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Notes on Spin Dynamics in Storage Rings

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Abstract

In the following report we present a collection of notes on spin dynamics in storage rings. The spin motion is described in terms of a pair of real canonical spin variables α and β and in four different spin dreibeins. The orbital motion is described by using the canonical variables $x,\ p_x,\ z,\ p_z,\ \sigma = s - v_0 \cdot t,\ p_\sigma = (1/\beta_0^2) \cdot \eta$ with $\eta = \Delta E/E_0$ of the fully 6–dimensional canonical formalism. Action–angle variables J_k, Φ_k of the linear coupled orbital motion are introduced by a canonical transformation. The equations thus obtained are valid for arbitrary velocity of the particles (below and above transition energy). The general periodic solution for spin motion, the $\vec{n}$–axis, is determined by the method of forced solution. Action–angle variables of spin motion and a dreibein which is a single valued function of the particle coordinates (J_k, Φ_k, s) on an arbitrary particle path are defined and the spin tune as a function of J_k is calculated. Finally, classical spin diffusion caused by radiation processes is investigated and a derivation of the depolarisation time presented.

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Contents

1 Introduction

In this paper we collect together a number of results and methods useful for studying spin–orbit motion in storage rings.

The starting point of our investigations is the Lorentz equation for orbital motion and the Thomas–BMT (T–BMT) equation for spin motion. These are studied in chapter 2.

Both equations are expressed in machine coordinates within the framework of the fully coupled 6–dimensional description of particle motion by using the variables x, p_x, z, p_z, σ, $p_\sigma = (1/\beta_0^2)\cdot\eta$ with $\eta = \Delta E/E_0$ which allows to handle the external magnetic forces in a consistent canonical manner and which includes consistently and canonically the synchrotron oscillations in the electric fields of the accelerating cavities. This description is summarised in chapter 3. The equations so derived are valid for arbitrary velocity of the particles (below and above transition energy).

In chapter 4 we introduce the 8–dimensional closed orbit for the combined spin–orbit system which leads to the periodic 6–dimensional closed orbit of particle motion and to the periodic orthonormal coordinate system $(\vec{n}_0, \vec{m}, \vec{l})$ for spin motion. In this paper the various systems of orthonormal coordinate vectors used for describing spin will be called "dreibeins".

In chapter 5 the spin motion is investigated in the $(\vec{n}_0, \vec{m}, \vec{l})$ dreibein by introducing two independent real canonical spin variables α and β.

We are then in a position to calculate the so called $\vec{n}$–axis by the method of forced solution (chapter 6).

The definition of the $\vec{n}$–axis allows now to introduce a new periodic dreibein and action–angle variables for spin motion and to calculate the spin tune as a function of orbital phase space variables (chapter 7).

Finally in chapter 8 the depolarisation time due to stochastic orbital motion is calculated.

A summary of the results is presented in chapter 9.

In this paper only the orbit to spin coupling is taken into account and the Stern–Gerlach forces [1] are neglected.

In the First Revision (February 1997) of this paper some minor changes were made to the text. In this Second Revision, some typographical errors are corrected and the argumentation in section 7.4 relating to the construction of the periodic reference vectors $(\hat{\vec{n}}_1, \hat{\vec{n}}_2)$ is reorganised and improved. Furthermore, figure 1 has been replaced by more explanation in the text.

2 Spin Motion in a Fixed Coordinate System

The starting point of our description of classical spin motion in storage rings will be the T–BMT equation [2, 3] combined with the Lorentz equation.

2.1 Orbital Motion (Lorentz Equation)

The equation of motion for a relativistic non–radiating charged particle in an electromagnetic field, the Lorentz equation, is:

$$e \cdot \vec{\varepsilon} + \frac{e}{c} \cdot \dot{\vec{r}} \times \vec{B} \;=\; \frac{d}{dt}\left(\frac{E}{c^2} \cdot \dot{\vec{r}}\right) \tag{2.1}$$

with

$$E = \frac{m_0 c^2}{\sqrt{1-(\dot{\vec{r}})^2/c^2}} = \gamma \cdot m_0 c^2 \tag{2.2}$$

(energy of the particle)

and the following definitions:

- e = charge of the particle ;
- m_0 = rest mass of the particle ;
- c = velocity of light;
- $\vec{\varepsilon}$ = electric field;
- $\vec{B}$ = magnetic field ;
- $\vec{r}$ = radius vector of the particle;
- $\gamma \;=\; E/m_0c^2$.

Equation (2.1) can be written in canonical form

$$\frac{d}{dt}\,X_1 \;=\; +\frac{\partial \mathcal{H}}{\partial p_x}\;;\quad \frac{d}{dt}\,P_1 = -\frac{\partial \mathcal{H}}{\partial X_1}\;; \tag{2.3a}$$

$$\frac{d}{dt}\,X_2 \;=\; +\frac{\partial \mathcal{H}}{\partial P_2}\;;\quad \frac{d}{dt}\,P_2 = -\frac{\partial \mathcal{H}}{\partial X_2}\;; \tag{2.3b}$$

$$\frac{d}{dt}\,X_3 \;=\; +\frac{\partial \mathcal{H}}{\partial P_3}\;;\quad \frac{d}{dt}\,P_3 = -\frac{\partial \mathcal{H}}{\partial X_3} \tag{2.3c}$$

using the Hamiltonian:

$$\hat{\mathcal{H}}(X_1, X_2, X_3;\, P_1, P_2, P_3;\, t) = c \cdot \left\{\vec{\pi}^{\,2} + m_0^2 c^2\right\}^{1/2} + e\phi \tag{2.4}$$

with

$$\vec{\pi} \;=\; \vec{P} - \frac{e}{c}\vec{A} \;\equiv\; m_0\gamma\,\dot{\vec{r}} \quad \text{(kinetic momentum vector)} \tag{2.5}$$

where X_1, X_2, X_3 and P_1, P_2, P_3 are canonical orbital position and momentum variables in a fixed Cartesian coordinate system ($\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$) and where $\vec{A}$ and ϕ are the vector and scalar potentials from which the electric field $\vec{\epsilon}$ and the magnetic field $\vec{B}$ are derived as

$$\vec{\epsilon} = -\text{grad}\ \phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\ ; \qquad (2.6a)$$

$$\vec{B} = \text{curl}\ \vec{A}\ . \qquad (2.6b)$$

2.2 Spin Motion (T–BMT Equation)

2.2.1 The T–BMT Equation

The equation of relativistic classical spin motion, the T–BMT equation, reads as [2, 3]:

$$\frac{d}{dt}\vec{\xi} = \vec{\Omega}_0 \times \vec{\xi} \qquad (2.7a)$$

with

$$\vec{\Omega}_0 = \frac{e}{m_0 c} \cdot \left[-\left(\frac{1}{\gamma} + a\right) \cdot \vec{B} + \frac{a\gamma}{1+\gamma} \cdot \frac{1}{c^2} \cdot (\dot{\vec{r}}\,\vec{B}) \cdot \dot{\vec{r}} + \left(a + \frac{1}{1+\gamma}\right) \dot{\vec{r}} \times \frac{\vec{\epsilon}}{c} \right] \qquad (2.7b)$$

and where $\vec{r}$ and $\vec{P}$ (the canonical orbital position and momentum variables) are determined by the Lorentz equation.

The following abbreviations have been used:

- $\vec{\xi}$ = classical spin angular momentum vector in the rest frame of the particle, of length 1 ;
- a = $(g-2)/2$ (0.00116 for electrons, 1.793 for protons) and quantifies the anomalous spin g factor .

In terms of the three unit cartesian coordinate vectors in the fixed laboratory frame, $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ we can write $\vec{r}$, $\vec{P}$, $\vec{\Omega}_0$ and $\vec{\xi}$ as:

$$\vec{r} = X_1 \cdot \vec{e}_1 + X_2 \cdot \vec{e}_2 + X_3 \cdot \vec{e}_3\ ; \qquad (2.8a)$$

$$\vec{P} = P_1 \cdot \vec{e}_1 + P_2 \cdot \vec{e}_2 + P_3 \cdot \vec{e}_3\ ; \qquad (2.8b)$$

$$\vec{\Omega}_0 = \Omega_1 \cdot \vec{e}_1 + \Omega_2 \cdot \vec{e}_2 + \Omega_3 \cdot \vec{e}_3\ ; \qquad (2.8c)$$

$$\vec{\xi} = \xi_1 \cdot \vec{e}_1 + \xi_2 \cdot \vec{e}_2 + \xi_3 \cdot \vec{e}_3\ . \qquad (2.8d)$$

It follows from eqn. (2.7a) that for two spins $\vec{\xi}_1$ and $\vec{\xi}_2$ the scalar product

$$\vec{\xi}_1(t) \cdot \vec{\xi}_2(t)$$

is a constant of motion:

$$\begin{aligned} \frac{d}{dt} \left(\vec{\xi}_1(t) \cdot \vec{\xi}_2(t)\right) &= 0 \; ; \\ \Longrightarrow \quad \vec{\xi}_1(t) \cdot \vec{\xi}_2(t) &= \text{const} \end{aligned}$$

i.e. the modulus of $\vec{\xi}$ and the angle between $\vec{\xi}_1$ and $\vec{\xi}_2$ are invariants:

$$|\vec{\xi}(t)| = \text{const} \; ; \tag{2.9a}$$

$$\sphericalangle \left(\vec{\xi}_1(t), \vec{\xi}_2(t)\right) = \text{const} \; . \tag{2.9b}$$

Introducing the matrix:

$$\underline{\Omega}_0 = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \tag{2.10a}$$

the T–BMT equation (2.7a) can also be written as:

$$\frac{d}{dt} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \underline{\Omega}_0 \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \; . \tag{2.10b}$$

As may be seen from eqn. (2.9a), the components ξ_1, ξ_2, ξ_3 of the spin vector $\vec{\xi}$ are not independent, since they fulfil the condition:

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = 1 \; .$$

Eliminating the variable ξ_3:

$$\xi_3 = \sqrt{1 - \xi_1^2 - \xi_2^2} \; ,$$

we get a pair of nonlinear coupled differential equations:

$$\frac{d}{dt}\, \xi_1(t) = \Omega_2 \cdot \sqrt{1 - \xi_1^2 - \xi_2^2} - \Omega_3 \cdot \xi_2 \; ; \tag{2.11a}$$

$$\frac{d}{dt}\, \xi_2(t) = \Omega_3 \cdot \xi_1 - \Omega_1 \cdot \sqrt{1 - \xi_1^2 - \xi_2^2} \; . \tag{2.11b}$$

2.2.2 The Spin Hamiltonian

Introducing a pair of independent spin variables α and β by the equations:

$$\xi_1 = \alpha \cdot \sqrt{1 - \frac{1}{4}(\alpha^2 + \beta^2)}\,; \tag{2.12a}$$

$$\xi_2 = \beta \cdot \sqrt{1 - \frac{1}{4}(\alpha^2 + \beta^2)}\,; \tag{2.12b}$$

$$\Longrightarrow \quad \xi_3 = \sqrt{1 - \xi_1^2 - \xi_2^2} = 1 - \frac{1}{2}\left(\alpha^2 + \beta^2\right) \tag{2.12c}$$

or (inverting (2.12a, b)):

$$\alpha = +\sqrt{\frac{2}{1+\xi_3}} \cdot \xi_1\ ; \tag{2.13a}$$

$$\beta = +\sqrt{\frac{2}{1+\xi_3}} \cdot \xi_2 \tag{2.13b}$$

the T–BMT equation (2.7a) can also be written in canonical form:

$$\frac{d}{dt}\,\alpha = +\frac{\partial \mathcal{H}_{spin}}{\partial \beta}\ ; \tag{2.14a}$$

$$\frac{d}{dt}\,\beta = -\frac{\partial \mathcal{H}_{spin}}{\partial \alpha} \tag{2.14b}$$

if we define the spin Hamiltonian $\mathcal{H}_{spin}$ as:

$$\begin{aligned}\mathcal{H}_{spin}(\alpha,\ \beta;\ s) &= \Omega_1 \cdot \xi_1 + \Omega_2 \cdot \xi_2 + \Omega_3 \cdot \xi_3 \\ &= \sqrt{1 - \frac{1}{4}(\alpha^2 + \beta^2)} \cdot [\Omega_1 \cdot \alpha + \Omega_2 \cdot \beta] \\ &\qquad + \left[1 - \frac{1}{2}\left(\alpha^2 + \beta^2\right)\right] \cdot \Omega_3\end{aligned} \tag{2.15}$$

(see Appendix A and Ref. [1]).

We then obtain from (2.14) and (2.15) the relations:

$$\frac{d}{dt}\,\alpha = +\frac{-\beta}{4\sqrt{1 - \frac{1}{4}(\alpha^2 + \beta^2)}} \cdot [\Omega_1 \cdot \alpha + \Omega_2 \cdot \beta]$$

$$+\sqrt{1-\frac{1}{4}\left(\alpha^2+\beta^2\right)}\cdot\Omega_2-\beta\cdot\Omega_3\ ; \tag{2.16a}$$

$$\begin{aligned}\frac{d}{dt}\,\beta \;=\; & -\frac{-\alpha}{4\sqrt{1-\frac{1}{4}\left(\alpha^2+\beta^2\right)}}\cdot\left[\Omega_1\cdot\alpha+\Omega_2\cdot\beta\right] \\ & -\sqrt{1-\frac{1}{4}\left(\alpha^2+\beta^2\right)}\cdot\Omega_1+\alpha\cdot\Omega_3\end{aligned} \tag{2.16b}$$

which are equivalent with eqn. (2.7a) (see Appendix A).

2.2.3 Introduction of the Canonical Variables J and ψ

Alternatively may introduce a second pair of canonical spin variables $(J,\ \psi)$ via the relations [4]:

$$\alpha = \sqrt{2\left(1-J\right)}\cdot\cos\psi\ ; \tag{2.17a}$$

$$\beta = \sqrt{2\left(1-J\right)}\cdot\sin\psi\ . \tag{2.17b}$$

From this definition we have:

$$\frac{\beta}{\alpha} = \tan\psi\ ; \tag{2.18a}$$

$$J = 1-\frac{1}{2}\left(\alpha^2+\beta^2\right) \tag{2.18b}$$

and

$$\begin{aligned}\xi_1 &= \alpha\cdot\sqrt{1-\frac{1}{4}\left(\alpha^2+\beta^2\right)} \\ &= \sqrt{2\left(1-J\right)}\,\cos\psi\cdot\sqrt{1-\frac{1}{2}(1-J)} \\ &= \sqrt{1-J^2}\cdot\cos\psi\ ;\end{aligned} \tag{2.19a}$$

$$\begin{aligned}\xi_2 &= \beta\cdot\sqrt{1-\frac{1}{4}\left(\alpha^2+\beta^2\right)} \\ &= \sqrt{1-J^2}\cdot\sin\psi\ ;\end{aligned} \tag{2.19b}$$

$$\begin{aligned}\xi_3 &= 1-\frac{1}{2}\left(\alpha^2+\beta^2\right) \\ &= J\ .\end{aligned} \tag{2.19c}$$

The transformation

$$\alpha,\ \beta \quad \Longrightarrow \quad \psi,\ J$$

can be obtained from the generating function

$$F_1(\alpha,\ \psi) = \frac{1}{2}\alpha^2 \cdot \tan\psi - \psi. \tag{2.20}$$

The transformation formulae are then:

$$\beta \ = \ +\frac{\partial F_1}{\partial \alpha} \ = \ \alpha \cdot \tan\psi\ ; \tag{2.21a}$$

$$\begin{aligned} J \ = \ -\frac{\partial F_1}{\partial \psi} \ &= \ -\frac{1}{2}\alpha^2 \cdot \left(1 + \tan^2\psi\right) + 1 \\ &= \ -\frac{1}{2}\alpha^2 \cdot \left(1 + \frac{\beta^2}{\alpha^2}\right) + 1 \\ &= \ -\frac{1}{2}\left(\alpha^2 + \beta^2\right) + 1\ ; \end{aligned} \tag{2.21b}$$

$$\begin{aligned} \mathcal{H}_{spin} \quad \longrightarrow \quad \mathcal{K}_{spin}(\psi,\ J) \ &= \ \mathcal{H}_{spin} + \frac{\partial F_1}{\partial s} \ = \ \mathcal{H}_{spin} \\ &= \ \Omega_1 \cdot \xi_1 + \Omega_2 \cdot \xi_2 + \Omega_3 \cdot \xi_3 \\ &= \ \sqrt{1 - J^2} \cdot [\Omega_1 \cdot \cos\psi + \Omega_2 \cdot \sin\psi] + \Omega_3 \cdot J\ , \end{aligned} \tag{2.21c}$$

and one sees that (2.21a, b) lead back to eqn. (2.18a, b). Thus ψ, J are indeed canonical variables [5].

2.3 Transition to a New Dreibein $\vec{u}_1,\ \vec{u}_2,\ \vec{u}_3$

We now consider the transformation [4]:

$$\vec{e}_1, \vec{e}_2, \vec{e}_3 \quad \longrightarrow \quad \vec{u}_1, \vec{u}_2, \vec{u}_3$$

with

$$\frac{d}{dt}\ \vec{u}_k(t) \ = \ \vec{U}(t) \times \vec{u}_k(t) \quad \Longrightarrow \quad \vec{U} \ = \ \frac{1}{2}\sum_{k=1}^{3} \vec{u}_k \times \frac{d}{dt}\ \vec{u}_k \tag{2.22}$$

and

$$\begin{aligned} \vec{\xi} \ &= \ \xi_1 \cdot \vec{e}_1 + \xi_2 \cdot \vec{e}_2 + \xi_3 \cdot \vec{e}_3 \\ &= \ \tilde{\xi}_1 \cdot \vec{u}_1 + \tilde{\xi}_2 \cdot \vec{u}_2 + \tilde{\xi}_3 \cdot \vec{u}_3\ . \end{aligned} \tag{2.23}$$

From (2.7a), (2.22) and (2.23) we obtain:

$$\begin{aligned}
\frac{d}{dt}\,\vec{\xi} &= \sum_{k=1}^{3} \vec{e}_k \times \frac{d}{dt}\,\xi_k \;=\; \vec{\Omega}_0 \times \vec{\xi} \\
&= \sum_{k=1}^{3} \tilde{\xi}_k \cdot \frac{d}{dt}\,\vec{u}_k + \sum_{k=1}^{3} \vec{u}_k \cdot \frac{d}{dt}\,\tilde{\xi}_k \\
&= \sum_{k=1}^{3} \tilde{\xi}_k \cdot \left[\vec{U} \times \vec{u}_k\right] + \sum_{k=1}^{3} \vec{u}_k \cdot \frac{d}{dt}\,\tilde{\xi}_k
\end{aligned}$$

and thus

$$\begin{aligned}
\sum_{k=1}^{3} \vec{u}_k \cdot \frac{d}{dt}\,\tilde{\xi}_k &= \vec{\Omega}_0 \times \vec{\xi} - \sum_{k=1}^{3} \tilde{\xi}_k \cdot \left[\vec{U} \times \vec{u}_k\right] \\
&= \vec{\Omega}_0 \times \vec{\xi} - \vec{U} \times \sum_{k=1}^{3} \tilde{\xi}_k\,\vec{u}_k \\
&= \vec{\Omega}_0 \times \vec{\xi} - \vec{U} \times \vec{\xi} \\
&= \left[\vec{\Omega}_0 - \vec{U}\right] \times \vec{\xi}\;. \qquad (2.24)
\end{aligned}$$

Therefore in the new dreibein the equation of spin motion is:

$$\Longrightarrow \quad \frac{d}{dt}\,\tilde{\xi}_k \;=\; \vec{u}_k \cdot \left\{\left[\vec{\Omega}_0 - \vec{U}\right] \times \vec{\xi}\right\}\;.$$

Writing:

$$\tilde{\xi}_1 = \tilde{\alpha} \cdot \sqrt{1 - \frac{1}{4}\left(\tilde{\alpha}^2 + \tilde{\beta}^2\right)}\,; \qquad (2.25a)$$

$$\tilde{\xi}_2 = \tilde{\beta} \cdot \sqrt{1 - \frac{1}{4}\left(\tilde{\alpha}^2 + \tilde{\beta}^2\right)}\,; \qquad (2.25b)$$

$$\tilde{\xi}_3 = 1 - \frac{1}{2}\left(\tilde{\alpha}^2 + \tilde{\beta}^2\right) \qquad (2.25c)$$

and

$$\vec{\Omega}_0 = \tilde{\Omega}_1 \cdot \vec{u}_1 + \tilde{\Omega}_2 \cdot \vec{u}_2 + \tilde{\Omega}_3 \cdot \vec{u}_3\;; \qquad (2.26a)$$

$$\vec{U} = \tilde{U}_1 \cdot \vec{u}_1 + \tilde{U}_2 \cdot \vec{u}_2 + \tilde{U}_3 \cdot \vec{u}_3 \qquad (2.26b)$$

the new Hamiltonian reads as:

$$
\begin{aligned}
\tilde{\mathcal{H}}_{spin} &= \left[\tilde{\Omega}_1 - \tilde{U}_1\right] \cdot \tilde{\xi}_1 + \left[\tilde{\Omega}_2 - \tilde{U}_2\right] \cdot \tilde{\xi}_2 + \left[\tilde{\Omega}_3 - \tilde{U}_3\right] \cdot \tilde{\xi}_3 \\
&= \sqrt{1 - \frac{1}{4}\left(\tilde{\alpha}^2 + \tilde{\beta}^2\right)} \left\{ \left[\tilde{\Omega}_1 - \tilde{U}_1\right] \cdot \tilde{\alpha} + \left[\tilde{\Omega}_2 - \tilde{U}_2\right] \cdot \tilde{\beta} \right\} \\
&\qquad + \left[\tilde{\Omega}_3 - \tilde{U}_3\right] \cdot \left[1 - \frac{1}{2}\left(\tilde{\alpha}^2 + \tilde{\beta}^2\right)\right] \ . \qquad (2.27)
\end{aligned}
$$

It follows that:

$$
\begin{aligned}
\frac{d}{dt}\,\tilde{\alpha} &= +\frac{\partial \tilde{\mathcal{H}}_{spin}}{\partial \tilde{\beta}} \\
&= +\frac{-\tilde{\beta}}{4\sqrt{1 - \frac{1}{4}\left(\tilde{\alpha}^2 + \tilde{\beta}^2\right)}} \cdot \left\{ \left[\tilde{\Omega}_1 - \tilde{U}_1\right] \cdot \tilde{\alpha} + \left[\tilde{\Omega}_2 - \tilde{U}_2\right] \cdot \tilde{\beta} \right\} \\
&\qquad + \sqrt{1 - \frac{1}{4}\left(\tilde{\alpha}^2 + \tilde{\beta}^2\right)} \cdot \left[\tilde{\Omega}_2 - \tilde{U}_2\right] - \tilde{\beta} \cdot \left[\tilde{\Omega}_3 - \tilde{U}_3\right] ; \qquad (2.28a)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\,\tilde{\beta} &= -\frac{\partial \tilde{\mathcal{H}}_{spin}}{\partial \tilde{\alpha}} \\
&= -\frac{-\tilde{\alpha}}{4\sqrt{1 - \frac{1}{4}\left(\tilde{\alpha}^2 + \tilde{\beta}^2\right)}} \cdot \left\{ \left[\tilde{\Omega}_1 - \tilde{U}_1\right] \cdot \tilde{\alpha} + \left[\tilde{\Omega}_2 - \tilde{U}_2\right] \cdot \tilde{\beta} \right\} \\
&\qquad - \sqrt{1 - \frac{1}{4}\left(\tilde{\alpha}^2 + \tilde{\beta}^2\right)} \cdot \left[\tilde{\Omega}_1 - \tilde{U}_1\right] + \tilde{\alpha} \cdot \left[\tilde{\Omega}_3 - \tilde{U}_3\right] . \qquad (2.28b)
\end{aligned}
$$

Introducing (as in eqn. (2.17) or (2.19)) the spin variables $(\tilde{J},\, \tilde{\psi})$ via the relations:

$$\tilde{\alpha} = \sqrt{2\,(1 - \tilde{J})} \cdot \cos\tilde{\psi} \ ; \qquad (2.29a)$$

$$\tilde{\beta} = \sqrt{2\,(1 - \tilde{J})} \cdot \sin\tilde{\psi} \qquad (2.29b)$$

or

$$\tilde{\xi}_1 = \sqrt{1 - \tilde{J}^2} \cdot \cos\tilde{\psi} \ ; \qquad (2.30a)$$

$$\tilde{\xi}_2 = \sqrt{1 - \tilde{J}^2} \cdot \sin\tilde{\psi} \ ; \qquad (2.30b)$$

$$\tilde{\xi}_3 = \tilde{J} \qquad (2.30c)$$

we get:

$$\tilde{\mathcal{K}}_{spin}(\tilde{J},\ \tilde{\psi}) \;=\; \sqrt{1-\tilde{J}^2}\cdot\Big\{\Big[\tilde{\Omega}_1-\tilde{U}_1\Big]\cos\tilde{\psi}+\Big[\tilde{\Omega}_2-\tilde{U}_2\Big]\sin\tilde{\psi}\Big\}+\Big[\tilde{\Omega}_3-\tilde{U}_3\Big]\cdot\tilde{J} \tag{2.31}$$

and

$$\begin{aligned}
\frac{d}{dt}\,\tilde{\psi} &= +\frac{\partial}{\partial\tilde{J}}\,\tilde{\mathcal{K}}_{spin}(\tilde{J},\ \tilde{\psi}) \\
&= -\frac{\tilde{J}}{\sqrt{1-\tilde{J}^2}}\cdot\Big\{\Big[\tilde{\Omega}_1-\tilde{U}_1\Big]\cos\tilde{\psi}+\Big[\tilde{\Omega}_2-\tilde{U}_2\Big]\sin\tilde{\psi}\Big\}+\Big[\tilde{\Omega}_3-\tilde{U}_3\Big]\,; \qquad (2.32a)\\
\frac{d}{dt}\,\tilde{J} &= -\frac{\partial}{\partial\tilde{\psi}}\,\tilde{\mathcal{K}}_{spin}(\tilde{J},\ \tilde{\psi}) \\
&= \sqrt{1-\tilde{J}^2}\cdot\Big\{-\Big[\tilde{\Omega}_1-\tilde{U}_1\Big]\sin\tilde{\psi}+\Big[\tilde{\Omega}_2-\tilde{U}_2\Big]\cos\tilde{\psi}\Big\}\ . \qquad (2.32b)
\end{aligned}$$

2.4 A Special Dreibein Based on a Solution of the T–BMT Equation

Equation (2.29a, b) represents the most general form of spin motion in an arbitrary rotating dreibein

$$(\vec{u}_1(t),\ \vec{u}_2(t),\ \vec{u}_3(t))\ .$$

If we require that $\vec{u}_3(t)$ is a solution of the T–BMT equation, then

$$\tilde{\alpha}=\tilde{\beta}=0$$

must be a solution of eqn. (2.29a, b).

Equation (2.29a, b) then leads to:

$$\begin{aligned}
\Big[\tilde{\Omega}_1-\tilde{U}_1\Big] &= 0\,; \qquad (2.33a)\\
\Big[\tilde{\Omega}_2-\tilde{U}_2\Big] &= 0 \qquad (2.33b)
\end{aligned}$$

and thus:

$$\begin{aligned}
\frac{d}{dt}\,\tilde{\alpha} &= -\tilde{\beta}\cdot\Big[\tilde{\Omega}_3-\tilde{U}_3\Big]\ ; \qquad (2.34a)\\
\frac{d}{dt}\,\tilde{\beta} &= +\tilde{\alpha}\cdot\Big[\tilde{\Omega}_3-\tilde{U}_3\Big] \qquad (2.34b)
\end{aligned}$$

or

$$\frac{d}{dt}\,(\tilde\alpha + i\,\tilde\beta) \;=\; i\,\left[\tilde\Omega_3 - \tilde U_3\right]\cdot(\tilde\alpha + i\,\tilde\beta) \tag{2.35}$$

$$\Longrightarrow \left[\tilde\alpha(t) + i\,\tilde\beta(t)\right] \;=\; \left[\tilde\alpha(t_0) + i\,\tilde\beta(t_0)\right]\cdot e^{\,i\,\cdot\,\int_0^t dt'\,\cdot\,\left[\tilde\Omega_3(t') - \tilde U_3(t')\right]}\;, \tag{2.36}$$

i.e. an arbitrary spin $\vec\xi$ precesses around the $\vec u_3$–axis. This result is in agreement with eqn. (2.9a, b).

Equations (2.32a, b) take the form:

$$\frac{d}{dt}\,\tilde\psi \;=\; \left[\tilde\Omega_3 - \tilde U_3\right]; \tag{2.37a}$$

$$\frac{d}{dt}\,\tilde J \;=\; 0\;. \tag{2.37b}$$

Choosing

$$\tilde U_3(t) = \tilde\Omega_3(t) - \frac{2\pi}{L}\,Q_{rot}\;, \tag{2.38}$$

where Q_{rot} denotes an arbitrary constant number, the precession becomes uniform with respect to the dreibein

$$(\vec u_1(t),\,\vec u_2(t),\,\vec u_3(t))$$

and from eqns. (2.37a, b) we get:

$$\frac{d}{dt}\,\tilde\psi \;=\; \frac{2\pi}{L}\,Q_{rot}\;; \tag{2.39a}$$

$$\frac{d}{dt}\,\tilde J \;=\; 0\;, \tag{2.39b}$$

i.e. $\tilde\psi$ and $\tilde J$ become action–angle variables for the spin Hamiltonian.

Remarks:

1) We will use these results in chapter 7 to define action–angle variables for spin motion in storage rings, where we shall introduce a special dreibein $(\vec n_1,\,\vec n_2,\,\vec n)$ for an arbitrary orbit reflecting the periodicity properties of the orbit. By construction this dreibein is unique except at spin orbit resonances.

2) The results of sections 2.2, 2.3 and 2.4 remain valid if we introduce the arc length s of the design orbit as independent variable (chapter 3) instead of the time t and if we change the coordinate system $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ by orthogonal coordinate transformations, since the structure of the T–BMT equation (2.7a) is unaffected by these procedures [1].

