Electrons are not protons:

electron polarisation in rings, decoherence and spin matching

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Abstract

Although depolarisation in proton and electron beams in storage rings and ring accelerators is rooted in the spin–orbit coupling embodied in the Thomas–BMT equation, the details of the depolarisation mechanisms are very different. In particular the polarisation of a high energy proton beam depends on its history whereas the polarisation of a high energy electron beam can depend strongly on the depolarising effects of synchrotron radiation. In both cases the spin distributions are most efficiently described in terms of the invariant spin field. The invariant spin field also provides the best framework for quantifying the differences. A good example of the differences is provided by the use of a Siberian Snake in an electron storage ring.

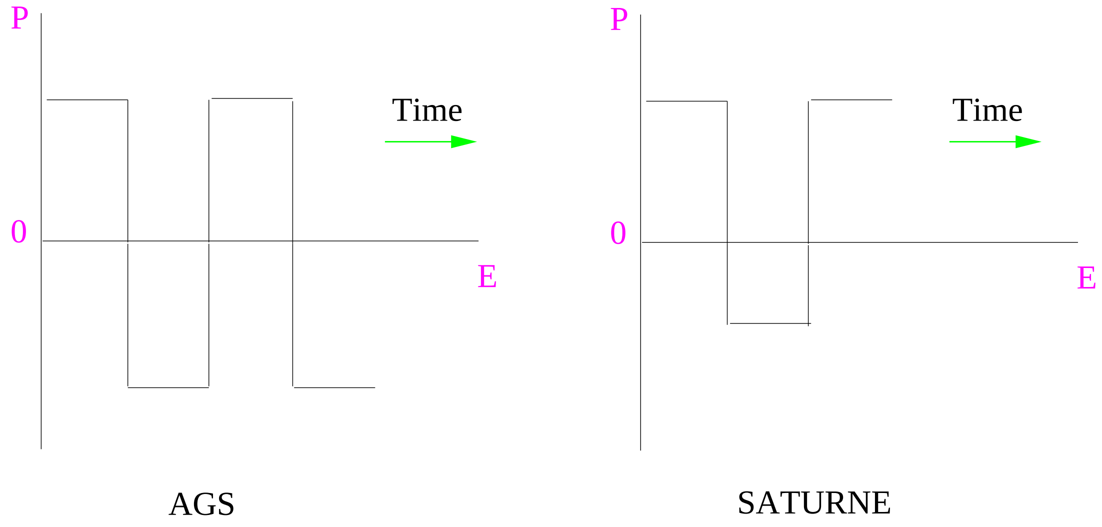
Snakes are essential for preserving proton spin polarisation during acceleration to high energy and can help to stabilize spin motion at the top energy. But snakes can be inappropriate for stored high energy electron beams which are self–polarised via the Sokolov–Ternov effect or prepolarised before injection at the full energy. For example, snakes can, in effect, "switch off" the Sokolov–Ternov effect and at high energy a single snake, installed to constrain the equilibrium polarisation direction ($\vec{n}_0$) to the machine plane can lead to a prohibitive increase in radiative depolarisation. The latter point will be demonstrated with a simple, exactly solvable model of spin decoherence and the result will be compared with that from the standard Derbenev–Kondratenko–Mane (DKM) calculation based on an exact expression for the invariant spin field. The model is a useful pedagogical tool for demonstrating the meaning and limitations of the DKM approach and for demonstrating the danger of horizontal $\vec{n}_0$.

Depolarisation of electrons by synchrotron radiation increases strongly with energy and can be especially strong if the ring is misaligned or has spin rotators to provide longitudinal polarisation at interaction points. But the depolarisation can be reduced by "linear spin matching", i.e. by a careful choice of the optics in sections of the ring. Spin matching is conveniently carried out in terms of the 8×8 spin-orbit transfer matrices of the SLIM formalism. This approach emphasizes the locality of the required "spin transparency", is convenient for diagnosis and allows computer algebra to be used.

The central differences:

- Proton depolarisation during acceleration by resonance crossing: memory, deterministic, "reversible".

Proton spin INFORMATION preservation during resonance crossing.



The final polarisation depends on the history.

- Electron depolarisation by synchrotron radiation "noise", irreversible, short memory, independence of history.

Stationary spin–orbit states in rings

- We don't discuss particle dynamics by sitting on the closed orbit.
- We also shouldn't discuss spin dynamics by sitting on the closed orbit — we must get out into phase space.

  And understand STATIONARY SPIN–ORBIT STATES:

  ===> "Invariant spin field".

- Essential for understanding/calculating high order $e^{\pm}$ depolarisation.

  And indispensible for understanding proton spin dynamics at very high energy (e.g. HERA at 800 GeV).

  Can then compare the two phenomenologies very easily.

- ===> Maximum attainable polarisation

- ===> Starting point for perturbation theory — if needed, e.g. noise, non-linear fields, beam-beam....

Invariant fields: phase space
Protons

.

- Canonical particle coordinates: $\vec{u} \equiv (x,\ p_x,\ y,\ p_y,\ z, p_z)$ Indep. var. = azimuth, s

- For electrons at high energy: $\vec{u} \equiv (x,\ p_x,\ y,\ p_y,\ \sigma,\ \eta = \delta E/E_0)$

- Phase space density, $\rho(\vec{u}; s)$: Liouville: ρ constant along paricle orbits =====>

$$\frac{\partial \rho}{\partial s} = \{H_{orb}, \rho\}$$

- Stationarity: $\rho(\vec{u}; s) = \rho(\vec{u}; s + C)$
  i.e. 1–turn periodicity of the (statistical) **scalar FIELD** $\rho(\vec{u}; s)$

  although individual particles MOVE AROUND IN PHASE SPACE.

Spin motion in electric and magnetic fields:

The T-BMT spin precession equation:

$$\frac{d\vec{S}}{ds} = \vec{\Omega} \times \vec{S}$$

$\vec{S}$: spin expectation value

$\vec{\Omega}$: depends on $\vec{B}, \vec{E}, \vec{\beta}, \gamma$

In transverse magnetic fields:

$$\Omega \ \propto \ (a + 1/\gamma) \cdot B$$

$a = (g - 2)/2$ where g is the relevant g factor.

$a = 1.793...$ for protons.

$a = -0.143$ for deuterons.

($a = 0.00115...$ for electrons.)

Invariant fields: spin

How can a proton beam be fully polarised but the polarimeter gives ZERO?

Invariant fields: spin
Protons

- Local spin polarisation $\vec{P}(\vec{u};s)$: T-BMT. =====> PARTIAL diferential equation:

$$\frac{\partial \vec{P}}{\partial s} = \{H_{orb}, \vec{P}\} + \vec{\Omega}(\vec{u};s) \times \vec{P}$$

- Stationarity: $\vec{P}(\vec{u};s) = \vec{P}(\vec{u};s+C)$
  i.e. 1–turn periodicity of the (statistical) **vector FIELD** $\vec{P}(\vec{u};s)$
  although individual particles MOVE AROUND IN PHASE SPACE AND THEIR SPINS MOVE TOO.

