{E}_{f,H})) , \tag{5.63}$$

is a H-reduction of $\lambda_{SOT(d)}$. Furthermore $\widehat{MAIN}_{\lambda_{SOT(d)},H}$ is a bijection from $\mathcal{C}(\mathbb{R}^d, SO(3)/H)$ onto $RED_H(\lambda_{SOT(d)})$. In particular, every H-reduction of $\lambda_{SOT(d)}$ is of the form (5.63).

Proof of Proposition 5.3: See Section E.28. □

While Proposition 5.3 states a one-one correspondence between $RED_H(\lambda_{SOT(d)})$ and $\mathcal{C}(\mathbb{R}^d, SO(3)/H)$ there is also a one-one correspondence between $RED_H(\lambda_{SOT(d)})$ and the cross sections of the associated bundle $\lambda_{SOT(d)}[SO(3)/H, L_{SO(3)/H}]$ where the left $SO(3)$-action $L_{SO(3)/H}$ is defined by (D.62). In fact it follows from Theorem D.2b in Section D.6.6 that the function $MAIN_{\lambda_{SOT(d)},H} : \Gamma(\lambda_{SOT(d)}[SO(3)/H, L_{SO(3)/H}]) \to RED_H(\lambda_{SOT(d)})$, which is defined by (D.159), is a bijection onto $RED_H(\lambda_{SOT(d)})$. However we here do not need $MAIN_{\lambda_{SOT(d)},H}$ but rather focus on $\widehat{MAIN}_{\lambda_{SOT(d)},H}$.

The following proposition builds up on the fact that $SO_3(2)$ is a closed topological subgroup of $SO(3)$ (see Definition B.2).

Proposition 5.4 *a) The function $F : SO(3)/SO_3(2) \to \mathbb{S}^2$, defined for $R \in SO(3)$ by*

$$F(RSO_3(2)) := L^{(3D)}(R; e^3) = Re^3 , \tag{5.64}$$

is a homeomorphism onto $\mathbb{S}^2$ where $RSO_3(2)) := \{RR' : R' \in SO_3(2))\}$ and where $L^{(3D)}$ is defined by (5.31). Moreover for $S \in \mathbb{S}^2, R, R' \in SO(3)$

$$F(L_{SO(3)/SO_3(2)}(R'; RSO_3(2))) = L^{(3D)}(R'; F(RSO_3(2))) , \tag{5.65}$$

$$F^{-1}(L^{(3D)}(R; S)) = L_{SO(3)/SO_3(2)}(R; F^{-1}(S)) , \tag{5.66}$$

where $L_{SO(3)/SO_3(2)}$ is defined by (D.62).

b) For every $f \in \mathcal{C}(\mathbb{R}^d, SO(3)/SO_3(2))$ we have

$$\check{E}_{f,SO_3(2)} = \{(\phi, R) \in \mathbb{R}^d \times SO(3) : (F \circ f)(\phi) = Re^3\} . \tag{5.67}$$

The function $\overline{MAIN}_{\lambda_{SOT(d)},SO_3(2)} : \mathcal{C}(\mathbb{R}^d, \mathbb{S}^2) \to RED_{SO_3(2)}(\lambda_{SOT(d)})$, defined, for $G \in \mathcal{C}(\mathbb{R}^d, \mathbb{S}^2)$, by

$$\overline{MAIN}_{\lambda_{SOT(d)},SO_3(2)}(G) := \widehat{MAIN}_{\lambda_{SOT(d)},SO_3(2)}(F^{-1} \circ G) , \tag{5.68}$$

is a bijection onto $RED_{SO_3(2)}(\lambda_{SOT(d)})$ where $\widehat{MAIN}_{\lambda_{SOT(d)},SO_3(2)}$ is defined by (5.63).

Proof of Proposition 5.4: See Section E.29. □

We recall from Proposition 5.3 that $\widehat{MAIN}_{\lambda_{SOT(d)},H}$ is a bijection from $\mathcal{C}(\mathbb{R}^d, SO(3)/H)$ onto $RED_{SO_3(2)}(\lambda_{SOT(d)})$ whence every H-reduction of $\lambda_{SOT(d)}$ is of the form $\widehat{MAIN}_{\lambda_{SOT(d)},SO_3(2)}(f)$. We now define $RED_{H,per}(\lambda_{SOT(d)})$ by

$$RED_{H,per}(\lambda_{SOT(d)}) := \{\widehat{MAIN}_{\lambda_{SOT(d)},H}(f) : f \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3)/H)\} . \tag{5.69}$$