3 Introduction of Machine Coordinates

3.1 Reference Trajectory and Coordinate Frame

The position vector $\vec{r}$ of the spin particle in eqns. (2.1) and (2.7) refers to a fixed coordinate system with the coordinates X_1, X_2 and X_3. However, in accelerator physics, it is useful to describe the motion in terms of the natural coordinates x, z, s in a suitable curvilinear coordinate system by introducing as usual [6]:

a) the closed design orbit (a piecewise flat path of a particle with constant energy E_0) which will in the following be described by the vector $\vec{r}_0(s)$ where s is the length along this ideal orbit;

b) an orthogonal coordinate system accompanying the particles which travels along the design orbit and comprises [7]:

$$\text{the unit tangent vector} \quad \vec{e}_s(s) = \frac{d}{ds}\vec{r}_0(s) \equiv \vec{r}_0\,'(s) \; ;$$

$$\text{a unit vector} \quad \vec{e}_x(s) \quad \text{perpendicular to } \vec{e}_s \text{ in the horizontal plane}$$

$$\text{and the unit vector} \quad \vec{e}_z(s) = \vec{e}_s(s) \times \vec{e}_x(s) \; .$$

The Serret–Fresnet formulae for the orthonormal triad $(\vec{e}_s, \vec{e}_x, \vec{e}_z)$ read as:

$$\frac{d}{ds}\,\vec{e}_x(s) = +K_x(s)\cdot\vec{e}_s(s) \; ; \tag{3.1a}$$

$$\frac{d}{ds}\,\vec{e}_z(s) = +K_z(s)\cdot\vec{e}_s(s) \; ; \tag{3.1b}$$

$$\frac{d}{ds}\,\vec{e}_s(s) = -K_x(s)\cdot\vec{e}_x(s) - K_z(s)\cdot\vec{e}_z(s) \tag{3.1c}$$

with the assumption that

$$K_x(s)\cdot K_z(s) = 0$$

(piecewise no torsion) and where $K_x(s)$, $K_z(s)$ designate the curvatures in the x–direction and in the z–direction respectively.

In this natural coordinate system an arbitrary orbit vector $\vec{r}(s)$ can be written in the form

$$\vec{r}(x,z,s) = \vec{r}_0(s) + x(s)\cdot\vec{e}_x(s) + z(s)\cdot\vec{e}_z(s) \ . \tag{3.2}$$

Note that the sign of $K_x(s)$ and $K_z(s)$ is fixed by eqns. (3.1).

3.2 Orbital Motion

3.2.1 The Orbital Hamiltonian

The variables x and z in eqn. (3.2) describe the amplitude of transverse motion.

In order to describe also the longitudinal motion (synchrotron oscillations) we have to introduce two additional small and oscillating variables σ and p_σ [1] with

$$\sigma \ = \ s - v_0\cdot t \tag{3.3}$$

and

$$p_\sigma \ = \ \frac{1}{\beta_0^2}\cdot\eta \tag{3.4}$$

where v_0 and η are given by

$$v_0 \ = \ \text{design speed} \ = \ c\beta_0 \ ; \ \ \beta_0 \ = \ \sqrt{1-\left(\frac{m_0c^2}{E_0}\right)^2}$$

and

$$\eta \ = \ \frac{\Delta E}{E_0} \ . \tag{3.5}$$

The variable σ denotes the delay in arrival time at position s of a particle and is the longitudinal separation of the particle from the centre of the bunch. The quantity η is the energy deviation of the particle.

Using this complete set of variables we are in a position to provide an analytical description for the orbital motion by a simultaneous treatment of longitudinal and transverse oscillations.

Starting then from the orbital Hamiltonian (2.4) for the motion of a charged particle in an electromagnetic field and introducing the length s along the design orbit as the independent variable (instead of the time t), we can construct the Hamiltonian of the orbital motion with respect to the new variables x, z, σ by a succession of canonical transformations and a scale transformation [8, 9, 1].

Choosing a gauge with $\phi = 0$ (e.g. Coulomb gauge) we then obtain:

$$\begin{aligned}\mathcal{H}(x,p_x,z,p_z,\sigma,p_\sigma;s) \ &= \ p_\sigma - (1+\hat{\eta})\cdot[1+K_x\cdot x+K_z\cdot z]\times \\ &\qquad \left\{1-\frac{(p_x-\frac{e}{p_0\cdot c}A_x)^2+(p_z-\frac{e}{p_0\cdot c}A_z)^2}{(1+\hat{\eta})^2}\right\}^{1/2} \\ &- \ [1+K_x\cdot x+K_z\cdot z]\cdot\frac{e}{p_0\cdot c}A_s\end{aligned} \tag{3.6}$$

with $\hat{\eta}$ defined by:

$$(1+\hat{\eta}) = \frac{1}{\beta_0}\sqrt{(1+\eta)^2-(\frac{m_0c^2}{E_0})^2} = \frac{1}{\beta_0}\cdot\frac{p\cdot c}{E_0} = \frac{p}{p_0}\ ; \tag{3.7a}$$

$$\hat{\eta} = \frac{p}{p_0}-1 = \frac{p-p_0}{p_0} = \frac{\Delta p}{p_0} \tag{3.7b}$$

$(p = m_0\gamma v)$.

The corresponding canonical equations read as :

$$\frac{d}{ds}\,x = +\frac{\partial\mathcal{H}}{\partial p_x}\ ; \quad \frac{d}{ds}\,p_x = -\frac{\partial\mathcal{H}}{\partial x}\ ; \tag{3.8a}$$

$$\frac{d}{ds}\,z = +\frac{\partial\mathcal{H}}{\partial p_z}\ ; \quad \frac{d}{ds}\,p_z = -\frac{\partial\mathcal{H}}{\partial z}\ ; \tag{3.8b}$$

$$\frac{d}{ds}\,\sigma = +\frac{\partial\mathcal{H}}{\partial p_\sigma}\ ; \quad \frac{d}{ds}\,p_\sigma = -\frac{\partial\mathcal{H}}{\partial \sigma} \tag{3.8c}$$

or, using a matrix form:

$$\frac{d}{ds}\,\vec{y} = -\underline{S}\cdot\frac{\partial\mathcal{H}}{\partial\vec{y}} \tag{3.9}$$

with

$$\vec{y}^{\,T} = (x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma) \tag{3.10}$$

where the matrix $\underline{S}$ is given by

$$\underline{S} = \begin{pmatrix} \underline{S}_2 & \underline{0} & \underline{0} \\ \underline{0} & \underline{S}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_2 \end{pmatrix}\ ; \quad \underline{S}_2 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}\ . \tag{3.11}$$

In order to utilise this Hamiltonian, the electric field $\vec{\epsilon}$ and the magnetic field $\vec{B}$ or the corresponding vector potential,

$$\vec{A} = \vec{A}(x,z,s), \tag{3.12}$$

for the cavities and for commonly occurring types of accelerator magnets must be given. Once $\vec{A}$ is known the fields $\vec{\varepsilon}$ and $\vec{B}$ may be found using the relations (2.6a, b). Expressed in the variables $x,\ z,\ s,\ \sigma$, eqns. (2.6a, b) become (with $\phi = 0$):

$$\vec{\varepsilon} = \beta_0\cdot\frac{\partial}{\partial\sigma}\ \vec{A} \tag{3.13}$$

and

$$B_x = \frac{1}{(1+K_x\cdot x+K_z\cdot z)}\cdot\left\{\frac{\partial}{\partial z}\left[(1+K_x\cdot x+K_z\cdot z)\cdot A_s\right]-\frac{\partial}{\partial s}A_z\right\} \ ; \tag{3.14a}$$

$$B_z = \frac{1}{(1+K_x\cdot x+K_z\cdot z)}\cdot\left\{\frac{\partial}{\partial s}A_x-\frac{\partial}{\partial x}\left[(1+K_x\cdot x+K_z\cdot z)\cdot A_s\right]\right\} \ ; \tag{3.14b}$$

$$B_s = \frac{\partial}{\partial x}A_z-\frac{\partial}{\partial z}A_x \ . \tag{3.14c}$$

We assume that the ring consists of bending magnets, quadrupoles, skew quadrupoles, solenoids, cavities and dipoles. Then the vector potential $\vec{A}$ can be written as [10] :

$$\begin{aligned}\frac{e}{p_0\cdot c}A_s &= -\frac{1}{2}\left[1+K_x\cdot x+K_z\cdot z\right]+\frac{1}{2}g\cdot(z^2-x^2)+N\cdot xz\\ &\quad -\frac{1}{\beta_0^2}\cdot\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right]\\ &\quad +\frac{e}{p_0\cdot c}\cdot(\Delta B_x\cdot z-\Delta B_z\cdot x) \ ;\end{aligned} \tag{3.15a}$$

$$\frac{e}{p_0\cdot c}A_x = -H\cdot z \ ; \quad \frac{e}{p_0\cdot c}A_z = +H\cdot x \tag{3.15b}$$

(h = harmonic number) with the following abbreviations:

$$g = \frac{e}{p_0\cdot c}\cdot\left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \ ; \tag{3.16a}$$

$$N = \frac{1}{2}\cdot\frac{e}{p_0\cdot c}\cdot\left(\frac{\partial B_x}{\partial x}-\frac{\partial B_z}{\partial z}\right)_{x=z=0} \ ; \tag{3.16b}$$

$$H = \frac{1}{2}\cdot\frac{e}{p_0\cdot c}\cdot B_s(0,0,s) \ ; \tag{3.16c}$$

$$K_x = +\frac{e}{p_0\cdot c}\cdot B_z(0,0,s) \ ; \quad K_z = -\frac{e}{p_0\cdot c}\cdot B_x(0,0,s) \ . \tag{3.16d}$$

In detail, one has:

a)	$g \neq 0$;	$N = K_x = K_z = H = V = \Delta B_x = \Delta B_z = 0$:	quadrupole;
b)	$N \neq 0$;	$g = K_x = K_z = H = V = \Delta B_x = \Delta B_z = 0$:	skew quadrupole;
c)	$K_x^2 + K_z^2 \neq 0$;	$g = N = H = V = \Delta B_x = \Delta B_z = 0$:	bending magnet;
d)	$H \neq 0$;	$g = N = K_x = K_z = V = \Delta B_x = \Delta B_z = 0$:	solenoid;
e)	$V \neq 0$;	$g = K_x = K_z = N = H = \Delta B_x = \Delta B_z = 0$:	cavity;
f)	$\Delta B_x^2 + \Delta B_z^2 \neq 0$;	$g = K_x = K_z = N = H = V = 0$:	dipole.

Thus the Hamiltonian (3.6) takes the form:

$$\begin{aligned}\mathcal{H}(x,p_x,z,p_z,\sigma,p_\sigma;s) &= p_\sigma - (1+\hat{\eta})\cdot[1+K_x\cdot x+K_z\cdot z]\times \\ &\qquad \left\{1-\frac{[p_x+H\cdot z]^2+[p_z-H\cdot x]^2}{(1+\hat{\eta})^2}\right\}^{1/2} \\ &+\frac{1}{2}\cdot[1+K_x\cdot x+K_z\cdot z]^2-\frac{1}{2}\cdot g\cdot(z^2-x^2)-N\cdot xz \\ &+\frac{1}{\beta_0^2}\cdot\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right] \ . \\ &-\frac{e}{p_0\cdot c}\cdot(\Delta B_x\cdot z-\Delta B_z\cdot x) \ . \end{aligned} \tag{3.17}$$

Furthermore, for the magnetic field $\vec{B}$ we get:

$$\frac{e}{E_0}B_x = \beta_0\left[-K_z+\frac{e}{p_0\cdot c}\Delta B_x+(N-H')\cdot x+g\cdot z\right] \ ; \tag{3.18a}$$

$$\frac{e}{E_0}B_z = \beta_0\left[+K_x+\frac{e}{p_0\cdot c}\Delta B_z-(N+H')\cdot z+g\cdot x\right] \ ; \tag{3.18b}$$

$$\frac{e}{E_0}B_s = \beta_0\cdot 2H \tag{3.18c}$$

and for the electric field $\vec{\epsilon}$ we have:

$$\begin{aligned}\epsilon_s &= V(s)\sin\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right] \\ &= V(s)\sin\varphi+\sigma(s)\cdot h\cdot\frac{2\pi}{L}\cdot V(s)\cos\varphi+\cdots \ ;\end{aligned} \tag{3.19a}$$

$$\epsilon_x = \epsilon_z = 0 \tag{3.19b}$$

(see eqns. (3.13), (3.14) and (3.15)).

Remark:

Equation (3.17) is valid only for protons. For electrons we need the extra term in the Hamiltonian

$$\mathcal{H}_{rad} = \frac{1}{\beta_0^2}\cdot C_1\cdot[K_x^2+K_z^2]\cdot\sigma \tag{3.20}$$

$$\left(\text{where } \ C_1=\frac{2}{3}\,e^2\frac{\gamma_0^4}{E_0}\right)$$

(for $v_0 \approx c$) in order to describe the energy loss by radiation in the bending magnets [7, 11]. In this case, the cavity phase φ in (3.15a) and (3.17) is determined by the need to replace the energy radiated in the bending magnets. Thus:

$$\underbrace{\int_{s_0}^{s_0+L} ds \cdot eV(s) \cdot \sin\varphi}_{\text{average energy uptake in the cavities ;}} \quad = \quad \underbrace{\int_{s_0}^{s_0+L} ds \cdot E_0 \cdot C_1 \cdot [K_x^2 + K_z^2] \ .}_{\text{average energy loss due to radiation}} \tag{3.21}$$

Note, that the $\mathcal{H}_{rad}$ term only accounts for the average energy loss. Deviations from this average due to stochastic radiation effects and damping introduce non–symplectic terms into the equation of motion.

For proton storage rings, where radiation effects can be neglected, one has:

$$\sin\varphi = 0 \quad \Longrightarrow \quad \varphi = 0,\ \pi \tag{3.22}$$

(no average energy gain in the cavities) and the choice for φ is determined by the stability condition for synchrotron motion [9, 10]:

$$\begin{cases} \varphi = 0 & \text{above “transition” ;} \\ \varphi = \pi & \text{below “transition” .} \end{cases}$$

3.2.2 Series Expansion of the Orbital Hamiltonian

Since

$$\begin{aligned} |p_x + H \cdot z| &\ll 1 \ ; \\ |p_z - H \cdot x| &\ll 1 \end{aligned}$$

the square root

$$\left[1 - \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{(1+\hat{\eta})^2}\right]^{1/2}$$

in (3.17) may be expanded in a series :

$$\left[1 - \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{(1+\hat{\eta})^2}\right]^{1/2} = \\ 1 - \frac{1}{2} \cdot \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{(1+\hat{\eta})^2} + ... \tag{3.23}$$

and the same can be done with the term

$$\frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right]$$

resulting from the cavity field :

$$\begin{aligned}
\frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] &= \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cos\varphi \\
&- \sigma \cdot \frac{eV(s)}{E_0} \sin\varphi \\
&- \frac{1}{2}\sigma^2 \cdot h \cdot \frac{2\pi}{L} \cdot \frac{eV(s)}{E_0} \cos\varphi + \cdots . \qquad (3.24)
\end{aligned}$$

Furthermore, for the quantity

$$\hat{\eta} \equiv f(\eta)$$

one obtains from eqn. (3.7a) :

$$\begin{aligned}
\hat{\eta} &\equiv f(\eta) \\
&= f(0) + f'(0) \cdot \eta + f''(0) \cdot \frac{1}{2}\eta^2 + \cdots \\
&= \frac{1}{\beta_0^2} \cdot \eta - \frac{1}{\beta_0^4 \cdot \gamma_0^2} \cdot \frac{1}{2}\eta^2 + \cdots \\
&= p_\sigma - \frac{1}{\gamma_0^2} \cdot \frac{1}{2} p_\sigma^2 \pm \cdots . \qquad (3.25)
\end{aligned}$$

Thus in practice the orbital motion can be conveniently calculated to various orders of approximation.

From (3.7b) and (3.25) we obtain to the first order:

$$p_\sigma \approx \frac{\Delta p}{p_0} .$$

If we wish to obtain a symplectic linearised treatment of synchro–betatron motion we expand the Hamiltonian up to second order in the orbit variables $x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma$. Then we obtain from (3.17) and (3.20):

$$\begin{aligned}
\mathcal{H} = &\frac{1}{2} \cdot \frac{1}{\gamma_0^2} \cdot p_\sigma^2 - [K_x \cdot x + K_z \cdot z] \cdot p_\sigma \\
&+\frac{1}{2} \cdot \left\{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2\right\} \\
&+\frac{1}{2} \cdot \left\{[K_x^2 + g] \cdot x^2 + [K_z^2 - g] \cdot z^2 - 2N \cdot xz\right\} \\
&-\frac{1}{2} \cdot \frac{1}{\beta_0^2} \cdot \sigma^2 \cdot \frac{eV(s)}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \cos\varphi \\
&-\frac{1}{\beta_0^2} \cdot \sigma \cdot \frac{eV(s)}{E_0} \cdot \sin\varphi + \frac{1}{\beta_0^2} \cdot C_1 \cdot [K_x^2 + K_z^2] \cdot \sigma \\
&-\frac{e}{p_0 \cdot c} \cdot (\Delta B_x \cdot z - \Delta B_z \cdot x) \qquad (3.26)
\end{aligned}$$

(constant terms in the Hamiltonian with no influence in the motion have been dropped).

The Hamiltonian (3.26) now leads to the (linearised) canonical equations:

$$\frac{d}{ds}\,x = p_x + H\cdot z\;; \tag{3.27a}$$

$$\frac{d}{ds}\,p_x = -[K_x^2+g]\cdot x + N\cdot z + K_x\cdot p_\sigma + [p_z - H\cdot x]\cdot H - \frac{e}{p_0\cdot c}\,\Delta B_z\;; \tag{3.27b}$$

$$\frac{d}{ds}\,z = p_z - H\cdot x\;; \tag{3.27c}$$

$$\frac{d}{ds}\,p_z = -[K_z^2-g]\cdot z + N\cdot x + K_z\cdot p_\sigma - [p_x + H\cdot z]\cdot H + \frac{e}{p_0\cdot c}\,\Delta B_x\;; \tag{3.27d}$$

$$\frac{d}{ds}\,\sigma = \frac{1}{\gamma_0^2}\cdot p_\sigma - [K_x\cdot x + K_z\cdot z]\;; \tag{3.27e}$$

$$\frac{d}{ds}\,p_\sigma = \frac{1}{\beta_0^2}\cdot\sigma\cdot\frac{eV(s)}{E_0}\cdot h\cdot\frac{2\pi}{L}\cdot\cos\varphi + \frac{1}{\beta_0^2}\cdot\frac{eV(s)}{E_0}\cdot\sin\varphi - \frac{1}{\beta_0^2}\cdot C_1\cdot[K_x^2+K_z^2] \tag{3.27f}$$

or in matrix form:

$$\frac{d}{ds}\vec{y} = \underline{A}\cdot\vec{y} + \vec{c}_0 + \vec{c}_1 \tag{3.28}$$

with

$$\underline{A} = \begin{pmatrix} 0 & 1 & H & 0 & 0 & 0 \\ -[K_x^2+g+H^2] & 0 & N & H & 0 & K_x \\ -H & 0 & 0 & 1 & 0 & 0 \\ N & -H & [K_z^2-g+H^2] & 0 & 0 & K_z \\ -K_x & 0 & -K_z & 0 & 0 & 1/\gamma_0^2 \\ 0 & 0 & 0 & 0 & \frac{1}{\beta_0^2}\cdot\frac{eV(s)}{E_0}\cdot\frac{2\pi h}{L}\cos\varphi_0 & 0 \end{pmatrix} \tag{3.29}$$

and

$$\vec{y}^{\,T} = (x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma)\;; \tag{3.30a}$$

$$\vec{c}_0^{\,T} = \frac{1}{\beta_0^2}\cdot\left(0,\ 0,\ 0,\ 0,\ 0,\ \frac{eV}{E_0}\sin\varphi - C_1\cdot[K_x^2+K_z^2]\right)\;; \tag{3.30b}$$

$$\vec{c}_1^{\,T} = \beta_0\cdot\left(0,\ -\frac{e}{E_0}\,\Delta B_z,\ 0\ +\frac{e}{E_0}\,\Delta B_x,\ 0,\ 0\right)\,. \tag{3.30c}$$

Note that the linear transverse oscillations (eqns. (3.27a - d)) and the longitudinal motion (eqns. (3.27e, f)) are coupled by the term

$$-[K_x \cdot x + K_z \cdot z] \tag{3.31}$$

appearing in (3.27e) which depends on the curvature of the orbit in the bending magnets.

3.3 Spin Motion

Introducing again the arc length s of the design orbit as independent variable and using the relationship:

$$\frac{d}{dt} \;=\; \frac{ds}{dt} \cdot \frac{d}{ds} \;=\; \dot{s} \cdot \frac{d}{ds} \tag{3.32}$$

the T–BMT equation (2.7a) becomes:

$$\frac{d}{ds}\vec{\xi} \;=\; \frac{1}{\dot{s}} \cdot (\vec{\Omega}_0 \times \vec{\xi}) \; . \tag{3.33}$$

Representing the spin vector $\vec{\xi}$ in the form

$$\vec{\xi} = \xi_s \cdot \vec{e}_s + \xi_x \cdot \vec{e}_x + \xi_z \cdot \vec{e}_z \tag{3.34}$$

and using eqn. (3.1) we have:

$$\begin{aligned}
\frac{d}{ds}\,\vec{\xi} \;&=\; \xi_s' \cdot \vec{e}_s + \xi_x' \cdot \vec{e}_x + \xi_z' \cdot \vec{e}_z + \xi_x \cdot \frac{d}{ds}\,\vec{e}_x + \xi_s \cdot \frac{d}{ds}\,\vec{e}_s + \xi_z \cdot \frac{d}{ds}\,\vec{e}_z \\
&=\; \xi_s' \cdot \vec{e}_s + \xi_x' \cdot \vec{e}_x + \xi_z' \cdot \vec{e}_z - \xi_s \cdot (K_x \cdot \vec{e}_x + K_x \cdot \vec{e}_z) + \xi_x \cdot K_x \vec{e}_s + \xi_z \cdot K_z \vec{e}_s \\
&=\; \xi_s' \cdot \vec{e}_s + \xi_x' \cdot \vec{e}_x + \xi_z' \cdot \vec{e}_z - \vec{\xi} \times (K_z \cdot \vec{e}_x - K_x \cdot \vec{e}_z)
\end{aligned} \tag{3.35}$$

so that eqn. (3.33) can be rewritten as:

$$\xi_s' \cdot \vec{e}_s + \xi_x' \cdot \vec{e}_x + \xi_z' \cdot \vec{e}_z = \vec{\Omega} \times \vec{\xi} \tag{3.36a}$$

with

$$\vec{\Omega} = \frac{1}{\dot{s}} \cdot \vec{\Omega}_0 - K_z \cdot \vec{e}_x + K_x \cdot \vec{e}_z \; . \tag{3.36b}$$

In order to get the components $\Omega_s,\ \Omega_x,\ \Omega_z$ of the vector $\vec{\Omega}$:

$$\begin{aligned}
\vec{\Omega} \;&=\; \Omega_1 \cdot \vec{e}_1 + \Omega_2 \cdot \vec{e}_2 + \Omega_3 \cdot \vec{e}_3 \\
&=\; \Omega_s \cdot \vec{e}_s + \Omega_x \cdot \vec{e}_x + \Omega_z \cdot \vec{e}_z
\end{aligned}$$

with respect to the triad $(\vec{e}_s,\, \vec{e}_x,\, \vec{e}_z)$ we use the relations:

$$\begin{aligned} \frac{d}{dt}\,\vec{r} &= \dot{s}\cdot\frac{d}{ds}\,\vec{r} \\ &= \dot{s}\cdot\{[1+K_x\cdot x+K_z\cdot z]\cdot\vec{e}_s + x'\cdot\vec{e}_x + z'\cdot\vec{e}_z\} \end{aligned} \tag{3.37}$$

where the quantities x' and z' are given by (see (3.17)):

$$\begin{aligned} x' &\equiv \frac{\partial\mathcal{H}}{\partial p_x} \\ &= [1+K_x\cdot x+K_z\cdot z] \\ &\qquad \times\left\{(1+\hat{\eta})^2-(p_x+H\cdot z)^2-(p_z-H\cdot x)^2\right\}^{-1/2}\cdot(p_x+H\cdot z)\ ; \end{aligned} \tag{3.38a}$$

$$\begin{aligned} z' &\equiv \frac{\partial\mathcal{H}}{\partial p_z} \\ &= [1+K_x\cdot x+K_z\cdot z] \\ &\qquad \times\left\{(1+\hat{\eta})^2-(p_x+H\cdot z)^2-(p_z-H\cdot x)^2\right\}^{-1/2}\cdot(p_z-H\cdot x) \end{aligned} \tag{3.38b}$$

and where the term $\dot{s}$ can be obtained from the relation:

$$\begin{aligned} \frac{\partial\mathcal{H}}{\partial p_\sigma} &\equiv \sigma' \\ &= 1-v_0\cdot\frac{d}{ds}\,t(s) \\ &= 1-v_0\cdot\frac{1}{\dot{s}} \\ \Longrightarrow\quad \dot{s} &= \frac{v_0}{1-\dfrac{\partial\mathcal{H}}{\partial p_\sigma}} \\ &= \frac{v_0}{[1+K_x\cdot x+K_z\cdot z]}\cdot\frac{1}{1+\eta} \\ &\qquad \times\left\{(1+\hat{\eta})^2-(p_x+H\cdot z)^2-(p_z-H\cdot x)^2\right\}^{1/2} \\ &= \frac{v_0}{[1+K_x\cdot x+K_z\cdot z]}\cdot\frac{1+\hat{\eta}}{1+\eta} \\ &\qquad \times\left\{1-\frac{(p_x+H\cdot z)^2+(p_z-H\cdot x)^2}{(1+\hat{\eta})^2}\right\}^{1/2}\ . \end{aligned} \tag{3.39}$$

Furthermore, for the quantities $(\dot{\vec{r}}\,\vec{B})$ and $\dot{\vec{r}} \times \vec{\varepsilon}$ appearing in eqn. (2.7b) we get:

$$(\dot{\vec{r}}\,\vec{B}) = \dot{s} \cdot \{[1 + K_x \cdot x + K_z \cdot z] \cdot B_s + x' \cdot B_x + z' \cdot B_z\} \; ; \tag{3.40a}$$

$$\dot{\vec{r}} \times \vec{\varepsilon} = \dot{s} \cdot \{z' \cdot \vec{e}_x - x' \cdot \vec{e}_z\} \cdot \varepsilon_s \; . \tag{3.40b}$$

Thus eqn. (3.36b) leads to:

$$\Omega_s = \frac{1}{\dot{s}} \cdot \Omega_{0s} \; ; \tag{3.41a}$$

$$\Omega_x = \frac{1}{\dot{s}} \cdot \Omega_{0x} - K_z \; ; \tag{3.41b}$$

$$\Omega_z = \frac{1}{\dot{s}} \cdot \Omega_{0z} + K_x \tag{3.41c}$$

with

$$\Omega_{0s} = \frac{e}{m_0 c} \cdot \left\{ -\left(\frac{1}{\gamma} + a\right) \cdot B_s + \frac{a\gamma}{1+\gamma} \cdot \frac{1}{c^2} \cdot (\dot{\vec{r}}\,\vec{B}) \cdot \dot{s}\,[1 + K_x \cdot x + K_z \cdot z] \right\} \; ; \tag{3.42a}$$

$$\begin{aligned} \Omega_{0x} = & \frac{e}{m_0 c} \cdot \left\{ -\left(\frac{1}{\gamma} + a\right) \cdot B_x + \frac{a\gamma}{1+\gamma} \cdot \frac{1}{c^2} \cdot (\dot{\vec{r}}\,\vec{B}) \cdot \dot{s}\,\frac{\partial \mathcal{H}}{\partial p_x} \right. \\ & \left. + \left(a + \frac{1}{1+\gamma}\right) \frac{\dot{s}}{c} \cdot \varepsilon_s \cdot \frac{\partial \mathcal{H}}{\partial p_z} \right\} \; ; \end{aligned} \tag{3.42b}$$

$$\begin{aligned} \Omega_{0z} = & \frac{e}{m_0 c} \cdot \left\{ -\left(\frac{1}{\gamma} + a\right) \cdot B_z + \frac{a\gamma}{1+\gamma} \cdot \frac{1}{c^2} \cdot (\dot{\vec{r}}\,\vec{B}) \cdot \dot{s}\,\frac{\partial \mathcal{H}}{\partial p_z} \right. \\ & \left. - \left(a + \frac{1}{1+\gamma}\right) \frac{\dot{s}}{c} \cdot \varepsilon_s \cdot \frac{\partial \mathcal{H}}{\partial p_x} \right\} \; . \end{aligned} \tag{3.42c}$$

Writing for the term $\left(\frac{1}{1+\gamma}\right)$ appearing in eqn. (3.42):

$$\begin{aligned} \frac{1}{1+\gamma} &= \frac{1}{(1+\gamma_0) + \gamma_0 \cdot \eta} \\ &= \frac{1}{1+\gamma_0} \cdot \left[1 - \frac{\gamma_0}{1+\gamma_0} \cdot \eta\right] + \cdots \end{aligned}$$

and taking into account eqns. (3.18), (3.19), (3.23), (3.25) and (3.37 - 40), the precession vector $\vec{\Omega}$ reads in linear approximation as:

$$\Omega_s = -2H \cdot (1 + a)$$

$$
\begin{aligned}
&+2H \cdot (1+a) \cdot p_\sigma \\
&-\beta_0^2 \cdot p_x \cdot \frac{a\gamma_0^2}{1+\gamma_0} \left[K_z - \frac{e}{p_0 \cdot c} \cdot \Delta B_x \right] \\
&+\beta_0^2 \cdot p_z \cdot \frac{a\gamma_0^2}{1+\gamma_0} \left[K_x + \frac{e}{p_0 \cdot c} \cdot \Delta B_z \right] \;;
\end{aligned}
\tag{3.43a}
$$

$$
\begin{aligned}
\Omega_x \;=\;& K_z \cdot a\gamma_0 - (1+a\gamma_0) \cdot \frac{e}{p_0 \cdot c} \Delta B_x \\
&-(1+a\gamma_0) \cdot [(N-H') \cdot x - (K_z^2 - g) \cdot z] \\
&+\frac{a\gamma_0^2}{1+\gamma_0} \cdot 2H \cdot \beta_0^2 \cdot [p_x + H \cdot z] \\
&+\left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0} \right] \cdot \frac{eV(s)}{E_0} \sin\varphi \cdot p_z \\
&-\left[1 + \frac{1+a\gamma_0}{\gamma_0^2 - 1} \right] \cdot \left(K_z - \frac{e}{p_0 \cdot c} \Delta B_x \right) \cdot \beta_0^2 \cdot p_\sigma \;;
\end{aligned}
\tag{3.43b}
$$

$$
\begin{aligned}
\Omega_z \;=\;& -K_x \cdot a\gamma_0 - (1+a\gamma_0) \cdot \frac{e}{p_0 \cdot c} \Delta B_z \\
&+(1+a\gamma_0) \cdot [(N+H') \cdot z - (K_x^2 + g) \cdot x] \\
&+\frac{a\gamma_0^2}{1+\gamma_0} \cdot 2H \cdot \beta_0^2 \cdot [p_z - H \cdot x] \\
&-\left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0} \right] \cdot \frac{eV(s)}{E_0} \sin\varphi \cdot p_x \\
&+\left[1 + \frac{1+a\gamma_0}{\gamma_0^2 - 1} \right] \cdot \left(K_x + \frac{e}{p_0 \cdot c} \Delta B_z \right) \cdot \beta_0^2 \cdot p_\sigma
\end{aligned}
\tag{3.43c}
$$

(no solenoid field in the bending magnets and in the cavities $\Longrightarrow$ $K_x \cdot H = K_z \cdot H = 0$; $V \cdot H = 0$).

