Spin transport, spin diffusion and Bloch equations in electron storage rings *

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We show how, beginning with the Fokker–Planck equation for electrons emitting synchrotron radiation in a storage ring, the corresponding equation for spin motion can be constructed. This is an equation of the Bloch type for the polarisation density.

1. INTRODUCTION and MOTIVATION

In this paper we show how to derive an evolution equation, related to the orbital Fokker–Planck equation, for spin transport in the presence of synchrotron radiation in electron storage rings. The key is to work with the phase space density of spin angular momentum. In this way, we are able to reproduce the spin diffusion terms of a full semiclassical treatment although our approach is largely classical. We begin by outlining some basic aspects of spin polarisation in electron storage rings.

Relativistic electrons circulating in a storage ring emit synchrotron radiation and a tiny fraction of the photons can cause spin flip from up to down and vice versa. However, the up–to–down and down–to–up rates differ so that the beam can become spin polarised antiparallel to the guide field, reaching a maximum polarisation, $P_{\rm ST}$, of $8/(5\sqrt{3}) = 0.924$ in a perfectly aligned flat ring without solenoids. This is the so–called Sokolov–Ternov (ST) effect [1]. The time constant for the exponential build–up is usually in the range of a few minutes to a few hours and decreases with the fifth power of the energy.

However real rings can have misalignments, solenoids and spin rotators and this together with the fact that spins precess in magnetic fields has important consequences for the polarisation which we now outline.

In the absence of spin flip, spin motion for electrons (and protons) moving in electric and magnetic fields is described by the T–BMT equation [2,3] $d\vec{S}/ds = \vec{\Omega} \times \vec{S}$ where $\vec{S}$ is the rest frame spin expectation value of the particle, s is the distance around the ring and $\vec{\Omega}$ depends on the electric and magnetic fields, the velocity and the energy. In magnetic fields $\vec{\Omega} = -e/(mc\gamma)\left[(1+a)\vec{B}_{||} + (1+a\gamma)\vec{B}_{\perp}\right]$ where $\vec{B}_{||}$ and $\vec{B}_{\perp}$ are the laboratory magnetic fields parallel and perpendicular to the trajectory. The gyromagnetic anomaly, $a = (g-2)/2$, for electrons is about 0.0011597. For protons it is about 1.7928. For both species $|\vec{S}| = \hbar/2$. The other symbols have their usual meanings. Thus for motion transverse to the magnetic field, $\vec{S}$ precesses around the field at a rate $(1 + a\gamma)$ faster than the rate of change of the orbit direction [4,5]. For electrons at 27.5 GeV at HERA [6] the *spin enhancement factor* $a\gamma$, is about 62.5. In the following the spin expectation value $\vec{S}$ will, for convenience, sometimes simply be called the "spin".

Synchrotron radiation not only creates polarisation but also produces other effects. In particular, the stochastic element of photon emission together with accompanying damping determines the equilibrium phase space density distribution and the beam can be described by a Fokker–Planck (FP) equation. This is traditionally derived by simulating the stochastic photon

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emission with Gaussian white noise [7–10]. The same photon emission also imparts a stochastic element to $\vec{\Omega}$ and then, via the T–BMT equation, spin diffusion (and thus depolarisation) can occur in the inhomogeneous fields of the ring [11–13]. Thus synchrotron radiation can create polarisation but can also lead to its destruction! The ratio: (depolarisation rate / polarisation rate) increases with the spin enhancement factor. The equilibrium polarisation is the result of a balance between the Sokolov–Ternov effect and this radiative depolarisation so that the attainable polarisation $P_{\rm eq}$ is less than $P_{\rm ST}$. In the approximation that the orbital motion is linear, the *value* of the polarisation is essentially the same at each point in phase space and in s and the polarisation is aligned along the Derbenev–Kondratenko (DK) vector $\hat{n}$ [14,12]. The unit *vector field* $\hat{n}$ depends on s and the position in phase space defined by $\vec{u} \equiv (x, p_x, y, p_y, \Delta s, \delta = \Delta E/E_0)$ [15]. $\hat{n}(\vec{u}(s); s)$ satisfies the T–BMT equation along any orbit $\vec{u}(s)$ and it is periodic in azimuth: $\hat{n}(\vec{u}; s) = \hat{n}(\vec{u}; s + C)$ where C is the ring circumference. On the closed orbit ($\vec{u} = \vec{0}$), $\hat{n}(\vec{u}; s)$ is denoted by $\hat{n}_0(s)$.