If $(\varphi, \bar{\varphi}) \in \mathfrak{Aut}_{Bun(SO(3))}(\lambda_{SOT(d)})$ and if $f \in \mathcal{C}(\mathbb{R}^d, SO(3)/H)$ then, recalling Section D.5, $\widehat{MAIN}_{\lambda_{SOT(d)},H}(f)$ is called invariant under $(\varphi, \bar{\varphi})$ if the total space, $\check{E}_{f,H}$, of $\widehat{MAIN}_{\lambda_{SOT(d)},H}(f)$ is invariant under φ, i.e., $\varphi(\check{E}_{f,H}) = \check{E}_{f,H}$ where $\check{E}_{f,H}$ is defined by (5.62). Furthermore if $(\omega, \Psi) \in SOT(d)$ and $f \in \mathcal{C}(\mathbb{R}^d, SO(3)/H)$ then $\widehat{MAIN}_{\lambda_{SOT(d)},H}(f)$ is called invariant under the group $\Phi_{\omega,\Psi}(\mathbb{Z})$ if it is invariant under each $\Phi_{\omega,\Psi}(n)$. Of course, by the special structure of $\mathbb{Z}$ and since $\Phi_{\omega,\Psi}$ is a group homomorphism, $\widehat{MAIN}_{\lambda_{SOT(d)},H}(f)$ is invariant under the group $\Phi_{\omega,\Psi}(\mathbb{Z})$ iff it is invariant under $\Phi_{\omega,\Psi}(1)$, i.e., iff $\varphi_{\omega,\Psi}(1; \check{E}_{f,H}) = \check{E}_{f,H}$ where $\varphi_{\omega,\Psi}$ is defined by (5.22).

Part a) of the following theorem is a special case of Zimmer's reduction theorem [Fer].

Theorem 5.5 *Let $(\omega, \Psi) \in SOT(d)$. Then the following hold.*

a) Let H be a closed topological subgroup of $SO(3)$ and let $f \in \mathcal{C}(\mathbb{R}^d, SO(3)/H)$. Then the H-reduction $\widehat{MAIN}_{\lambda_{SOT(d)},H}(f)$ of $\lambda_{SOT(d)}$ is invariant under the group $\Phi_{\omega,\Psi}(\mathbb{Z})$ iff, for every $\phi \in \mathbb{R}^d$,

$$f(L_\omega(1; \phi)) = L_{SO(3)/H}(\Psi(1; \phi); f(\phi)) , \tag{5.70}$$

where $L_{SO(3)/H}$ is defined by (D.62).

b) Let $G \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$. Then the $SO_3(2)$-reduction $\widehat{MAIN}_{\lambda_{SOT(d)},SO_3(2)}(F^{-1} \circ G)$ of $\lambda_{SOT(d)}$ is invariant under $\Phi_{\omega,\Psi}(\mathbb{Z})$ iff S_G is an invariant spin field of (ω, Ψ). In particular (ω, Ψ) has an invariant spin field iff $\lambda_{SOT(d)}$ has a 2π-periodic $SO_3(2)$-reduction which is invariant under $\Phi_{\omega,\Psi}(\mathbb{Z})$.

Proof of Theorem 5.5: See Section E.30. □

Note by (5.63),(5.67) and Theorem 5.5b that if $(\omega,\Psi) \in \mathcal{SOT}(d)$ and $\mathcal{S}_G$ is an invariant spin field of (ω,Ψ) then the total space of the invariant $SO_3(2)$-reduction $\widehat{MAIN}_{\lambda_{SOT(d)},SO_3(2)}(F^{-1}\circ G)$ of $\lambda_{SOT(d)}$ has the form

$$\check{E}_{F^{-1}\circ G,SO_3(2)} = \{(\phi,R) \in \mathbb{R}^d \times SO(3) : G(\phi) = Re^3\} \,. \qquad (5.71)$$

Thus (5.71) represents the invariant spin field $\mathcal{S}_G$ by a subset of $\mathbb{R}^d \times SO(3)$, i.e., we have a 'geometrization' of invariant spin fields.

Another aspect of Theorem 5.5 is the following. While, by Theorem 5.5b, invariant spin fields are linked to 2π-periodic invariant $SO_3(2)$-reductions of $\lambda_{SOT(d)}$, it is easy to show, by Theorem 5.5a, that spin-orbit resonances of first kind are linked to 2π-periodic invariant H-reductions of $\lambda_{SOT(d)}$ where H is the trivial subgroup of $SO(3)$.

5.3.6 Closing remarks on $\lambda_{SOT(d)}$

This completes the coverage of principal bundles since our only objective in this regard was to show how the principal $SO(3)$-bundle $\lambda_{SOT(d)}$ underlies the theory of $\mathcal{SOT}(d)$.

Following the Feres machinery one could extend our study. However this would go beyond the scope of the present work so I just mention four points. Firstly, by using the linearity of $L^{(3D)}(R;S)$ in S, one can extend the structure group from $SO(3)$ to $GL(3)$ and study, by a 'prolongation' of the principal $SO(3)$-bundle $\lambda_{SOT(d)}$ to a principal $GL(3)$-bundle, the $\mathbb{Z}$-actions $L_{\omega,\Psi}$ and $L^{(PF)}_{\omega,\Psi}$ in terms of vector bundle techniques ($GL(3)$ denotes the group of real nonsingular 3×3–matrices). Secondly, one can go beyond Theorem 5.5 to study invariant H-reductions of $\lambda_{SOT(d)}$ in a more general way by asking what closed subgroups H of $SO(3)$ allow for 2π-periodic H-

reductions which are invariant under a given spin-orbit torus in $\mathcal{SOT}(d)$. For such a study the 'algebraic hull' is an important tool which was introduced by Zimmer in the 1980's. Thirdly one can apply rigidity theorems which allow to discuss properties which are stable (='rigid') under the extension of the time group $\mathbb{Z}$. Fourthly, the choice of $\lambda_{SOT(d)}$ is not unique. For example an alternative choice is to employ $\mathbb{T}^d$ rather than $\mathbb{R}^d$ in the definition of the total resp. base space of the principal $SO(3)$-bundle. In fact this alternative choice is very convenient when one would go deeper into the matter of spin-orbit tori but for the purposes of the present work the choice of $\lambda_{SOT(d)}$ is sufficient and leads to analogous results as if one would use $\mathbb{T}^d$ instead of $\mathbb{R}^d$.