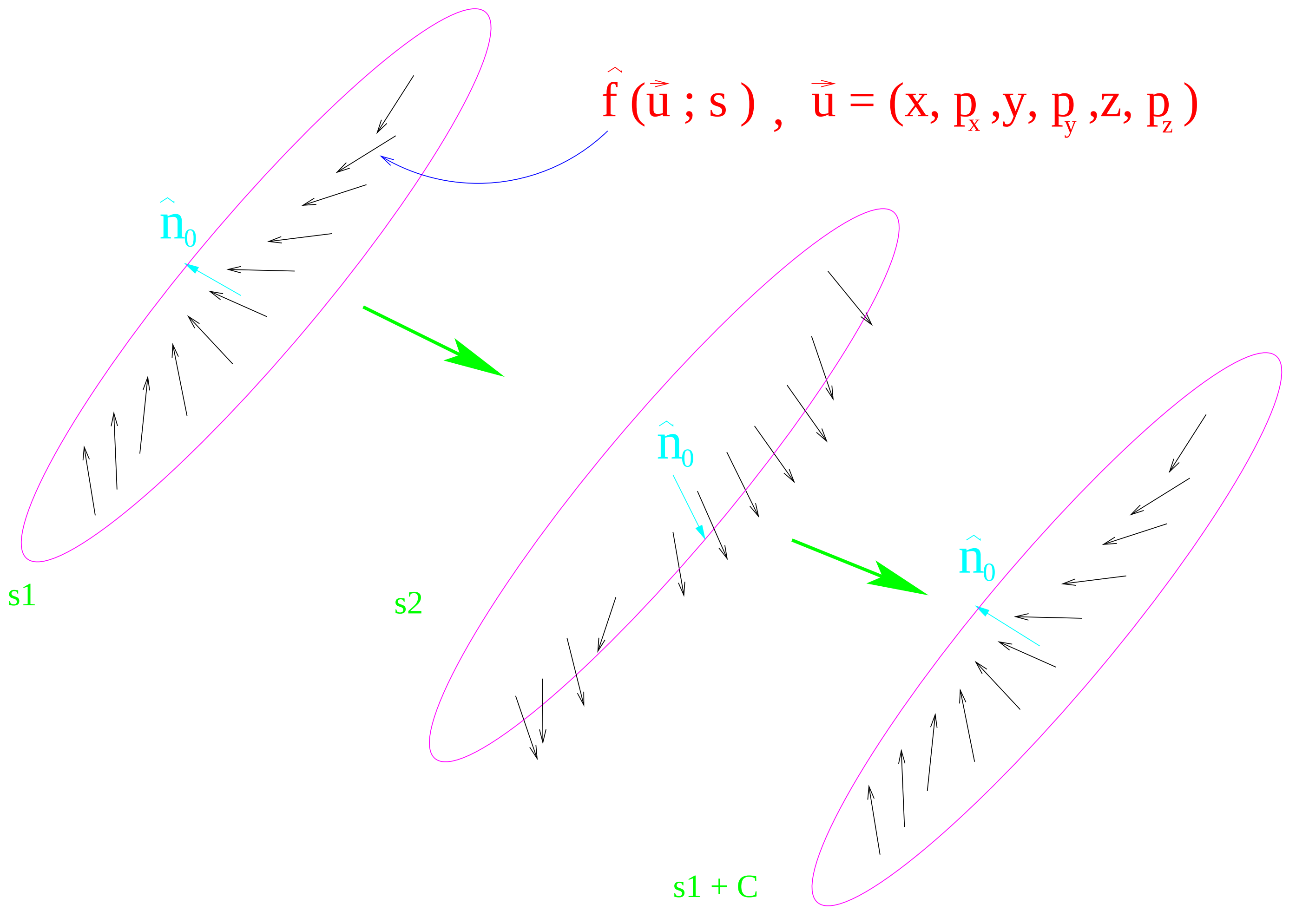
- $|\vec{P}|$ is constant along orbits: ===> $\hat{n}(\vec{u};s) = \vec{P}/|\vec{P}|$

$$\frac{\partial \hat{n}}{\partial s} = \{H_{orb}, \hat{n}\} + \vec{\Omega}(\vec{u};s) \times \hat{n}$$

- Stationarity: $\hat{n}(\vec{u};s) = \hat{n}(\vec{u};s+C)$ ===> $\hat{n}$ is called the INVARIANT SPIN FIELD.
- **Non–trivial** T–BMT solution satisfying CONSTRAINTS.
- Solutions obeying these constraints are unstable (illdefined) at spin–orbit resonances.

The invariant spin field (n–axis, Derbenev–Kondratenko vector)

A pre−established s−periodic unit vector field at each phase space point



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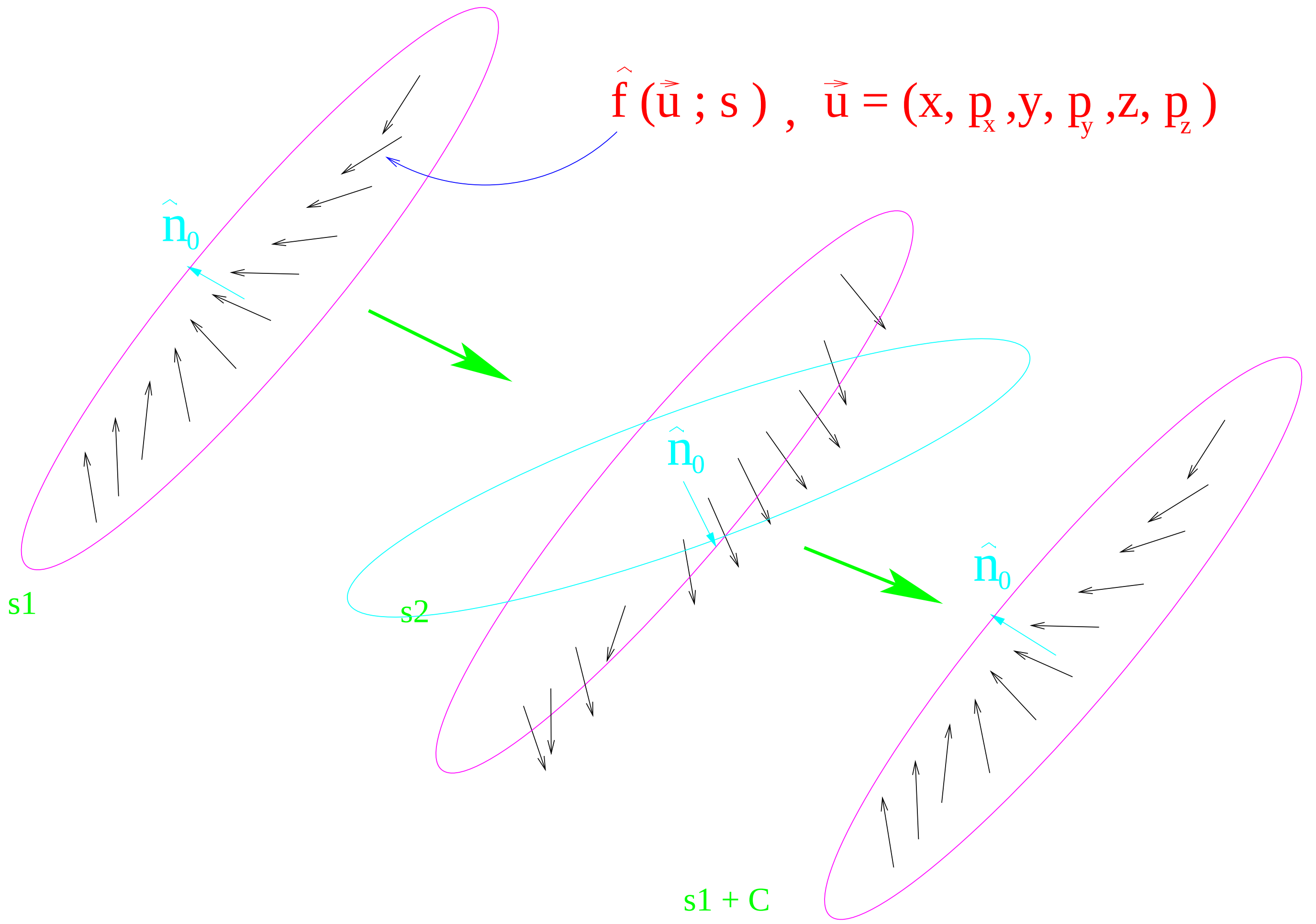