4 Introduction of a Periodic Reference Orbit for the Combined Spin–Orbit System

As can be seen from (3.26) and (3.43), the series expansion for $\mathcal{H}$ contains terms linear in the orbital coordinates x, p_x, z, p_z, σ, p_σ and $\vec{\Omega}$ contains terms independent of the orbital coordinates. These and the linear terms can be eliminated by introducing a new reference orbit for the combined spin–orbit system (8-dimensional closed orbit).

4.1 Definition of the 8–Dimensional Closed Orbit

We begin by defining the 8–dimensional closed orbit:

$$\left(\vec{y}_0(s), \vec{\xi}_0(s)\right)$$

containing a periodic orbital part

$$\vec{y}_0^{\,T} = (x_0,\ p_{x0};\ z_0,\ p_{z0};\ \sigma_0,\ p_{\sigma 0}),$$

with

$$\vec{y}_0(s+L) \ = \ \vec{y}_0(s) \tag{4.1a}$$

and a spin part $\vec{\xi}_0(s)$ which defines a periodic spin vector

$$\vec{\xi}_0(s) \ = \ \xi_{0s} \cdot \vec{e}_s + \xi_{0x} \cdot \vec{e}_x + \xi_{0z} \cdot \vec{e}_z$$

with

$$\vec{\xi}_0(s+L) \ = \ \vec{\xi}_0(s) \tag{4.1b}$$

whereby the equations of motion read as:

$$\frac{d}{ds}\,\vec{y}_0 \ = \ -\underline{S} \cdot \frac{\partial}{\partial \vec{y}_0}\,\mathcal{H}(\vec{y}_0; s)\ ; \tag{4.2a}$$

$$\vec{e}_s \cdot \frac{d}{ds}\,\xi_{0s} + \vec{e}_x \cdot \frac{d}{ds}\,\xi_{0x} + \vec{e}_z \cdot \frac{d}{ds}\,\xi_{0z} \ = \ \vec{\Omega}^{(0)} \times \vec{\xi}_0 \tag{4.2b}$$

(see eqns. (3.9) and (3.36a)) with

$$\vec{\Omega}^{(0)} \equiv \vec{\Omega}(\vec{y}_0, s) \tag{4.3}$$

and $\underline{S}$ is given by eqn. (3.11). Thus $\left(\vec{y}_0(s), \vec{\xi}_0(s)\right)$ is a periodic solution of the combined equations (3.9) and (3.36) of spin–orbit motion.

The components of the precession vector $\vec{\Omega}^{(0)}$ are given by (eqns. (3.43a, b, c)):

$$\begin{aligned}
\Omega_s^{(0)} &= -2H \cdot (1+a) \\
&\quad +2H \cdot (1+a) \cdot p_{\sigma 0} \\
&\quad -\beta_0^2 \cdot p_{x0} \cdot \frac{a\gamma_0^2}{1+\gamma_0} \left[K_z - \frac{e}{p_0 \cdot c} \cdot \Delta B_x \right] \\
&\quad +\beta_0^2 \cdot p_{z0} \cdot \frac{a\gamma_0^2}{1+\gamma_0} \left[K_x + \frac{e}{p_0 \cdot c} \cdot \Delta B_z \right] \; ; \qquad (4.4a)
\end{aligned}$$

$$\begin{aligned}
\Omega_x^{(0)} &= K_z \cdot a\gamma_0 - (1+a\gamma_0) \cdot \frac{e}{p_0 \cdot c} \Delta B_x \\
&\quad -(1+a\gamma_0) \cdot \left[(N-H') \cdot x_0 - (K_z^2 - g) \cdot z_0 \right] \\
&\quad +\frac{a\gamma_0^2}{1+\gamma_0} \cdot 2H \cdot \beta_0^2 \cdot [p_{x0} + H \cdot z_0] \\
&\quad +\left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0} \right] \cdot \frac{eV(s)}{E_0} \sin\varphi \cdot p_{z0} \\
&\quad -\left[1 + \frac{1+a\gamma_0}{\gamma_0^2 - 1} \right] \cdot \left(K_z - \frac{e}{p_0 \cdot c} \Delta B_x \right) \cdot \beta_0^2 \cdot p_{\sigma 0} \; ; \qquad (4.4b)
\end{aligned}$$

$$\begin{aligned}
\Omega_z^{(0)} &= -K_x \cdot a\gamma_0 - (1+a\gamma_0) \cdot \frac{e}{p_0 \cdot c} \Delta B_z \\
&\quad +(1+a\gamma_0) \cdot \left[(N+H') \cdot z_0 - (K_x^2 + g) \cdot x_0 \right] \\
&\quad +\frac{a\gamma_0^2}{1+\gamma_0} \cdot 2H \cdot \beta_0^2 \cdot [p_{z0} - H \cdot x_0] \\
&\quad -\left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0} \right] \cdot \frac{eV(s)}{E_0} \sin\varphi \cdot p_{x0} \\
&\quad +\left[1 + \frac{1+a\gamma_0}{\gamma_0^2 - 1} \right] \cdot \left(K_x + \frac{e}{p_0 \cdot c} \Delta B_z \right) \cdot \beta_0^2 \cdot p_{\sigma 0} \; . \qquad (4.4c)
\end{aligned}$$

With the help of this precession vector

$$\vec{\Omega}^{(0)}(s) \;=\; \Omega_s^{(0)} \cdot \vec{e}_s + \Omega_x^{(0)} \cdot \vec{e}_x + \Omega_z^{(0)} \cdot \vec{e}_z \qquad (4.5)$$

which describes the spin motion along the closed orbit $\vec{y}_0(s)$ we can construct a suitable periodic reference frame for spin

$$\left[\vec{n}_0(s+L),\ \vec{m}(s+L),\ \vec{l}(s+L)\right] = \left[\vec{n}_0(s),\ \vec{m}(s),\ \vec{l}(s)\right]$$

(see Appendix B and Refs. [1, 12]) with

$$\begin{aligned}
\vec{n}_0 &= \vec{\xi}_0/|\vec{\xi}_0| \ ; & (4.6a)\\
\vec{n}_0(s) &\perp \vec{m}(s) \perp \vec{l}(s) \ ; & (4.6b)\\
\vec{n}_0(s) &= \vec{m}(s) \times \vec{l}(s) \ ; & (4.6c)\\
|\vec{n}_0(s)| &= |\vec{m}(s)| \;=\; |\vec{l}(s)| \;=\; 1 & (4.6d)
\end{aligned}$$

and

$$\begin{aligned}
\vec{e}_s \cdot \frac{d}{ds}\, n_{0s} + \vec{e}_x \cdot \frac{d}{ds}\, n_{0x} + \vec{e}_z \cdot \frac{d}{ds}\, n_{0z} &= \vec{\Omega}^{(0)} \times \vec{n}_0(s) \ ; & (4.7a)\\
\vec{e}_s \cdot \frac{d}{ds}\, m_s + \vec{e}_x \cdot \frac{d}{ds}\, m_x + \vec{e}_z \cdot \frac{d}{ds}\, m_z &= \vec{\Omega}^{(0)} \times \vec{m}(s) + \vec{l}\,(s) \cdot \frac{d}{ds}\, \psi_{spin}(s) \ ; & (4.7b)\\
\vec{e}_s \cdot \frac{d}{ds}\, l_s + \vec{e}_x \cdot \frac{d}{ds}\, l_x + \vec{e}_z \cdot \frac{d}{ds}\, l_z &= \vec{\Omega}^{(0)} \times \vec{l}\,(s) - \vec{m}(s) \cdot \frac{d}{ds}\, \psi_{spin}(s) & (4.7c)
\end{aligned}$$

and

$$\psi_{spin}(s+L) - \psi_{spin}(s) \;=\; 2\pi \cdot Q_{spin} \qquad (4.8)$$

in which the unit spin vector $\vec{\xi}$ may be represented as

$$\vec{\xi} = \sqrt{1 - \xi_m^2 - \xi_l^2} \cdot \vec{n}_0 + \xi_m \cdot \vec{m} + \xi_l \cdot \vec{l}\ . \qquad (4.9)$$

With the condition (B.16b), spins on the closed orbit precess at the constant rate ψ'_{spin} with respect to $(\vec{m}, \vec{l})$. Furthermore, $J = \vec{\xi} \cdot \vec{n}_0$ is constant. Thus with (B.16b) and the corresponding orthonormal vectors $(\vec{m}, \vec{l})$ we can consider (J, ψ_{spin}) to be action–angle variables for motion on the closed orbit. As explained in Appendix B, the spin frequency Q_{spin} is arbitrary up to an integer. There is a different $(\vec{m}, \vec{l})$ pair corresponding to each integer part.

In the case that Q_{spin} is an integer (the fractional part is zero) not only $\vec{n}_0$ but also $\vec{m}_0$ and $\vec{l}_0$ are periodic solutions of the T–BMT equation. Thus in this case the $(\vec{n}_0\, \vec{m}_0\, \vec{l}_0)$ dreibein is not unique. Alternatively we can note that (3×3) spin transfer matrix on the closed orbit becomes a unit matrix with degenerate eigenvectors. Given one choice of $(\vec{n}_0\, \vec{m}_0\, \vec{l}_0)$ or $(\vec{n}_0\, \vec{m}\, \vec{l})$, equally valid dreibeins can then be generated by arbitrary orthogonal transformations.

The vector $\vec{n}_0$ obeys the T–BMT equation, but off resonance $(\vec{m}, \vec{l})$ are not T–BMT solutions. On resonance this distinction is lost.

4.2 Transformation of the Orbital Variables

4.2.1 Canonical Transformation

The orbit vector $\vec{y}(s)$ can now be separated into two components

$$\vec{y}(s) = \vec{y}_0(s) + \tilde{\vec{y}}(s) \ , \tag{4.10}$$

where the vector $\tilde{\vec{y}}(s)$ describes the synchro–betatron oscillations about the new closed equilibrium trajectory $\vec{y}_0(s)$.

The transformation

$$\vec{y} \quad \Longrightarrow \quad \tilde{\vec{y}} \tag{4.11}$$

can be obtained from the generating function

$$\begin{aligned} F_2(x, \tilde{p}_x; z, \tilde{p}_z; \sigma, \tilde{p}_\sigma; s) \;\; &= \;\; (x - x_0) \cdot (\tilde{p}_x + p_{x0}) + (z - z_0) \cdot (\tilde{p}_z + p_{z0}) \\ &\qquad + (\sigma - \sigma_0) \cdot (\tilde{p}_\sigma + p_{\sigma 0}) + f(s) \ . \end{aligned} \tag{4.12}$$

The transformation equations read as:

$$p_x \;\; = \;\; \frac{\partial F_2}{\partial x} \;\; = \;\; \tilde{p}_x + p_{x0} \ ; \quad \tilde{x} \;\; = \;\; \frac{\partial F_2}{\partial \tilde{p}_x} \;\; = \;\; x - x_0 \ ; \tag{4.13a}$$

$$p_z \;\; = \;\; \frac{\partial F_2}{\partial z} \;\; = \;\; \tilde{p}_z + p_{z0} \ ; \quad \tilde{x} \;\; = \;\; \frac{\partial F_2}{\partial \tilde{p}_x} \;\; = \;\; x - x_0 \ ; \tag{4.13b}$$

$$p_\sigma \;\; = \;\; \frac{\partial F_2}{\partial \sigma} \;\; = \;\; \tilde{p}_\sigma + p_{\sigma 0} \ ; \quad \tilde{\sigma} \;\; = \;\; \frac{\partial F_2}{\partial \tilde{p}_\sigma} \;\; = \;\; \sigma - \sigma_0 \tag{4.13c}$$

which reproduce the defining equation (4.10) for $\tilde{\vec{y}}$.

The term $f(s)$ in eqn. (4.12) is an arbitrary function. Choosing $f(s)$ such that

$$\frac{d}{ds}\ f(s) = x_0(s) \cdot \frac{d}{ds}\ p_{x0}(s) + z_0(s) \cdot \frac{d}{ds}\ p_{z0}(s) + \sigma_0(s) \cdot \frac{d}{ds}\ p_{\sigma 0}(s)$$

we furthermore have:

$$\begin{aligned} \frac{\partial F_2}{\partial s} \;\; &= \;\; -\frac{dx_0}{ds} \cdot p_x + \frac{dp_{x0}}{ds} \cdot x - \frac{dz_0}{ds} \cdot p_z + \frac{dp_{z0}}{ds} \cdot z - \frac{d\sigma_0}{ds} \cdot p_\sigma + \frac{dp_{\sigma 0}}{ds} \cdot \sigma \\ &= \;\; -p_x \cdot \left(\frac{\partial \mathcal{H}}{\partial p_x}\right)_{\vec{y}=\vec{y}_0} - x \cdot \left(\frac{\partial \mathcal{H}}{\partial x}\right)_{\vec{y}=\vec{y}_0} \\ &\qquad -p_z \cdot \left(\frac{\partial \mathcal{H}}{\partial p_z}\right)_{\vec{y}=\vec{y}_0} - z \cdot \left(\frac{\partial \mathcal{H}}{\partial z}\right)_{\vec{y}=\vec{y}_0} \\ &\qquad -p_\sigma \cdot \left(\frac{\partial \mathcal{H}}{\partial p_\sigma}\right)_{\vec{y}=\vec{y}_0} - \sigma \cdot \left(\frac{\partial \mathcal{H}}{\partial \sigma}\right)_{\vec{y}=\vec{y}_0} \\ &= \;\; -\vec{y} \cdot \left(\frac{\partial \mathcal{H}}{\partial \vec{y}}\right)_{\vec{y}=\vec{y}_0} \end{aligned}$$

and therefore

$$\begin{aligned}\mathcal{H} \quad \longrightarrow \quad \tilde{\mathcal{H}} &\equiv \mathcal{H} + \frac{\partial F_2}{\partial s} \\ &= \mathcal{H} - \vec{y} \cdot \left(\frac{\partial \mathcal{H}}{\partial \vec{y}}\right)_{\vec{y}=\vec{y}_0} \quad . \end{aligned} \tag{4.14}$$

4.2.2 The Linearised Equations of Orbital Motion

The orbital Hamiltonian $\tilde{\mathcal{H}}$ in the linear case (see eqn. (3.26)) takes the form :

$$\begin{aligned} &\tilde{\mathcal{H}}(\tilde{x}, \tilde{z}, \tilde{\sigma}; \tilde{p}_x, \tilde{p}_z, \tilde{p}_\sigma; s) \\ &= \frac{1}{2} \cdot \frac{1}{\gamma_0^2} \cdot \tilde{p}_\sigma^2 - [K_x \cdot \tilde{x} + K_z \cdot \tilde{z}] \cdot \tilde{p}_\sigma \\ &+ \frac{1}{2} \cdot \left\{ [\tilde{p}_x + H \cdot \tilde{z}]^2 + [\tilde{p}_z - H \cdot \tilde{x}]^2 \right\} \\ &+ \frac{1}{2} \cdot \left\{ \left[K_x^2 + g\right] \cdot \tilde{x}^2 + \left[K_z^2 - g\right] \cdot \tilde{z}^2 - 2N \cdot \tilde{x}\tilde{z} \right\} \\ &- \frac{1}{2}\, \tilde{\sigma}^2 \cdot \frac{1}{\beta_0^2} \cdot \frac{eV(s)}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \cos\varphi \ . \end{aligned} \tag{4.15}$$

The corresponding canonical equations read as:

$$\frac{d}{ds}\,\tilde{\vec{y}} \quad = \quad \underline{A}(s) \cdot \tilde{\vec{y}} \tag{4.16}$$

with $\underline{A}$ given by (3.29).

Due to the linearity of the equations of motion (4.16) the solution may be written in the form:

$$\tilde{\vec{y}}(s) \quad = \quad \underline{M}(s,\, s_0)\, \tilde{\vec{y}}(s_0)$$

which defines the transfer matrix $\underline{M}(s,\, s_0)$.

Since the variables $\tilde{x}, \tilde{z}, \tilde{\sigma}; \tilde{p}_x, \tilde{p}_z, \tilde{p}_\sigma$ are canonical, the transfer matrix is symplectic [13] :

$$\underline{M}^T(s, s_0) \cdot \underline{S} \cdot \underline{M}(s, s_0) = \underline{S} \ . \tag{4.17}$$

The symplecticity condition (4.17) ensures that the transfer matrix , $\underline{M}(s, s_0)$, contains complete information about the stability of the (linear) synchro–betatron oscillations.

As a result of this condition, we are able to introduce action–angle variables for the orbital motion.

To do that we remark, that the orbit vector $\tilde{\vec{y}}$ can be represented as a linear combination of the (normalised) eigenvectors $\vec{v}_{\pm k}(s)$ $(k = I,\ II,\ III)$ of the revolution matrix $\underline{M}(s+L, s)$ and may thus be written as:

$$\tilde{\vec{y}}\,(s) = \sum_{k=I,II,III} \{A_k \cdot \vec{v}_k\,(s) + A_{-k} \cdot \vec{v}_{-k}\,(s)\} \tag{4.18}$$

with

$$\underline{M}(s+L\ ,s)\,\vec{v}_\mu(s) \;\;=\;\; e^{-i\,\cdot\,2\pi Q_\mu} \cdot \vec{v}_\mu(s)\ ; \tag{4.19a}$$

$$Q_{-k} \;\;=\;\; -Q_k\ ;\quad (k = I,\ II,\ III) \tag{4.19b}$$

and

$$\left\{ \begin{array}{l} \vec{v}_k^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_k(s) = -\vec{v}_{-k}^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_{-k}(s) = i\ ; \\ \\ \vec{v}_\mu^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_\nu(s) = 0 \quad \text{otherwise} \end{array} \right. \tag{4.20}$$

whereby we have assumed, that the stability condition:

$$Q_\mu \quad \text{real number} \tag{4.21}$$

is satisfied.

Note that the eigenvectors $\vec{v}_\mu(s)$ represent special solutions of the equation of motion (4.16):

$$\vec{v}_\mu(s) \;\;=\;\; \underline{M}(s, s_0)\,\vec{v}_\mu(s_0)\ .$$

If we put

$$\vec{v}_\mu(s) = \hat{\vec{v}}_\mu(s) \cdot e^{-i\,\cdot\,2\pi Q_\mu\,\cdot\,(s/L)} \tag{4.22a}$$

we obtain from (4.19):

$$\hat{\vec{v}}_\mu(s+L) = \hat{\vec{v}}_\mu(s)\ . \tag{4.22b}$$

Equation (4.22) is a statement of the Floquet theorem : vectors $\vec{v}_\mu(s)$ are special solutions of the equations of motion (4.16) which can be expressed as the product of a periodic function $\hat{\vec{v}}_\mu(s)$ and a harmonic function

$$e^{-i\,\cdot\,2\pi Q_\mu\,\cdot\,(s/L)}\ .$$

The "Floquet vectors" $\hat{\vec{v}}_\mu(s)$ fulfil the same relationships as the eigenvectors $\vec{v}_\mu(s)$:

$$\left\{ \begin{array}{l} \hat{\vec{v}}_k^{\,+}(s) \cdot \underline{S} \cdot \hat{\vec{v}}_k(s) = -\hat{\vec{v}}_{-k}^{\,+}(s) \cdot \underline{S} \cdot \hat{\vec{v}}_{-k}(s) = i\ ; \\ \\ \hat{\vec{v}}_\mu^{\,+}(s) \cdot \underline{S} \cdot \hat{\vec{v}}_\nu(s) = 0 \quad \text{otherwise}\ . \end{array} \right. \tag{4.23}$$

Using these results we are now able to introduce a new set of canonical variables. For this we write for the coefficients $A_k,\ A_{-k}\ \ (k = I, II, III)$ in eqn. (4.18) :

$$A_k = \sqrt{J_k} \cdot e^{-i[\Phi_k - 2\pi Q_k \cdot s/L]} \ ; \tag{4.24a}$$

$$A_{-k} = \sqrt{J_k} \cdot e^{+i[\Phi_k - 2\pi Q_k \cdot s/L]} \ . \tag{4.24b}$$

Then eqn. (4.18) takes the form:

$$\tilde{\vec{y}}(s) = \sum_{k=I,II,III} \sqrt{J_k} \cdot \left\{ \tilde{\vec{v}}_k(s) \cdot e^{-i\Phi_k} + \tilde{\vec{v}}_{-k}(s) \cdot e^{+i\Phi_k} \right\} \ . \tag{4.25}$$

From (4.25) we now have:

$$\frac{\partial \tilde{\vec{y}}}{\partial \Phi_k} = -i \cdot \sqrt{J_k} \cdot \left\{ \tilde{\vec{v}}_k(s) \cdot e^{-i\Phi_k} - \tilde{\vec{v}}_{-k}(s) \cdot e^{+i\Phi_k} \right\} \ ; \tag{4.26a}$$

$$\frac{\partial \tilde{\vec{y}}}{\partial J_k} = +\frac{1}{2\sqrt{J_k}} \cdot \left\{ \tilde{\vec{v}}_k(s) \cdot e^{-i\Phi_k} + \tilde{\vec{v}}_{-k}(s) \cdot e^{+i\Phi_k} \right\} \ . \tag{4.26b}$$

Taking into account the relations (4.23) one obtains the equations [14]:

$$\frac{\partial \tilde{\vec{y}}}{\partial J_k}^T \cdot \underline{S} \cdot \frac{\partial \tilde{\vec{y}}}{\partial \Phi_l} = -\frac{\partial \tilde{\vec{y}}}{\partial \Phi_l}^T \cdot \underline{S} \cdot \frac{\partial \tilde{\vec{y}}}{\partial J_k} = \delta_{kl} \ ; \tag{4.27a}$$

$$\frac{\partial \tilde{\vec{y}}}{\partial J_k}^T \cdot \underline{S} \cdot \frac{\partial \tilde{\vec{y}}}{\partial J_l} = \frac{\partial \tilde{\vec{y}}}{\partial \Phi_k}^T \cdot \underline{S} \cdot \frac{\partial \tilde{\vec{y}}}{\partial \Phi_l} = 0 \tag{4.27b}$$

which can be combined into the matrix form

$$\underline{\mathcal{J}}^T \cdot \underline{S} \cdot \underline{\mathcal{J}} = \underline{S} \tag{4.28}$$

where $\underline{\mathcal{J}}$ signifies the Jacobian matrix

$$\underline{\mathcal{J}} = \left(\frac{\partial \tilde{\vec{y}}}{\partial \Phi_I}, \frac{\partial \tilde{\vec{y}}}{\partial J_I}, \frac{\partial \tilde{\vec{y}}}{\partial \Phi_{II}}, \frac{\partial \tilde{\vec{y}}}{\partial J_{II}}, \frac{\partial \tilde{\vec{y}}}{\partial \Phi_{III}}, \frac{\partial \tilde{\vec{y}}}{\partial J_{III}} \right) \tag{4.29}$$

being a 6×6–matrix just written as a row of column vectors $(\partial \tilde{\vec{y}}/\partial \Phi_I)$ etc.

Equation (4.28) proves that eqn. (4.25) represents a canonical transformation

$$\tilde{x},\ \tilde{p}_x,\ \tilde{z},\ \tilde{p}_z,\ \tilde{\sigma},\ \tilde{p}_\sigma \ \longrightarrow\ \Phi_I,\ J_I,\ \Phi_{II},\ J_{II},\ \Phi_{III},\ J_{III} \tag{4.30}$$

and that $\Phi_k,\ J_k\ (k = I, II, III)$ are indeed canonical variables which can now be interpreted as action–angle variables since

$$\frac{dJ_k}{ds} = 0 \ ; \tag{4.31a}$$

$$\frac{d\Phi_k}{ds} = \frac{2\pi}{L} Q_k \ . \tag{4.31b}$$

The way to find the Hamiltonian in terms of J_k, Φ_k is explained in Ref. [14].

The orbit vector $\tilde{\vec{y}}(s)$ in (4.25) is thus an explicit function of the canonical variables J_k and Φ_k and of the longitudinal variable s, via the eigenvectors, $\tilde{\vec{v}}_k(s)$.

4.3 Spin Motion

In analogy to the separation of the oscillation amplitude $\vec{y}$ into two parts we can divide the precession vector $\vec{\Omega}$ (see eqn. (3.36b)) into two components by writing:

$$\vec{\Omega}\,(\vec{y}) = \vec{\Omega}^{(0)} + \vec{\omega} \tag{4.32}$$

with

$$\vec{\omega} \equiv \vec{\Omega} - \vec{\Omega}^{(0)} = \vec{\omega}\,(\vec{\tilde{y}})\ . \tag{4.33}$$

Writing

$$\vec{\omega} \;=\; \omega_s \cdot \vec{e}_s + \omega_x \cdot \vec{e}_x + \omega_z \cdot \vec{e}_z \tag{4.34}$$

we obtain from eqns. (3.43a, b, c) for the linearised components ω_s, ω_x, ω_z, of the vector $\vec{\omega}$:

$$\begin{aligned}
\omega_s \;=\; & +2H \cdot (1+a) \cdot \tilde{p}_\sigma \\
& -\beta_0^2 \cdot \tilde{p}_x \cdot \frac{a\gamma_0^2}{1+\gamma_0} \left[K_z - \frac{e}{p_0 \cdot c}\,\Delta B_x\right] \\
& +\beta_0^2 \cdot \tilde{p}_z \cdot \frac{a\gamma_0^2}{1+\gamma_0} \left[K_x + \frac{e}{p_0 \cdot c}\,\Delta B_z\right]\ ;
\end{aligned} \tag{4.35a}$$

$$\begin{aligned}
\omega_x \;=\; & -(1+a\gamma_0) \cdot \left[(N-H') \cdot \tilde{x} - (K_z^2 - g) \cdot \tilde{z}\right] \\
& + \frac{a\gamma_0^2}{1+\gamma_0} \cdot 2H \cdot \beta_0^2 \cdot [\tilde{p}_x + H \cdot \tilde{z}] \\
& + \frac{1}{\beta_0^2} \cdot \left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}\right] \cdot \frac{eV(s)}{E_0} \sin\varphi \cdot \tilde{p}_z \\
& - \left[1 + \frac{1+a\gamma_0}{\gamma_0^2 - 1}\right] \cdot \left(K_z - \frac{e}{p_0 \cdot c}\,\Delta B_x\right) \cdot \beta_0^2 \cdot \tilde{p}_\sigma\ ;
\end{aligned} \tag{4.35b}$$

$$\begin{aligned}
\omega_z \;=\; & +(1+a\gamma_0) \cdot \left[(N+H') \cdot \tilde{z} - (K_x^2 + g) \cdot \tilde{x}\right] \\
& + \frac{a\gamma_0^2}{1+\gamma_0} \cdot 2H \cdot \beta_0^2 \cdot [\tilde{p}_z - H \cdot \tilde{x}] \\
& - \frac{1}{\beta_0^2} \cdot \left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}\right] \cdot \frac{eV(s)}{E_0} \sin\varphi \cdot \tilde{p}_x \\
& + \left[1 + \frac{1+a\gamma_0}{\gamma_0^2 - 1}\right] \cdot \left(K_x + \frac{e}{p_0 \cdot c}\,\Delta B_z\right) \cdot \beta_0^2 \cdot \tilde{p}_\sigma\ .
\end{aligned} \tag{4.35c}$$

Furthermore, the vector

$$\begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix}$$

with the linearised components $\omega_s,\ \omega_x,\ \omega_z$ can be written as:

$$\begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} = \underline{F}_{\ (3\times 6)} \cdot \begin{pmatrix} \tilde{x} \\ \tilde{p}_x \\ \tilde{z} \\ \tilde{p}_z \\ \tilde{\sigma} \\ \tilde{p}_\sigma \end{pmatrix} \tag{4.36}$$

with

$$\begin{aligned}
F_{12} &= -a(\gamma_0 - 1) \cdot \left[K_z - \frac{e}{p_0 \cdot c} \cdot \Delta B_x \right] \ ; \\
F_{14} &= +a(\gamma_0 - 1) \cdot \left[K_x + \frac{e}{p_0 \cdot c} \cdot \Delta B_z \right] \ ; \\
F_{16} &= +2H \cdot (1 + a) \ ; \\
F_{21} &= -(1 + a\gamma_0) \cdot (N - H') \ ; \\
F_{22} &= +a(\gamma_0 - 1) \cdot 2H \ ; \\
F_{23} &= +(1 + a\gamma_0) \cdot (K_z^2 - g) + 2a(\gamma_0 - 1) \cdot H^2 \ ; \\
F_{24} &= \left[a\gamma_0 + \frac{\gamma_0}{1 + \gamma_0} \right] \cdot \frac{e}{E_0} V(s) \sin\varphi \ ; \\
F_{26} &= -\left[1 + \frac{a}{\gamma_0} \right] \cdot \left(K_z - \frac{e}{p_0 \cdot c} \Delta B_x \right) \ ; \\
F_{31} &= -(1 + a\gamma_0) \cdot (K_x^2 + g) - 2a(\gamma_0 - 1) \cdot H^2 \ ; \\
F_{32} &= -\left[a\gamma_0 + \frac{\gamma_0}{1 + \gamma_0} \right] \cdot \frac{e}{E_0} V(s) \sin\varphi \ ;
\end{aligned}$$

$$
\begin{aligned}
F_{33} &= +(1 + a\gamma_0) \cdot (N + H') \ ; \\
F_{34} &= +a(\gamma_0 - 1) \cdot 2H \ ; \\
F_{36} &= + \left[1 + \frac{a}{\gamma_0} \right] \cdot \left(K_x + \frac{e}{p_0 \cdot c} \Delta B_z \right) \ ; \\
F_{ik} &= 0 \quad \text{otherwise} \ .
\end{aligned}
\tag{4.37}
$$

The precession vector $\vec{\omega}$ describes the spin motion in the $(\vec{n}_0,\, \vec{m},\, \vec{l}\,)$ spin frame, as shall be shown in the next chapter.