Chapter 6

Summary of spin-orbit theory and conclusions

As pointed out in the Introduction, this work studies the combined system of discrete time equations of motion (2.1),(2.2) for spin and orbit which plays a central role in the benchmark study of polarized beams in storage rings.

From a technical point of view a distinguishing feature of the present work is to formulate all concepts and properties in mathematical terms. Accordingly the mathematical notion of spin-orbit torus is introduced and a number of properties of spin-orbit tori are derived. Most of the definitions I employ are destillations of established concepts from polarized beam physics into the language of mathematics. To my knowledge some of the results are completely new (e.g., Theorem 4.17 on the impact of Homotopy Theory on spin tunes) and some results which are not new (e.g., Yokoya's uniqueness theorem 3.13) were never formulated in mathematical precise terms whence were never rigorously proved before.

From a conceptional point of view a distinguishing feature of the present work is to employ a transformation theory of spin-orbit tori (see Section 3) with the purpose

to clarify established concepts like 'spin tune'.

For a more detailed outline of this work see Section 1.2. Avenues for further work are of course plentiful. In addition to those mentioned in Section 1.3, one topic of further studies could be the continuation of the work of Section 5.3. In fact as outlined in Section 5.3.6, there are further applications of the principal $SO(3)$-bundle $\lambda_{SOT(d)}$ in wait which may shed further light into the matter of spin-orbit tori.

What would the new avenues bring? Physics

Appendices

Appendix A

Group actions

If X is a set, G a group with identity e_G and $L : G \times X \to X$ a function satisfying, for $g, h \in G, x \in X$,

$$L(e_G; x) = x \tag{A.1}$$

$$L(gh; x) = L(g; L(h; x)) , \tag{A.2}$$

then L is called a left G-action on X and the pair (X, L) is called a left G-space. G-spaces are often called 'transformation groups'. Note that the group law of G is written multiplicatively in (A.2) and it is obvious how (A.2) would read if the group law of G is written additively (the latter convention is common if the group G is Abelian). It follows from (A.1),(A.2) that each $L(g; \cdot)$ is a bijection from X onto X. A left G-action L on X is called 'transitive' if for every pair of elements x, y of X a $g \in G$ exists such that $L(g; x) = y$. If G, G' are groups and $\Phi : G \to G'$ is a group homomorphism and if (X, L') is a left G'-space then (X, L) is a left G-space where we define, for $g \in G, x \in X$,

$$L(g; x) := L'(\Phi(g); x) . \tag{A.3}$$

In this work a topological group is defined in the common, broad sense as in [Hus]. If X is a topological space, G is a topological group, and (X, L) is a left G-space such that the L is continuous, then (X, L) is called a topological left G-space. Of course in that case each $L(g; \cdot)$ is a homeomorphism from X onto X. In the important subcase when the topology of G is discrete (e.g., when $G = \mathbb{Z}$) the condition that L is continuous is equivalent to $L(g; \cdot)$ being continuous for all $g \in G$.

If $(X, L), (X', L')$ are left G-spaces and if $f : X \to X'$ is a function satisfying, for $g \in G, x \in X$,

$$f(L(g; x)) = L'(g; f(x)) , \tag{A.4}$$

then f is called a G-map from (X, L) to (X', L'). G-maps are often called 'equivariant'. One calls $(X, L), (X', L')$ "conjugate" if the G-map f is a bijection onto X'. In the special case $G = \mathbb{Z}$ the function f is a G-map iff (A.4) holds just for $g = 1, x \in X$.

If the G-map f is onto X' then the left G-space (X, L) is called an extension of the left G-space (X', L'). In the special case where the extension (X, L) has the form $(X' \times Y, L)$ for some set Y and if f is the natural projection from from $X' \times Y$ onto X', then the left G-space (X, L) is called a skew product of the left G-space (X', L').

Remark:

(1) Let (X', L'), $(X' \times Y, L)$ be left G-spaces and let $(X' \times Y, L)$ be a skew product of (X', L'). This is a strong restriction on L, as follows.

By eq. (A.2), we have, for $g \in G, x' \in X', y \in Y$,

$$L(g; x', y) = \begin{pmatrix} L'(g; x') \\ L''(g; x', y) \end{pmatrix} , \tag{A.5}$$

where the function $L'' : G \times X' \times Y \to Y$ satisfies, for $g, h \in G, x' \in X', y \in Y$,

$$L''(e_G; x', y) = y , \tag{A.6}$$

$$L''(gh; x', y) = L''(g; L'(h; x'), L''(h; x', y)) , \tag{A.7}$$

which is the announced restriction on L. □

If $(X, L), (X', L')$ are topological left G-spaces and if a continuous G-map f exists from (X, L) to (X', L') which is a homeomorphism onto X', then the topological left G-spaces $(X, L), (X', L')$ are called "conjugate". If $(X, L), (X', L')$ are topological left G-spaces and if a continuous G-map f exists from (X, L) to (X', L') such that f is onto X', then the topological left G-space (X, L) is called an extension of the topological left G-space (X', L'). In the special case where the extension (X, L) has the form $(X' \times Y, L)$ for some topological space Y and if f is the natural projection from $X' \times Y$ onto X', then the topological left G-space (X, L) is called a skew product of the topological left G-space (X', L') (note that $X' \times Y$ is equipped with the product topology).