The Invariant Spin Field, $\hat{n}$

- $\vec{n}(M(\vec{u};s);s) = R_{3\times3}(\vec{u};s)\vec{n}(\vec{u};s)$

  This is NOT the eigenproblem $\vec{N}(\vec{u};s) = R_{3\times3}(\vec{u};s)\vec{N}(\vec{u};s)$

  $\hat{n}$ is NOT a "closed spin solution"!!!

  Instead, the field seen **AS A WHOLE** is invariant.

- On the closed orbit $\hat{n}(\vec{u};s) \longrightarrow \hat{n}(\vec{0};s) \equiv \hat{n}_0(s)$.
- ===> $\hat{n}$ and $\hat{n}_0(s)$ should not be confused!!!

- The invariant spin field for 1 plane of orbit motion is a smooth closed vector curve.

- For 3 planes of orbit motion $\hat{n}$ is on a smooth surface but is not closed.

The invariant spin field (ISF):

defines one axis of a local orthonormal coordinate system
at each point in phase space and azimuth for describing spin motion

— **Pre-established** at each $s, \vec{u}, \gamma_0$ independently of the presence of particles or spins.

For protons: the invariant spin field
defines the maximum attainable equilibrium polarisation.

$$\vec{P}_{eq}(\vec{J},\vec{\phi};s) = P(\vec{J})\ \hat{n}(\vec{J},\vec{\phi};s)$$

$$|\vec{P}_{meas}(s)| \quad = \quad |<P(\vec{J})\ \hat{n}(\vec{J},\vec{\phi};s)>_s| \quad \leq \quad |<\hat{n}(\vec{J},\vec{\phi};s)>_s|$$

Over one turn, the particles of an equilibrium phase space distribution replace each other, and spins set parallel to the local $\hat{n}$'s replace each other too.

Even if the spin field is very complicated: once in equilibrium, stay in equilibrium — but small $\vec{P}_{meas}$.

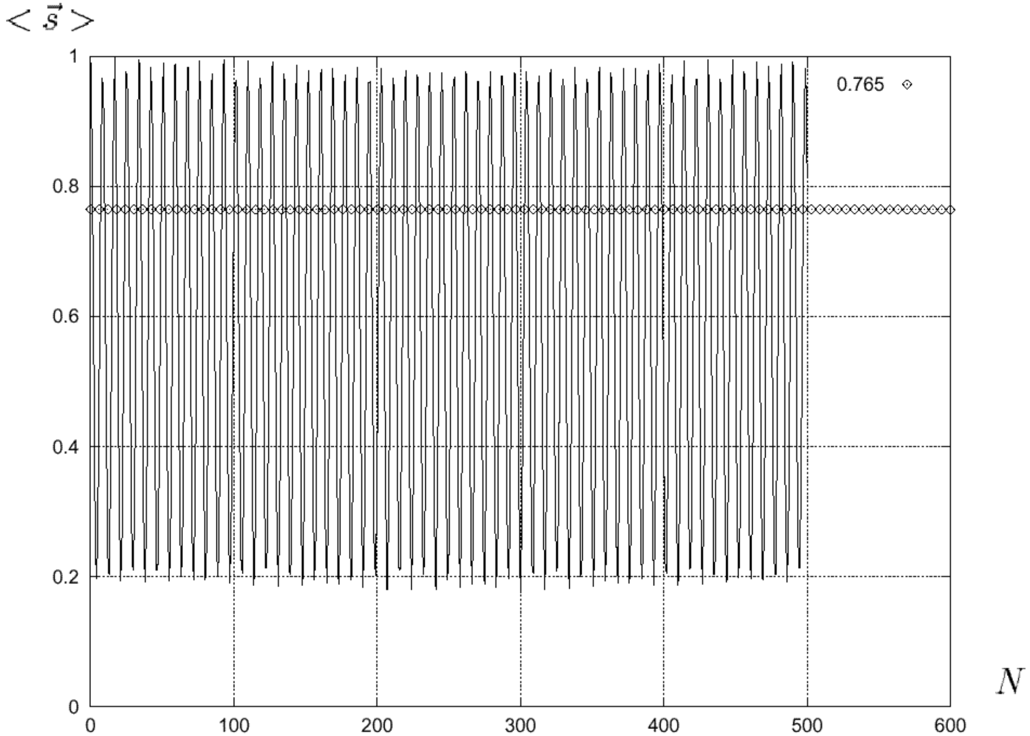


Figure 1: HERA protons at about 800 GeV: propagation of a beam that is initially completely polarised parallel to $\vec{n}_0$ leads to a fluctuating polarisation. For another beam in which the spins are initially parallel to their local $\vec{n}$ the polarisation stays constant, in this case equal to 0.765.

The stable spin direction?

- The ISF gives the stable POLARISATION directionSSSSSSSSSSSS.

- $\hat{n}_0$ gives the stable spin direction on the closed orbit.
  BUT THERE IS ONLY A TINY FRACTION OF PARTICLES ON OR NEAR THE CLOSED ORBIT!

- At very high energy
  $< \hat{n}(\vec{J}, \vec{\phi}; s) >_s$ and $< P(\vec{J})\ \hat{n}(\vec{J}, \vec{\phi}; s) >_s$ need not be parallel to $\hat{n}_0(s)$

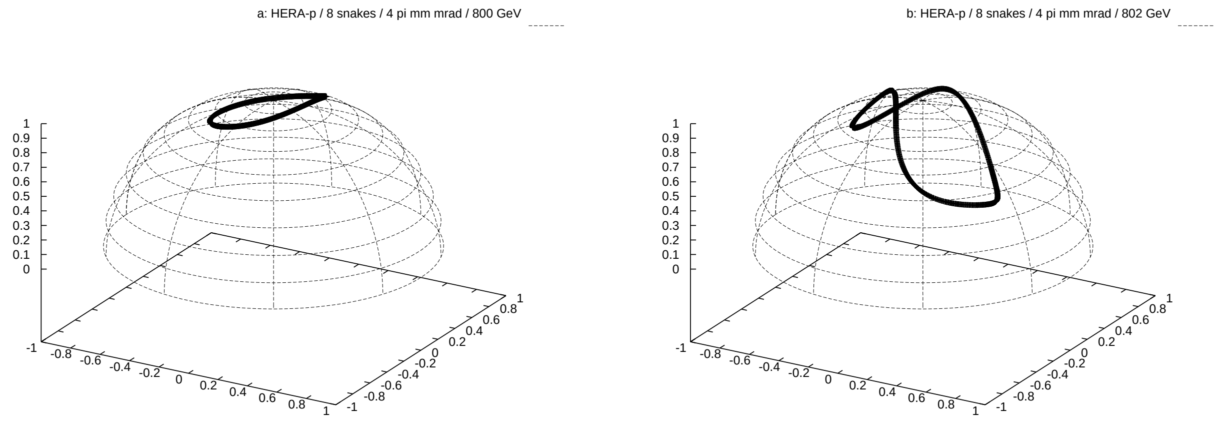


Figure 2: The $\hat{n}$–vector for the 4π mm mrad ellipse at 800 GeV (left) and 802 GeV (right).