Taking into account radiative depolarisation due to photon–induced longitudinal recoils, the equilibrium electron polarisation along the $\hat{n}$ field as given by Derbenev and Kondratenko and by Mane is [14,16]

$$P_{\rm dkm} = -\frac{8}{5\sqrt{3}} \frac{\oint ds \left\langle |K(s)|^3 \hat{b} \cdot (\hat{n} - \frac{\partial \hat{n}}{\partial \delta}) \right\rangle_s}{\oint ds \left\langle |K(s)|^3 (1 - \frac{2}{9}(\hat{n} \cdot \hat{s})^2 + \frac{11}{18} \left(\frac{\partial \hat{n}}{\partial \delta}\right)^2) \right\rangle_s} \qquad (1)$$

where $\hat{b}$ and $\hat{s}$ denote the magnetic field direction and direction of motion respectively, and where $< >_s$ denotes an average over phase space at azimuth s. The quantity $K(s)$ is the orbit curvature due to the magnetic fields. The term $\frac{11}{18}(\partial \hat{n}/\partial \delta)^2$, encapsulates the spin diffusion. The ensemble average of the polarisation is $\vec{P}_{\rm eq}(s) = P_{\rm dkm} \ \langle \hat{n} \rangle_s$ and $\langle \hat{n} \rangle_s$ is very nearly aligned along $\hat{n}_0(s)$. For the perfectly aligned flat ring without solenoids mentioned at the beginning, $\partial \hat{n}/\partial \delta$ vanishes so that $P_{\rm dkm} = 0.924$. Further details on this formalism can be found in Barber and Ripken [12,13].

The Derbenev–Kondratenko–Mane (DKM) formula is based on the reasonable and justifiable assumption that at spin–orbit equilibrium the polarisation is locally parallel to $\hat{n}$ [12]. But it would be more satisfying to have access to a more basic approach free of assumptions. For example, it would be good to have a kind of spin–orbit FP equation which would allow non–equilibrium spin–orbit systems to be studied, and to be able to obtain the DKM result as a special case. An equation of this type for spin has already been derived by Derbenev and Kondratenko [17] using semiclassical radiation theory beginning with the quantum mechanical density operator for the spin–orbit system. This equation includes the effects of spin diffusion, the ST effect and also some "cross terms". At the same time they obtained the FP equation for the orbital motion with the same form as in Barber et al. [7–9]. The derivations based on the semiclassical radiation theory in DK [17] are very arduous but unavoidable for the ST part of the picture. However, one is tempted to try to obtain the *pure spin diffusion* part via the traditional route based on Gaussian white noise in analogy to the description of orbital motion. This would lead to a better appreciation of the results in DK [17] and to more insights. We have succeeded in this approach and proceed by developing our arguments within a purely classical framework using the fact that the spin expectation value $\vec{S}$ for a particle following a classical orbit obeys a classical equation of motion, namely the T-BMT equation. We therefore postpone further discussion of the semiclassical calculation until later.

2. SPIN–ORBIT TRANSPORT WITHOUT RADIATION

In the absence of radiation and other non–Hamiltonian effects and with the orbital Hamiltonian $h_{\rm orb}$, the orbital phase space density $W_{\rm orb}$ evolves according to an equation of the Liouville type:

$$\frac{\partial W_{\rm orb}}{\partial s} = \{h_{\rm orb}, W_{\rm orb}\}_{\vec{u}} \qquad (2)$$

where $\{,\}_{\vec{u}}$ is the Poisson bracket involving derivatives w.r.t. the components of $\vec{u}$. We normalise the density to unity: $\int d^6u \; W_{\rm orb}(\vec{u};s) = 1$.