New
see remark

If (X, L) is a topological left G-space and H is a topological group then a function $f \in \mathcal{C}(G \times X, H)$ is called a continuous H-cocycle over the topological left G-space (X, L) if, for $g, g' \in G, x \in X$,

$$f(gg', x) = f(g, L(g'; x))f(g', x) . \tag{A.8}$$

We define, for given X, G, H, the set $COC(X, G, H)$ as the collection of pairs (L, f) with the property that (X, L) is a topological left G-space and that f is a continuous H-cocycle over (X, L). For literature on cocycles, see, e.g., [HK1, KR, Zim1]. Note also that two conventions for the definition of cocycles are used: ours and the 'dual' one. In the latter convention (see e.g. [KR, Zim1]) $(f(g, x))^{-1}$, not $f(g, x)$, is a cocycle. However for convenience we stick to our convention which is the same as in [HK1].

Right G-actions are defined in direct analogy to left G-actions. In fact, if X is a set, G a group with identity e_G and $R : G \times X \to X$ a map satisfying, for $g, h \in G, x \in X$,

$$R(e_G; x) = x , \tag{A.9}$$

$$R(gh; x) = R(h; R(g; x)) , \tag{A.10}$$

then R is called a right G-action on X and the pair (X, R) is called a right G-space. Due to the close analogy of the concepts of right G-action and left G-action it is obvious how a topological right G-space, a G-map etc. are defined.

As is common, we will often skip the word 'left', i.e., we often call a left G-action a G-action, and a left G-space a G-space etc. This convention is especially useful if G is Abelian since in that case left and right G-actions are the same. Thus, for example, the terminology '$\mathbb{Z}$-action' is a stringent alternative to the equally valid but less stringent terminology 'left $\mathbb{Z}$-action'.

The following facts about right G-spaces are important for principal bundles (the latter are treated in Appendix D) so let (X, R) be a right G-space. Let the set $X^\bullet$ be defined by $X^\bullet := \{(x, R(g;x)) : g \in G, x \in X\}$ and the function $\sigma_R : G \times X \to X^\bullet$ be defined by $\sigma_R(g,x) := (x, R(g;x))$. Clearly σ_R is onto $X^\bullet$. The right G-action R is called free if, for all $x \in X$, the equality: $R(g;x) = x$ implies: $g = e_G$. It is easy to see that R is free iff σ_R is one-one. In fact, if $\sigma_R(g,x) = \sigma_R(g',x')$ then $(x, R(g;x)) = (x', R(g';x'))$ whence, if R is free, $x = x', g = g'$ so that σ_R is one-one. Conversely, let $R(g;x) = x$. Thus $\sigma_R(g,x) = (x, R(g;x)) = (x,x) = (x, R(e_G;x)) = \sigma_R(e_G, x)$ whence, if σ_R is one-one, $g = e_G$ so that R is free. We thus have shown that R is free iff σ_R is one-one. Therefore, since σ_R is onto $X^\bullet$, R is free iff σ_R is a bijection from $G \times X$ onto $X^\bullet$. Of course if R is free the inverse σ_R^{-1} is well defined and one then defines the function $\tau_R : X^\bullet \to G$ by $\tau_R := pr_1 \circ \sigma_R^{-1}$ where $pr_1(g,x) := g$. If R is free one calls τ_R the "translation function" of R. Note that if R is free then for $g \in G, x \in X$ we have $R(\tau_R(x, R(g,x)), x) = R((pr_1 \circ \sigma_R^{-1})(x, R(g,x)), x) = R(pr_1(g,x), x) = R(g,x)$ whence for $x, x' \in X$ we have $R(\tau_R(x,x'), x) = x'$. Of course if R is free then τ_R is the only function $\tau : X^\bullet \to G$ which satisfies, for $x, x' \in X$, $R(\tau(x,x'), x) = x'$. A topological right G-space (X, R) is called "principal" if R is free and if τ_R is continuous.

If (X, R) is a right G-space and $x \in X$ then the set $\{R(g;x) : g \in G\}$ is called

the orbit of x under R. The set of orbits under R is denoted by X/R and the map $p_R : X \to X/R$ is defined by

$$p_R(x) := \{R(g;x) : g \in G\} = \bigcup_{g \in G} \{R(g;x)\}\,. \tag{A.11}$$

Clearly p_R is onto X/R. Note that, for $x, y \in X$, we have that $p_R(x) = p_R(y)$ iff $y \in p_R(x)$. Thus, for $x \in X$,

$$\begin{aligned} p_R^{-1}(p_R(\{x\})) &= p_R^{-1}(\{p_R(x)\}) = \{y \in X : p_R(y) \in \{p_R(x)\}\} \\ &= \{y \in X : p_R(y) = p_R(x)\} = \{y \in X : y \in p_R(x)\} = p_R(x)\,. \end{aligned} \tag{A.12}$$