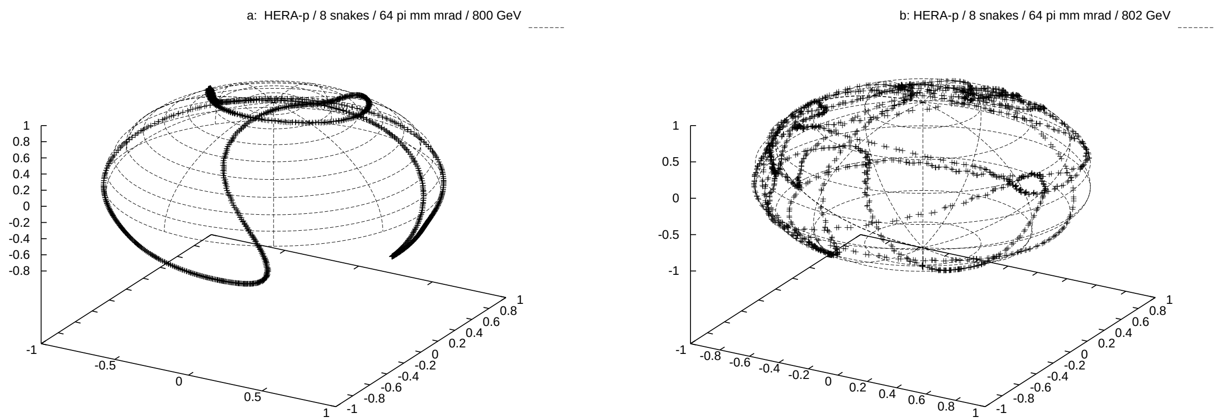


Figure 3: The $\hat{n}$–vector for the 64π mm mrad ellipse at 800 GeV (left) and 802 GeV (right).

The spin tune:

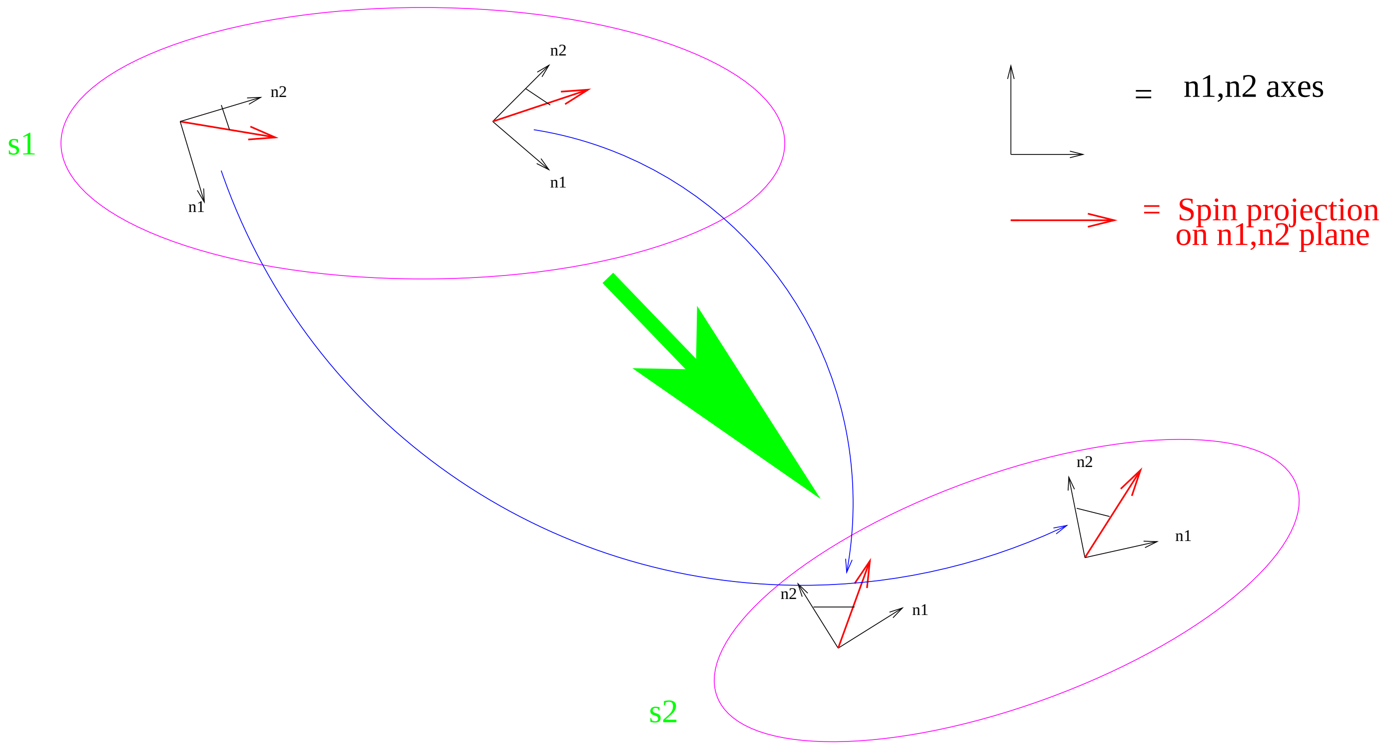
In transverse magnetic fields:

$$\delta\theta_{spin} = a\gamma \cdot \delta\theta_{orbit}$$

- $a\gamma$ is called the **“naive spin tune”**:
- It is a natural spin frequency of the system.
- At 27.5 GeV for electrons $a\gamma = 62.5$
- At 920 GeV for protons $a\gamma = 1759$–BIG!!
- ===> 1 mrad of orbit deviation causes $> \pi/2$
  of spin precession!!!!
  High fields=====> extreme sensitivity.

The real spin tune: measures rate of precession around $\hat{n}$

Attaching coordinate axes to each phase space point



Spin precession rate w.r.t. n1, n2 is the same at all phase space points with same J_x , J_y , J_z .

→ Amplitude dependent spin tune! $\nu_{spin}(J)$

The real spin tune:

Not a single number, but an **equivalence class**

with elements related by "gauge transformations" of the **local** coordinate systems.

Without snakes, the real spin tune $\nu(\vec{J})$ does NOT oscillate with synchrotron motion: although $a\gamma$ does.

Spin–orbit resonance.

- Interleaved vertical and horizontal (quad and imperfection) fields.
- Rotations around different axes don't commute.
- If the spin and (linear) orbit motion are in resonance:

  $\nu_{spin}(\vec{J}) = m + m_x \cdot Q_x + m_z \cdot Q_z + m_s \cdot Q_s$

  ====> **CRAZY** spin field:

- High order resonances even for perfectly linear spin motion. (non–commutation).
- Two main groups of resonances:
  - Integer resonances due to motion along the **distorted** periodic orbit ===> strong tilt of $\hat{n}_0$ from ideal.
  - Synchro-beta ('intrinsic') resonances due to **synchro-beta oscillations** AROUND the distorted periodic orbit.