5 The Equations of Spin Motion in the $(\vec{n}_0,\, \vec{m},\, \vec{l}\,)$ System

5.1 Spin Motion in the $(\vec{n}_0,\, \vec{m},\, \vec{l}\,)$ System

By eqns. (2.8) and (4.9) we have:

$$
\begin{aligned}
\vec{\xi} &= \xi_s \cdot \vec{e}_s + \xi_x \cdot \vec{e}_x + \xi_z \cdot \vec{e}_z \\
&= \xi_n \cdot \vec{n}_0 + \xi_m \cdot \vec{m} + \xi_l \cdot \vec{l} \ ,
\end{aligned}
\tag{5.1}
$$

which we rewrite as:

$$
\begin{pmatrix} \xi_n \\ \xi_m \\ \xi_l \end{pmatrix} = \begin{pmatrix} \vec{n}_0\, \vec{e}_s & \vec{n}_0\, \vec{e}_x & \vec{n}_0\, \vec{e}_z \\ \vec{m}\, \vec{e}_s & \vec{m}\, \vec{e}_x & \vec{m}\, \vec{e}_z \\ \vec{l}\, \vec{e}_s & \vec{l}\, \vec{e}_x & \vec{l}\, \vec{e}_z \end{pmatrix} \begin{pmatrix} \xi_s \\ \xi_x \\ \xi_z \end{pmatrix} \ ; \tag{5.2a}
$$

$$
\begin{pmatrix} \xi_s \\ \xi_x \\ \xi_z \end{pmatrix} = \begin{pmatrix} \vec{n}_0\, \vec{e}_s & \vec{m}\, \vec{e}_s & \vec{l}\, \vec{e}_s \\ \vec{n}_0\, \vec{e}_x & \vec{m}\, \vec{e}_x & \vec{l}\, \vec{e}_x \\ \vec{n}_0\, \vec{e}_z & \vec{m}\, \vec{e}_z & \vec{l}\, \vec{e}_z \end{pmatrix} \begin{pmatrix} \xi_n \\ \xi_m \\ \xi_l \end{pmatrix} \ . \tag{5.2b}
$$

The equations of motion for spin in the $(\vec{e}_x,\, \vec{e}_z,\, \vec{e}_s)$ system read in matrix form as (see eqn. (3.36)):

$$
\frac{d}{ds} \begin{pmatrix} \xi_s \\ \xi_x \\ \xi_z \end{pmatrix} = \underline{\Omega} \begin{pmatrix} \xi_s \\ \xi_x \\ \xi_z \end{pmatrix} \tag{5.3}
$$

and (see eqns. (4.7a, b, c)):

$$\frac{d}{ds}\begin{pmatrix} \vec{n}_0\,\vec{e}_s \\ \vec{n}_0\,\vec{e}_x \\ \vec{n}_0\,\vec{e}_z \end{pmatrix} = \underline{\Omega}^{(0)}\begin{pmatrix} \vec{n}_0\,\vec{e}_s \\ \vec{n}_0\,\vec{e}_x \\ \vec{n}_0\,\vec{e}_z \end{pmatrix} \;;$$

$$\frac{d}{ds}\begin{pmatrix} \vec{m}\,\vec{e}_s \\ \vec{m}\,\vec{e}_x \\ \vec{m}\,\vec{e}_z \end{pmatrix} = \underline{\Omega}^{(0)}\begin{pmatrix} \vec{m}\,\vec{e}_s \\ \vec{m}\,\vec{e}_x \\ \vec{m}\,\vec{e}_z \end{pmatrix} + \psi'_{spin}(s)\cdot\begin{pmatrix} \vec{l}\,\vec{e}_s \\ \vec{l}\,\vec{e}_x \\ \vec{l}\,\vec{e}_z \end{pmatrix} \;;$$

$$\frac{d}{ds}\begin{pmatrix} \vec{l}\,\vec{e}_s \\ \vec{l}\,\vec{e}_x \\ \vec{l}\,\vec{e}_z \end{pmatrix} = \underline{\Omega}^{(0)}\begin{pmatrix} \vec{l}\,\vec{e}_s \\ \vec{l}\,\vec{e}_x \\ \vec{l}\,\vec{e}_z \end{pmatrix} - \psi'_{spin}(s)\cdot\begin{pmatrix} \vec{m}\,\vec{e}_s \\ \vec{m}\,\vec{e}_x \\ \vec{m}\,\vec{e}_z \end{pmatrix}$$

$$\Longrightarrow \frac{d}{ds}\begin{pmatrix} \vec{n}_0\,\vec{e}_s & \vec{n}_0\,\vec{e}_x & \vec{n}_0\,\vec{e}_z \\ \vec{m}\,\vec{e}_s & \vec{m}\,\vec{e}_x & \vec{m}\,\vec{e}_z \\ \vec{l}\,\vec{e}_s & \vec{l}\,\vec{e}_x & \vec{l}\,\vec{e}_z \end{pmatrix} = \begin{pmatrix} \vec{n}_0\,\vec{e}_s & \vec{n}_0\,\vec{e}_x & \vec{n}_0\,\vec{e}_z \\ \vec{m}\,\vec{e}_s & \vec{m}\,\vec{e}_x & \vec{m}\,\vec{e}_z \\ \vec{l}\,\vec{e}_s & \vec{l}\,\vec{e}_x & \vec{l}\,\vec{e}_z \end{pmatrix} [\underline{\Omega}^{(0)}]^T + \begin{pmatrix} 0 & 0 & 0 \\ \vec{l}\,\vec{e}_s & \vec{l}\,\vec{e}_x & \vec{l}\,\vec{e}_z \\ -\vec{m}\,\vec{e}_s & -\vec{m}\,\vec{e}_x & -\vec{m}\,\vec{e}_z \end{pmatrix}\cdot\psi'_{spin}(s) \qquad (5.4)$$

with

$$\underline{\Omega} \equiv \begin{pmatrix} 0 & -\Omega_z & \Omega_x \\ \Omega_z & 0 & -\Omega_s \\ -\Omega_x & \Omega_s & 0 \end{pmatrix} = \underline{\Omega}^{(0)} + \underline{\omega} \,. \qquad (5.5)$$

From (5.2), (5.3) and (5.4) we obtain:

$$\frac{d}{ds}\begin{pmatrix} \xi_n \\ \xi_m \\ \xi_l \end{pmatrix} =$$

$$\left\{\begin{pmatrix} \vec{n}_0\,\vec{e}_s & \vec{n}_0\,\vec{e}_x & \vec{n}_0\,\vec{e}_z \\ \vec{m}\,\vec{e}_s & \vec{m}\,\vec{e}_x & \vec{m}\,\vec{e}_z \\ \vec{l}\,\vec{e}_s & \vec{l}\,\vec{e}_x & \vec{l}\,\vec{e}_z \end{pmatrix} [\underline{\Omega}^{(0)}]^T + \begin{pmatrix} 0 & 0 & 0 \\ \vec{l}\,\vec{e}_s & \vec{l}\,\vec{e}_x & \vec{l}\,\vec{e}_z \\ -\vec{m}\,\vec{e}_s & -\vec{m}\,\vec{e}_x & -\vec{m}\,\vec{e}_z \end{pmatrix}\cdot\psi'_{spin}(s)\right\}\begin{pmatrix} \xi_s \\ \xi_x \\ \xi_z \end{pmatrix}$$

$$+ \begin{pmatrix} \vec{n}_0\,\vec{e}_s & \vec{n}_0\,\vec{e}_x & \vec{n}_0\,\vec{e}_z \\ \vec{m}\,\vec{e}_s & \vec{m}\,\vec{e}_x & \vec{m}\,\vec{e}_z \\ \vec{l}\,\vec{e}_s & \vec{l}\,\vec{e}_x & \vec{l}\,\vec{e}_z \end{pmatrix} \underline{\Omega} \begin{pmatrix} \xi_s \\ \xi_x \\ \xi_z \end{pmatrix} \qquad (5.6)$$

or (since $[\underline{\Omega}^{(0)}]^T = -\underline{\Omega}^{(0)}$ and $\underline{\Omega} - \underline{\Omega}^{(0)} = \underline{\omega}$) :

$$\frac{d}{ds}\begin{pmatrix} \xi_n \\ \xi_m \\ \xi_l \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} \vec{n}_0\,\vec{e}_s & \vec{n}_0\,\vec{e}_x & \vec{n}_0\,\vec{e}_z \\ \vec{m}\,\vec{e}_s & \vec{m}\,\vec{e}_x & \vec{m}\,\vec{e}_z \\ \vec{l}\,\vec{e}_s & \vec{l}\,\vec{e}_x & \vec{l}\,\vec{e}_z \end{pmatrix} \underline{\omega} + \begin{pmatrix} 0 & 0 & 0 \\ \vec{l}\,\vec{e}_s & \vec{l}\,\vec{e}_x & \vec{l}\,\vec{e}_z \\ -\vec{m}\,\vec{e}_s & -\vec{m}\,\vec{e}_x & -\vec{m}\,\vec{e}_z \end{pmatrix} \cdot \psi'_{spin}(s) \right\} \begin{pmatrix} \xi_s \\ \xi_x \\ \xi_z \end{pmatrix}$$

$$= \begin{pmatrix} \vec{n}_0\,\vec{e}_s & \vec{n}_0\,\vec{e}_x & \vec{n}_0\,\vec{e}_z \\ \vec{m}\,\vec{e}_s & \vec{m}\,\vec{e}_x & \vec{m}\,\vec{e}_z \\ \vec{l}\,\vec{e}_s & \vec{l}\,\vec{e}_x & \vec{l}\,\vec{e}_z \end{pmatrix} \begin{pmatrix} 0 & -\omega_z & \omega_x \\ \omega_z & 0 & -\omega_s \\ -\omega_x & \omega_s & 0 \end{pmatrix} \begin{pmatrix} \xi_s \\ \xi_x \\ \xi_z \end{pmatrix}$$

$$+\psi'_{spin}(s) \cdot \begin{pmatrix} 0 & 0 & 0 \\ \vec{l}\,\vec{e}_s & \vec{l}\,\vec{e}_x & \vec{l}\,\vec{e}_z \\ -\vec{m}\,\vec{e}_s & -\vec{m}\,\vec{e}_x & -\vec{m}\,\vec{e}_z \end{pmatrix} \begin{pmatrix} \vec{n}_0\,\vec{e}_s & \vec{m}\,\vec{e}_s & \vec{l}\,\vec{e}_s \\ \vec{n}_0\,\vec{e}_x & \vec{m}\,\vec{e}_x & \vec{l}\,\vec{e}_x \\ \vec{n}_0\,\vec{e}_z & \vec{m}\,\vec{e}_z & \vec{l}\,\vec{e}_z \end{pmatrix} \begin{pmatrix} \xi_n \\ \xi_m \\ \xi_l \end{pmatrix}$$

$$= \begin{pmatrix} \vec{n}_0\,\vec{e}_s & \vec{n}_0\,\vec{e}_x & \vec{n}_0\,\vec{e}_z \\ \vec{m}\,\vec{e}_s & \vec{m}\,\vec{e}_x & \vec{m}\,\vec{e}_z \\ \vec{l}\,\vec{e}_s & \vec{l}\,\vec{e}_x & \vec{l}\,\vec{e}_z \end{pmatrix} \begin{pmatrix} -\omega_z\xi_x + \omega_x\xi_z \\ +\omega_z\xi_s - \omega_s\xi_z \\ -\omega_x\xi_s + \omega_s\xi_x \end{pmatrix} + \psi'_{spin}(s) \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \xi_n \\ \xi_m \\ \xi_l \end{pmatrix}$$

$$= \begin{pmatrix} \vec{n}_0\,\vec{e}_s & \vec{n}_0\,\vec{e}_x & \vec{n}_0\,\vec{e}_z \\ \vec{m}\,\vec{e}_s & \vec{m}\,\vec{e}_x & \vec{m}\,\vec{e}_z \\ \vec{l}\,\vec{e}_s & \vec{l}\,\vec{e}_x & \vec{l}\,\vec{e}_z \end{pmatrix} \cdot \left[\begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} \times \begin{pmatrix} \xi_s \\ \xi_x \\ \xi_z \end{pmatrix} \right] + \psi'_{spin}(s) \cdot \begin{pmatrix} 0 \\ +\xi_l \\ -\xi_m \end{pmatrix}$$

$$= \left[\begin{pmatrix} \omega_n \\ \omega_m \\ \omega_l \end{pmatrix} \times \begin{pmatrix} \xi_n \\ \xi_m \\ \xi_l \end{pmatrix} \right] + \psi'_{spin}(s) \cdot \begin{pmatrix} 0 \\ +\xi_l \\ -\xi_m \end{pmatrix}$$

$$= \begin{pmatrix} \omega_m\,\xi_l - \omega_l\,\xi_m \\ \omega_l\,\xi_n - \omega_n\,\xi_l \\ \omega_n\,\xi_m - \omega_m\,\xi_n \end{pmatrix} + \psi'_{spin}(s) \cdot \begin{pmatrix} 0 \\ +\xi_l \\ -\xi_m \end{pmatrix}$$

$$= \begin{pmatrix} (\vec{m}\cdot\vec{\omega})\cdot\xi_l - (\vec{l}\cdot\vec{\omega})\cdot\xi_m \\ (\vec{l}\cdot\vec{\omega})\cdot\xi_n - (\vec{n}_0\cdot\vec{\omega})\cdot\xi_l \\ (\vec{n}_0\cdot\vec{\omega})\cdot\xi_m - (\vec{m}\cdot\vec{\omega})\cdot\xi_n \end{pmatrix} + \psi'_{spin}(s) \cdot \begin{pmatrix} 0 \\ +\xi_l \\ -\xi_m \end{pmatrix} \tag{5.7}$$

with

$$\begin{aligned} \vec{\omega} &= \omega_s\cdot\vec{e}_s + \omega_x\cdot\vec{e}_x + \omega_z\cdot\vec{e}_z \\ &= \omega_n\cdot\vec{n}_0 + \omega_m\cdot\vec{m} + \omega_l\cdot\vec{l}\,. \end{aligned} \tag{5.8}$$

Using the notation:

$$\underline{\hat{\Omega}} = \begin{pmatrix} 0 & -\hat{\Omega}_l & \hat{\Omega}_m \\ \hat{\Omega}_l & 0 & -\hat{\Omega}_n \\ -\hat{\Omega}_m & \hat{\Omega}_n & 0 \end{pmatrix} \tag{5.9}$$

with

$$\begin{aligned}\hat{\Omega}_n &= (\vec{n}_0 \cdot \vec{\omega}) - \psi'_{spin} \; ; & (5.10a)\\ \hat{\Omega}_m &= (\vec{m} \cdot \vec{\omega}) \; ; & (5.10b)\\ \hat{\Omega}_l &= (\vec{l} \cdot \vec{\omega}) & (5.10c)\end{aligned}$$

eqn. (5.7) can be rewritten in matrix form as:

$$\frac{d}{ds}\begin{pmatrix}\xi_n\\ \xi_m\\ \xi_l\end{pmatrix} = \underline{\hat{\Omega}}\begin{pmatrix}\xi_n\\ \xi_m\\ \xi_l\end{pmatrix} . \tag{5.11}$$

Equation (5.11) may be solved by methods as described in Appendix C.

5.2 The Spin Hamiltonian

As described in section 2.2.2, we now introduce canonical spin variables α and β for the spin vector $\vec{\xi}$:

$$\vec{\xi} = \xi_n \cdot \vec{n}_0 + \xi_m \cdot \vec{m} + \xi_l \cdot \vec{l}$$

by the relations:

$$\begin{aligned}\xi_n &= 1 - \frac{1}{2}\left(\alpha^2 + \beta^2\right) \; ; & (5.12a)\\ \xi_m &= \alpha \cdot \sqrt{1 - \frac{1}{4}(\alpha^2 + \beta^2)} \; ; & (5.12b)\\ \xi_l &= \beta \cdot \sqrt{1 - \frac{1}{4}(\alpha^2 + \beta^2)} \; ; & (5.12c)\end{aligned}$$

$$\Longrightarrow \quad \begin{cases}\alpha = +\sqrt{\dfrac{2}{1+\xi_n}} \cdot \xi_m \; ; \\ \beta = +\sqrt{\dfrac{2}{1+\xi_n}} \cdot \xi_l \; .\end{cases} \tag{5.13}$$

If the orbit vector $\tilde{\vec{y}}(s)$ is known, we then can calculate the spin motion from the equations:

$$\begin{aligned}\frac{d}{ds}\,\alpha &= +\frac{\partial \mathcal{H}_{spin}}{\partial \beta} \; ; & (5.14a)\\ \frac{d}{ds}\,\beta &= -\frac{\partial \mathcal{H}_{spin}}{\partial \alpha} & (5.14b)\end{aligned}$$

which may be rewritten by using the notation

$$\vec{\zeta} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

as:

$$\frac{d}{ds}\,\vec{\zeta} \;=\; -\underline{S}_2 \cdot \frac{d}{d\vec{\zeta}}\mathcal{H}_{spin}\ . \tag{5.15}$$

The spin Hamiltonian $\mathcal{H}_{spin}$ reads (see eqns. (2.15), (5.10) and (5.11)) as:

$$\begin{aligned} \mathcal{H}_{spin}(\alpha,\beta;\ s) \;=\; & \left(-\frac{1}{2}\left(\alpha^2+\beta^2\right),\ \alpha\cdot\sqrt{1-\frac{1}{4}\left(\alpha^2+\beta^2\right)},\ \beta\cdot\sqrt{1-\frac{1}{4}\left(\alpha^2+\beta^2\right)}\right) \\ & \times \begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix} \begin{pmatrix} \omega_s(s) \\ \omega_x(s) \\ \omega_z(s) \end{pmatrix} \\ & +\frac{1}{2}\left(\alpha^2+\beta^2\right)\cdot\frac{d}{ds}\ \psi_{spin}(s) \end{aligned} \tag{5.16}$$

with [1]

$$\begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} \;=\; \underline{F}_{(3\times 6)}\cdot\tilde{\vec{y}} \tag{5.17}$$

or

$$\begin{aligned} & \mathcal{H}_{spin}(\alpha,\beta;\ s) \\ = \; & \sqrt{1-\frac{1}{4}\left(\alpha^2+\beta^2\right)}\cdot(\alpha,\beta)\begin{pmatrix} m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix}\cdot\underline{F}_{(3\times 6)}(s)\cdot\tilde{\vec{y}}(s) \\ & +\frac{1}{2}\left(\alpha^2+\beta^2\right)\cdot\left\{\frac{d}{ds}\ \psi_{spin}(s)-\left(n_{0s}(s),\ n_{0x}(s),\ n_{0z}(s)\right)\cdot\underline{F}_{(3\times 6)}(s)\cdot\tilde{\vec{y}}(s)\right\} \\ = \; & F(\alpha,\,\beta)\cdot(\beta,\,-\alpha)\cdot\underline{G}_0\cdot\tilde{\vec{y}}(s) \\ & +\frac{1}{2}\left(\alpha^2+\beta^2\right)\cdot\left\{\frac{d}{ds}\ \psi_{spin}(s)-g(s)\right\} \end{aligned} \tag{5.18}$$

with

$$\underline{G}_0(s) \;=\; \begin{pmatrix} l_s(s) & l_x(s) & l_z(s) \\ -m_s(s) & -m_x(s) & -m_z(s) \end{pmatrix}\cdot\underline{F}_{(3\times 6)}(s)\ ; \tag{5.19a}$$

$$g(s) = (n_{0s},\ n_{0x},\ n_{0z}) \cdot \underline{F}_{(3\times 6)}(s) \cdot \tilde{\vec{y}}(s) \ ; \tag{5.19b}$$

$$\begin{aligned} F(\alpha,\,\beta) &= \sqrt{1 - \frac{1}{4}\,(\alpha^2 + \beta^2)} \\ &= \sum_{n=0}^{\infty} \begin{pmatrix} 1/2 \\ n \end{pmatrix} \cdot (-1)^n \cdot \left[\frac{1}{4}\left(\alpha^2 + \beta^2\right)\right]^n \\ &= 1 - \frac{1}{8}\left(\alpha^2 + \beta^2\right) + \cdots \ . \end{aligned} \tag{5.19c}$$

From (5.14) and (5.18) we obtain the equation of spin motion in the form:

$$\begin{aligned} \frac{d}{ds}\,\vec{\zeta} &= \begin{pmatrix} \frac{\partial}{\partial\beta}\,[\,\beta \cdot F(\alpha,\,\beta)\,] & 0 \\ 0 & \frac{\partial}{\partial\alpha}\,[\,\alpha \cdot F(\alpha,\,\beta)\,] \end{pmatrix} \cdot \underline{G}_0(s) \cdot \tilde{\vec{y}}\,(s) \\ &\quad + \underline{D}_0(s) \cdot \vec{\zeta} + g(s) \cdot \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \cdot \vec{\zeta} \end{aligned} \tag{5.20}$$

with

$$\underline{D}_0(s) = \begin{pmatrix} 0 & +\psi'_{spin}(s) \\ -\psi'_{spin}(s) & 0 \end{pmatrix} \ . \tag{5.21}$$

5.3 The Linearised Equations of Spin Motion Combined with the Equations of Orbital Motion

5.3.1 The Spin Motion in Linear Order

To prepare for later investigations we consider in particular the linear order of spin motion:

$$\alpha \ll 1, \quad \beta \ll 1 \quad \Longrightarrow \quad F(\alpha,\,\beta) = 1$$

which leads to the Hamiltonian :

$$\begin{aligned} \mathcal{H}_{spin}(\alpha,\beta;\ s) &= (\alpha,\ \beta) \cdot \begin{pmatrix} m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix} \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} \\ &\quad + \frac{1}{2}\left[\alpha^2 + \beta^2\right] \cdot \frac{d}{ds}\ \psi_{spin}(s) \ . \end{aligned} \tag{5.22}$$

The corresponding canonical equations for α and β read (see eqn. (5.20)) :

$$\frac{d}{ds}\,\vec{\zeta} = \underline{G}_0(s)\cdot\vec{y}(s) + \underline{D}_0(s)\cdot\vec{\zeta}\,. \tag{5.23}$$

Here the matrix $\underline{G}_0$ describes the linear spin–orbit coupling and the function $\psi_{spin}(s)$, appearing in $\underline{D}_0(s)$, designates the spin phase function.

In this form the relation (5.23) is the basic equation for spin motion used in the computer program SLIM [15, 7]. We have thus derived the SLIM formalism from canonical equations based on a polynomial expansion of a spin Hamiltonian.

Remark:

The first order approximation of eqn. (5.7) in the limit

$$\begin{aligned} |\xi_m| &\ll 1\,; \\ |\xi_l| &\ll 1\,; \\ |\xi_n| &\approx 1 \end{aligned}$$

takes the form:

$$\alpha' = +(\vec{l}\cdot\vec{\omega}) + \beta\,\psi'_{spin}\,; \tag{5.24a}$$

$$\beta' = -(\vec{m}\cdot\vec{\omega}) - \alpha\,\psi'_{spin} \tag{5.24b}$$

with

$$\xi_m \equiv \alpha\,; \tag{5.25a}$$

$$\xi_l \equiv \beta\,. \tag{5.25b}$$

This is equivalent to eqn. (5.23).

5.3.2 The Combined Form of Orbital Motion with Linear Spin Motion

By combining the orbital part $\tilde{\vec{y}}$ and the spin part $\vec{\zeta}$ into an 8–dimensional vector as first done by A. Chao [15]:

$$\vec{u} = \begin{pmatrix} \tilde{\vec{y}} \\ \vec{\zeta} \end{pmatrix} \tag{5.26}$$

we can rewrite the orbital equation (4.16) and the spin equation (5.23) in a compact matrix notation as follows:

$$\frac{d}{ds}\vec{u} = \underline{\hat{A}}\cdot\vec{u} \tag{5.27}$$

with

$$\underline{\hat{A}} \;=\; \begin{pmatrix} \underline{A} & \underline{0} \\ \underline{G}_0 & \underline{D}_0 \end{pmatrix} \;. \tag{5.28}$$

Since eqn. (5.27) is linear and homogeneous, the solution can be written in the form:

$$\vec{u}(s) = \underline{\hat{M}}(s, s_0) \cdot \vec{u}(s_0) \tag{5.29}$$

which defines the 8–dimensional transfer matrix $\underline{\hat{M}}(s, s_0)$ of spin–orbit motion.

By eqn. (5.27), $\underline{\hat{M}}(s, s_0)$ is determined by the differential equation

$$\frac{d}{ds}\, \underline{\hat{M}}(s, s_0) \;=\; \underline{\hat{A}}(s) \cdot \underline{\hat{M}}(s, s_0) \;; \tag{5.30a}$$

$$\underline{\hat{M}}(s_0, s_0) \;=\; \underline{1} \;. \tag{5.30b}$$

If we write $\underline{\hat{M}}$ as

$$\underline{\hat{M}} \;=\; \begin{pmatrix} \underline{M} & \underline{0} \\ \underline{G} & \underline{D} \end{pmatrix} \tag{5.31}$$

we obtain from eqn. (5.30) [16]:

$$\frac{d}{ds} \begin{pmatrix} \underline{M} & \underline{0} \\ \underline{G} & \underline{D} \end{pmatrix} \;=\; \begin{pmatrix} \underline{A} & \underline{0} \\ \underline{G}_0 & \underline{D}_0 \end{pmatrix} \begin{pmatrix} \underline{M} & \underline{0} \\ \underline{G} & \underline{D} \end{pmatrix} \;=\; \begin{pmatrix} \underline{AM} & \underline{0} \\ \underline{G_0M} + \underline{D_0G} & \underline{D_0D} \end{pmatrix} \tag{5.32a}$$

and

$$\begin{pmatrix} \underline{M}(s_0, s_0) & \underline{0} \\ \underline{G}(s_0, s_0) & \underline{D}(s_0, s_0) \end{pmatrix} \;=\; \underline{1} \tag{5.32b}$$

or

$$I) \;\; \frac{d}{ds}\, \underline{M}(s, s_0) \;=\; \underline{A}(s) \cdot \underline{M}(s, s_0) \;;\; \underline{M}(s_0, s_0) = \underline{1} \;; \tag{5.33}$$

$$(\underline{M}(s, s_0) = \text{transfer matrix for the orbit}) \;;$$

$$II) \;\; \frac{d}{ds}\, \underline{D}(s, s_0) \;=\; \underline{D}_0(s) \cdot \underline{D}(s, s_0) \;;\; \underline{D}(s_0, s_0) = \underline{1}$$

$$\Rightarrow \;\; \underline{D}\,(s, s_0) \;=\; \begin{pmatrix} \cos\left[\psi_{spin}(s) - \psi_{spin}(s_0)\right] & \sin\left[\psi_{spin}(s) - \psi_{spin}(s_0)\right] \\ -\sin\left[\psi_{spin}(s) - \psi_{spin}(s_0)\right] & \cos\left[\psi(s)_{spin} - \psi_{spin}(s_0)\right] \end{pmatrix} \;; \tag{5.34}$$

$$III)\ \frac{d}{ds}\,\underline{G}(s,s_0) \;=\; \underline{G_0}(s)\cdot\underline{M}(s,s_0)+\underline{D_0}(s)\cdot\underline{G}(s,s_0)\ ;\ \underline{G}(s_0,s_0)=\underline{0}$$

$$\Rightarrow\ \underline{G}(s,s_0) \;=\; \underline{D}(s,s_0)\cdot\int_{s_0}^{s} d\tilde{s}\cdot\underline{D}(s_0,\tilde{s})\cdot\underline{G_0}(\tilde{s})\cdot\underline{M}(\tilde{s},s_0)$$
$$=\; \int_{s_0}^{s} d\tilde{s}\cdot\underline{D}(s,\tilde{s})\cdot\underline{G_0}(\tilde{s})\cdot\underline{M}(\tilde{s},s_0)\ . \tag{5.35}$$

By eqns. (5.33 - 35) the transfer matrix $\underline{\hat{M}}(s,s_0)$ is determined in a unique way.

In particular, one finds the following expressions for the revolution matrix $\underline{\hat{M}}(s_0+L,s_0)$:

$$\underline{\hat{M}}(s_0+L,s_0) \;=\; \begin{pmatrix} \underline{M}(s_0+L,s_0) & \underline{0} \\ \underline{G}(s_0+L,s_0) & \underline{D}(s_0+L,s_0) \end{pmatrix} \tag{5.36}$$

with

$$\underline{D}(s_0+L,s_0) \;=\; \begin{pmatrix} \cos\left[2\pi Q_{spin}\right] & \sin\left[2\pi Q_{spin}\right] \\ -\sin\left[2\pi Q_{spin}\right] & \cos\left[2\pi Q_{spin}\right] \end{pmatrix} \tag{5.37}$$

where the quantity Q_{spin} defines the (linear) spin tune on the closed orbit (see eqn. (4.8)).