It follows from (A.11),(A.12) that for $A \subset X$

$$\begin{aligned} p_R^{-1}(p_R(A)) &= p_R^{-1}(p_R(\bigcup_{x \in A} \{x\})) = p_R^{-1}\Big(\bigcup_{x \in A} p_R(\{x\})\Big) = \bigcup_{x \in A} p_R^{-1}(p_R(\{x\})) \\ &= \bigcup_{x \in A} p_R(x) = \bigcup_{x \in A} \bigcup_{g \in G} \{R(g;x)\} = \bigcup_{g \in G} \bigcup_{x \in A} \{R(g;x)\} = \bigcup_{g \in G} R(g;A)\,. \end{aligned} \tag{A.13}$$

If X is a topological space and (X, R) is a topological right G-space then one equips X/R with the quotient topology w.r.t. p_R, i.e., a subset U of X/R is open iff $p_R^{-1}(U)$ is open in X. Thus the map p_R is identifying and one calls X/R an "orbit space". To show that p_R is open, let U be open in X whence, by (A.13),

$$p_R^{-1}(p_R(U)) = \bigcup_{g \in G} R(g;U)\,. \tag{A.14}$$

Since each $R(g;\cdot)$ is a homeomorphism from X onto X we have that $R(g;U)$ is open in X whence $\bigcup_{g \in G} R(g;U)$ is open in X. Thus, by (A.14), $p_R^{-1}(p_R(U))$ is open in X. Since the topology of X/R is the quotient topology w.r.t. p_R we have that $p_R(U)$ is open in X/R whence p_R is open.

There are many textbook treatments of group action. Two useful textbooks, dedicated to group actions, are [Die2, Kaw].

Appendix B

Topological concepts and facts

In this section we provide some concepts and facts from Topology, in particular some know-how about *liftings* and *factors* of *bundles* and *fiber structures* (see Definition B.1). This know-how is especially useful for continuous and 2π-periodic functions like $\Psi(n;\cdot)$ arising in the study of spin-orbit tori (ω, Ψ). The concept of bundle is also of importance for us in Section D where we refine it to the concept of principal bundle. As in Section A, we present the material in such detail that it is essentially self contained.

Hurewicz fibrations (see Definition B.5) are fiber structures which satisfy a certain condition. In fact, for our purposes, a Hurewicz fibration has sufficient structure to obtain from a continuous function a lifting which is a continuous function as well. While liftings provide a tool to obtain continuous functions, factors provide another tool to obtain continuous functions (namely to turn 2π-periodic functions on $\mathbb{R}^k$ into functions on the *k-torus* $\mathbb{T}^k$ defined below). For these matters we introduce with Definition B.2 four well-known fiber structures and demonstrate in Section B.1 that all four of them are Hurewicz fibrations. They will be used for liftings and one of them will be used for factors. Three of the four *projections* (see Definition B.1) are

covering maps (see Definition B.7). Note that fiber structures (and even Hurewicz fibrations) are pretty simple concepts which do not involve any group actions. Thus in this section we neither employ the machinery of principal bundles nor do we need Category theory (see however Appendix D). The know-how we use about liftings and Hurewicz fibrations can be found in [Dug, Spa] and the know-how about factors in [SZ]. See also [Bre, Di2, Rot, Die1]. Our terminology is close to [Dug, Hus].

B.1 Bundles, fiber structures and Hurewicz fibrations

In this section we choose our four fiber structures and show that they are Hurewicz fibrations. The search for liftings w.r.t. our fiber structures is the content of Sections B.2 and B.3. In Section B.3 this search will be facilitated by the use of *factors* (see Definition B.1) w.r.t. one of the four fiber structures (the latter fiber structure is also used in Section C.2).

Definition B.1 *(Bundle, fiber structure, lifting, factor, cross section, locally trivial) Given topological spaces X, Y, we denote the set of continuous functions from X into Y by $\mathcal{C}(X,Y)$ and the set of homeomorphisms from X onto Y by $HOMEO(X,Y)$.*

A triple (E,p,B) is called a bundle *if E and B are topological spaces and if p is in $\mathcal{C}(E,B)$. A bundle (E,p,B) is called a* fiber structure *if p is onto B. One calls E the* total space, *B the* base space *and p the* projection *of the bundle. For $b \in B$, $p^{-1}(b)$ is called the* fibre of p over b *and its topology is defined as the relative topology from E.*

If $\xi = (E,p,B)$ is a bundle, X is a topological space and $g \in \mathcal{C}(X,B)$, then $f \in \mathcal{C}(X,E)$ is called a lifting *of f w.r.t. the bundle ξ if $g = p \circ f$. If $g \in \mathcal{C}(E,X)$*

then a $f \in C(B, X)$ *is called a* factor *of* g *w.r.t. the bundle* ξ *if* $g = f \circ p$. *If* $\sigma \in C(B, E)$ *satisfies* $id_B = p \circ \sigma$, *where* id_B *is the identity map on* B, *then one calls* σ *a* cross section *of* ξ. *The set of cross sections of* ξ *is denoted by* $\Gamma(\xi)$.