    ===> $|\hat{n}(\vec{u}; s) - \hat{n}_0(s)|$ LARGE.

    ===> $| < \hat{n}(\vec{J}, \vec{\phi}; s) >_s |$ SMALL — geometry.

    e.g. $\approx 60^\circ$ ===> $P_{meas} \approx 0.5$!!!!

SPIN98 + M. Vogt thesis 2000.

With no snakes, spin tune rises with energy and resonances are crossed.

With snakes REAL spin tune $\neq 1/2$ and can still hit resonances even with perfect alignment!!!!

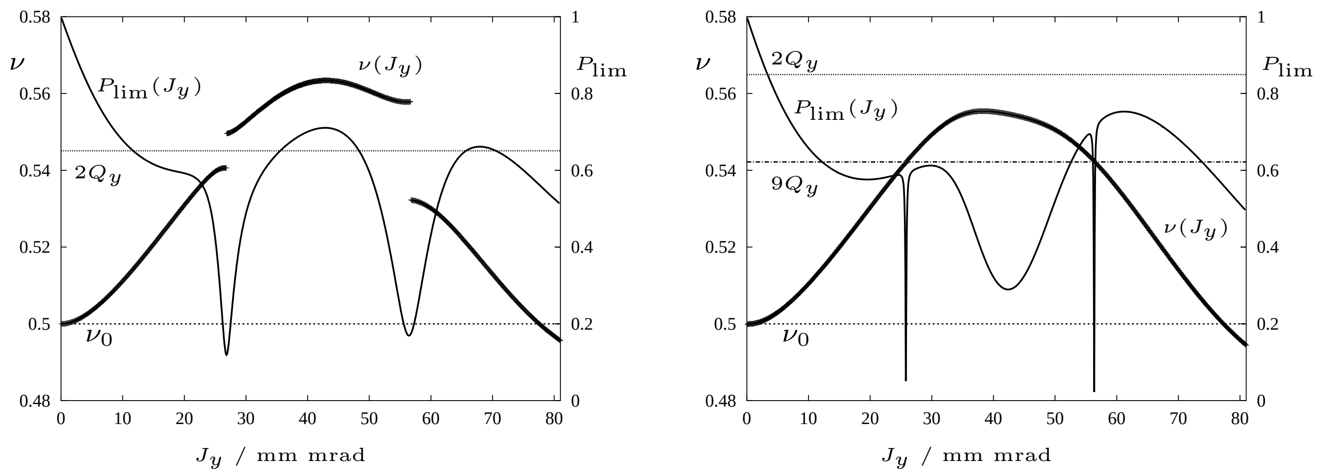
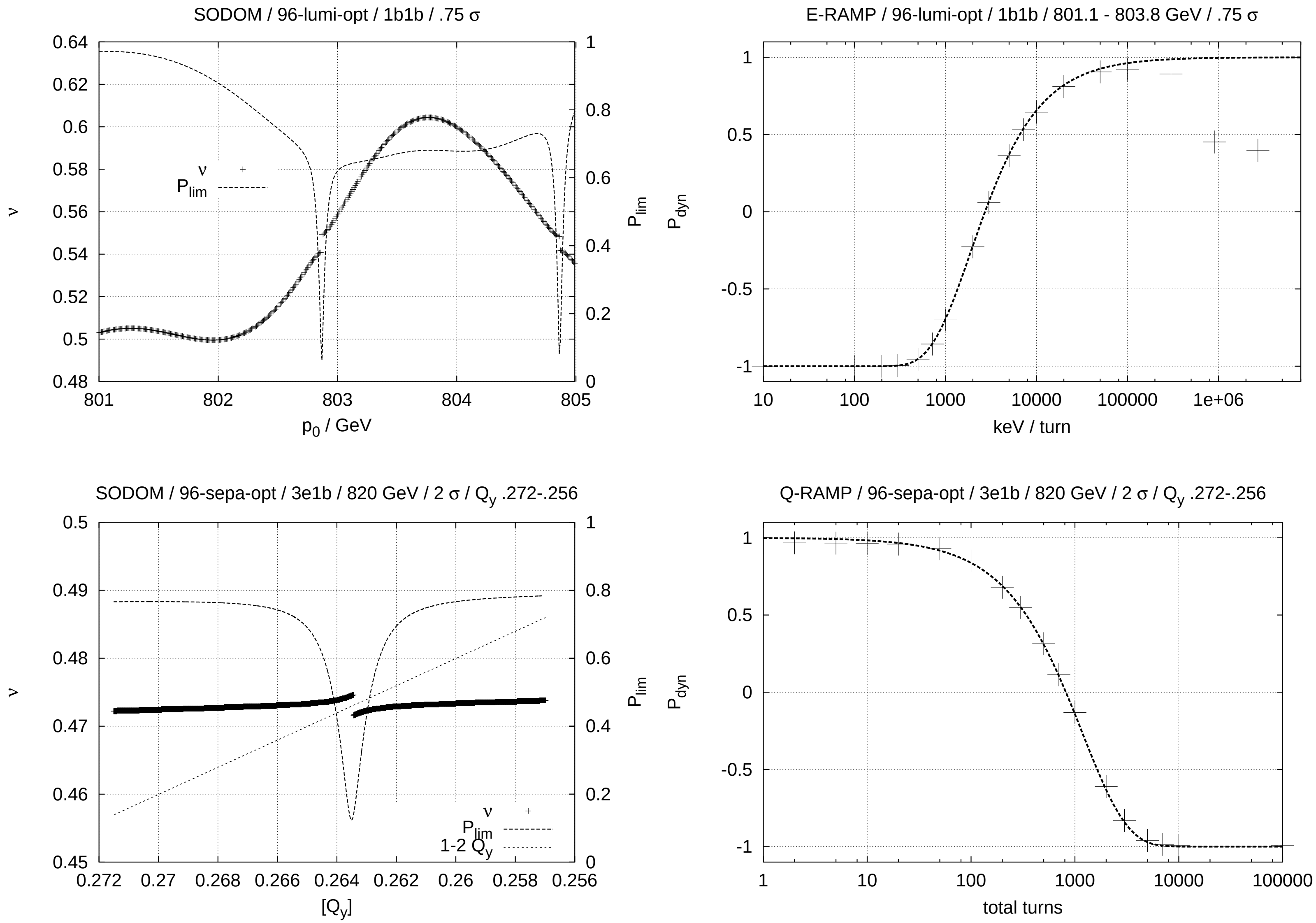


Figure 4: The amplitude dependent spin tune ν and the static polarisation limit $P_{\rm lim}$ vs. vertical orbital action J_y as calculated with `SPRINT` for the HERA–p. Left: vertical tune $Q_y = 32.2725$, right: $Q_y = 32.2825$.

SPIN2000 + M. Vogt thesis 2000.



- Top left:
  Energy scan of P_{lim} and ν for HERA–p with flatteners and a 4 snake scheme (rad., 45°, rad., 45°) with purely vertical motion at 0.75 σ.

- Top right:
  The dependence of the final P_{dyn} after ramping through the resonance at approximately 802.7 GeV on the energy gain per turn.

- Bottom left:
  Tune scan of P_{lim} and ν for HERA–p with flatteners and a 4 snake scheme (long., −45°, rad., 45°) with purely vertical motion at 2 σ.

- Bottom right:
  The dependence of the final P_{dyn} after ramping through the resonance at $[Q_y] \approx 0.2635$ on the total number of turns.