The eigenvectors of the whole 8–dimensional revolution matrix $\underline{\hat{M}}(s_0+L,s_0)$ for spin and orbit which are defined by

$$\underline{\hat{M}}(s_0+L,s_0)\cdot\vec{q}_\mu = \hat{\lambda}_\mu\cdot\vec{q}_\mu \tag{5.38}$$

can now be written in the form:

$$\vec{q}_k(s_0) \;=\; \begin{pmatrix} \vec{v}_k(s_0) \\ \vec{w}_k(s_0) \end{pmatrix}\ ;\ \vec{q}_{-k}(s_0) = [\vec{q}_k(s_0)]^* \tag{5.39a}$$

$$\text{for } k = I,\ II,\ III$$

and

$$\vec{q}_{IV}(s_0) \;=\; \begin{pmatrix} \vec{0}_6(s_0) \\ \vec{w}_{IV}(s_0) \end{pmatrix}\ ;\ \vec{q}_{-IV}(s_0) = [\vec{q}_{IV}(s_0)]^* \tag{5.39b}$$

$$\text{for } k = IV\ .$$

By combining eqns. (5.38), (5.39), (5.36), (5.37) and (4.19) we obtain for the 2–dimensional vectors $\vec{w}_k(s_0)$ $(k = I,\ II,\ III)$ and $\vec{w}_{IV}(s_0)$:

$$\underline{G}\,(s+L,s)\ \vec{v}_k(s)+\underline{D}\,(s+L,s)\ \vec{w}_k(s) = \lambda_k\cdot\vec{w}_k(s)$$

$$\Longrightarrow\qquad \vec{w}_k(s_0) = -\left[\underline{D}(s_0+L,s_0)-\hat{\lambda}_k\right]^{-1}\cdot\underline{G}(s_0+L,s_0)\cdot\vec{v}_k(s_0) \tag{5.40a}$$

$$\text{for } k = I,\ II,\ III\ ;$$

$$\vec{w}_{IV}(s_0) \;=\; \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot e^{-i\,\psi_{spin}(s_0)} \tag{5.40b}$$

$$\text{for } k = IV$$

and

$$\vec{w}_{-k}(s_0) = [\vec{w}_k(s_0)]^* ; \quad (k = I,\ II,\ III,\ IV) \tag{5.41}$$

($\vec{v}_k(s_0)$ being defined in (4.19a)).

The corresponding eigenvalues are

$$\hat{\lambda}_k = \lambda_k = e^{-i \cdot 2\pi Q_k} \; ; \quad (k = I,\ II,\ III) \tag{5.42a}$$

and

$$\hat{\lambda}_{IV} = e^{-i \cdot 2\pi Q_{IV}} \quad \text{with} \quad Q_{IV} = Q_{spin} \; . \tag{5.42b}$$

For the the eigenvectors $\vec{q}_\mu(s)$ of the transfer matrix $\underline{\hat{M}}(s+L,s)$ (initial position s):

$$\underline{\hat{M}}(s+L,s) \cdot \vec{q}_\mu(s) = \hat{\lambda}_\mu(s) \cdot \vec{q}_\mu(s) \tag{5.43}$$

we also have:

$$\vec{q}_\mu(s) = \underline{\hat{M}}(s,s_0)\; \vec{q}_\mu(s_0) \equiv \begin{pmatrix} \vec{v}_k(s) \\ \vec{w}_k(s) \end{pmatrix} \; . \tag{5.44}$$

In particular we get

$$\vec{q}_{IV}(s) \;=\; \begin{pmatrix} \vec{0}_6 \\ \vec{w}_{IV}(s) \end{pmatrix} \; ; \quad \vec{q}_{-IV}(s) = [\vec{q}_{IV}(s)]^* \tag{5.45a}$$

with

$$\vec{w}_{IV}(s) = \underline{D}(s,s_0)\, \vec{w}_{IV}(s_0) = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot e^{-i\,\psi_{spin}(s)} \; ; \quad \vec{w}_{-IV}(s) = [\vec{w}_{IV}(s)]^*. \tag{5.45b}$$

The eigenvalues are independent of s:

$$\hat{\lambda}_\mu(s) = \hat{\lambda}_\mu(s_0) \; . \tag{5.46}$$

As may be seen by (5.31) and (5.35), the solution of eqn. (5.23) can be written as:

$$\begin{aligned} \vec{\zeta}\,(s) &= \underline{D}\,(s,s_0) \cdot \vec{\zeta}\,(s_0) + \int_{s_0}^{s} d\tilde{s} \cdot \underline{D}\,(s,\tilde{s}) \cdot \underline{G}\,_0(\tilde{s}) \cdot \underline{M}\,(\tilde{s},s_0) \cdot \vec{y}(s_0) \\ &= \underline{D}\,(s,s_0) \cdot \left\{ \vec{\zeta}\,(s_0) + \int_{s_0}^{s} d\tilde{s} \cdot \underline{D}\,(s_0,\tilde{s}) \cdot \underline{G}\,_0(\tilde{s}) \cdot \underline{M}\,(\tilde{s},s_0) \cdot \vec{y}(s_0) \right\} \; . \end{aligned} \tag{5.47}$$

Remark:

Note, that the components $\vec{w}_k$ in eqn. (5.44)

$$\begin{aligned} \vec{w}_k(s) &= -\left[\underline{D}\,(s+L,s) - \lambda_k \cdot \underline{1}\right]^{-1} \cdot \underline{G}\,(s+L,s)\ \vec{v}_k(s) \\ &= -\left[\underline{D}\,(s+L,s) - \lambda_k \cdot \underline{1}\right]^{-1} \cdot \int_s^{s+L} d\tilde{s} \cdot \underline{D}\,(s+L,\tilde{s}) \cdot \underline{G}\,_0(\tilde{s}) \cdot \vec{v}_k(\tilde{s}) \end{aligned} \tag{5.48}$$

for $(k = I,\ II,\ III)$ are solutions of eqn. (5.23) with $\vec{y}(s) = \vec{v}_k(s)$:

$$\begin{aligned} \frac{d}{ds}\ \vec{w}_k(s) &= -\left[\underline{D}\,(s+L,s) - \lambda_k \cdot \underline{1}\right]^{-1} \cdot \int_s^{s+L} d\tilde{s} \cdot \frac{d}{ds}\underline{D}\,(s+L,\tilde{s}) \cdot \underline{G}\,_0(\tilde{s}) \cdot \vec{v}_k(\tilde{s}) \\ &\quad -\left[\underline{D}\,(s+L,s) - \lambda_k \cdot \underline{1}\right]^{-1} \cdot \underline{D}\,(s+L,s+L) \cdot \underline{G}\,_0(s+L) \cdot \vec{v}_k(s+L) \\ &\quad +\left[\underline{D}\,(s+L,s) - \lambda_k \cdot \underline{1}\right]^{-1} \cdot \underline{D}\,(s+L,s) \cdot \underline{G}\,_0(s) \cdot \vec{v}_k(s) \\ &= -\left[\underline{D}\,(s+L,s) - \lambda_k \cdot \underline{1}\right]^{-1} \cdot \int_s^{s+L} d\tilde{s} \cdot \underline{D}_0(s) \cdot \underline{D}\,(s+L,\tilde{s}) \cdot \underline{G}\,_0(\tilde{s}) \cdot \vec{v}_k(\tilde{s}) \\ &\quad -\left[\underline{D}\,(s+L,s) - \lambda_k \cdot \underline{1}\right]^{-1} \cdot \underline{1} \cdot \underline{G}\,_0(s) \cdot \lambda_k \vec{v}_k(s) \\ &\quad +\left[\underline{D}\,(s+L,s) - \lambda_k \cdot \underline{1}\right]^{-1} \cdot \underline{D}\,(s+L,s) \cdot \underline{G}\,_0(s) \cdot \vec{v}_k(s) \\ &= \underline{D}_0(s) \cdot \vec{w}_k(s) + \left[\underline{D}\,(s+L,s) - \lambda_k \cdot \underline{1}\right]^{-1} \cdot \left[\underline{D}\,(s+L,s) - \lambda_k \cdot \underline{1}\right] \cdot \underline{G}\,_0(s) \cdot \vec{v}_k(s) \\ &= \underline{D}_0(s) \cdot \vec{w}_k(s) + \underline{G}_0(s) \cdot \vec{v}_k(s)\ . \end{aligned}$$

This result is in agreement with the definition of $\vec{w}_k(s)$ by eqn. (5.44), i.e. the spin–orbit vector $\vec{q}_\mu(s)$ defined by (5.44) is a solution of eqn. (5.27), which represents the combined form of spin–orbit motion.

6 Calculation of the $\vec{n}$–Axis

We are now in a position to calculate the so–called $\vec{n}$–axis [17, 4] which represents a special solution of the T–BMT equation on the 6–dimensional particle orbit having the same periodicity properties as the particle orbit, namely:

$$\begin{aligned}\vec{n}(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III}, s) &= \vec{n}(J_I, J_{II}, J_{III}, \Phi_I + 2\pi, \Phi_{II}, \Phi_{III}, s) \\ &= \vec{n}(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II} + 2\pi, \Phi_{III}, s) \\ &= \vec{n}(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III} + 2\pi, s) \\ &= \vec{n}(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III}, s + L) \qquad (6.1)\end{aligned}$$

where the variables J_k and Φ_k $(k = I,\ II,\ III)$ are the action–angle variables of the orbital motion [12].

The $\vec{n}$–axis is the key object in the definition of combined action–angle variables for spin and orbit on arbitrary orbits and for describing spin kinematics in electron storage rings [4, 17, 18].

6.1 Definition of the $\vec{n}$–Axis

In order to obtain the $\vec{n}$–axis, we introduce a damping term

$$-\gamma \cdot \vec{\zeta} \quad \text{with} \quad \gamma > 0$$

on the r.h.s. of eqn. (5.14) for spin motion:

$$\frac{d}{ds}\,\vec{\zeta} \;=\; -\underline{S}_2 \cdot \frac{d}{d\vec{\zeta}}\mathcal{H}_{spin} - \gamma \cdot \vec{\zeta}\ . \qquad (6.2)$$

Denoting then the solution of eqn. (6.2) by

$$\vec{\zeta} \;=\; \vec{F}\,(\vec{\zeta}_0,\, \gamma;\, s) \qquad (6.3)$$

with

$$\vec{\zeta}_0 \equiv \vec{\zeta}\,(s_0)\ ,$$

the $\vec{n}$–axis is calculated from the relation:

$$\vec{n}\,(s) \;=\; \lim_{0<\gamma\to 0}\ \lim_{s_0\to -\infty}\ \vec{F}\,(\vec{\zeta}_0,\, \gamma;\, s)\ . \qquad (6.4)$$

Thus we are using the fact that the asymptotic forced solution of a damped oscillator has the periodicity of the driving force [19]. In this case the driving force is the vector $\vec{\omega}$ which has the periodicity (6.1) and we introduced the damping via the term $\gamma \cdot \vec{\zeta}$. By subsequently allowing $\gamma\cdot\vec{\zeta}$ to approach zero we recover the solution to the T–BMT equation with the desired periodicity properties.

6.2 Perturbation Theory

The $\vec{n}$–axis shall now be determined in a perturbation theory.

Using (4.18) we write eqn. (5.19b) in the form:

$$g(s) = \sum_{k=I,II,III} \{A_k \cdot g_k(s) + A_{-k} \cdot g_{-k}(s)\} \tag{6.5}$$

with

$$g_\mu(s) = (n_{0s}(s),\ n_{0x}(s),\ n_{0z}(s)) \cdot \underline{F}_{(3\times 6)} \cdot \vec{v}_\mu(s). \tag{6.6}$$

Then we consider the (small) coefficients $A_{\pm k}$ as perturbation parameters which lead to a perturbation series for the spin vector $\vec{\zeta}$:

$$\vec{\zeta} = \vec{\zeta}^{(0)} + \vec{\zeta}^{(1)} + \vec{\zeta}^{(2)} + \cdots + \vec{\zeta}^{(N)} + \cdots . \tag{6.7}$$

The different orders of the $\vec{n}$–axis may then be constructed in a systematic manner using the method of forced solution as described by eqns. (6.2 - 4).

6.2.1 The $\vec{n}$–Axis in 0^{th} Order

From eqns. (5.20) and (6.2) we have:

$$\frac{d}{ds}\vec{\zeta}^{(0)} = \underline{D}_0(s) \cdot \vec{\zeta}^{(0)} - \gamma \cdot \vec{\zeta}^{(0)} . \tag{6.8}$$

with the solution :

$$\vec{\zeta}^{(0)}(s) = e^{-\gamma \cdot (s - s_0)} \cdot \underline{D}(s, s_0) \cdot \vec{\zeta}^{(0)}(s_0)$$

and with $\underline{D}(s, s_0)$ given by (5.34).
The forced solution is:

$$\begin{aligned} \vec{n}^{(0)}(s) &= \lim_{0<\gamma\to 0} \lim_{s_0\to -\infty} e^{-\gamma \cdot (s - s_0)} \cdot \underline{D}(s, s_0) \cdot \vec{\zeta}^{(0)}(s_0) \\ &= 0 , \end{aligned} \tag{6.9}$$

i.e. the $\vec{n}$–axis coincides with $\vec{n}_0$ in 0^{th} order.

Note, that $\vec{n}^{(0)}(s)$ satisfies the periodicity condition (6.1).

6.2.2 The $\vec{n}$–Axis in Linear Order

In first order we obtain from (5.20) and (6.2):

$$\frac{d}{ds}\vec{\zeta}^{(1)} = \underline{G}_0(s) \cdot \tilde{\vec{y}}(s) + \underline{D}_0(s) \cdot \vec{\zeta}^{(1)} - \gamma \cdot \vec{\zeta}^{(1)} \, . \tag{6.10}$$

The solution of eqn. (6.10) reads as:

$$\begin{aligned}
\vec{\zeta}^{(1)}(s) &= e^{-\gamma \cdot (s - s_0)} \cdot \underline{D}(s, s_0) \\
&\qquad \times \left\{ \vec{\zeta}^{(1)}(s_0) + \int_{s_0}^{s} d\tilde{s} \cdot \underline{D}(s_0, \tilde{s}) \cdot \underline{G}_0(\tilde{s}) \cdot \tilde{\vec{y}}(\tilde{s}) \cdot e^{+\gamma \cdot (\tilde{s} - s_0)} \right\} \, .
\end{aligned}$$

and the forced solution is:

$$\begin{aligned}
\vec{n}^{(1)}(s) &= \lim_{0<\gamma\to 0} \lim_{s_0\to -\infty} e^{-\gamma \cdot (s - s_0)} \cdot \underline{D}(s, s_0) \\
&\qquad \times \left\{ \vec{\zeta}^{(1)}(s_0) + \int_{s_0}^{s} d\tilde{s} \cdot \underline{D}(s_0, \tilde{s}) \cdot \underline{G}_0(\tilde{s}) \cdot \tilde{\vec{y}}(\tilde{s}) \cdot e^{+\gamma \cdot (\tilde{s} - s_0)} \right\} \\
&= \lim_{0<\gamma\to 0} \int_{-\infty}^{s} d\tilde{s} \cdot \underline{D}(s, \tilde{s}) \cdot \underline{G}_0(\tilde{s}) \cdot \tilde{\vec{y}}(\tilde{s}) \cdot e^{+\gamma \cdot \tilde{s}} \\
&= \lim_{0<\gamma\to 0} \sum_{n=0}^{\infty} \int_{s-(n+1)\cdot L}^{s-n\cdot L} d\tilde{s} \cdot \underline{D}(s, \tilde{s}) \cdot \underline{G}_0(\tilde{s}) \cdot \sum_{\mu=1}^{6} A_\mu \, \vec{v}_\mu(\tilde{s}) \cdot e^{+\gamma \cdot \tilde{s}} \\
&= \lim_{0<\gamma\to 0} \sum_{n=0}^{\infty} \int_{s}^{s+L} ds' \cdot \underline{D}(s, s') \cdot \underline{D}(s', s' - [n+1]\,L) \cdot \underline{G}_0(s' - [n+1]\,L) \\
&\qquad \times \sum_{\mu=1}^{6} A_\mu \, \vec{v}_\mu(s' - [n+1]\,L) \cdot e^{+\gamma \cdot (s' - [n+1]\,L)} \\
&= \lim_{0<\gamma\to 0} \sum_{n=0}^{\infty} \int_{s}^{s+L} ds' \cdot \underline{D}(s, s') \cdot \underline{D}^{n+1}(s + L, s) \cdot \underline{G}_0(s') \\
&\qquad \times \sum_{\mu=1}^{6} A_\mu \, \vec{v}_\mu(s') \cdot \lambda_\mu^{-(n+1)} \cdot e^{-(n+1) \cdot \gamma L} \\
&= \lim_{0<\gamma\to 0} \sum_{\mu=1}^{6} A_\mu \sum_{n=0}^{\infty} \underline{D}^{n+1}(s + L, s) \cdot \lambda_\mu^{-(n+1)} \cdot e^{-(n+1) \cdot \gamma L} \\
&\qquad \times \int_{s}^{s+L} ds' \cdot \underline{D}(s, s') \cdot \underline{G}_0(s') \cdot \vec{v}_\mu(s') \\
&= \sum_{\mu=1}^{6} A_\mu \cdot \lim_{0<\gamma\to 0} \left[\underline{1} - \lambda_\mu^{-1} \cdot e^{-\gamma L} \cdot \underline{D}(s + L, s) \right]^{-1} \cdot \lambda_\mu^{-1} \cdot e^{-\gamma L} \cdot \underline{D}(s + L, s)
\end{aligned}$$

$$\times \int_s^{s+L} ds' \cdot \underline{D}\,(s,s') \cdot \underline{G}_0(s') \cdot \vec{v}_\mu(s')$$

$$= \sum_{\mu=1}^{6} A_\mu \cdot [\lambda_\mu \cdot \underline{1} - \underline{D}\,(s+L,s)]^{-1}$$

$$\times \int_s^{s+L} d\tilde{s} \cdot \underline{D}\,(s+L,\tilde{s}) \cdot \underline{G}_0(\tilde{s}) \cdot \vec{v}_\mu(\tilde{s})$$

$$= \sum_{\mu=1}^{6} A_\mu \cdot \vec{w}_\mu(s) \tag{6.11}$$

(see also Ref. [20]) whereby eqn. (5.48) and the following relations have been used:

$$\underline{D}\,(s', s'-[n+1]\,L) = \underline{D}^{(n+1)}\,(s'+L,s') = \underline{D}^{(n+1)}\,(s+L,s) \; ;$$

$$\underline{G}_0(s'-[n+1]\,L) = \underline{G}_0(s') \; ;$$

$$\vec{v}_\mu(s'-[n+1]\,L) = \lambda_\mu^{-(n+1)} \cdot \vec{v}_\mu(s') \; ;$$

$$\underline{D}\,(s,s') \cdot \underline{D}^{(n+1)}\,(s+L,s) = \underline{D}^{(n+1)}\,(s+L,s) \cdot \underline{D}\,(s,s') \; ;$$

$$\sum_{n=0}^{\infty} \underline{D}^n\,(s+L,s) \cdot \lambda_\mu^{-n} \cdot e^{-n \cdot \gamma L} = \left[\underline{1} - \lambda_\mu^{-1} \cdot e^{-\gamma L} \cdot \underline{D}\,(s+L,s)\right]^{-1} .$$

Equation (6.11) can also be written in the form:

$$\vec{n}^{\,(1)}\,(s) = \sum_{\mu=1}^{6} \hat{A}_\mu \cdot \hat{\vec{w}}_\mu(s) \tag{6.12}$$

with

$$\hat{\vec{w}}_\mu(s) = e^{+i \cdot 2\pi Q_\mu \cdot (s/L)} \cdot \vec{w}_\mu(s) \tag{6.13}$$

and

$$\hat{A}_\mu = e^{-i \cdot 2\pi Q_\mu \cdot (s/L)} \cdot A_\mu$$

$$= \sqrt{J_\mu} \cdot e^{-i\Phi_\mu}$$

$$= \hat{A}_\mu(J_\mu, \Phi_\mu) \; . \tag{6.14}$$

It follows from (6.12) that $\vec{n}^{\,(1)}(s)$ satisfies the periodicity condition (6.1) since

$$\vec{\tilde{w}}_\mu(s+L) = \vec{\tilde{w}}_\mu(s) \tag{6.15}$$

and

$$\hat{A}_\mu(J_\mu, \Phi_\mu + 2\pi) \;=\; \hat{A}_\mu(J_\mu, \Phi_\mu)\ . \tag{6.16}$$

<u>Remark:</u>

If we transform the rotation matrix $\underline{D}(s+L,s)$

$$\underline{D}\,(s+L,s) \;=\; \begin{pmatrix} \cos\,[2\pi\, Q_{spin}] & \sin\,[2\pi\, Q_{spin}] \\ -\sin\,[2\pi\, Q_{spin}] & \cos\,[2\pi\, Q_{spin}] \end{pmatrix}$$

into principle axes:

$$\underline{U}^{-1}\cdot\underline{D}\,(s+L,s)\cdot\underline{U} \;=\; \underline{J}\ ;\quad \underline{D}\,(s+L,s) \;=\; \underline{U}\cdot\underline{J}\cdot\underline{U}^{-1}\ ; \tag{6.17a}$$

$$\underline{U} \;=\; \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\ ; \tag{6.17b}$$

$$\underline{J} \;=\; \begin{pmatrix} e^{+i\,\cdot\,2\pi Q_{spin}} & 0 \\ 0 & e^{-i\,\cdot\,2\pi Q_{spin}} \end{pmatrix} \tag{6.17c}$$

the vector $\vec{w}_k$ can be put into the form:

$$\begin{aligned}
\vec{w}_k(s) &= -\left[\underline{U}\cdot\underline{J}\cdot\underline{U}^{-1} - e^{-i\,\cdot\,2\pi Q_k}\cdot\underline{1}\right]^{-1}\cdot\underline{G}\,(s+L,s)\ \vec{v}_k(s) \\
&= -\underline{U}\cdot\left[\underline{J} - e^{-i\,\cdot\,2\pi Q_k}\cdot\underline{1}\right]^{-1}\cdot\underline{U}^{-1}\cdot\underline{G}\,(s+L,s)\ \vec{v}_k(s) \\
&= \underline{U}\cdot\begin{pmatrix} \dfrac{i}{2\sin\pi[Q_k+Q_{spin}]}\cdot e^{+i\pi[Q_k-Q_{spin}]} & 0 \\ 0 & \dfrac{i}{2\sin\pi[Q_k-Q_{spin}]}\cdot e^{+i\pi[Q_k+Q_{spin}]} \end{pmatrix} \\
&\qquad \times\underline{U}^{-1}\cdot\underline{G}\,(s+L,s)\ \vec{v}_k(s)\ .
\end{aligned}$$

This equation shows that the components of $\vec{w}_k$ become infinitely large for

$$Q_k \pm Q_{spin} \;\longrightarrow\; \text{integer}\ .$$

6.2.3 The $\vec{n}$–Axis at Second Order

$$\frac{d}{ds}\,\vec{\zeta}^{\,(2)} \;=\; \underline{D}_{\,0}(s)\cdot\vec{\zeta}^{\,(2)} + \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\cdot g(s)\cdot\vec{n}^{\,(1)}(s) - \gamma\cdot\vec{\zeta}^{\,(2)} \qquad (6.18)$$

with

$$\begin{aligned} \vec{n}^{\,(1)}(s) &= \sum_{k=I,II,III} \{A_k\cdot\vec{w}_k(s) + A_{-k}\cdot\vec{w}_{-k}(s)\} \\ &= \sum_{\mu} A_\mu\cdot\vec{w}_\mu(s) \end{aligned} \qquad (6.19)$$

and

$$g(s) \;=\; \sum_{\mu} A_\mu\cdot g_\mu(s)\ ; \qquad (6.20a)$$

$$g_\mu(s) \;=\; (n_s(s),\ n_x(s),\ n_z(s))\cdot\,\underline{F}_{\,(3\times 6)}(s)\cdot\vec{v}_\mu(s)\ . \qquad (6.20b)$$

The solution of eqn. (6.18)

$$\vec{\zeta}^{(2)}(s) \equiv\ \vec{F}^{\,(2)}\,(\vec{\zeta}_0^{(2)},\,\gamma;\,s)$$

reads as:

$$\begin{aligned} \vec{F}^{\,(2)}\,(\vec{\zeta}_0^{(2)},\,\gamma;\,s) \;&=\; e^{-\gamma\,\cdot\,(s\,-\,s_0)}\cdot\underline{D}\,(s,s_0)\cdot\vec{\zeta}^{\,(2)}\,(s_0) \\ &+e^{-\gamma\,\cdot\,(s\,-\,s_0)}\cdot\underline{D}\,(s,s_0) \\ &\times\int_{s_0}^{s} d\tilde{s}\cdot\underline{D}\,(s_0,\tilde{s})\cdot\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\cdot g(\tilde{s})\cdot\vec{n}^{\,(1)}(\tilde{s})\cdot e^{+\gamma\,\cdot\,(\tilde{s}\,-\,s_0)} \end{aligned}$$

The forced solution is:

$$\begin{aligned} \vec{n}^{\,(2)}(s) \;&=\; \lim_{0<\gamma\to 0}\ \lim_{s_0\to-\infty}\ \vec{F}^{\,(2)}\,(\vec{\zeta}_0^{(2)},\,\gamma;\,s) \\ &= \lim_{0<\gamma\to 0}\int_{-\infty}^{s} d\tilde{s}\cdot\underline{D}\,(s,\tilde{s})\cdot\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\cdot g(\tilde{s})\cdot\vec{n}^{\,(1)}(\tilde{s})\cdot e^{+\gamma\,\cdot\,\tilde{s}} \\ &= \sum_{\mu_1}\sum_{\mu_2} A_{\mu_1}A_{\mu_2}\cdot\ \lim_{0<\gamma\to 0}\int_{-\infty}^{s} d\tilde{s}\cdot\underline{D}\,(s,\tilde{s})\cdot\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\cdot g_{\,\mu_1}(\tilde{s})\cdot\vec{w}_{\,\mu_2}(\tilde{s})\cdot e^{+\gamma\,\cdot\,\tilde{s}} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mu_1}\sum_{\mu_2} A_{\mu_1}A_{\mu_2}\cdot \lim_{0<\gamma\to 0}\sum_{n=0}^{\infty}\int_{s-(n+1)\cdot L}^{s-n\cdot L} d\tilde{s}\cdot \underline{D}(s,\tilde{s}) \\
&\qquad \times\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\cdot g_{\mu_1}(\tilde{s})\cdot \vec{w}_{\mu_2}(\tilde{s})\cdot e^{+\gamma\cdot\tilde{s}} \\
&= \sum_{\mu_1}\sum_{\mu_2} A_{\mu_1}A_{\mu_2}\cdot \lim_{0<\gamma\to 0}\sum_{n=0}^{\infty}\int_{s}^{s+L} ds'\cdot \underline{D}(s,s')\cdot \underline{D}(s',s'-[n+1]\,L) \\
&\qquad \times\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\cdot g_{\mu_1}(s'-[n+1]\,L)\cdot \vec{w}_{\mu_2}(s'-[n+1]\,L) \\
&\qquad \times e^{+\gamma\cdot(s'-[n+1]\,L)} \\
&= \sum_{\mu_1}\sum_{\mu_2} A_{\mu_1}A_{\mu_2}\cdot \lim_{0<\gamma\to 0}\sum_{n=0}^{\infty}\int_{s}^{s+L} ds'\cdot \underline{D}(s,s')\cdot \underline{D}^{n+1}(s+L,s) \\
&\qquad \times\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\cdot \lambda_{\mu_1}^{-(n+1)}\, g_{\mu_1}(s')\cdot \lambda_{\mu_2}^{-(n+1)}\,\vec{w}_{\mu_2}(s') \\
&\qquad \times e^{+\gamma\cdot(s'-[n+1]\,L)} \\
&= \sum_{\mu_1}\sum_{\mu_2} A_{\mu_1}A_{\mu_2}\cdot \lim_{0<\gamma\to 0}\sum_{n=0}^{\infty}\underline{D}^{n+1}(s+L,s)\cdot \lambda_{\mu_1}^{-(n+1)}\cdot\lambda_{\mu_2}^{-(n+1)}\cdot e^{-(n+1)\cdot\gamma L} \\
&\quad \times\int_{s}^{s+L} ds'\cdot \underline{D}(s,s')\cdot\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\cdot g_{\mu_1}(s')\cdot \vec{w}_{\mu_2}(s')\cdot e^{+\gamma\cdot s'} \\
&= \sum_{\mu_1}\sum_{\mu_2} A_{\mu_1}A_{\mu_2}\cdot \lim_{0<\gamma\to 0}\left[\underline{1}-\lambda_{\mu_1}^{-1}\cdot\lambda_{\mu_2}^{-1}\cdot e^{-\gamma L}\cdot\underline{D}(s+L,s)\right]^{-1} \\
&\quad \times\lambda_{\mu_1}^{-1}\cdot\lambda_{\mu_2}^{-1}\cdot e^{-\gamma L}\cdot\ \underline{D}(s+L,s) \\
&\quad \times\int_{s}^{s+L} ds'\cdot \underline{D}(s,s')\cdot\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\cdot g_{\mu_1}(s')\cdot \vec{w}_{\mu_2}(s') \\
&= \sum_{\mu_1}\sum_{\mu_2} A_{\mu_1}A_{\mu_2}\cdot[\lambda_{\mu_1}\lambda_{\mu_2}\cdot\underline{1}-\ \underline{D}(s+L,s)]^{-1} \\
&\quad \times\int_{s}^{s+L} ds'\cdot g_{\mu_1}(s')\cdot\underline{D}(s+L,s')\cdot\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\cdot\vec{w}_{\mu_2}(s')\ . \qquad (6.21)
\end{aligned}
$$

Clearly the components of $\vec{n}^{\,(2)}$ become infinitely large when

$$Q_{\mu_1}+Q_{\mu_2} \;=\; integer+Q_{spin}\ .$$

(For the general resonance relation see eqn. (6.27)).