Since the T–BMT equation is linear in spin, the local polarisation $\vec{P}_{\rm loc}(\vec{u};s)$, which is proportional to an average of the $\vec{S}$ vectors in an infinitesimal packet of phase space at $(\vec{u};s)$, obeys the T–BMT equation along any orbit $\vec{u}(s)$. If $\vec{P}_{\rm loc}(\vec{u};s)$ is a smooth function of $(\vec{u};s)$ we can rewrite this as

$$\frac{\partial \vec{P}_{\rm loc}}{\partial s} \;=\; \{h_{\rm orb}, \vec{P}_{\rm loc}\}_{\vec{u}} + \vec{\Omega} \times \vec{P}_{\rm loc} \; . \tag{3}$$

3. SPIN–ORBIT TRANSPORT WITH RADIATION

To include radiation we model the photon emission as a Gaussian white noise process overlaid onto smooth radiation damping. Then eq. (2) is replaced by a FP equation:

$$\frac{\partial W_{\rm orb}}{\partial s} = \mathcal{L}_{\rm FP,orb} \; W_{\rm orb} \; , \tag{4}$$

where the orbital FP operator can be decomposed into the form:

$$\mathcal{L}_{\rm FP,orb} = \mathcal{L}_{\rm ham} + \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 \tag{5}$$

and where $\mathcal{L}_{\rm ham}$ would result in eq. (2) and $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$ are terms due to damping and noise containing zeroth, first and second order derivatives w.r.t. the components of $\vec{u}$ respectively. The detailed forms for the $\mathcal{L}$'s can be found in Barber et al. [7–9] but are not important for the argument that follows. After a few damping times $W_{\rm orb}$ approaches an equilibrium form.

But how can we write the analogue of eq. (4) for polarisation? After all, to obtain an equation of the FP type we need a density and polarisation is not a density. But we do have the spin angular momentum density. In particular we have the *density in phase space, per particle, of spin angular momentum* $\vec{\mathcal{S}}$ [3], and its close relative the *polarisation density* $\vec{\mathcal{P}} = 2/\hbar \; \vec{\mathcal{S}}$. This latter can be written as

$$\vec{\mathcal{P}}(\vec{u};s) = \vec{P}_{\rm loc}(\vec{u};s) \; W_{\rm orb}(\vec{u};s) \; . \tag{6}$$

[3] i.e. the phase space density of $\vec{S}$ divided by the number of particles in the bunch.

By combining eqs. (2) and (3) we then obtain

$$\frac{\partial \vec{\mathcal{P}}}{\partial s} = \{h_{\rm orb}, \vec{\mathcal{P}}\}_{\vec{u}} + \vec{\Omega} \times \vec{\mathcal{P}} \; . \tag{7}$$

This equation for the polarisation density has the same form as eq. (2) for the phase space density except for the precession term and since eq. (2) is just the radiationless version of eq. (4) we can now guess how the extension of eq. (7) to include radiation will look.

To come further we parametrise the components of $\vec{S}$ in terms of the canonical variables J and ψ defined by the relations $S_1 = \cos(\psi)\sqrt{\hbar^2/4 - J^2}$, $S_2 = \sin(\psi)\sqrt{\hbar^2/4 - J^2}$ and $S_3 = J$ and having the Poisson bracket $\{\psi, J\}_{\psi,J} = 1$ [18,19,15]. These lead to the standard Poisson brackets for angular momentum: $\{S_j, S_k\} = \sum_{m=1}^{3} \varepsilon_{jkm} \; S_m$. The spin variables commute with the orbital variables.