Acceleration: evolution through stationary states?

At fixed γ_0:

$$\frac{d\ \vec{S}\cdot\hat{n}}{ds} = 0$$

along an orbit (angle between 2 T-BMT solutions is constant).

During acceleration (using pre–established) $\hat{n}(\vec{u}; s, \gamma_0)$: $\frac{d\ \vec{S}\cdot\hat{n}(\vec{u};s,\gamma_0)}{ds} \neq 0$

If

$$\frac{d\gamma_0}{ds} \quad \text{and} \quad \frac{\partial\hat{n}(\vec{u};s,\gamma_0)}{d\gamma_0}$$

are small enough $\vec{S}\cdot\hat{n}$ is an adiabatic invariant and a stationary spin distribution transforms to a new stationary spin distribution with the same $P(\vec{J})$!!! Spin can follow $\hat{n}$!!!

If a $\vec{J}$ dependent resonance is crossed, $P(\vec{J})$ can change but $\vec{P}(\vec{J},\vec{\phi} : s)$ is still parallel to $\hat{n}(\vec{J},\vec{\phi}; s)$

$$|\vec{P}_{meas}(s)| \quad = \quad |< P(\vec{J})\ \hat{n}(\vec{J},\vec{\phi}; s) >_s| \quad \leq \quad |< \hat{n}(\vec{J},\vec{\phi}; s) >_s|$$

$$|P(\vec{J})| \quad \leq \quad 1$$

HISTORY!

The Froissart–Stora formula for crossing resonances

$$\frac{P_{\text{final}}}{P_{\text{initial}}} = 2\ e^{-\frac{\pi|\epsilon|^2}{2\alpha}} - 1$$

- ϵ is the "resonance strength", a measure of the dominant spin perturbation at resonance (Fourier component),

- α expresses the rate of resonance crossing.

Very fast resonance crossing: Small $\frac{|\epsilon|^2}{2\alpha}$: polarisation preserved.

Very slow resonance crossing: Large $\frac{|\epsilon|^2}{2\alpha}$: adiabatic invariance ===> full spin flip without polarisation loss.

Electrons

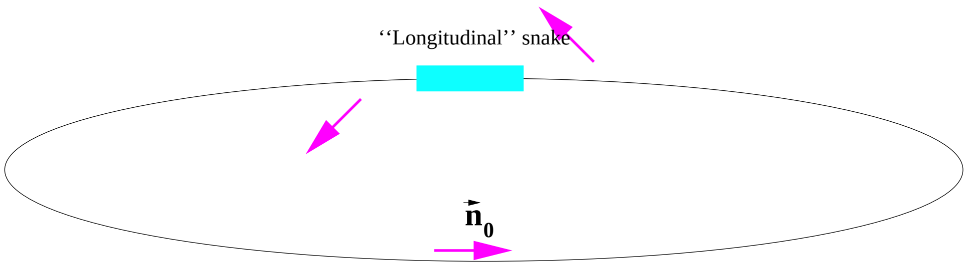
- Synchrotron radiation: ===> polarisation build up by the Sokolov–Ternov effect!!!
- Synchrotron radiation: ===> noise and damping.
- ===> Stochastic orbital motion in the magnetic fields
- ===> Spin diffusion ===> depolarisation!!!!
- The resulting polarisation comes from a balance of polarisation and depolarisation.
- How to calculate???

For an overview of polarised electron phenomenology see:

"Electron polarisation in rings", D.P. Barber, Snowmass 2001, Working Group M5

at http://snowmassserver.snowmass2001.org/

A simple model example: a single Siberian Snake in a perfect flat smooth ring.



Ideal snake: no syncho−beta dependence ⟹ $\vec{\mathbf{n}}_0$ horizontal everywhere

Synchrotron phase space (σ, η) , smooth dispersion and quads.

$$\hat{n}_0(s) \quad \equiv \quad \cos\Big(g_6(s)\Big)\hat{e}_1 + \sin\Big(g_6(s)\Big)\hat{e}_2,$$

$$\hat{n} \quad \equiv \quad \cos(f)\hat{e}_1 + \sin(f)\hat{e}_2$$

$f(\sigma, \eta; s) = g_6(s) + \sigma g_{19}(s) + \eta g_{20}(s) \Longrightarrow$ (T − BMT solution along orbit $\sigma(s), \eta(s)$)

At HERA, 27.5 GeV, $|\hat{n}(\sigma, \eta; s) - \hat{n}_0(s)| \Longleftrightarrow 200$ mrad $===> | < \hat{n} > | \approx 1$

Simple model continued:

The corresponding stochastic differential equation for the spin–orbit motion in the arc.

$$\begin{pmatrix} \sigma'(s) \\ \eta'(s) \\ \psi'(s) \end{pmatrix} = \begin{pmatrix} 0 & -\kappa & 0 \\ \Omega_s^2/\kappa & -2\ \alpha_s/C & 0 \\ 0 & 2\pi\nu_0/C & 0 \end{pmatrix} \cdot \begin{pmatrix} \sigma(s) \\ \eta(s) \\ \psi(s) \end{pmatrix} + \sqrt{\omega} \cdot \begin{pmatrix} 0 \\ \zeta(s) \\ 0 \end{pmatrix}$$

For notation:

K. Heinemann, DESY Report 97–166 (1997) and Los Alamos archive: physics/9709025.

D.P. Barber, M. Böge, K. Heinemann, H. Mais, G. Ripken, Proc. 11th Int. Symp. High Energy Spin Physics, Bloomington, Indiana (1994). AIP Proceedings 343.

K. Heinemann, D.P. Barber 1996

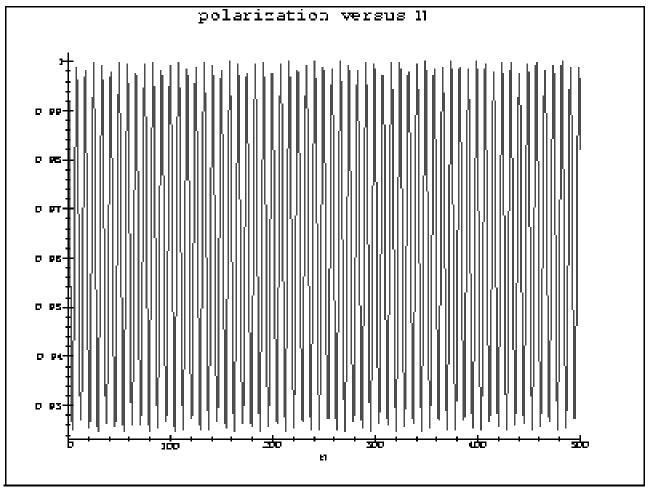


Figure 5: No radiation, spins initially set parallel to $\hat{n}_0$, 27.5 GeV HERA: initial state not in equilibrium ===> oscillating polarisation.