A fiber structure (E, p, B) *is called* locally trivial *if for every* $b \in B$ *an open neighborhood* U *of* b, *a topological space* Y *and a homeomorphism* $\varphi : U \times Y \to p^{-1}(U)$ *onto* $p^{-1}(U)$ *exist such that, for all* $x \in U, y \in Y$, $p \circ \varphi(x, y) = x$ *where* $U \times Y$ *has the product topology,* U *has the relative topology from* B *and* $p^{-1}(U)$ *has the relative topology from* E. □

Remark:

(1) Our notion of 'bundle' is from [Hus] and our notion of 'fiber structure' is from [Dug] and all concrete examples of bundles in this work are fiber structures. Note that a bundle which has a cross section is a fiber structure. If $\xi = (E, p, B)$ is a fiber structure and X a topological space then, since p is onto B, every $g \in C(E, X)$ has at most one factor w.r.t. ξ.

Clearly the concepts of bundle and fiber structure are trivial and the topologies of the fibres in a fiber structure are in general largely unrelated - in particular they are in general not homeomorphic. However a fiber structure has a lot of structure if it is locally trivial. In particular for locally trivial fiber structure (E, p, B), every $b \in B$ has an open neighborhood U such that the fibres $p^{-1}(u)$ with $u \in U$ are homeomorphic. We will see that the four fiber structures to be introduced in this section are locally trivial, a circumstance which makes it easy to show, again in this section, that all four of them are Hurewicz fibrations. □

Definition B.2 *A function on* $\mathbb{R}^k$ *is called* 2π*-periodic if it is* 2π*-periodic in all* k *arguments. If* Y *is a topological space, we denote the set of* 2π*-periodic functions in* $C(\mathbb{R}^k, Y)$ *by* $C_{per}(\mathbb{R}^k, Y)$. *The set* $SO(3)$ *consists of those real* 3×3*-matrices* R *with*

$\det(R) = 1$ *for which* $R^T R = I_{3\times3}$ *where* R^T *denotes the transpose of* R *and* $I_{3\times3}$ *the* 3×3 *unit matrix. We define*

$$\mathcal{J} := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad SO_3(2) := \{\exp(2\pi x \mathcal{J}) : x \in \mathbb{R}\} \subset SO(3), \quad \text{(B.1)}$$

and consider $SO_3(2)$ *as a topological subspace of* $SO(3)$. *Denoting the fractional part of a real number* x *by* $\lfloor x \rfloor$, *we obtain for* $x \in \mathbb{R}$

$$\exp(2\pi x \mathcal{J}) = \exp(2\pi \lfloor x \rfloor \mathcal{J}) = \begin{pmatrix} \cos(2\pi x) & -\sin(2\pi x) & 0 \\ \sin(2\pi x) & \cos(2\pi x) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \text{(B.2)}$$

Thus $SO_3(2)$ *is, under matrix multiplication, an Abelian subgroup of* $SO(3)$. *Clearly for every* $R \in SO_3(2)$ *a unique* $r \in [0, 1)$ *exists such that* $p_1(2\pi r) = \exp(2\pi r \mathcal{J})$ *and we abbreviate* $PH(R) := r$ *and call* $PH(R)$ *the "phase" of* R. *The function* $p_1 : \mathbb{R} \to SO_3(2)$, *defined by* $p_1(y) := \exp(y\mathcal{J})$, *clearly belongs to* $C_{per}(\mathbb{R}, SO_3(2))$ *and is onto* $SO_3(2)$ *whence* $(\mathbb{R}, p_1, SO_3(2))$ *is a fiber structure.*

We define the k*-sphere* $\mathbb{S}^k := \{x \in \mathbb{R}^{k+1} : |x| = 1\}$ *(*k *positive integer) and equip it with the relative topology from* $\mathbb{R}^{k+1}$. *We define the function* $p_2 : \mathbb{S}^3 \to SO(3)$ *by* $p_2(\bar{r})x := (2r_0^2 - 1)x + 2r(r^T x) + 2r_0(r \times x)$, *where* $\bar{r} =: (r_0, r) \in \mathbb{S}^3, r_0 \in \mathbb{R}, r \in \mathbb{R}^3$ *and* $x \in \mathbb{R}^3$. *Since the topology of* $SO(3)$ *is defined as the relative topology from* $\mathbb{R}^{3\times3}$, $p_2 \in C(\mathbb{S}^3, SO(3))$. *Note that the trace of* $p_2(\bar{r})$ *reads as* $Tr[p_2(\bar{r})] = 4r_0^2 - 1$. *On* $\mathbb{S}^3$ *one introduces a multiplication by* $(r_0, r)(s_0, s) = (r_0 s_0 - r^T s, r_0 s + s_0 r + r \times s)$ *where* $r_0, s_0 \in \mathbb{R}, r, s \in \mathbb{R}^3$. *One observes that* $\mathbb{S}^3$ *is a topological group whose unit element is* $(1, 0, 0, 0)^T$. *The inverse of* (r_0, r) *is* $(r_0, -r)$. *Moreover* p_2 *is a group homomorphism, i.e.* $p_2(\bar{r}\bar{s}) = p_2(\bar{r})p_2(\bar{s})$. *It is thus easy to show that* p_2 *is onto* $SO(3)$ *hence* $(\mathbb{S}^3, p_2, SO(3))$ *is a fiber structure.*