In terms of the combined spin–orbit Hamiltonian $h = h_{orb} + \vec{S}\cdot\vec{\Omega}$ the T–BMT equation can now be written as $d\vec{S}/ds = \{\vec{S}, h\}_{\vec{u},\psi,J}$ and the equations of radiationless orbital motion $d\vec{u}/ds = \{\vec{u}, h\}_{\vec{u},\psi,J}$, are the usual equations of orbital motion except for additional terms accounting for Stern–Gerlach (SG) forces.

We now need the joint spin–orbit density $W(\vec{u},\vec{S};s)$. This contains a factor $\delta(\hbar/2 - |\vec{S}|)$ to account for the fact that we wish to describe processes for which $|\vec{S}| = \hbar/2$ and we normalise W to unity: $\int d^6u \; d^3S \; W(\vec{u},\vec{S};s) = 1$. Moreover $\int d^3S \; W(\vec{u},\vec{S};s) = W_{\rm orb}(\vec{u};s)$.

Equation (6) for the polarisation density can then be written as

$$\vec{\mathcal{P}}(\vec{u};s) = \int d^3S \; \frac{\vec{S}}{|\vec{S}|} \; W(\vec{u},\vec{S};s) \; . \tag{8}$$

The polarisation of the whole beam as measured by a polarimeter at azimuth s is $\int d^6u \; \vec{\mathcal{P}}(\vec{u};s)$.

Since here, spin is a spectator, being only indirectly affected by the radiation through the orbital motion, the FP equation for the combined orbit and spin density is [4]

[4] The critical energy for synchrotron radiation is usually tens of KeV but the SG energy is many orders of magnitude smaller [12]. Therefore the influence of spin motion

$$\frac{\partial W}{\partial s} = \mathcal{L}_{\text{FP,orb}} W - (\vec{\Omega} \times \vec{S}) \cdot (\vec{\nabla}_{\vec{S}} W) = \qquad (9)$$
$$\mathcal{L}_{\text{FP,orb}} W + \vec{\Omega} \cdot \{\vec{S}\,, W\}_{\psi,J}$$

where $\vec{\nabla}_{\vec{S}} W$ is the gradient of W w.r.t. the three components of spin. The spin part of the corresponding Langevin equation has no noise terms.

Using eq. (9) we can write

$$\int d^3S \; \frac{\vec{S}}{|\vec{S}|} \; \frac{\partial W}{\partial s} =$$
$$\int d^3S \; \frac{\vec{S}}{|\vec{S}|} \left(\mathcal{L}_{\text{FP,orb}} W + \vec{\Omega} \cdot \{\vec{S}\,, W\}_{\psi,J} \right) \quad (10)$$

and then with eq. (8) and by integrating the term containing $\vec{\Omega}$ we obtain

$$\frac{\partial \vec{\mathcal{P}}}{\partial s} \; = \; \mathcal{L}_{\text{FP,orb}} \vec{\mathcal{P}} + \vec{\Omega} \times \vec{\mathcal{P}} \; . \qquad (11)$$

This is the extension of eq. (7) to include radiation that we have been seeking and we see that it is an obvious generalisation of eq. (7). If we switch off the radiation, we of course obtain eq. (7) but by introducing $W(\vec{u}, \vec{S}; s)$ we avoid the heuristic derivation of eq. (3). We call eqs. (7) and (11) "Bloch" equations following the usage for equations of this general form in the nuclear magnetic resonance literature. Concrete examples of eqs. (4)–(11) for simple exactly solvable models can be found in Heinemann [20].