6.2.4 The $\vec{n}$–Axis at k^{th} Order

For an arbitrary (the k^{th}) order we can in general write:

$$\frac{d}{ds}\,\vec{\zeta}^{(k)} = \underline{D}_0(s)\cdot\vec{\zeta}^{(k)} + \sum_{\mu_1,\mu_2,\cdots,\mu_k} A_{\mu_1}A_{\mu_2}\cdots A_{\mu_k}\cdot\vec{K}^{(k)}_{\mu_1\mu_2\cdots\mu_k}(s) - \gamma\cdot\vec{\zeta}^{(k)}\,, \tag{6.22}$$

whereby the term $\vec{K}^{(k)}_{\mu_1\mu_2\cdots\mu_k}(s)$ depends on $\vec{n}^{(\nu)}$ for $\nu < k$.

For instance we have:

$$\vec{K}^{(0)} = 0\,; \tag{6.23a}$$

$$\vec{K}^{(1)}_{\mu} = \underline{G}_0(s)\cdot\vec{v}_\mu(s)\,; \tag{6.23b}$$

$$\vec{K}^{(2)}_{\mu_1\mu_2} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\cdot\vec{w}_{\mu_1}(s)\cdot(n_s,\,n_x,\,n_z)\cdot\underline{F}_{(3\times 6)}(s)\,\vec{v}_{\mu_2}(s)\,. \tag{6.23c}$$

(see eqns. (6.8), (6.10), (6.18)). The general form of $\vec{K}^{(k)}_{\mu_1\mu_2\cdots\mu_k}(s)$ may be found from eqn. (5.20) by iteration.

<u>Assumption</u>:

$$\vec{K}^{(k)}_{\mu_1\mu_2\cdots\mu_k}(s+L) = \lambda_{\mu_1}\lambda_{\mu_2}\cdots\lambda_{\mu_k}\cdot\vec{K}^{(k)}_{\mu_1\mu_2\cdots\mu_k}(s)\,. \tag{6.24}$$

This relation is fulfilled for $k = 0,\,1,\,2$ as may be seen from eqn. (6.23a, b, c), and in the following shall be proved by induction.

From eqn. (6.22) we obtain:

$$\frac{d}{ds}\,\vec{\zeta}^{(k)} = \underline{D}_0(s)\cdot\vec{\zeta}^{(k)} + \sum_{\mu_1,\mu_2,\cdots,\mu_k} A_{\mu_1}A_{\mu_2}\cdots A_{\mu_k}\cdot\vec{K}^{(k)}_{\mu_1\mu_2\cdots\mu_k}(s) - \gamma\cdot\vec{\zeta}^{(k)}\,. \tag{6.25}$$

The solution of eqn. (6.25)

$$\vec{\zeta}^{(k)} \equiv \vec{F}^{(k)}(\vec{\zeta}^{(k)}_0,\,\gamma;\,s)$$

reads as:

$$\vec{\zeta}^{(1)}(s) = e^{-\gamma\cdot(s-s_0)}\cdot\underline{D}(s,s_0)$$

$$\times\left\{\vec{\zeta}^{(k)}(s_0) + \int_{s_0}^{s} d\tilde{s}\cdot\underline{D}(s_0,\tilde{s})\cdot\sum_{\mu_1,\mu_2,\cdots,\mu_k} A_{\mu_1}A_{\mu_2}\cdots A_{\mu_k}\cdot\vec{K}^{(k)}_{\mu_1\mu_2\cdots\mu_k}(s)\cdot e^{+\gamma\cdot(\tilde{s}-s_0)}\right\}\,.$$

The forced solution is:

$$
\begin{aligned}
\vec{n}^{(k)} &= \lim_{0<\gamma\to 0}\ \lim_{s_0\to -\infty}\ \vec{F}^{(k)}(\vec{\zeta}_0^{(k)},\ \gamma;\ s) \\
&= \lim_{0<\gamma\to 0}\int_{-\infty}^{s} d\tilde{s}\cdot \underline{D}(s,\tilde{s})\cdot \sum_{\mu_1,\mu_2,\cdots,\mu_k} A_{\mu_1}A_{\mu_2}\cdots A_{\mu_k}\cdot \vec{K}^{(k)}_{\mu_1\mu_2\cdots\mu_k}(s)\cdot e^{+\gamma\cdot\tilde{s}} \\
&= \sum_{\mu_1,\mu_2,\cdots\mu_k} A_{\mu_1}A_{\mu_2}\cdots A_{\mu_k}\cdot \lim_{0<\gamma\to 0}\int_{-\infty}^{s} d\tilde{s}\cdot \underline{D}(s,\tilde{s})\cdot \vec{K}^{(k)}_{\mu_1\mu_2\cdots\mu_k}(\tilde{s})\cdot e^{+\gamma\cdot\tilde{s}} \\
&= \sum_{\mu_1,\mu_2,\cdots\mu_k} A_{\mu_1}A_{\mu_2}\cdots A_{\mu_k}\cdot \lim_{0<\gamma\to 0}\sum_{n=0}^{\infty}\int_{s-(n+1)\cdot L}^{s-n\cdot L} d\tilde{s}\cdot \underline{D}(s,\tilde{s})\cdot \vec{K}^{(k)}_{\mu_1\mu_2\cdots\mu_k}(\tilde{s})\cdot e^{+\gamma\cdot\tilde{s}} \\
&= \sum_{\mu_1,\mu_2,\cdots\mu_k} A_{\mu_1}A_{\mu_2}\cdots A_{\mu_k}\cdot \lim_{0<\gamma\to 0}\sum_{n=0}^{\infty}\int_{s}^{s+L} ds'\cdot \underline{D}(s,s')\cdot \underline{D}(s',s'-[n+1]\,L) \\
&\qquad \times\vec{K}^{(k)}_{\mu_1\mu_2\cdots\mu_k}(s'-[n+1]\,L)\cdot e^{+\gamma\cdot(s'-[n+1]\,L)} \\
&= \sum_{\mu_1,\mu_2,\cdots\mu_k} A_{\mu_1}A_{\mu_2}\cdots A_{\mu_k}\cdot \lim_{0<\gamma\to 0}\sum_{n=0}^{\infty}\int_{s}^{s+L} ds'\cdot \underline{D}(s,s')\cdot \underline{D}^{n+1}(s+L,s) \\
&\qquad \times\left[\lambda_{\mu_1}^{-(n+1)}\lambda_{\mu_2}^{-(n+1)}\cdots\lambda_{\mu_k}^{-(n+1)}\right]\cdot \vec{K}^{(k)}_{\mu_1\mu_2\cdots\mu_k}(s')\cdot e^{+\gamma\cdot(s'-[n+1]\,L)} \\
&= \sum_{\mu_1,\mu_2,\cdots\mu_k} A_{\mu_1}A_{\mu_2}\cdots A_{\mu_k}\cdot \lim_{0<\gamma\to 0}\sum_{n=0}^{\infty}\underline{D}^{n+1}(s+L,s)\cdot\left[\lambda_{\mu_1}^{-(n+1)}\lambda_{\mu_2}^{-(n+1)}\cdots\lambda_{\mu_k}^{-(n+1)}\right] \\
&\qquad \times e^{-(n+1)\cdot\gamma L}\cdot\int_{s}^{s+L} ds'\cdot \underline{D}(s,s')\cdot \vec{K}^{(k)}_{\mu_1\mu_2\cdots\mu_k}(s')\cdot e^{+\gamma\cdot s'} \\
&= \sum_{\mu_1,\mu_2,\cdots\mu_k} A_{\mu_1}A_{\mu_2}\cdots A_{\mu_k}\cdot \lim_{0<\gamma\to 0}\left[\underline{1}-(\lambda_{\mu_1}\lambda_{\mu_2}\cdots\lambda_{\mu_k})^{-1}\cdot e^{-\gamma L}\cdot\underline{D}(s+L,s)\right]^{-1} \\
&\qquad \times(\lambda_{\mu_1}\lambda_{\mu_2}\cdots\lambda_{\mu_k})^{-1}\cdot e^{-\gamma L}\cdot\ \underline{D}(s+L,s)\cdot\int_{s}^{s+L} ds'\cdot \underline{D}(s,s')\cdot \vec{K}^{(k)}_{\mu_1\mu_2\cdots\mu_k}(s') \\
&= \sum_{\mu_1,\mu_2,\cdots\mu_k} A_{\mu_1}A_{\mu_2}\cdots A_{\mu_k}\cdot[\lambda_{\mu_1}\lambda_{\mu_2}\cdots\lambda_{\mu_k}\cdot\underline{1}-\ \underline{D}(s+L,s)]^{-1} \\
&\qquad \times\int_{s}^{s+L} ds'\cdot \vec{K}^{(k)}_{\mu_1\mu_2\cdots\mu_k}(s')\ .
\end{aligned}
\tag{6.26}
$$

Transforming the rotation matrix $\underline{D}(s+L,s)$ into principle axis (see eqns. (6.17a, b, c)), one can easily show that the term

$$
[\lambda_{\mu_1}\lambda_{\mu_2}\cdots\lambda_{\mu_k}\cdot\underline{1}-\ \underline{D}(s+L,s)]^{-1}
$$

in (6.26) becomes infinite for

$$Q_{\mu_1} + Q_{\mu_2} + \cdots + Q_{\mu_k} \;=\; integer + Q_{spin}\ . \tag{6.27}$$

Finally we remark that the vector $\vec{n}^{(k)}$ in (6.26) obeys the periodicity relation (6.1):

$$\begin{aligned}\vec{n}^{(k)}(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III}, s) &= \vec{n}^{(k)}(J_I, J_{II}, J_{III}, \Phi_I + 2\pi, \Phi_{II}, \Phi_{III}, s)\\ &= \vec{n}^{(k)}(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II} + 2\pi, \Phi_{III}, s)\\ &= \vec{n}^{(k)}(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III} + 2\pi, s)\\ &= \vec{n}^{(k)}(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III}, s + L)\end{aligned} \tag{6.28}$$

as may be seen by using eqns. (6.14), (6.16) and (6.24).

With the help of (6.26) and (6.24) we now can prove that the relation (6.24) is also fulfilled for the next order $(k+1)$. It follows then that eqn. (6.24) and thus also the periodicity condition (6.28) is valid for all orders of $\vec{n}(s)$.

Remarks:

1) In this paper we have used the canonical variables α and β to calculate the $\vec{n}$–axis. It is also possible to work with the components ξ_n, ξ_m and ξ_l of the spin vector $\vec{\xi}$ by using an iterative technique applied to equations of spin motion as represented in the form of eqn. (2.10). This is essentially the method used in the SMILE formalism introduced by S. Mane [18]. It is also possible to calculate $\vec{n}$ iteratively in terms of $(\xi_m,\, \xi_l)$ by expanding eqn. (4.9) [21].

2) The vector $\vec{n}$ is unique except at spin–orbit resonances and in the neighbourhood of resonances $\vec{n}$ depends very sensitively on the phase space position. These statements are non–perturbative and are analogous to the lack of uniqueness of $\vec{n}_0$ on the closed orbit (see section 4.2.1 and 7.3). However, when calculating $\vec{n}$ perturbatively, we find that the components $(\tilde{\alpha},\, \tilde{\beta})$ diverge at resonances. This is the perturbative manifestation of the instability and non–uniqueness that would be found in a non–perturbative calculation.

3) As a result of the periodicity relations (6.1), the $\vec{n}$–axis can be expanded into a Fourier series:

$$\begin{aligned}\vec{n}\,(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III};\, s) \;=& \sum_{m_1, m_2, m_3, q} \vec{N}_{m_1 m_2 m_3\, q}\,(J_k)\\ &\times e^{-i\,\{m_1\,\Phi_I(\tilde{s}) + m_2\,\Phi_{II}(\tilde{s}) + m_3\,\Phi_{III}(\tilde{s}) + q\cdot 2\pi\,(\tilde{s}/L)\}}\ .\end{aligned} \tag{6.29}$$

Using the ansatz (6.29), we obtain an alternative method to determine the $\vec{n}$–axis by solving the equation of spin motion (5.14). (See also Ref. [4]).

4) The solution (6.26) diverges not only at the resonance (6.27), but also at the resonances that were found at lower order.

7 Spin Motion in the $(\vec{n}_1, \vec{n}_2, \vec{n})$ System

7.1 Introduction of the Dreibein $(\vec{n}_1, \vec{n}_2, \vec{n})$

In this section we demonstrate how to use the $\vec{n}$–axis to construct a special dreibein $(\vec{n}_1, \vec{n}_2, \vec{n})$ on an arbitrary 6–dimensional orbit.

In the following we shall introduce a compact matrix notation by rewriting an arbitrary vector

$$\vec{A} \;=\; A_n \cdot \vec{n}_0 + A_m \cdot \vec{m} + A_l \cdot \vec{l}$$

as a column vector with components A_n, A_m, A_l:

$$A_n \cdot \vec{n}_0 + A_m \cdot \vec{m} + A_l \cdot \vec{l} \;\equiv\; \begin{pmatrix} A_n \\ A_m \\ A_l \end{pmatrix}$$

and defining the derivative of a column vector with respect to the arc length s as the derivative of the corresponding components A_i but not of the unit vectors:

$$\frac{d}{ds}\begin{pmatrix} A_n \\ A_m \\ A_l \end{pmatrix} \;\equiv\; \vec{n}_0 \cdot \frac{d}{ds}\, A_n + \vec{m} \cdot \frac{d}{ds}\, A_m + \vec{l} \cdot \frac{d}{ds}\, A_l \; .$$

In particular we then have:

$$\vec{m} \;\equiv\; \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \; ; \; \vec{l} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and thus in this notation:

$$\frac{d}{ds}\, \vec{m} \;=\; \frac{d}{ds}\, \vec{l} \;=\; 0 \; .$$

Furthermore we construct the unit vectors $\vec{n}_1$ and $\vec{n}_2$ as:

$$\vec{n}_1 \;\equiv\; \begin{pmatrix} (\vec{n}_1\, \vec{n}_0) \\ (\vec{n}_1\, \vec{m}) \\ (\vec{n}_1\, \vec{l}) \end{pmatrix} \;=\; \frac{\vec{n} \times \vec{m}}{|\vec{n} \times \vec{m}|}\; ; \tag{7.1a}$$

$$\vec{n}_2 \equiv \begin{pmatrix} (\vec{n}_2\,\vec{n}_0) \\ (\vec{n}_2\,\vec{m}) \\ (\vec{n}_2\,\vec{l}) \end{pmatrix} = \vec{n} \times \vec{n}_1$$

$$= \frac{\vec{n} \times [\vec{n} \times \vec{m}]}{|\vec{n} \times \vec{m}|}$$

$$= \frac{\vec{n} \cdot (\vec{n}\,\vec{m}) - \vec{m}}{|\vec{n} \times \vec{m}|} \; ; \tag{7.1b}$$

$$\Longrightarrow \quad \vec{n}_1 \times \vec{n}_2 = \vec{n} \; .$$

By construction $\vec{n}_1$ and $\vec{n}_2$ obey the same periodicity condition (6.1) as $\vec{n}$ and form a mutually orthogonal set [1] .

Since

$$\frac{d}{ds}\,\frac{\vec{a}(s)}{|\vec{a}(s)|} = \frac{\vec{a}'}{|\vec{a}|} - \frac{\vec{a}}{|\vec{a}|^2} \cdot \frac{d}{ds}\,\sqrt{\vec{a}^2}$$

$$= \frac{\vec{a}'}{|\vec{a}|} - \frac{\vec{a}}{|\vec{a}|^2} \cdot \frac{\vec{a} \cdot \vec{a}'}{|\vec{a}|}$$

$$= \frac{\vec{a}' \cdot (\vec{a}\,\vec{a}) - \vec{a} \cdot (\vec{a}\,\vec{a}')}{|\vec{a}|^3}$$

$$= \frac{\vec{a} \times [\vec{a}' \times \vec{a}]}{|\vec{a}|^3}$$

we obtain with

$$\vec{a} = \vec{n} \times \vec{m} \tag{7.2}$$

and $(\vec{a}/|\vec{a}|) \equiv \vec{n}_1$:

$$\frac{d}{ds}\,\vec{n}_1 = \frac{\vec{n}_1 \times [\vec{a}' \times \vec{n}_1]}{|\vec{a}|}$$

$$= \frac{\vec{n}_1 \times \left\{ \left(\vec{m} \times \left[\vec{n} \times \hat{\vec{\Omega}} \right] \right) \times \vec{n}_1 \right\}}{|\vec{a}|} \; ,$$

where $\hat{\vec{\Omega}}$ is the spin precession vector in the $(\vec{n}_0,\, \vec{m},\, \vec{l})$ frame.

Writing

$$\left\{ \vec{n}_1 \times \left[\vec{m} \times (\vec{n} \times \hat{\vec{\Omega}}) \right] \right\} = A \cdot \vec{n} + B \cdot \vec{n}_1 + C \cdot \vec{n}_2$$

[1] For $\vec{n} = \vec{n}_0$ we have $\vec{n}_1 = \vec{l}$ and $\vec{n}_2 = -\vec{m}$.

$$
\begin{aligned}
\Longrightarrow \quad |\vec{a}| \cdot \frac{d}{ds}\,\vec{n}_1 &= \{A \cdot \vec{n} + B \cdot \vec{n}_1 + C \cdot \vec{n}_2\} \times \vec{n}_1 \\
&= \{A \cdot \vec{n} + C \cdot \vec{n}_2\} \times \vec{n}_1 \\
&= +A \cdot \vec{n}_2 - C \cdot \vec{n}
\end{aligned}
$$

it follows that:

$$
\begin{aligned}
A &= \vec{n} \cdot \left\{\vec{n}_1 \times \left[\vec{m} \times (\vec{n} \times \hat{\vec{\Omega}})\right]\right\} \\
&= \left[\vec{m} \times (\vec{n} \times \hat{\vec{\Omega}})\right] \cdot (\vec{n} \times \vec{n}_1) \\
&= \left[\vec{m} \times (\vec{n} \times \hat{\vec{\Omega}})\right] \cdot \vec{n}_2 \\
&= (\vec{n} \times \hat{\vec{\Omega}}) \cdot (\vec{n}_2 \times \vec{m}) \\
&= \hat{\vec{\Omega}} \cdot [\vec{n} \times (\vec{m} \times \vec{n}_2] \\
&= \hat{\vec{\Omega}}\ \{\vec{m} \cdot (\vec{n}\,\vec{n}_2) - \vec{n}_2 \cdot (\vec{n}\,\vec{m})\} \\
&= -(\hat{\vec{\Omega}}\,\vec{n}_2) \cdot (\vec{n}\,\vec{m})\ ;
\end{aligned}
$$

$$
\begin{aligned}
C &= \vec{n}_2 \cdot \left\{\vec{n}_1 \times \left[\vec{m} \times (\vec{n} \times \hat{\vec{\Omega}})\right]\right\} \\
&= \left[\vec{m} \times (\vec{n} \times \hat{\vec{\Omega}})\right] \cdot (\vec{n}_2 \times \vec{n}_1) \\
&= \left[\vec{m} \times (\vec{n} \times \hat{\vec{\Omega}}\right] \cdot (-\vec{n}) \\
&= -(\vec{n} \times \hat{\vec{\Omega}}) \cdot (\vec{n} \times \vec{m}) \\
&= -(\vec{n} \times \hat{\vec{\Omega}}) \cdot |\vec{a}| \cdot \vec{n}_1 \\
&= -|\vec{a}| \cdot \hat{\vec{\Omega}} \cdot (\vec{n}_1 \times \vec{n})\ ; \\
&= +|\vec{a}| \cdot (\hat{\vec{\Omega}}\,\vec{n}_2)
\end{aligned}
$$

and thus:

$$
\begin{aligned}
\frac{d}{ds}\,\vec{n}_1 &= \left\{\frac{A}{|\vec{a}|} \cdot \vec{n} + \frac{C}{|\vec{a}|} \cdot \vec{n}_2\right\} \times \vec{n}_1 \\
&= \left\{\frac{A}{|\vec{a}|} \cdot \vec{n} + (\hat{\vec{\Omega}}\,\vec{n}_2)\right\} \times \vec{n}_1\ . \qquad (7.2a)
\end{aligned}
$$

Furthermore we get:

$$
\frac{d}{ds}\,\vec{n}_2 = \frac{d}{ds}\,(\vec{n} \times \vec{n}_1)
$$

$$
\begin{aligned}
&= \frac{d\vec{n}}{ds} \times \vec{n}_1 + \vec{n} \times \frac{d\vec{n}_1}{ds} \\
&= \left[\hat{\vec{\Omega}} \times \vec{n}\right] \times \vec{n}_1 + \vec{n} \times \left[\frac{A}{|\vec{a}|} \cdot \vec{n}_2 - \frac{C}{|\vec{a}|} \cdot \vec{n}\right] \\
&= (\hat{\vec{\Omega}}\,\vec{n}_1) \cdot \vec{n} - (\vec{n}_1\,\vec{n}) \cdot \hat{\vec{\Omega}} + \frac{A}{|\vec{a}|} \cdot \vec{n} \times \vec{n}_2 \\
&= (\hat{\vec{\Omega}}\,\vec{n}_1) \cdot \vec{n} + \frac{A}{|\vec{a}|} \cdot \vec{n} \times \vec{n}_2 \\
&= \left\{\frac{A}{|\vec{a}|} \cdot \vec{n} + (\hat{\vec{\Omega}}\,\vec{n}_1) \cdot \vec{n}_1\right\} \times \vec{n}_2
\end{aligned}
\tag{7.2b}
$$

and

$$
\begin{aligned}
\frac{d}{ds}\,\vec{n} &= \hat{\vec{\Omega}} \times \vec{n} \\
&= \left\{(\hat{\vec{\Omega}}\,\vec{n}) \cdot \vec{n} + (\hat{\vec{\Omega}}\,\vec{n}_1) \cdot \vec{n}_1 + (\hat{\vec{\Omega}}\,\vec{n}_2) \cdot \vec{n}_2\right\} \times \vec{n} \\
&= \left\{(\hat{\vec{\Omega}}\,\vec{n}_1) \cdot \vec{n}_1 + (\hat{\vec{\Omega}}\,\vec{n}_2) \cdot \vec{n}_2\right\} \times \vec{n}\ .
\end{aligned}
\tag{7.2c}
$$

Using the relation:

$$
\vec{x} \times \vec{x} = 0
$$

for an arbitrary vector $\vec{x}$, eqns. (7.2a, b, c) can be gathered in the form:

$$
\left\{
\begin{aligned}
\frac{d}{ds}\,\vec{n} &= \vec{U} \times \vec{n}\ ; \\
\frac{d}{ds}\,\vec{n}_1 &= \vec{U} \times \vec{n}_1\ ; \\
\frac{d}{ds}\,\vec{n}_2 &= \vec{U} \times \vec{n}_2
\end{aligned}
\right.
\tag{7.3}
$$

with

$$
\vec{U} = \frac{A}{|\vec{a}|} \cdot \vec{n} + (\hat{\vec{\Omega}}\,\vec{n}_1) \cdot \vec{n}_1 + (\hat{\vec{\Omega}}\,\vec{n}_2) \cdot \vec{n}_2\ .
\tag{7.4}
$$

We can obtain the same expression for $\vec{U}$ by calculating $\frac{1}{2}\sum_{k=1}^{3} \vec{n}_k \times \frac{d}{ds}\,\vec{n}_k$ with $\vec{n}_3 \equiv \vec{n}$ [4, 1].

7.2 Equations of Motion with Respect to the Dreibein $(\vec{n}_1,\, \vec{n}_2,\, \vec{n})$

In the following we use the formalism of sections (2.3, 2.4). Decomposing the spin vector $\vec{\xi}$ with respect to the dreibein $(\vec{n}_1,\, \vec{n}_2,\, \vec{n})$

$$\vec{\xi} \;=\; \tilde{\xi}_0 \cdot \vec{n} + \tilde{\xi}_1 \cdot \vec{n}_1 + \tilde{\xi}_2 \cdot \vec{n}_2 \ ,$$

we obtain from eqn. (5.11):

$$\begin{aligned} \frac{d}{ds}\,\vec{\xi} &= \hat{\vec{\Omega}} \times \vec{\xi} \\ &= \Big\{ (\hat{\vec{\Omega}}\,\vec{n}) \cdot \vec{n} + (\hat{\vec{\Omega}}\,\vec{n}_1) \cdot \vec{n}_1 + (\hat{\vec{\Omega}}\,\vec{n}_2) \cdot \vec{n}_2 \Big\} \times \vec{\xi} \ ; \end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad \hat{\vec{\Omega}} \times \Big\{ \tilde{\xi}_0 \cdot \vec{n} + \tilde{\xi}_1 \cdot \vec{n}_1 + \tilde{\xi}_2 \cdot \vec{n}_2 \Big\} &= \frac{d\tilde{\xi}_0}{ds} \cdot \vec{n} + \frac{d\tilde{\xi}_1}{ds} \cdot \vec{n}_1 + \frac{d\tilde{\xi}_2}{ds} \cdot \vec{n}_2 \\ &\quad + \tilde{\xi}_0 \cdot \frac{d}{ds}\,\vec{n} + \tilde{\xi}_1 \cdot \frac{d}{ds}\,\vec{n}_1 + \tilde{\xi}_2 \cdot \frac{d}{ds}\,\vec{n}_2 \\ &= \frac{d\tilde{\xi}_0}{ds} \cdot \vec{n} + \frac{d\tilde{\xi}_1}{ds} \cdot \vec{n}_1 + \frac{d\tilde{\xi}_2}{ds} \cdot \vec{n}_2 \\ &\quad + \vec{U} \times \Big\{ \tilde{\xi}_0 \cdot \vec{n} + \tilde{\xi}_1 \cdot \vec{n}_1 + \tilde{\xi}_2 \cdot \vec{n}_2 \Big\} \ ; \end{aligned}$$

$$\Longrightarrow \quad \frac{d\tilde{\xi}_0}{ds} \cdot \vec{n} + \frac{d\tilde{\xi}_1}{ds} \cdot \vec{n}_1 + \frac{d\tilde{\xi}_2}{ds} \cdot \vec{n}_2 \;=\; [\,\hat{\vec{\Omega}} - \vec{U}\,] \times \Big(\tilde{\xi}_0 \cdot \vec{n} + \tilde{\xi}_1 \cdot \vec{n}_1 + \tilde{\xi}_2 \cdot \vec{n}_2 \Big) \ .$$

Since $\vec{n}$ obeys the T–BMT equation (see eqn. (2.33)) , the new precession vector $[\,\hat{\vec{\Omega}} - \vec{U}\,]$ is parallel to $\vec{n}$:

$$[\,\hat{\vec{\Omega}} - \vec{U}\,] \;=\; k(\Phi_k, J_k;\, s) \cdot \vec{n} \ ,$$

and in this frame the spins precess around $\vec{n}$ with constant $J = \vec{n} \cdot \vec{\xi}$.

Our notation reflects to the fact that $\Omega,\ \vec{n}_1,\ \vec{n}_2,\ \vec{n}$ and thus k all depend on

$$(J_k,\, \Phi_k,\, s) \ .$$

Furthermore k, like $\Omega,\ \vec{n},\ \vec{n}_1,\ \vec{n}_2$, is periodic in $(\Phi_k,\, s)$.

Then:

$$\begin{aligned} \frac{d\tilde{\xi}_0}{ds} \cdot \vec{n} + \frac{d\tilde{\xi}_1}{ds} \cdot \vec{n}_1 + \frac{d\tilde{\xi}_2}{ds} \cdot \vec{n}_2 &= [k(\Phi_k, J_k;\, s) \cdot \vec{n}] \times \Big(\tilde{\xi}_0 \cdot \vec{n} + \tilde{\xi}_1 \cdot \vec{n}_1 + \tilde{\xi}_2 \cdot \vec{n}_2 \Big) \\ &= k(\Phi_k, J_k;\, s) \cdot \Big(\tilde{\xi}_1 \cdot \vec{n}_2 - \tilde{\xi}_2 \cdot \vec{n}_1 \Big) \end{aligned}$$

with (see 2.7b)

$$
\begin{aligned}
k(\Phi_k, J_k;\, s) &= (\hat{\vec{\Omega}}\, \vec{n}) - \frac{A}{|\vec{a}|} \\
&= (\hat{\vec{\Omega}}\, \vec{n}) + \vec{n}_1 \frac{d\vec{n}_2}{ds} \qquad (7.5)
\end{aligned}
$$

leads to

$$
\frac{d}{ds}\begin{pmatrix} \tilde{\xi}_0 \\ \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -k(\Phi_k, J_k;\, s)\cdot\tilde{\xi}_2 \\ +k(\Phi_k, J_k;\, s)\cdot\tilde{\xi}_1 \end{pmatrix} \; ; \qquad (7.6)
$$

$$
\frac{d}{ds}\left[\tilde{\xi}_1 + i\cdot\tilde{\xi}_2\right] = i\, k(\Phi_k, J_k;\, s)\cdot\left[\tilde{\xi}_1 + i\cdot\tilde{\xi}_2\right] \; ; \qquad (7.7)
$$

$$
\left[\tilde{\xi}_1(s) + i\cdot\tilde{\xi}_2(s)\right] = \left[\tilde{\xi}_1(s_0) + i\cdot\tilde{\xi}_2(s_0)\right]\cdot\exp\left\{i\cdot\int_{s_0}^{s} d\tilde{s}\cdot k(\Phi_k, J_k;\, \tilde{s})\right\}. \qquad (7.8)
$$

The interpretation of (7.5) is obvious: w.r.t. $\vec{n}_1$ and $\vec{n}_2$ the rate of spin precession around $\vec{n}$ is composed of the projection of $\hat{\vec{\Omega}}$ onto $\vec{n}$ and the rate of rotation of $\vec{n}_1$ and $\vec{n}_2$ around $\vec{n}$.

7.3 Spin Tune on an Arbitrary Orbit

We now define:

$$
\hat{Q}_{spin} = -\lim_{N\to\infty}\frac{1}{2\pi N}\int_{s_0}^{s_0+N\cdot L} d\tilde{s}\cdot k(\Phi_k, J_k;\, \tilde{s}) \; . \qquad (7.9)
$$

The quantity $\hat{Q}_{spin}$ is a measure of the average spin precession frequency in the $(\vec{n}_1, \vec{n}_2, \vec{n})$ frame on an arbitrary orbit. Note, that $\hat{Q}_{spin}$ is independent of the starting point s_0.