We define the function $p_3 : SO(3) \to \mathbb{S}^2$ *by* $p_3(R) := Re^3$, *where* e^3 *denotes the third unit vector, i.e.,* $e^3 = (0, 0, 1)^T$. *More generally,* e^i *denotes the* i*-th unit* *vector in any $\mathbb{R}^k$, i.e., $(e^i)_i := 1$ and, for $i \neq j$, $(e^i)_j := 0$. It is easy to see that $p_3 \in \mathcal{C}(SO(3), \mathbb{S}^2)$ and is onto $\mathbb{S}^2$ whence $(SO(3), p_3, \mathbb{S}^2)$ is a fiber structure.*

We define the complex unit circle $\mathbb{T} := \{x \in \mathbb{C} : |x| = 1\}$ and the k-torus $\mathbb{T}^k$, i.e., the k-fold cartesian product of $\mathbb{T}$ (whenever we write $\mathbb{T}^k$, this implies that k is a positive integer). We consider $\mathbb{T}$ as a topological subspace of $\mathbb{C}$ and $\mathbb{T}^k$ as the topological product of its k factors. Defining $p_{4,k} : \mathbb{R}^k \to \mathbb{T}^k$ by $p_{4,k}(\phi) := (\exp(i\phi_1), ..., \exp(i\phi_k))^T$ it is easy to see that $p_{4,k} \in \mathcal{C}_{per}(\mathbb{R}^k, \mathbb{T}^k)$ and is onto $\mathbb{T}^k$ whence $(\mathbb{R}^k, p_{4,k}, \mathbb{T}^k)$ is a fiber structure. □

Having defined our four fiber structures, the remaining task of this section is to show that all of them are Hurewicz fibrations. Since the notion of Hurewicz fibration is closely related to Homotopy Theory we first need

Definition B.3 *(Homotopic functions) Let X, Y be topological spaces and let $f_i \in \mathcal{C}(X, Y)$ be continuous maps where $i = 0, 1$. Then we write $f_0 \simeq_Y f_1$ if a $h \in \mathcal{C}(X \times [0,1], Y)$ exists such that $h(\cdot, 0) = f_0$ and $h(\cdot, 1) = f_1$ where $X \times [0,1]$ is equipped with the product topology and $[0,1]$ is equipped with the relative topology from $\mathbb{R}$. One then says that f_0, f_1 are* homotopic *w.r.t. Y. It is easily shown (see, e.g., [Rot, Spa]) that $\simeq_Y$ is an equivalence relation on $\mathcal{C}(X, Y)$ and we denote by $[X, Y]$ the set of all equivalence classes.*

Note that for cartesian products like $X \times [0,1]$ we choose the product topology if not mentioned otherwise. A $g \in \mathcal{C}(X, Y)$ is called nullhomotopic w.r.t. Y, if it is homotopic w.r.t. Y to a constant map in $\mathcal{C}(X, Y)$. □

If two functions have different domain then they cannot be homotopic. It is also clear that, in the notation of Definition B.3, always functions exist in $\mathcal{C}(X, Y)$ which are nullhomotopic w.r.t. Y. Note that continuous functions with common domain are often not homotopic. Note that the suffix in $\simeq_Y$ is important. In fact,

for every pair f_0, f_1 of continuous functions on a topological space X one can choose Y sufficiently large such that $f_0 \simeq_Y f_1$ [Dug, Section XV.1]. Nevertheless one often does not mention Y when the context is clear.

Proposition B.4 *a) Let X and Y be topological spaces and let $g_i \in \mathcal{C}(\mathbb{R}^k, X)$ and $f_i \in \mathcal{C}(X, Y)$ where $i = 0, 1$. If $f_0 \simeq_Y f_1$ and $g_0 \simeq_X g_1$ then $f_1 \circ g_1 \simeq_Y f_0 \circ g_0$.*

b) If X is a topological space and if $g \in \mathcal{C}(\mathbb{R}^k, X)$ then g is nullhomotopic w.r.t. X.

c) Let X and Y be topological spaces and let Y be path-connected. Then all $g \in \mathcal{C}(X, Y)$ which are nullhomotopic w.r.t. Y, are homotopic w.r.t. Y. In other words, all $g \in \mathcal{C}(X, Y)$, which are nullhomotopic w.r.t. Y, belong to the same element of $[X, Y]$.

Proof of Proposition B.4a: Let X and Y be topological spaces and let $g_i \in \mathcal{C}(\mathbb{R}^k, X)$ and $f_i \in \mathcal{C}(X, Y)$ where $i = 0, 1$. Thus a $F \in \mathcal{C}(X \times [0,1], Y)$ exists such that $F(\cdot, i) = f_i(\cdot)$ and a $G \in \mathcal{C}(\mathbb{R}^k \times [0,1], X)$ exists such that $G(\cdot, i) = g_i(\cdot)$. The function $H : \mathbb{R}^k \times [0,1] \to Y$, defined by $H(x,t) := F(G(x,t), t)$, is continuous and satisfies $H(x,i) = F(G(x,i), i) = F(g_i(x), i) = f_i(g_i(x))$. Thus $f_1 \circ g_1 \simeq_Y f_0 \circ g_0$. □

Proof of Proposition B.4b: See [Dug, Section XV.1]. □

Proof of Proposition B.4c: See [SZ, Section 2.1]. □

It follows from Proposition B.4 that if X is a path-connected topological space, then all $g \in \mathcal{C}(\mathbb{R}^k, X)$ are homotopic w.r.t. X.