4. DISCUSSION and CONCLUSION

The derivation of the Bloch equation for $\vec{\mathcal{P}}$ given here is independent of the source of noise and damping. In fact as soon as we have the $\mathcal{L}_{\text{FP,orb}}$ for a process we can write down the corresponding Bloch equation for $\vec{\mathcal{P}}$. Furthermore, providing that spin is a spectator, this approach can be applied to more general diffusion problems where the operator $\mathcal{L}_{\text{FP,orb}}$ is replaced by the appropriate form. For example, a Bloch equation can be written to describe the effect of intrabeam scattering without spin flip or the scattering of protons without spin flip off gas atoms and molecules. Note that the Bloch equation is valid far from spin–orbit equilibrium and that it is linear in $\vec{\mathcal{P}}$. Moreover, it is universal in the sense that it does not explicitly contain the orbital density W_{orb}. Surely this is the best place to begin discussions on spin diffusion. In the case of noise and damping due to synchrotron radiation and if the spin–orbit coupling term in eq. (11) were to vanish ($\vec{\Omega} = \vec{0}$), the three components of $\vec{\mathcal{P}}$ would each reach equilibrium forms proportional to the equilibrium form for W_{orb}. However $\vec{\Omega}$ does not vanish but instead mixes the components. This is the route, in this picture, by which $\vec{\Omega}$ causes depolarisation.

The corresponding evolution equation for the local polarisation $\vec{P}_{\text{loc}}$ can be found by substituting eq. (6) into eq. (11) and using eq. (4) but the resulting equation is complicated owing to the second derivative in $\mathcal{L}_2$ and it is *not* universal since it contains W_{orb}. So to extract $\vec{P}_{\text{loc}}$ one should first solve eqs. (4) and (11) separately and then use eq. (6).

The semiclassical calculation [17] involves writing the density operator in two component spin space as $\rho = \frac{1}{2}(\rho_{\text{orb}} + \vec{\sigma} \cdot \vec{\xi})$ where $\vec{\sigma}$ is the spin operator, ρ_{orb} is the density operator of the orbital motion and where the operator $\vec{\xi}$, which encodes information about the polarisation, is equivalent to $\vec{\mathcal{P}}$. In the quantum mechanical picture, all expectation values involving spin can depend only on $\vec{\xi}$ and different mixed spin states leading to the same $\vec{\xi}$ are equivalent. Correspondingly, the definition of $\vec{\mathcal{P}}(\vec{u}; s)$ (eqs. (6) and (8)) involves integration over the spin distribution at $(\vec{u}; s)$ so that in principle different spin distributions at $(\vec{u}; s)$ can lead to the same $\vec{\mathcal{P}}(\vec{u}; s)$. Thus $\vec{\mathcal{P}}$ provides not only an economical representation of the spin motion by virtue of its being an average over spin degrees of freedom, but even as an entity with classical properties, it also embodies the effective indistinguishability of equivalent spin distributions.

At zeroth order in $\hbar$ and in the absence of radi-

on the orbital motion can be neglected. To include the SG forces one replaces the Poisson bracket term in eq. (9) with $\{\vec{\Omega} \cdot \vec{S}\,, W\}_{\vec{u},\psi,J}$.

ation the Weyl transform of $\vec{\xi}$ fulfills eq. (7) as expected [12,21]. At higher order in $\hbar$ SG effects appear. The corresponding calculation in the presence of radiation [17] delivers terms equivalent to those on the r.h.s. of eq. (11), which are due to pure spin diffusion, together with terms due to the ST effect which are, not surprisingly, of the Baier–Katkov–Strakhovenko form [22,23]. There are also the cross terms. So starting with eq. (11) one could, on the basis of physical intuition, add in the ST terms by hand. But the cross terms can be very important [12,13] and they would be missed. So to obtain a complete description of spin motion a full quantum mechanical, or at least semiclassical, treatment of combined spin and orbital motion is unavoidable. Our work is a classical reconstruction of the pure noise and damping part of eq. (2) in DK [17]. Since the evolution equation for the orbital phase space density in DK [17] is the usual FP equation, one sees that the calculation in DK [17] provides a physical justification for using Gaussian white noise models for orbital motion. The use of eq. (11) and of eq. (2) in DK [17] and their connection with the DKM formula will be described in future papers but an outline of how to extract information for a spin–orbit system close to equilibrium is given in Barber and Heinemann [24].