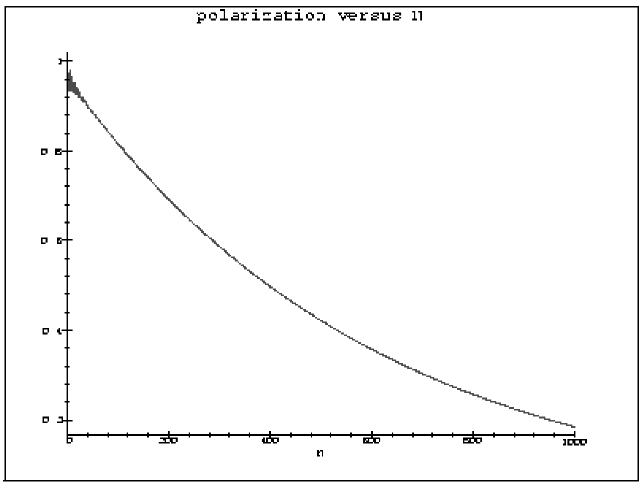
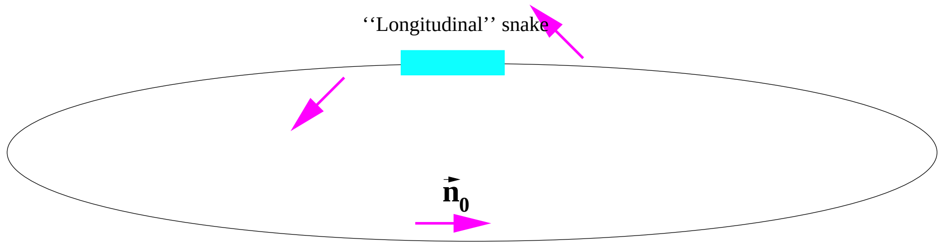


Figure 6: With radiation, spins initially set parallel to $\hat{n}_0$, 27.5 GeV HERA: after transients $\vec{P}(\sigma, \eta; s)$ parallel to $\hat{n}(\sigma, \eta; s)$ with the (σ, η) independent $|P|$ falling exponentially.

A single Siberian Snake in a perfect flat ring.



$\vec{n}_0$ horizontal everywhere:	HERA	$\tau_{dep} = 260$	millisecs at 27.5 GeV
	eRHIC	τ_{dep}	tens of seconds at 10 GeV
	MIT−Bates	τ_{dep}	hours at few hundred MeV

No Sokolov−Ternov ⟶ very exciting possibility to observe ''kinetic polarization'' at MIT−Bates ring.

Full 3–D spin motion

Particle transport in the presence of damping and diffusion.

Fokker–Planck equation:

$$\frac{\partial \rho}{\partial s} = \mathcal{L}_{\rm FP,orb}\ \rho$$

where with synchrotron photon emission modelled as additive noise the orbital Fokker–Planck operator can be decomposed into the form:

$$\mathcal{L}_{\rm FP,orb} = \underbrace{\mathcal{L}_{\rm ham}}_{\mathcal{L}_{\rm ham} \to \text{Liouville}} + \underbrace{\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2}_{\text{damping and noise}}\ .$$

Without the S–T terms, the corresponding form for the

Polarisation Density $\vec{\mathcal{P}}$:

$$\boxed{\frac{\partial \vec{\mathcal{P}}}{\partial s} = \mathcal{L}_{\mathrm{FP,orb}} \vec{\mathcal{P}} + \vec{\Omega}(\vec{u}; s) \times \vec{\mathcal{P}}}$$

Barber + Heinemann 1990's

$$\vec{P}(s) = \int d^6u \; \vec{\mathcal{P}}(\vec{u}; s).$$

This equation:

- can be derived in a classical picture,
- is homogeneous in $\vec{\mathcal{P}}$ i.e. it's "universal",
- is valid far from spin–orbit equilibrium,
- contains the whole of depolarisation!

After including the S–T terms, this becomes (Derbenev + Kondratenko, Barber + Heinemann):

$$\underbrace{\frac{\partial\vec{\mathcal{P}}}{\partial s} = \mathcal{L}_{\text{ham}}\vec{\mathcal{P}} + \vec{\Omega}(\vec{u};s)\times\vec{\mathcal{P}}}_{\equiv\text{Damping and noise free part}} + \underbrace{\mathcal{L}_0\,\vec{\mathcal{P}} + \mathcal{L}_1\,\vec{\mathcal{P}} + \mathcal{L}_2\,\vec{\mathcal{P}} + \underbrace{\frac{1}{\tau_0(\vec{u})}\left[\vec{\mathcal{P}} - \frac{2}{9}\hat{v}(\vec{\mathcal{P}}\cdot\hat{v}) + \frac{8\hat{b}(\vec{u})}{5\sqrt{3}}\rho\right]}_{\text{ST in BKS form}} + \underbrace{\text{X–terms}}_{\text{Kinetic pol.}}}_{\text{SMALL}}$$

$\Downarrow$

$\equiv$ T-BMT equation (BIG)

$\Downarrow$

Stationary state

$\Downarrow$

$\hat{n}$-axis (Invariant spin field) $\rightarrow$ DETERMINES DIRECTION

$\Downarrow$

Rate of polarisation loss $\propto$ Functional of $\hat{n}$, $\partial_{\vec{u}}\,\hat{n}$, $\partial^2_{\vec{u}}\,\hat{n}\ldots\ldots$ (e.g. DK formula).

$\Longrightarrow$ large near spin orbit resonances — since $\hat{n}$ is then very sensitive to $\vec{u}$.

The Derbenev–Kondratenko–Mane Formula: full 3–D.

$$P_{\mathrm{eq,DK}} = -\frac{8}{5\sqrt{3}} \frac{\oint ds \, \langle |K|^3 \hat{b} \cdot \left[\hat{n} - \frac{\partial \hat{n}}{\partial \eta}\right] \rangle_s}{\oint ds \langle \, |K|^3 \{1 - \frac{2}{9}(\hat{n} \cdot \hat{v})^2 + \frac{11}{18} \left|\frac{\partial \hat{n}}{\partial \eta}\right|^2 \} \rangle_s}$$

$$\tau_{\mathrm{dep}}^{-1} \; = \; \frac{5\sqrt{3}}{8} \frac{r_{\mathrm{e}} \gamma^5 \hbar}{m_{\mathrm{e}}} \frac{1}{C} \oint ds \left\langle |K|^3 \frac{11}{18} \left(\frac{\partial \hat{n}}{\partial \eta}\right)^2 \right\rangle_s$$

$\hat{b}$ field direction, K curvature

$\langle \quad \rangle_s$: ensemble average.

$$\vec{P}_{meas}(s) \; = \; P_{\mathrm{eq,DK}} \, \langle \hat{n} \rangle_s \; \approx \; P_{\mathrm{eq,DK}} \, \hat{n}_0 \quad \text{since} \quad |\hat{n}(\vec{u}; s) - \hat{n}_0(s)| \qquad \text{SMALL.}$$

Check the DKM formula for the Barber – Heinemann model:

Exact result of model:

$$\tau_{spin}^{-1} = \frac{d^2}{\lambda_0^2} \cdot \frac{\omega}{2\ c\ \lambda\ L} \cdot \frac{1}{\{\cosh(cL/2) + \cos(\lambda L)\}} \cdot \Big(2\lambda \sinh(cL/2) - c\sin(\lambda L)\Big)$$

DKM version using the expression for $\hat{n}(\sigma, \eta : s)$:

$$(c_0\ \tau_{\rm dep})^{-1} \quad = \quad \frac{d^2}{{\lambda_0}^2} \cdot \frac{\omega}{2\ \lambda_0\ L} \cdot \frac{1}{\{1 + \cos(\lambda_0 L)\}} \cdot \Big(\lambda_0 L - \sin(\lambda_0 L)\Big)\,.$$

- Resonance denominators
- BIG effect even way off resonance
- ===> Avoid $\hat{n}_0$ in horizontal plane!!!