On the closed orbit we obtain:

$$
\vec{n} \equiv \vec{n}_0 \; ; \; \tilde{\vec{y}} \equiv 0 \implies k(\Phi_k, J_k;\, s) = -\psi'_{spin}(s)
$$

and thus:

$$
\begin{aligned}
\hat{Q}_{spin} &= -\lim_{N\to\infty}\frac{1}{2\pi N}\int_{s_0}^{s_0+N\cdot L} d\tilde{s}\cdot[-\psi'_{spin}(\tilde{s})] \\
&= \frac{1}{2\pi N}\cdot\{\psi_{spin}(s_0 + N\cdot L) - \psi_{spin}(s_0)\} \\
&= \frac{\psi_{spin}(s_0 + L) - \psi_{spin}(s_0)}{2\pi} \equiv Q_{spin} \; . \qquad (7.10)
\end{aligned}
$$

Thus our $\hat{Q}_{spin}$ reduces to Q_{spin} on the closed orbit.

For the spin tune shift with respect to the closed orbit:

$$\delta Q_{spin} = \hat{Q}_{spin} - Q_{spin} \tag{7.11}$$

we get from (7.9) and (7.10):

$$\delta Q_{spin} = -\lim_{N\to\infty} \frac{1}{2\pi N} \int_{s_0}^{s_0+N\cdot L} d\tilde{s} \cdot \left\{ k(\Phi_k, J_k;\, \tilde{s}) + \psi'_{spin}(\tilde{s}) \right\} \ . \tag{7.12a}$$

Defining in (7.12):

$$\Delta k(\Phi_k, J_k;\, s) = k(\Phi_k, J_k;\, s) + \psi'_{spin}(s) \tag{7.12b}$$

which is periodic in Φ_k and s so that:

$$\begin{aligned} \Delta k(\Phi_I, \Phi_{II}, \Phi_{III}; J_I, J_{II}, J_{III};\, s) &= \Delta k(\Phi_I + 2\pi, \Phi_{II}, \Phi_{III}; J_I, J_{II}, J_{III};\, s) \\ &= \Delta k(\Phi_I, \Phi_{II} + 2\pi, \Phi_{III}; J_I, J_{II}, J_{III};\, s) \\ &= \Delta k(\Phi_I, \Phi_{II}, \Phi_{III} + 2\pi; J_I, J_{II}, J_{III};\, s) \\ &= \Delta k(\Phi_I, \Phi_{II}, \Phi_{III}; J_I, J_{II}, J_{III};\, s + L) \end{aligned}$$

we expand Δk as a Fourier expansion:

$$\Delta k(\Phi_k, J_k;\, s) = \sum_{m_1, m_2, m_3} R_{m_1 m_2 m_3}(J_k; s) \cdot e^{-i\,\{m_1\Phi_I + m_2\Phi_{II} + m_3\Phi_{III}\}}, \tag{7.13a}$$

where

$$\begin{aligned} R_{m_1 m_2 m_3}(J_k; s) &= \left\{\frac{1}{2\pi}\right\}^3 \cdot \int_0^{2\pi} d\Phi_I \int_0^{2\pi} d\Phi_{II} \int_0^{2\pi} d\Phi_{III} \\ &\quad \times \Delta k(\Phi_k, J_k;\, s) \cdot e^{+i\,\{m_1\Phi_I + m_2\Phi_{II} + m_3\Phi_{III}\}} \end{aligned} \tag{7.13b}$$

with

$$\begin{aligned} R_{m_1 m_2 m_3}\,(J_k;\, s + L) &= R_{m_1 m_2 m_3}\,(J_k;\, s) \\ &= \sum_q R_{m_1 m_2 m_3\, q}\,(J_k) \cdot e^{-i\,q\,\cdot\,2\pi\,(s/L)} \ ; \end{aligned} \tag{7.13c}$$

$$\Longrightarrow \quad R_{m_1 m_2 m_3\, q}\,(J_k) = \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot R_{m_1 m_2 m_3}\,(J_k;\, \tilde{s}) \cdot e^{+i\,q\,\cdot\,2\pi\,(\tilde{s}/L)} \ ;$$

$$R_{-m_1, -m_2, -m_3, -q}\,(J_k) = \left[R_{m_1 m_2 m_3\, q}\,(J_k)\right]^* \ .$$

Then we can in general write:

$$\delta Q_{spin} = -\lim_{N\to\infty} \frac{1}{2\pi N} \cdot \sum_{m_1, m_2, m_3, q} \int_{s_0}^{s_0+N\cdot L} d\tilde{s} \cdot R_{m_1 m_2 m_3\, q}(J_k) \times e^{-i\,\{m_1\Phi_I(\tilde{s}) + m_2\Phi_{II}(\tilde{s}) + m_3\Phi_{III}(\tilde{s}) + q\cdot 2\pi\,(\tilde{s}/L)\}} \,. \tag{7.14}$$

Equation (7.14) may be written in the form:

$$\begin{aligned}
\delta Q_{spin} &= -\sum_{m_1, m_2, m_3, q} \lim_{N\to\infty} \frac{1}{2\pi N} \cdot \sum_{\mu=1}^{N} \int_{s_0+(\mu-1)\cdot L}^{s_0+\mu\cdot L} d\tilde{s} \cdot R_{m_1 m_2 m_3\, q}(J_k) \\
&\qquad \times e^{-i\,\{m_1\,\Phi_I(\tilde{s}) + m_2\,\Phi_{II}(\tilde{s}) + m_3\,\Phi_{III}(\tilde{s}) + q\cdot 2\pi\,(\tilde{s}/L)\}} \\
&= -\sum_{m_1, m_2, m_3, q} \lim_{N\to\infty} \frac{1}{2\pi N} \cdot \sum_{\mu=1}^{N} \int_{s_0}^{s_0+L} ds' \cdot R_{m_1 m_2 m_3\, q}(J_k) \\
&\qquad \times e^{-i\, m_1 \cdot \Phi_I(s' + [\mu-1]\cdot L)} \\
&\qquad \times e^{-i\, m_2 \cdot \Phi_{II}(s' + [\mu-1]\cdot L)} \\
&\qquad \times e^{-i\, m_3 \cdot \Phi_{III}(s' + [\mu-1]\cdot L)} \\
&\qquad \times e^{-i\, q \cdot (2\pi/L)\,(s' - [\mu-1]\cdot L)} \\
&= -\sum_{m_1, m_2, m_3, q} \lim_{N\to\infty} \frac{1}{2\pi N} \cdot \sum_{\mu=1}^{N} e^{-2\pi i\cdot(\mu-1)\,\{m_1\,Q_I + m_2\,Q_{II} + m_3\,Q_{III} + q\}} \\
&\qquad \times \int_{s_0}^{s_0+L} ds' \cdot R_{m_1 m_2 m_3\, q}(J_k) \\
&\qquad \times e^{-i\,\{m_1\,\Phi_I(s') + m_2\,\Phi_{II}(s') + m_3\,\Phi_{III}(s') + q\cdot 2\pi\,(s'/L)\}} \\
&= -\sum_{q} \lim_{N\to\infty} \frac{1}{2\pi N} \sum_{\mu=1}^{N} e^{-2\pi i\cdot(\mu-1)\, q} \,.\, \int_{s_0}^{s_0+L} ds' \cdot R_{000q}(J_k) \cdot e^{-i\, q\cdot 2\pi\,(s'/L)} \\
&\quad - \sum_{\substack{m_1, m_2, m_3, q \\ m_1, m_2, m_3 \neq 0,0,0}} \lim_{N\to\infty} \frac{1}{2\pi N} \cdot \sum_{\mu=1}^{N} e^{-2\pi i\cdot(\mu-1)\,\{m_1\,Q_I + m_2\,Q_{II} + m_3\,Q_{III} + q\}} \\
&\qquad \times \int_{s_0}^{s_0+L} ds' \cdot R_{m_1 m_2 m_3\, q}(J_k) \\
&\qquad \times e^{-i\,\{m_1\,\Phi_I(s') + m_2\,\Phi_{II}(s') + m_3\,\Phi_{III}(s') + q\cdot 2\pi\,(s'/L)\}} \\
&= -\lim_{N\to\infty} \frac{1}{2\pi N} \cdot \sum_{\mu=1}^{N} \int_{s_0}^{s_0+L} ds' \cdot R_{0000}(J_k)
\end{aligned}$$

$$
\begin{aligned}
& - \sum_{\substack{m_1, m_2, m_3, q \\ m_1, m_2, m_3 \neq 0,0,0}} \lim_{N\to\infty} \frac{1}{2\pi N} \cdot \frac{1 - e^{-2\pi i \cdot N \cdot \{m_1 Q_I + m_2 Q_{II} + m_3 Q_{III} + q\}}}{1 - e^{-2\pi i \cdot \{m_1 Q_I + m_2 Q_{II} + m_3 Q_{III} + q\}}} \\
& \qquad \times \int_{s_0}^{s_0+L} ds' \cdot R_{m_1 m_2 m_3 q}(J_k) \\
& \qquad\qquad \times e^{-i\{m_1 \Phi_I(s') + m_2 \Phi_{II}(s') + m_3 \Phi_{III}(s') + q \cdot 2\pi (s'/L)\}} \\
= & - \frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} ds' \cdot R_{0000}(J_k) \\
= & - \frac{L}{2\pi} \cdot R_{0000}(J_k) \qquad (7.15)
\end{aligned}
$$

away from the orbital resonance

$$
\begin{aligned}
m_1 Q_I + m_2 Q_{II} + m_3 Q_{III} &= integer \; ; \\
(m_1, \, m_2, \, m_3, \neq \, 0, \, 0, \, 0) \; .
\end{aligned}
$$

On such a resonance the terms proportional to $R_{m_1 m_2 m_3 q}$ and its conjugate survive the averaging and $\hat{Q}_{spin}$ then depends on the orbit phase. In the following we will consistently exclude the case of orbital resonance since on resonance, $\hat{Q}_{spin}$ is not constant and cannot be used as a tune for action–angle variables of spin [4].

In eqn. (7.15) we have used the relation:

$$
\Phi_k(s) \;=\; \Phi_k(s_0) + \frac{2\pi}{L} Q_k \cdot (s - s_0) \qquad (7.16)
$$

(see eqn. (4.31b)). It follows from (7.15) that δQ_{spin} is independent of $\Phi_k \, (k = I, \, II, \, III)$:

$$
\delta Q_{spin} \;=\; \delta Q_{spin}(J_k) \; . \qquad (7.17)
$$

To get the spin tune shift at first order we replace $\vec{n}$ with $\vec{n}_0$ to obtain:

$$
\begin{aligned}
\vec{n} \to \vec{n}_0 \quad \Longrightarrow \quad k^{(1)}(\Phi_k, J_k; \, s) \;&=\; \hat{\vec{\Omega}} \, \vec{n}_0 \\
&=\; (\vec{n}_0 \cdot \vec{\omega}) - \psi'_{spin}(s)
\end{aligned}
$$

(see eqns. (7.5) and (5.10)) and thus:

$$
\Delta k^{(1)}(\Phi_k, J_k; \, s) \;=\; (\vec{n}_0 \cdot \vec{\omega})
$$

or taking into account (4.25) and (5.17):

$$
\Delta k^{(1)}(\Phi_k, J_k; \, s) = \sum_{k=I, II, III} \sqrt{J_k} \cdot e^{-i\Phi_k(\tilde{s})} \cdot (n_{0s}(s), \, n_{0x}(s), \, n_{0z}(s)) \; \underline{F}_{\,(3\times 6)} \, \vec{v}_k(s) \qquad (7.18)
$$

which leads to:

$$\delta Q_{spin}^{(1)} \;=\; 0, \tag{7.19}$$

i.e. the spin tune shift vanishes in 1st order.

7.4 Action–Angle Variables for Spin Motion on an Arbitrary Orbit

Since $k(J_k, \Phi_k; s)$ is not constant we now wish to find a new spin coordinate system in which spins precess uniformly and thereby construct action–angle variables for spin on arbitrary orbits. To this end we introduce the variables $(J,\ \psi)$ by the definition:

$$\left\{\begin{array}{lcl} \tilde{\xi}_0 & \equiv & J\ ; \\ \tilde{\xi}_1 & = & \sqrt{1-J^2}\cdot\cos\psi(s)\ ; \\ \tilde{\xi}_2 & = & \sqrt{1-J^2}\cdot\sin\psi(s)\ . \end{array}\right. \tag{7.20}$$

The spin Hamiltonian describing precession motion around $\vec{n}$ (see eqn. (7.7)) is:

$$\mathcal{H}_{spin} \;=\; k(\Phi_k, J_k;\ s)\cdot J \tag{7.21}$$

and the canonical equations of motion are then:

$$\frac{d}{ds}\,\psi \;=\; +\frac{\partial\mathcal{H}_{spin}}{\partial J} \;=\; k(\Phi_k, J_k;\ s)\ ; \tag{7.22a}$$

$$\frac{d}{ds}\,J \;=\; -\frac{\partial\mathcal{H}_{spin}}{\partial\psi} \;=\; 0\ . \tag{7.22b}$$

The solution of (7.22a, b) is given by:

$$\psi(s) \;=\; \int_{s_0}^{s} d\tilde{s}\cdot k(\Phi_k, J_k;\ \tilde{s})\ ; \tag{7.23a}$$

$$J \;=\; const\ . \tag{7.23b}$$

We now employ the canonical transformation

$$\psi;\ \ J \;\longrightarrow\; \hat{\psi} = -2\pi\hat{Q}_{spin}\cdot\frac{s}{L};\ \ \hat{J} = J\ . \tag{7.24}$$

based on the generating function

$$F_2(\psi, \hat{J};\ s) \;=\; -\hat{J}\cdot\left\{\frac{2\pi}{L}\,\hat{Q}_{spin}\cdot s + \int_{s_0}^{s} d\tilde{s}\cdot k(\Phi_k, J_k;\ \tilde{s})\right\} + \psi\cdot\hat{J} \tag{7.25}$$

which reads as:

$$\left\{ \begin{array}{lllll} \hat{\psi} & = & \dfrac{\partial F_2}{\partial \tilde{J}} & = & -\dfrac{2\pi}{L}\,\hat{Q}_{spin}\cdot s - \displaystyle\int_{s_0}^{s} d\tilde{s}\cdot k(\Phi_k, J_k;\; \tilde{s}) + \psi \;\;=\;\; -\dfrac{2\pi}{L}\,\hat{Q}_{spin}(J_k)\cdot s \;; \\ J & = & \dfrac{\partial F_2}{\partial \psi} & = & \hat{J}\;. \end{array} \right. \tag{7.26}$$

The new Hamiltonian

$$\hat{\mathcal{H}}_{spin} \;=\; \mathcal{H}_{spin} + \frac{\partial F_2}{\partial s} \tag{7.27}$$

is given by:

$$\begin{aligned} \hat{\mathcal{H}}_{spin} &= \mathcal{H}_{spin} + \frac{\partial F_2}{\partial s} \\ &= k(\Phi_k, J_k;\; s)\cdot J - \hat{J}\cdot\left\{\frac{2\pi}{L}\,\hat{Q}_{spin}(J_k) + k(\Phi_k, J_k;\; s)\right\} \\ &= -\frac{2\pi}{L}\,\hat{Q}_{spin}(J_k)\cdot\hat{J}\;. \end{aligned} \tag{7.28}$$

The corresponding canonical equations read as:

$$\frac{d}{ds}\,\hat{\psi} \;=\; +\frac{\partial\hat{\mathcal{H}}_{spin}}{\partial\tilde{J}} \;=\; -\frac{2\pi}{L}\,\hat{Q}_{spin}(J_k) \;=\; const\;; \tag{7.29a}$$

$$\frac{d}{ds}\,\hat{J} \;=\; -\frac{\partial\hat{\mathcal{H}}_{spin}}{\partial\tilde{\psi}} \;=\; 0\;. \tag{7.29b}$$

Thus we now have a spin precession frequency independent of Φ_k and s but dependent on J_k and identical to the average spin tune in the $(\vec{n}_1,\, \vec{n}_2,\, \vec{n})$ dreibein. The quantities $\hat{J}$, $\hat{\psi}$ are our action–angle variables for spin on an arbitrary 6–dimensional orbit.

Associated with these new variables $\hat{J}$, $\hat{\psi}$ is the new dreibein $(\hat{\vec{n}}_1,\, \hat{\vec{n}}_2,\, \vec{n})$ with $(\hat{\vec{n}}_1,\, \hat{\vec{n}}_2)$ given by:

$$\begin{aligned} \hat{\vec{n}}_1 &= +\vec{n}_1\cdot\cos\chi(s) + \vec{n}_2\cdot\sin\chi(s)\;; \\ \hat{\vec{n}}_2 &= -\vec{n}_1\cdot\sin\chi(s) + \vec{n}_2\cdot\cos\chi(s) \end{aligned}$$

or

$$\hat{\vec{n}}_1 + i\cdot\hat{\vec{n}}_2 \;=\; [\vec{n}_1 + i\cdot\vec{n}_2]\cdot\exp\left[-i\cdot\chi(s)\right] \tag{7.30a}$$

where by (7.26)

$$\begin{aligned} \chi(\Phi_k(s), J_k;\; s) &= \chi(\Phi_k(s_0), J_k;\; s_0) + \int_{s_0}^{s} d\tilde{s}\cdot k(\Phi_k(\tilde{s}), J_k;\; \tilde{s}) + \frac{2\pi}{L}\,\hat{Q}_{spin}\cdot(s - s_0) \\ &= \chi(\Phi_k(s_0), J_k;\; s_0) + \int_{s_0}^{s} d\tilde{s}\cdot\left\{k(\Phi_k(\tilde{s}), J_k;\; \tilde{s}) + \frac{2\pi}{L}\,\hat{Q}_{spin}\right\}\;. \end{aligned} \tag{7.30b}$$

Defining the spin components $\hat{\xi}_1,\ \hat{\xi}_2$ with respect to $(\hat{\vec{n}}_1,\ \hat{\vec{n}}_2)$ by the relation:

$$\vec{\xi} \;=\; \tilde{\xi}_0 \cdot \vec{n} + \hat{\xi}_1 \cdot \hat{\vec{n}}_1 + \hat{\xi}_2 \cdot \hat{\vec{n}}_2$$

we can write:

$$\frac{d}{ds}\ \left[\hat{\xi}_1 + i \cdot \hat{\xi}_2\right] \;=\; i\,\Omega(J_k) \cdot \left[\hat{\xi}_1 + i \cdot \hat{\xi}_2\right]$$

or

$$\left[\hat{\xi}_1(s) + i \cdot \hat{\xi}_2(s)\right] \;=\; \left[\hat{\xi}_1(s_0) + i \cdot \hat{\xi}_2(s_0)\right] \cdot \exp\left\{i\,\Omega(J_k) \cdot (s - s_0)\right\}$$

with

$$\Omega(J_k) \;=\; -\frac{2\pi}{L} \cdot \hat{Q}_{spin}$$

which expresses again the fact that the spins precess uniformly in the new dreibein.

Choosing for simplicity

$$\chi(\Phi_k(s_0), J_k;\ s_0) \;=\; 0\ ,$$

in (7.30b) we may write [2]:

$$\chi(\Phi_k(s), J_k;\ s) \;=\; \int_{s_0}^{s} d\tilde{s} \cdot \left\{k(\Phi_k(\tilde{s}), J_k;\ \tilde{s}) + \frac{2\pi}{L}\,\hat{Q}_{spin}\right\}\ . \tag{7.31}$$

Further insight into the properties of the new dreibein can be obtained by using (7.12b) and expanding the integrand in (7.31):

$$\begin{aligned}\Delta\hat{k}(\Phi_k, J_k;\ s) \;&=\; k(\Phi_k, J_k;\ s) + \frac{2\pi}{L}\,\hat{Q}_{spin}(J_k) \\ &=\; \Delta k(\Phi_k, J_k;\ s) + \frac{2\pi}{L}\,\delta Q_{spin}\,(J_k) + \left[\frac{2\pi}{L}\,Q_{spin} - \psi'_{spin}(s)\right]\end{aligned} \tag{7.32}$$

into a Fourier series:

$$\begin{aligned}\Delta\hat{k}(\Phi_I, \Phi_{II}, \Phi_{III}; J_I, J_{II}, J_{III};\ s) \;&=\; \Delta\hat{k}(\Phi_I + 2\pi, \Phi_{II}, \Phi_{III}; J_I, J_{II}, J_{III};\ s) \\ &=\; \Delta\hat{k}(\Phi_I, \Phi_{II} + 2\pi, \Phi_{III}; J_I, J_{II}, J_{III};\ s) \\ &=\; \Delta\hat{k}(\Phi_I, \Phi_{II}, \Phi_{III} + 2\pi; J_I, J_{II}, J_{III};\ s) \\ &=\; \Delta\hat{k}(\Phi_I, \Phi_{II}, \Phi_{III}; J_I, J_{II}, J_{III};\ s + L)\ ;\end{aligned}$$

[2]Note that the proof of the periodicity conditions (7.35) and (7.36) also works for an arbitrary choice of $\chi(\Phi_k(s_0), J_k;\ s_0)$.

$$\Longrightarrow \quad \Delta\hat{k}(\Phi_k, J_k;\, s) \;=\; \sum_{m_1,\, m_2,\, m_3} \hat{R}_{m_1 m_2 m_3}(s)\cdot e^{-i\,\{m_1\Phi_I + m_2\Phi_{II} + m_3\Phi_{III}\}}; \quad (7.33a)$$

$$\hat{R}_{m_1 m_2 m_3}(s) \;=\; \left(\frac{1}{2\pi}\right)^3 \cdot \int_0^{2\pi} d\Phi_I \int_0^{2\pi} d\Phi_{II} \int_0^{2\pi} d\Phi_{III}\cdot \Delta\hat{k}(\Phi_k, J_k;\, s) \times e^{+i\,\{m_1\Phi_I(s) + m_2\Phi_{II}(s) + m_3\Phi_{III}(s)\}} \quad (7.33b)$$

with

$$\begin{aligned}
\hat{R}_{m_1 m_2 m_3}(s+L) &= \hat{R}_{m_1 m_2 m_3}(s) \\
&= \sum_q \hat{R}_{m_1 m_2 m_3\, q}(J_k)\cdot e^{-i\, q\,\cdot\, 2\pi\,(s/L)}\ ; \\
\Longrightarrow \quad \hat{R}_{m_1 m_2 m_3\, q}(J_k) &= \frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s}\cdot \hat{R}_{m_1 m_2 m_3}(J_k;\, \tilde{s})\cdot e^{+i\, q\,\cdot\, 2\pi\,(\tilde{s}/L)}
\end{aligned}$$

which leads to:

$$\Delta\hat{k}(\Phi_k, J_k;\, s) \;=\; \sum_{\substack{m_1,\, m_2,\, m_3,\, q \\ (m_1, m_2, m_3, q)\,\neq\,(0,0,0,0)}} \hat{R}_{m_1 m_2 m_3\, q}(J_k) \times e^{-i\,\{m_1\,\Phi_I(s) + m_2\,\Phi_{II}(s) + m_3\,\Phi_{III}(s) + q\,\cdot\, 2\pi\,(\tilde{s}/L)\}}\ .$$

Here we have used the fact that the coefficient

$$\hat{R}_{0000} \;=\; \frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s}\cdot \left(\frac{1}{2\pi}\right)^3 \cdot \int_0^{2\pi} d\Phi_I \int_0^{2\pi} d\Phi_{II} \int_0^{2\pi} d\Phi_{III}\cdot \Delta\hat{k}(\Phi_k, J_k;\, \tilde{s})$$

vanishes:

$$\hat{R}_{0000} \;=\; 0$$

as may be derived from eqn. (7.32), since

$$\begin{aligned}
&\frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s}\cdot \left(\frac{1}{2\pi}\right)^3 \cdot \int_0^{2\pi} d\Phi_I \int_0^{2\pi} d\Phi_{II} \int_0^{2\pi} d\Phi_{III}\cdot \Delta k(\Phi_k, J_k;\, \tilde{s}) \\
&\quad = R_{0000} \;=\; -\frac{2\pi}{L}\,\delta Q_{spin}(J_k)
\end{aligned}$$

and

$$\frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s}\cdot \left[\frac{2\pi}{L}\, Q_{spin} - \psi'_{spin}(\tilde{s})\right] \;=\; 0\ .$$

Then, using eqn. (7.16), eqn. (7.31) can be written in the form:

$$
\begin{aligned}
& \chi(\Phi_k, J_k;\, s) \\
= \; & \int_{s_0}^{s} d\tilde{s} \cdot \sum_{\substack{m_1,\, m_2,\, m_3,\, q \\ m_1, m_2, m_3,\, q \neq 0,0,0,0}} \hat{R}_{m_1 m_2 m_3\, q}\,(J_k) \\
& \quad \times e^{-i\,\{m_1\,\Phi_I(\tilde{s}) + m_2\,\Phi_{II}(\tilde{s}) + m_3\,\Phi_{III}(\tilde{s}) + q \cdot 2\pi\,(\tilde{s}/L)\}} \\
= \; & \sum_{\substack{m_1,\, m_2,\, m_3,\, q \\ m_1, m_2, m_3,\, q \neq 0,0,0,0}} \hat{R}_{m_1 m_2 m_3\, q}\,(J_k) \cdot e^{-i\,\{m_1\,\Phi_I(s_0) + m_2\,\Phi_{II}(s_0) + m_3\,\Phi_{III}(s_0)\}} \\
& \qquad \times e^{2\pi i \cdot (s_0/L)\,\{m_1\,Q_I + m_2\,Q_{II} + m_3\,Q_{III}\}} \\
& \times \int_{s_0}^{s} ds' \cdot e^{-2\pi i \cdot (s'/L)\,\{m_1\,Q_I + m_2\,Q_{II} + m_3\,Q_{III} + q\;\}} \\
= \; & \sum_{\substack{m_1,\, m_2,\, m_3,\, q \\ m_1, m_2, m_3,\, q \neq 0,0,0,0}} \hat{R}_{m_1 m_2 m_3\, q}\,(J_k) \cdot e^{-i\,\{m_1\,\Phi_I(s_0) + m_2\,\Phi_{II}(s_0) + m_3\,\Phi_{III}(s_0)\}} \\
& \qquad \times e^{2\pi i \cdot (s_0/L)\,\{m_1\,Q_I + m_2\,Q_{II} + m_3\,Q_{III}\}} \\
& \qquad \times \left[\frac{e^{-2\pi i \cdot (s'/L)\,\{m_1\,Q_I + m_2\,Q_{II} + m_3\,Q_{III} + q\;\}}}{-2\pi i \cdot (1/L) \cdot \{m_1\,Q_I + m_2\,Q_{II} + m_3\,Q_{III} + q\}} \right]_{s'=s_0}^{s'=s} \\
= \; & \sum_{\substack{m_1,\, m_2,\, m_3,\, q \\ m_1, m_2, m_3,\, q \neq 0,0,0,0}} \hat{R}_{m_1 m_2 m_3\, q}\,(J_k) \cdot e^{-i\,\{m_1\,\Phi_I(s_0) + m_2\,\Phi_{II}(s_0) + m_3\,\Phi_{III}(s_0)\}} \\
& \qquad \times \frac{e^{2\pi i \cdot (s_0/L)\,\{m_1\,Q_I + m_2\,Q_{II} + m_3\,Q_{III}\}}}{-2\pi i \cdot (1/L) \cdot \{m_1\,Q_I + m_2\,Q_{II} + m_3\,Q_{III} + q\}} \\
& \qquad \times \Big\{ e^{-2\pi i \cdot (s/L)\,\{m_1\,Q_I + m_2\,Q_{II} + m_3\,Q_{III} + q\;\}} \\
& \qquad\qquad - e^{-2\pi i \cdot (s_0/L)\,\{m_1\,Q_I + m_2\,Q_{II} + m_3\,Q_{III} + q\;\}} \Big\} \\
= \; & \sum_{\substack{m_1,\, m_2,\, m_3,\, q \\ m_1, m_2, m_3,\, q \neq 0,0,0,0}} \hat{R}_{m_1 m_2 m_3\, q}\,(J_k) \cdot e^{-i\,\{m_1\,\Phi_I(s) + m_2\,\Phi_{II}(s) + m_3\,\Phi_{III}(s) + 2\pi q \cdot s/L\}} \\
& \qquad \times \frac{1}{-2\pi i \cdot (1/L) \cdot \{m_1\,Q_I + m_2\,Q_{II} + m_3\,Q_{III} + q\}}
\end{aligned}
$$

$$
\begin{aligned}
- & \sum_{\substack{m_1,\, m_2,\, m_3,\, q \\ m_1, m_2, m_3,\, q \neq 0,0,0,0}} \hat{R}_{m_1 m_2 m_3\, q}\,(J_k) \cdot e^{-i\,\{m_1\,\Phi_I(s_0) + m_2\,\Phi_{II}(s_0) + m_3\,\Phi_{III}(s_0) + 2\pi q \cdot s_0/L\}} \\
& \qquad \times \frac{1}{-2\pi i \cdot (1/L) \cdot \{m_1\,Q_I + m_2\,Q_{II} + m_3\,Q_{III} + q\}} \\
= & \sum_{\substack{m_1,\, m_2,\, m_3,\, q \\ m_1, m_2, m_3,\, q \neq 0,0,0,0}} \hat{R}_{m_1 m_2 m_3\, q}\,(J_k) \cdot e^{-i\,\{m_1\,\Phi_I(s) + m_2\,\Phi_{II}(s) + m_3\,\Phi_{III}(s) + 2\pi q \cdot s/L\}} \\
& \qquad \times \frac{1}{-2\pi i \cdot (1/L) \cdot \{m_1\,Q_I + m_2\,Q_{II} + m_3\,Q_{III} + q\}} \\
& \qquad - const. \qquad (7.34)
\end{aligned}
$$

From (7.34) it is clear that $\chi(\Phi_k, J_k;\, s)$ fulfils the periodicity conditions

$$
\begin{aligned}
\chi(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III}, s) &= \chi(J_I, J_{II}, J_{III}, \Phi_I + 2\pi, \Phi_{II}, \Phi_{III}, s) \\
&= \chi(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II} + 2\pi, \Phi_{III}, s) \\
&= \chi(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III} + 2\pi, s) \\
&= \chi(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III}, s + L)\,. \qquad (7.35)
\end{aligned}
$$

Then by (7.30a) we see that $\hat{\vec{n}}_1$ and $\hat{\vec{n}}_2$ fulfil the same periodicity relations as $\chi(\Phi_k(s), J_k;\, s)$ and thus as $\vec{n}_1$ and $\vec{n}_2$ (see eqn. (6.1)) :

$$
\begin{aligned}
\hat{\vec{n}}_\nu(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III}, s) &= \hat{\vec{n}}_\nu(J_I, J_{II}, J_{III}, \Phi_I + 2\pi, \Phi_{II}, \Phi_{III}, s) \\
&= \hat{\vec{n}}_\nu(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II} + 2\pi, \Phi_{III}, s) \\
&= \hat{\vec{n}}_\nu(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III} + 2\pi, s) \\
&= \hat{\vec{n}}_\nu(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III}, s + L)\ ; \qquad (7.36)
\end{aligned}
$$

with $\nu\ =\ 1,\, 2$ for an arbitrary orbit.