For a fiber structure (E, p, B) and a nonempty subset U of B the function $p|p^{-1}(U) : p^{-1}(U) \to U$ is onto U since p is onto B. Choosing for $p^{-1}(U)$ the relative topology from E and for U the relative topology from B, it is clear that $p|p^{-1}(U)$ is a continuous function hence $(p^{-1}(U), p|p^{-1}(U), U)$ is a fiber structure.

Definition B.5 *(Hurewicz fibration) Let X be a topological space. A fiber structure (E,p,B) is called a* fibration for X *if it has the following property: if $G \in \mathcal{C}(X \times [0,1], B)$ and if $G(\cdot,0)$ has a lifting f w.r.t. (E,p,B) then G has a lifting F w.r.t. (E,p,B) such that $f(\cdot) = F(\cdot,0)$.*

A fiber structure (E,p,B) is called a Hurewicz fibration *if it is a fibration for arbitrary topological spaces X.*

A fiber structure (E,p,B) is called a local Hurewicz fibration *if every $b \in B$ has a neighborhood U such that the fiber structure $(p^{-1}(U), p|p^{-1}(U), U)$ is a Hurewicz fibration. Recall that $p^{-1}(U)$ has the relative topology from E and that U has the relative topology from B.* □

Note that the concept of local Hurewicz fibration will play a role in the proof of Lemma B.6.

We see by Definition B.5 that liftings w.r.t. Hurewicz fibrations can be found by the following method. If (E,p,B) is a Hurewicz fibration and if one looks for a lifting of a continuous map $g : X \to B$ w.r.t. (E,p,B) then one just tries to find a continuous map $g' : X \to B$ with $g \simeq_B g'$ which is so simple that a lifting of g' w.r.t. (E,p,B) can be easily found. As a matter of fact, in Sections B.2, B.3 we will often apply this method.

To show that our four fiber structures are Hurewicz fibrations, the following lemma is crucial.

Lemma B.6 *(Homotopy lifting theorem) Let (E,p,B) be a fiber structure which is locally trivial and let B be a compact Hausdorff space. Then (E,p,B) is a Hurewicz fibration.*

Proof of Lemma B.6: Since B is a compact Hausdorff space, the claim follows by applying [Dug, Corollary XX.3.6] if (E,p,B) is a local Hurewicz fibration.

Thus we only have to show that (E,p,B) is a local Hurewicz fibration so let $b \in B$. By Definition B.1 an open neighborhood U of b, a topological space Y and a homeomorphism $\varphi : U \times Y \to p^{-1}(U)$ onto $p^{-1}(U)$ exist such that, for all $b \in U, y \in Y$, $p \circ \varphi(b,y) = b$. We only have to show that the fiber structure $(p^{-1}(U), p|p^{-1}(U), U)$ is a Hurewicz fibration. Thus let $G \in \mathcal{C}(X \times [0,1], U)$ and let $g(\cdot) := G(\cdot,0)$ have a *lifting* f w.r.t. $(p^{-1}(U), p|p^{-1}(U), U)$. We define the function $F : X \times [0,1] \to p^{-1}(U)$ by $F(x,t) := \varphi\Big(G(x,t), pr_2(\varphi^{-1}(f(x)))\Big)$ where pr_2 is the projection on the second factor, i.e., $pr_2(b,y) = y$. Since φ is a homeomorphism onto $p^{-1}(U)$, F is a continous function. Clearly $p(F(x,t)) = G(x,t)$ whence F is a lifting of G w.r.t. $(p^{-1}(U), p|p^{-1}(U), U)$. Furthermore, for every $e \in p^{-1}(U)$, we have $e = \varphi(\varphi^{-1}(e)) = \varphi\Big(pr_1(\varphi^{-1}(e)), pr_2(\varphi^{-1}(e))\Big) = \varphi\Big(p(e), pr_2(\varphi^{-1}(e))\Big)$ where pr_1 is the projection on the second factor, i.e., $pr_1(b,y) = b$. Hence $F(x,0) = \varphi\Big(G(x,0), pr_2(\varphi^{-1}(F(x,0)))\Big)$. Since also $F(x,0) = \varphi\Big(G(x,0), pr_2(\varphi^{-1}(f(x)))\Big)$ and since φ is a bijection we conclude that $F(\cdot,0) = f(\cdot)$. Since b and X were chosen arbitrarily we thus have shown that (E,p,B) is a local Hurewicz fibration. □

Since the base spaces $SO_3(2), SO(3), \mathbb{S}^2$ and $\mathbb{T}^k$ of our four fiber structures are compact Hausdorff spaces, we see by Lemma B.6 that our aim of proving that these fiber structures are Hurewicz fibrations reduces to showing that they are locally trivial.

We first introduce

Definition B.7 *(Covering map) Let X, Y be topological spaces and $p \in \mathcal{C}(X,Y)$ be onto Y. Then p is called a covering map w.r.t. X and Y if every point of Y has an open neighbourhood U such that $p^{-1}(U)$ is a disjoint union $\bigcup_{\lambda\in\Lambda} U_\lambda$ of open sets $U_\lambda \subset X$ with $p(U_\lambda) = U$ and such that every $p|U_\lambda : U_\lambda \to U$ is a homeomorphism*

Anoltte new Appendices.

See B12