  ===> Avoid that spin couples to dispersion (sync. phase space is BIG).

For electrons with radiation:

- The VALUE of the polarisation $P_{\mathrm{eq,DK}}$ is the same at all phase space points and azimuth s.
- The DIRECTION of the polarisation is parallel to $\hat{n}$
- At practical energies $|\hat{n}(\vec{u};s) - \hat{n}_0(s)|$ SMALL e.g. ≤ 100 mrad away from resonances.
- The rate of depolarisation depends on the DERIVATIVE $\partial\hat{n}/\partial\eta$
- An estimate for pure synchrotron motion: $\sigma_\eta \approx 10^{-3}$, $|\hat{n}(\vec{u};s) - \hat{n}_0(s)| \approx 1$ mrad $\Longrightarrow |\partial\hat{n}/\partial\eta| \approx 1 \Longrightarrow P_{\mathrm{eq,DK}} \approx 0.60!!!$
- Very close to resonances $\hat{n}(\vec{u};s)$ is a very sensitive function of $\vec{u}$ so that $\partial\hat{n}/\partial\eta$ can be large and the equilibrium $P_{\mathrm{eq,DK}}$ can be small.
- For electrons, even without Sokolov–Ternov build up, the equilibrium of the spin DIRECTIONS (along the spin field $\hat{n}$) is established by noise and damping.
- For protons, the equilibrium of the spin DIRECTIONS is established during acceleration.

Full calculation of $\hat{n}(\vec{u}; s)$ is HARD and needs big computing power

High order perturbation theory: Unitarity problems near resonances.

SMILE (S.R. Mane),

SpinLie (Yu. Eidelmann and V. Yakimenko).

Nonperturbative:

SODOM (K. Yokoya),

SPRINT (K. Heinemann, G. H. Hoffstaetter, M. Vogt).

===> linearize

SLIM/SLICK (A.W. Chao (D.P. Barber)) , SITF (J. Kewisch),

ASPIRIN (V. Ptitsin)

Linearization ignores most non–commutation ===> only first order resonances. Unitarity problems.

SLIM/SLICK/SITF I.

$$\vec{\Omega} = \vec{\Omega}^{\rm co} + \vec{\omega}^{\rm sb}$$

$\vec{\omega}^{\rm sb}$ is small (?)

In practical electron rings $\hat{n}(\vec{u};s)$ is close to $\hat{n}_0(s)$ so use:

$$\hat{n}(\vec{u};s) \;=\; \hat{n}_0(s) + \alpha(\vec{u};s)\hat{m}(s) + \beta(\vec{u};s)\hat{l}(s)$$

where $\sqrt{\alpha^2+\beta^2} \ll 1$

We write the components $\omega_s^{\rm sb}$, $\omega_x^{\rm sb}$, $\omega_y^{\rm sb}$ in the form

$$\begin{pmatrix} \omega_s^{\rm sb} \\ \omega_x^{\rm sb} \\ \omega_y^{\rm sb} \end{pmatrix} = \mathbf{F_{3\times 6}} \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \sigma \\ \eta \end{pmatrix}$$

SLIM/SLICK/SITF II.

In linear approximation the combined orbit and spin motion is described by 8×8 transport matrices of the form

$$\hat{\mathbf{M}} = \begin{pmatrix} \mathbf{M_{6\times6}} & \mathbf{0_{6\times2}} \\ \mathbf{G_{2\times6}} & \mathbf{D_{2\times2}} \end{pmatrix}$$

acting on the vector $(\vec{u}, \alpha, \beta)$,

SLIM/SLICK/SITF III.

The eigenvectors for one turn defined by $\hat{\mathbf{M}}(s_0 + C, s_0) \cdot \vec{q}_\mu = \hat{\lambda}_\mu \cdot \vec{q}_\mu$ are written in the form

$$\vec{q}_k(s_0) \quad = \quad \begin{pmatrix} \vec{v}_k(s_0) \\ \vec{w}_k(s_0) \end{pmatrix}, \quad \vec{q}_{-k}(s_0) = [\vec{q}_k(s_0)]^*$$

$$\text{for } k = I,\ II,\ III\ ;$$

Then with respect to the $(\hat{n}_0, \hat{m}, \hat{l})$ frame,

$$\frac{\partial \hat{n}}{\partial \eta} \quad \equiv \quad i \sum_{k=I,II,III} \{v_{k5}^* \vec{w}_k - v_{k5} \vec{w}_k^*\}$$

$$= \quad -2\ \mathrm{Im} \sum_{k=I,II,III} v_{k5}^* \vec{w}_k$$

Note that this is independent of the phase space vector and emittances!

The v_{k5}^* describe the coupling of the orbit to radiation.

$\frac{\partial \hat{n}}{\partial \delta} \equiv \sum_{3modes}$ coupling of spin to orbit $\times$ coupling of orbit to radiation

Spin matching I.

To minimize depolarisation, minimise the coupling of the spin to the orbit at dipoles where the coupling of orbit to radiation does not vanish.

$$\vec{w}_k(s_0) \quad = \quad -\left[\mathbf{D}(s_0+C,s_0)-\hat{\lambda}_k\right]^{-1}\mathbf{G}(s_0+C,s_0)\vec{v}_k(s_0)$$

$$\text{for } k = I,\ II,\ III\ ;$$

Minimize the appropriate parts of the 1–turn SPIN–ORBIT coupling matrix $\mathbf{G}(s_0+C,s_0)$ ===>

Minimize the appropriate parts of the SPIN–ORBIT coupling matrix $\mathbf{G}(s+\Delta,s)$ for strings of elements: SPIN TRANSPARENCY

Spin matching II.

The matrix approach to linear spin matching: minimize $\mathbf{G_{2\times 6}}$

Advantages:

- Direct connection to quantities appearing in SLIM (SLICK).
- Necessary for coupled systems (skew quads, solenoids).
- For a big ring:
  Evaluation (numerical) of integrals in a thick lens optimization program is too slow ===> analytic integration? ===> integrals already contained in $\mathbf{G_{2\times 6}}$.
- "Locality": once $\mathbf{G_{2\times 6}}$ is zero for a section of the ring it remains zero no matter what changes are made to the optics outside.
- Provides a systematic basis for investigation of the algebraic properties using e.g. REDUCE, MATHEMATICA, MAPLE.
- The interpretation is usually transparent, e.g. arbitrary string of quads and drifts.

Spin matching III.

The basic rules of self polarisation and spin matching.

- Keep $\hat{n}_0$ aligned to the field in as many of the ring dipoles as possible to drive S–T effect at full rate. E.g. minimize the regions around IPs where $\hat{n}_0$ is horizontal and there is radiation in dipoles.

- Minimize $\mathbf{G_{2\times 6}}$ across the regions around IPs where $\hat{n}_0$ is horizontal.

- Get a grip on the remaining effects of $\mathbf{G_{2\times 6}}$.

- Then do very good orbit correction to avoid the $\hat{n}_0$ tilts (resulting from misalignments) that couple spin to horizontal synchro–betatron motion and nullify the effect of good spin transparency.

Spin matching IV.

See the article by D.P. Barber and G. Ripken in the Handbook of Accelerator Physics and Engineering, Eds. A.W. Chao and M. Tigner, 2nd edition, World Scientific, 2002.

Higher order resonances. e.g. sync. side bands

Beam–beam forces!!!