It should now be clear that the polarisation density is the most natural polarisation–like quantity to use in FP–like descriptions of spin motion in accelerator physics. In fact in retrospect its (three component) equation of motion (eq. (11)) is an intuitively obvious generalisation of the (one component) equation for the particle density (eq. (4)) with an extra term to describe the T–BMT precession of the polarisation density [24]. Moreover, since the spin degrees of freedom have been integrated out, the problems of dealing with FP equations containing (spin) variables describing motion on the sphere and of enforcing periodicity conditions for the spin distribution, are bypassed. Perhaps some problems in condensed matter physics involving spin diffusion due to fluctuating magnetic fields could be conveniently handled by simulating the field fluctuations in terms of particle motion in an artificial "phase space" and then working with the accompanying artificial polarisation density.

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REFERENCES

1. A.A. Sokolov and I.M. Ternov, Sov. Phys. Dokl., **8** (1964) 1203.
2. L. Thomas, Philosophical Magazine, **3** (1927) 1.
3. V. Bargmann, L. Michel and V. L. Telegdi, Phys. Rev. Letts., **2** (1959) 435.
4. A.W. Chao, Nucl. Inst. Meth., **180** (1981) 29.
5. A.W. Chao, "Polarisation of a Stored Electron Beam", AIP Proceedings 87, edited by M. Month (1982).
6. D.P. Barber et al., Phys. Letts., **B343** (1995) 436.
7. D.P. Barber et al., DESY report 91-146 (1991).
8. J.M. Jowett, "Introductory Statistical Mechanics for Electron Storage Rings", AIP Proceedings 153, edited by M. Month (1987).
9. F. Ruggiero, E. Picasso and L. A. Radicati, Ann. Phys., **197** (1990) 396.
10. H. Risken, The Fokker-Planck equation: Methods of solution and applications, (Springer) 1989.
11. V.N. Baier and Yu. Orlov, Sov. Phys. Dokl., **10** (1966) 1145.
12. D.P. Barber et al., five articles in proceedings of ICFA workshop "Quantum Aspects of Beam Physics", Monterey, U.S.A., 1998, edited by P. Chen, World Scientific (1999). Also in extended form as DESY report 98-096 (1998) and Los Alamos archive: physics/9901038, 9901041, 9901042, 9901043, 9901044.
13. D.P. Barber and G. Ripken, Handbook of Accelerator Physics and Engineering, edited by A.W. Chao and M. Tigner, World Scientific (1999). Also in extended form as DESY report 99-095 (1999) and Los Alamos archive: physics/9907034.
14. Ya.S. Derbenev and A.M. Kondratenko, Sov. Phys. JETP., **37** (1973) 968.
15. D.P. Barber, K. Heinemann and G. Ripken, Zeitschrift f. Physik, **C64** (1994) 117–167.

16. S.R. Mane, Phys. Rev., **A36** (1987) 105–130.
17. Ya.S. Derbenev and A. M. Kondratenko, Sov. Phys. Dokl., **19** (1975) 438.
18. K. Yokoya, DESY report 86-57 (1986).
19. D.P. Barber, K. Heinemann and G. Ripken, DESY report M-92-04 (1992), second revised version, September 1999.
20. K. Heinemann, DESY report 97-166 (1997) and Los Alamos archive: physics/9709025.
21. K. Heinemann, thesis in preparation.
22. V.N. Baier, V.M. Katkov and V.M. Strakhovenko, Sov. Phys. JETP., **31** (1970) 908.
23. B. Montague, Physics Reports, **113** (1984).
24. D.P. Barber and K. Heinemann, "Red, blue and green electrons", in the proceedings of the workshop "Polarized Protons at High Energies – Accelerator Challenges and Physics Opportunities", DESY, May 1999, edited by A. De Roeck et al.
DESY report DESY–PROC–1999–03 (1999).