Thus we have found that the dreibein $(\hat{\vec{n}}_1,\, \hat{\vec{n}}_2,\, \vec{n})$ within which spin motion can be described in terms of action–angle variables, is fully periodic in $(\Phi_k,\, s)$. A numerically equivalent construction of $(\hat{\vec{n}}_1,\, \hat{\vec{n}}_2)$ was used first in the program SPRINT [22, 23, 24, 25]. The need for a periodic dreibein on arbitrary orbits is discussed in Ref. [4]. Recalling the behaviour of spins on the closed orbit, we note (see section 4.1 and Appendix B) that in the non–periodic $(\vec{n}_0,\, \vec{m}_0,\, \vec{l}_0)$ dreibein the spins are stationary, but in the $(\vec{n}_0,\, \vec{m},\, \vec{l})$ dreibein the phase function ψ can be chosen so as to vary linearly in s so that the spins precess at a fixed frequency Q_{spin}. The quantity Q_{spin} is arbitrary up to an integer corresponding to various choices of $(\vec{m},\, \vec{l})$.

Moreover the vectors $(\vec{m},\, \vec{l})$ are periodic in s. The construction that we have just carried out is the generalisation of this to arbitrary integrable orbits. On the closed orbit $(\hat{\vec{n}}_1,\, \hat{\vec{n}}_2)$ reduce to $(\vec{l},\, -\vec{m})$.

From (7.34) it is clear that our construction of the periodic $(\hat{\vec{n}}_1,\, \hat{\vec{n}}_2)$ becomes invalid on an orbit resonance

$$
\begin{aligned}
m_1\, Q_I + m_2\, Q_{II} + m_3\, Q_{III} + q \; &= \; 0\ ; \\
(m_1,\, m_2,\, m_3,\, q \neq 0,\, 0,\, 0,\, 0)
\end{aligned}
$$

for which $\hat{R}_{m_1 m_2 m_3\, q}$ does not vanish. But this case had to be excluded in order to make $\hat{Q}_{spin}$ independent of orbit phase. Away from spin orbit resonances the whole dreibein $(\hat{\vec{n}}_1,\, \hat{\vec{n}}_2,\, \vec{n})$ is a single valued function of the phase space point $(J_k,\, \Phi_k,\; s)$ and the vectors $(\vec{n}_1, \vec{n}_2)$ and $(\hat{\vec{n}}_1, \hat{\vec{n}}_2)$ oscillate with respect to each other. In the neighbourhood of an orbital resonance the oscillation amplitude becomes very large.

Remarks:

An alternative way, used in the earlier versions of this paper, to prove the periodicity of $(\hat{\vec{n}}_1,\, \hat{\vec{n}}_2)$ is to write (7.30b) in the special form

$$
\begin{aligned}
\chi(\Phi_k(\tilde{s}), J_k;\, s) \;&=\; \lim_{0<\gamma\to 0}\; \lim_{s_0\to -\infty} \int_{s_0}^{s} d\tilde{s}\cdot \Big\{ k(\Phi_k(\tilde{s}), J_k;\, \tilde{s}) + \frac{2\pi}{L}\, \hat{Q}_{spin}(J_k) \Big\} \cdot e^{+\gamma\,\cdot\,\tilde{s}} \\
&=\; \lim_{0<\gamma\to 0} \int_{-\infty}^{s} d\tilde{s}\cdot \Big\{ k(\Phi_k(\tilde{s}), J_k;\, \tilde{s}) + \frac{2\pi}{L}\, \hat{Q}_{spin}(J_k) \Big\} \cdot e^{+\gamma\,\cdot\,\tilde{s}}\,. \qquad (7.37)
\end{aligned}
$$

by making a special choice of $\chi(\Phi_k, J_k;\, s_0)$ and using the method of forced solution.
Then we have:

$$
\begin{aligned}
\chi(\Phi_k(s), J_k;\, s + L) \;&=\; \lim_{0<\gamma\to 0} \int_{-\infty}^{s+L} d\tilde{s}\cdot \Big\{ k(\Phi_k(\tilde{s} - L), J_k;\, \tilde{s}) + \frac{2\pi}{L}\, \hat{Q}_{spin}(J_k) \Big\} \cdot e^{+\gamma\,\cdot\,\tilde{s}}, \\
&=\; \lim_{0<\gamma\to 0} \int_{-\infty}^{s} ds'\cdot \Big\{ k(\Phi_k(s'), J_k;\, s' + L) + \frac{2\pi}{L}\, \hat{Q}_{spin}(J_k) \Big\} \cdot e^{+\gamma\,\cdot\,(s' + L)}, \\
&=\; \lim_{0<\gamma\to 0} \int_{-\infty}^{s} ds'\cdot \Big\{ k(\Phi_k(s'), J_k;\, s') + \frac{2\pi}{L}\, \hat{Q}_{spin}(J_k) \Big\} \cdot e^{+\gamma\,\cdot\,(s' + L)} \\
&=\; \lim_{0<\gamma\to 0} \int_{-\infty}^{s} ds'\cdot \Big\{ k(\Phi_k(s'), J_k;\, s') + \frac{2\pi}{L}\, \hat{Q}_{spin}(J_k) \Big\} \cdot e^{+\gamma\,\cdot\,s'} \\
&=\; \chi\,(\Phi_k(s), J_k;\, s)
\end{aligned}
$$

and

$$
\chi(\Phi_k + 2\pi, J_k;\, s) \;=\; \lim_{0<\gamma\to 0} \int_{-\infty}^{s} d\tilde{s}\cdot \Big\{ k(\Phi_k + 2\pi, J_k;\, \tilde{s}) + \frac{2\pi}{L}\, \hat{Q}_{spin}(J_k) \Big\} \cdot e^{+\gamma\,\cdot\,\tilde{s}}
$$

$$= \lim_{0<\gamma\to 0} \int_{-\infty}^{s} d\tilde{s} \cdot \left\{ k(\Phi_k, J_k;\, \tilde{s}) + \frac{2\pi}{L}\, \hat{Q}_{spin}(J_k) \right\} \cdot e^{+\gamma\,\cdot\,\tilde{s}}$$

$$= \chi(\Phi_k, J_k;\, s)\ ,$$

since

$$k(\Phi_k(s), J_k;\, s + L) \;=\; k(\Phi_k(s), J_k;\, s)$$

and

$$k(\Phi_k + 2\pi, J_k;\, s) \;=\; k(\Phi_k, J_k;\, s)\ .$$

Thus the function $\chi(\Phi_k, J_k;\, s)$ indeed obeys the periodicity relation (7.35). Note, that this analysis is not valid on resonances for which $\hat{R}_{m_1 m_2 m_3\, q}$ does not vanish.

The proof of the periodicity condition (7.35) remains valid, if we use the ansatz:

$$\chi(\Phi_k(s), J_k;\, s) \;=\; C + \lim_{0<\gamma\to 0} \int_{-\infty}^{s} d\tilde{s} \cdot \left\{ k(\Phi_k(\tilde{s}), J_k;\, \tilde{s}) + \frac{2\pi}{L}\, \hat{Q}_{spin}(J_k) \right\} \cdot e^{+\gamma\,\cdot\,\tilde{s}}\ ,$$

by introducing an arbitrary additional integration constant C in (7.31). Using

$$C \;=\; -\lim_{0<\gamma\to 0} \int_{-\infty}^{s_0} d\tilde{s} \cdot \left\{ k(\Phi_k(\tilde{s}), J_k;\, \tilde{s}) + \frac{2\pi}{L}\, \hat{Q}_{spin}(J_k) \right\} \cdot e^{+\gamma\,\cdot\,\tilde{s}}$$

we obtain (7.31) again:

$$\begin{aligned} \chi(\Phi_k(s), J_k;\, s) \;&=\; \lim_{0<\gamma\to 0} \int_{s_0}^{s} d\tilde{s} \cdot \left\{ k(\Phi_k(\tilde{s}), J_k;\, \tilde{s}) + \frac{2\pi}{L}\, \hat{Q}_{spin}(J_k) \right\} \cdot e^{+\gamma\,\cdot\,\tilde{s}} \\ &=\; \int_{s_0}^{s} d\tilde{s} \cdot \left\{ k(\Phi_k(\tilde{s}), J_k;\, \tilde{s}) + \frac{2\pi}{L}\, \hat{Q}_{spin}(J_k) \right\}\ . \end{aligned} \tag{7.38}$$

2) Having established the periodic dreibein $(\hat{\vec{n}}_1,\, \hat{\vec{n}}_2,\, \vec{n})$ in which spins precess uniformly, we can construct another dreibein $(\hat{\hat{\vec{n}}}_1,\, \hat{\hat{\vec{n}}}_2,\, \vec{n})$ as

$$\begin{aligned} \hat{\hat{\vec{n}}}_1 + i\cdot\hat{\hat{\vec{n}}}_2 \;&=\; \left[\hat{\vec{n}}_1 + i\cdot\hat{\vec{n}}_2\right] \\ &\quad\times \exp\left\{-i\left[m_1\Phi_I(s) + m_2\Phi_{II}(s) + m_3\Phi_{III}(s) + q\cdot 2\pi\,(s/L)\right]\right\}\ . \end{aligned} \tag{7.39}$$

Since the spins precess at the rate

$$\Omega(J_k) \;=\; -\frac{2\pi}{L}\,\hat{Q}_{spin}$$

with respect to $(\hat{\vec{n}}_1,\, \hat{\vec{n}}_2,\, \vec{n})$ then they precess at the rate

$$\begin{aligned} &\Omega(J_k) - \frac{2\pi}{L}\left[q + m_1\cdot Q_I + m_2\cdot Q_{II} + m_3\cdot Q_{III}\right] \\ =\;& -\frac{2\pi}{L}\left[\hat{Q}_{spin} + q + m_1\cdot Q_I + m_2\cdot Q_{II} + m_3\cdot Q_{III}\right] \end{aligned}$$

with respect to $(\hat{\hat{\vec{n}}}_1, \hat{\hat{\vec{n}}}_2, \vec{n})$.

By construction $(\hat{\hat{\vec{n}}}_1, \hat{\hat{\vec{n}}}_2)$ is again periodic and the precession rate is uniform. Thus the $(\hat{\hat{\vec{n}}}_1, \hat{\hat{\vec{n}}}_2, \vec{n})$ dreibein has the same status as $(\hat{\vec{n}}_1, \hat{\vec{n}}_2, \vec{n})$ and is a perfectly valid dreibein. On the resonance,

$$\hat{Q}_{spin} + q + m_1 \cdot Q_I + m_2 \cdot Q_{II} + m_3 \cdot Q_{III} \quad = \quad 0 \ , \tag{7.40}$$

the spins are stationary in the "resonance dreibein" $(\hat{\hat{\vec{n}}}_1, \hat{\hat{\vec{n}}}_2, \vec{n})$ constructed using the

$$(m_1, m_2, m_3, q).$$

Thus at resonance these $(\hat{\hat{\vec{n}}}_1, \hat{\hat{\vec{n}}}_2, \vec{n})$ all obey the T–BMT equation and the distinction between $\vec{n}$ and $(\hat{\hat{\vec{n}}}_1, \hat{\hat{\vec{n}}}_2)$ is lost. Then at a spin–orbit resonance (7.38) a dreibein can always be found in which spins are stationary and for which the choice of $(\hat{\hat{\vec{n}}}_1, \hat{\hat{\vec{n}}}_2, \vec{n})$ is not unique. This is analogous to the situation with the closed orbit discussed in section 4.1 (see also Ref. [4]).

3) In arriving at action–angle variables for spin, either on the closed orbit or an arbitrary trajectory, we can see an analogy with the case of the construction of action–angle variables for orbital motion. In the latter case the orbital Hamiltonian is s–dependent (section 3.2.1) but the s–dependence can be absorbed by applying an s–dependent canonical transformation involving orbital eigenvectors (section 4.2.2) and the motion from then on is described with respect to a basis of orbital eigenvectors with amplitudes that are written in terms of action–angle variables. Actually, we use the "Floquetized" orbital eigenvectors periodic in s and a uniformly increasing phase function. In the case of spin on the closed orbit we come to action–angle variables by an s–dependent canonical transformation involving a spin phase function ψ increasing linearly in s (section 4.1) and the corresponding s dependent orthonormal pair $(\vec{m}, \vec{l})$ which is periodic in s. On an arbitrary orbit, we arrive at action–angle variables via a canonical transformation to a dreibein which is periodic in Φ_k as well as in s.

For an analysis of spin–orbit motion and the introduction of spin tune in terms of Floquet theory see [26].

4) To construct $(\hat{\vec{n}}_1, \hat{\vec{n}}_2)$ we began by using the periodic vector product $\vec{n} \times \vec{m}$. We could instead have begun with the vector product $\vec{n} \times \vec{e}_x$ or with $\vec{n} \times \vec{v}$ ($\vec{v}$=unit particle direction vector) since $\vec{e}_x$ is periodic in s and $\vec{v}$ is periodic in s and Φ_k. Then we could have calculated $k(J_k, \Phi_k, s)$ for these cases and proceeded with the canonical transformation. The resulting various $(\hat{\hat{\vec{n}}}_1, \hat{\hat{\vec{n}}}_2)$ are presumably related by a transformation of the form (7.39).

5) Note that $\hat{Q}_{spin}$ depends on the orbital actions J_k and that an ensemble spans a range of J_k. Thus more than one orbit in an ensemble can in principle be on resonance and some orbits will have no associated unique dreibein i.e. no unique $\vec{n}$–axis.

8 Calculation of the Depolarisation Time

Having determined the $\vec{n}$–axis and investigated the spin motion in the special coordinate system

$$(\hat{\vec{n}}_1,\, \hat{\vec{n}}_2,\, \vec{n})$$

associated with the $\vec{n}$–axis, we are now in a position to study classical spin diffusion and to calculate the depolarisation time.

The spin diffusion is caused by radiation processes which can be characterised by a stochastic differential equation for the orbital coefficients A_k [12, 7]:

$$A'_k(s) = +i \cdot v^*_{k5}(s) \cdot \sqrt{\omega(s)} \cdot \mathcal{P}(s) \; ; \tag{8.1a}$$
$$A'_{-k}(s) = [A'_k(s)]^* \; ; \tag{8.1b}$$
$$(k = I,\, II,\, III)$$

with

$$< \mathcal{P}(s), \mathcal{P}(s') > \; = \; \delta(s - s') \; . \tag{8.2}$$

and

$$\omega(s) = (|K_x|^3 + |K_z|^3) \cdot C_2 \tag{8.3}$$

where the coefficient C_2 is given by:

$$C_2 = \frac{55 \cdot \sqrt{3}}{48} \cdot C_1 \cdot \Lambda \cdot \gamma_0^2 \quad \text{with} \quad \Lambda = \frac{\hbar}{m_0 c} \quad \text{and} \quad C_1 = \frac{2}{3} e^2 \cdot \frac{\gamma_0^4}{E_0} \; . \tag{8.4}$$

Equations (8.1a, b) may be written in a combined form as:

$$A'_\mu(s) = +V_\mu(s) \cdot \sqrt{\omega(s)} \cdot \mathcal{P}(s) \tag{8.5}$$

with

$$V_k(s) = +i \cdot v^*_{k5}(s) \; ; \tag{8.6a}$$
$$V_{-k}(s) = [V_k(s)]^* \; . \tag{8.6b}$$

In the following we shall use the notation:

$$\begin{aligned} \vec{n} &= \vec{n}\,(\hat{\alpha}, \hat{\beta}) \\ &= \left\{1 - \frac{1}{2}\left(\hat{\alpha}^2 + \hat{\beta}^2\right)\right\} \cdot \vec{n}_0 + \hat{\alpha}\sqrt{1 - \frac{1}{4}\left(\hat{\alpha}^2 + \hat{\beta}^2\right)} \cdot \vec{m} + \hat{\beta}\sqrt{1 - \frac{1}{4}\left(\hat{\alpha}^2 + \hat{\beta}^2\right)} \cdot \vec{l} \end{aligned}$$

for the $\vec{n}$–axis, denoting the α– and β–component of $\vec{n}$ by $\hat{\alpha}$ and $\hat{\beta}$.

8.1 Depolarisation Degree due to an Orbital Jump

We consider the case where at each point in phase space the spins are initially all parallel to the respective $\vec{n}$–axes and that orbital equilibrium characterised by the distribution function $W_{orbit}^{(stat)}(J_l, \Phi_l)$:

$$W_{orbit}^{(stat)}(J_l, \Phi_l) = \frac{1}{(2\pi)^3} \cdot \frac{1}{\hat{J}_I \cdot \hat{J}_{II} \cdot \hat{J}_{III}} \cdot e^{-\left[J_I/\hat{J}_I + J_{II}/\hat{J}_{II} + J_{III}/\hat{J}_{III}\right]} \tag{8.7}$$

with

$$\begin{aligned} \hat{J}_k &= \frac{M_k}{2a_k} \\ &= \frac{1}{2\alpha_k} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot |v_{k5}(\tilde{s})|^2 \cdot \omega(\tilde{s}) \ ; \quad (k = I,\ II,\ III) \ . \end{aligned} \tag{8.8}$$

(see Ref. [12]) has already been established.

After a jump to a new position in phase space

$$\begin{aligned} J_k &\longrightarrow J_k + \Delta J_k \ ; \\ \Phi_k &\longrightarrow \Phi_k + \Delta\Phi_k \end{aligned}$$

caused by the emission of a photon, the spin of the particle has the direction of the old $\vec{n}$–axis, $\vec{n}_{old}$, characterised by the components $\hat{\alpha}$ and $\hat{\beta}$ but the spin now precesses around the new $\vec{n}$–axis, $\vec{n}_{new} = \vec{n}_{old} + \Delta\vec{n}$, with components $\hat{\alpha} + \Delta\hat{\alpha}$, $\hat{\beta} + \Delta\hat{\beta}$ corresponding to the new orbital position. Only the projection of $\vec{n}_{old}$ onto $\vec{n}_{new}$ survives, since the spin component perpendicular to $\vec{n}_{new}$ $(= \vec{n}(\hat{\alpha} + \Delta\hat{\alpha}, \hat{\beta} + \Delta\hat{\beta})$) averages away due to the uniform precession of the spin with respect to the dreibein

$$(\hat{\vec{n}}_1,\ \hat{\vec{n}}_2,\ \vec{n})_{new} \ .$$

Thus the depolarisation takes place along the latest $\vec{n}$–axis, $\vec{n}_{new}$, and by simple geometrical considerations one finds that the degree of depolarisation due to the jump

$$\begin{aligned} \hat{\alpha} &\longrightarrow \hat{\alpha} + \Delta\hat{\alpha} \ ; \\ \hat{\beta} &\longrightarrow \hat{\beta} + \Delta\hat{\beta} \end{aligned}$$

reads exactly as [27] :

$$\Delta P = \frac{1}{2} \cdot |\Delta\vec{n}|^2 \ . \tag{8.9}$$

By writing $\Delta\vec{n}$ as

$$\Delta\vec{n} = \frac{\partial\vec{n}}{\partial\hat{\alpha}} \cdot \Delta\hat{\alpha} + \frac{\partial\vec{n}}{\partial\hat{\beta}} \cdot \Delta\hat{\beta} \tag{8.10}$$

with

$$
\begin{aligned}
\frac{\partial \vec{n}}{\partial \hat{\alpha}} &= -\hat{\alpha} \cdot \vec{n}_0 \\
&+ \left\{ \sqrt{1 - \frac{1}{4} \left(\hat{\alpha}^2 + \hat{\beta}^2 \right)} - \frac{\hat{\alpha}^2}{4 \cdot \sqrt{1 - \frac{1}{4} \left(\hat{\alpha}^2 + \hat{\beta}^2 \right)}} \right\} \cdot \vec{m} \\
&- \frac{\hat{\alpha}\, \hat{\beta}}{4 \cdot \sqrt{1 - \frac{1}{4} \left(\hat{\alpha}^2 + \hat{\beta}^2 \right)}} \cdot \vec{l}\,; \qquad (8.11a)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \vec{n}}{\partial \hat{\beta}} &= -\hat{\beta} \cdot \vec{n}_0 \\
&- \frac{\hat{\alpha}\, \hat{\beta}}{4 \cdot \sqrt{1 - \frac{1}{4} \left(\hat{\alpha}^2 + \hat{\beta}^2 \right)}} \cdot \vec{m} \\
&+ \left\{ \sqrt{1 - \frac{1}{4} \left(\hat{\alpha}^2 + \hat{\beta}^2 \right)} - \frac{\hat{\beta}^2}{4 \cdot \sqrt{1 - \frac{1}{4} \left(\hat{\alpha}^2 + \hat{\beta}^2 \right)}} \right\} \cdot \vec{l}\,. \qquad (8.11b)
\end{aligned}
$$

and putting (8.11a, b) into (8.10), we then obtain:

$$
\Delta P = \frac{1}{2} \left\{ \left(\frac{\partial \vec{n}}{\partial \hat{\alpha}} \right)^2 \cdot \Delta \hat{\alpha}^{\,2} + 2 \left(\frac{\partial \vec{n}}{\partial \hat{\alpha}} \cdot \frac{\partial \vec{n}}{\partial \hat{\beta}} \right) \cdot \Delta \hat{\alpha}\, \Delta \hat{\beta} + \left(\frac{\partial \vec{n}}{\partial \hat{\beta}} \right)^2 \cdot \Delta \hat{\beta}^2 \right\} \qquad (8.12)
$$

with

$$
\left(\frac{\partial \vec{n}}{\partial \hat{\alpha}} \right)^2 = 1 + \frac{1}{4} (\hat{\alpha}^2 - \hat{\beta}^2) + \frac{\hat{\alpha}^2 \left(\hat{\alpha}^2 + \hat{\beta}^2 \right)}{16 \cdot \left[1 - \frac{1}{4} \left(\hat{\alpha}^2 + \hat{\beta}^2 \right) \right]}\,; \qquad (8.13a)
$$

$$
\left(\frac{\partial \vec{n}}{\partial \hat{\alpha}} \cdot \frac{\partial \vec{n}}{\partial \hat{\beta}} \right) = \frac{1}{2}\, \hat{\alpha} \hat{\beta} + \frac{\hat{\alpha} \hat{\beta} \left(\hat{\alpha}^2 + \hat{\beta}^2 \right)}{16 \cdot \left[1 - \frac{1}{4} \left(\hat{\alpha}^2 + \hat{\beta}^2 \right) \right]}\,; \qquad (8.13b)
$$

$$
\left(\frac{\partial \vec{n}}{\partial \hat{\beta}} \right)^2 = 1 - \frac{1}{4} (\hat{\alpha}^2 - \hat{\beta}^2) + \frac{\hat{\beta}^2 \left(\hat{\alpha}^2 + \hat{\beta}^2 \right)}{16 \cdot \left[1 - \frac{1}{4} \left(\hat{\alpha}^2 + \hat{\beta}^2 \right) \right]}\,. \qquad (8.13c)
$$

8.2 Stochastic Behaviour of Spin Motion

The stochastic behaviour of the orbital motion characterised by eqn. (8.5) is transferred to the spin motion. Thus we have:

$$\begin{aligned}
\left(\frac{d}{ds}\,\hat{\alpha}\right)_{rad} &= \sum_{\mu} \frac{\partial \hat{\alpha}}{\partial A_{\mu}} \cdot A'_{\mu}(s) \\
&= \sum_{\mu} \frac{\partial \hat{\alpha}}{\partial \hat{A}_{\mu}} \cdot \frac{\partial \hat{A}_{\mu}}{\partial A_{\mu}} \cdot V_{\mu}(s) \cdot \sqrt{\omega(s)} \cdot \mathcal{P}(s) \\
&= \sum_{\mu} \frac{\partial \hat{\alpha}}{\partial \hat{A}_{\mu}} \cdot e^{-i\,\cdot\,2\pi Q_{\mu}\,\cdot\, s/L} \cdot V_{\mu}(s) \cdot \sqrt{\omega(s)} \cdot \mathcal{P}(s) \\
&= \sum_{\mu} \frac{\partial \hat{\alpha}}{\partial \hat{A}_{\mu}} \cdot \hat{V}_{\mu}(s) \cdot \sqrt{\omega(s)} \cdot \mathcal{P}(s) \\
&= F_1(s) \cdot \sqrt{\omega(s)} \cdot \mathcal{P}(s)\ ; \qquad (8.14a)
\end{aligned}$$

$$\left(\frac{d}{ds}\,\hat{\beta}\right)_{rad} = F_2(s) \cdot \sqrt{\omega(s)} \cdot \mathcal{P}(s) \qquad (8.14b)$$

with

$$\begin{aligned}
F_1(s) &= \sum_{\mu} \frac{\partial \hat{\alpha}}{\partial \hat{A}_{\mu}} \cdot \hat{V}_{\mu}(s)\ ; \qquad (8.15a) \\
F_2(s) &= \sum_{\mu} \frac{\partial \hat{\beta}}{\partial \hat{A}_{\mu}} \cdot \hat{V}_{\mu}(s) \qquad (8.15b)
\end{aligned}$$

and

$$\begin{aligned}
\hat{V}_{\mu}(s) &\equiv e^{-i\,\cdot\,2\pi Q_{\mu}\,\cdot\, s/L} \cdot V_{\mu}(s) \\
&= \hat{V}_{\mu}(s+L)\ . \qquad (8.16)
\end{aligned}$$

Integration of eqn. (8.14) leads to:

$$\begin{aligned}
\left[\hat{\alpha}(s_1) - \hat{\alpha}(s_0)\right]_{rad} &= \int_{s_0}^{s_1} d\tilde{s} \cdot \delta c(\tilde{s}) \cdot F_1(\tilde{s})\ ; \qquad (8.17a) \\
\left[\hat{\beta}(s_1) - \hat{\beta}(s_0)\right]_{rad} &= \int_{s_0}^{s_1} d\tilde{s} \cdot \delta c(\tilde{s}) \cdot F_2(\tilde{s})\ . \qquad (8.17b)
\end{aligned}$$

From eqn. (8.17) we obtain:

$$< \left[\hat{\alpha}(s_0) - \hat{\alpha}(s_1)\right]_{rad} \cdot \left[\hat{\beta}(s_0) - \hat{\beta}(s_1)\right]_{rad} >$$

$$
\begin{aligned}
&= \int_{s_0}^{s_1} ds' \cdot \int_{s_0}^{s_1} ds'' \cdot < \delta c(s')\, \delta c(s'') > \cdot F_1(s') \cdot F_2(s'') \\
&= \int_{s_0}^{s_1} ds' \cdot \int_{s_0}^{s_1} ds'' \cdot \omega(s') \cdot \delta(s' - s'') \cdot F_1(s') \cdot F_2(s'') \\
&= \int_{s_0}^{s_1} ds' \cdot \omega(s') \cdot F_1(s') \cdot F_2(s') \qquad (8.18a)
\end{aligned}
$$

and

$$
< \left(\left[\hat{\alpha}(s_0) - \hat{\alpha}(s_1) \right]_{rad} \right)^2 > \;=\; \int_{s_0}^{s_1} ds' \cdot \omega(s') \cdot F_1(s')^2 \; ; \qquad (8.18b)
$$

$$
< \left(\left[\hat{\beta}(s_0) - \hat{\beta}(s_1) \right]_{rad} \right)^2 > \;=\; \int_{s_0}^{s_1} ds' \cdot \omega(s') \cdot F_2(s')^2 \; . \qquad (8.18c)
$$

8.3 Depolarisation Time

For the stochastic average of ΔP in (8.12) we get, using eqns. (8.18a, b, c) :

$$
\begin{aligned}
< \Delta P(J_k, \Phi_k; s) > = \; & \Delta s \cdot \frac{1}{2} \omega(s) \\
& \times \left\{ \left(\frac{\partial \vec{n}}{\partial \hat{\alpha}} \right)^2 \cdot F_1(s)^2 + 2 \left(\frac{\partial \vec{n}}{\partial \hat{\alpha}} \cdot \frac{\partial \vec{n}}{\partial \hat{\beta}} \right) \cdot F_1(s)\, F_2(s) + \left(\frac{\partial \vec{n}}{\partial \hat{\beta}} \right)^2 \cdot F_2(s)^2 \right\} . \qquad (8.19)
\end{aligned}
$$

Taking an average of $< \Delta P(J_k, \Phi_k; s) >$ over one revolution and over the orbital phase space by using the distribution function (8.7), eqn. (8.19) then leads to the characteristic spin depolarisation time τ_D for the diffusion of spins resulting from stochastic orbit motion given by:

$$
\begin{aligned}
\tau_D^{-1} &= \frac{c}{L} \cdot \int_s^{s+L} d\tilde{s} \cdot \int_0^{2\pi} d\Phi_I \int_0^{2\pi} d\Phi_{II} \int_0^{2\pi} d\Phi_{III} \int_0^{\infty} dJ_I \int_0^{\infty} dJ_{II} \int_0^{\infty} dJ_{III} \\
&\qquad \times W_{orbit}^{(stat)}(J_l, \Phi_l) \cdot \frac{< \Delta P(J_k, \Phi_k; s) >}{\Delta s} \\
&= \frac{c}{2L} \int_s^{s+L} d\tilde{s} \cdot \omega(\tilde{s}) \cdot R(\tilde{s}) \\
&= \frac{c}{2L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot \omega(\tilde{s}) \cdot R(\tilde{s}) \qquad (8.20)
\end{aligned}
$$

with

$$
R(s) \;=\; \int_0^{\infty} dJ_I \int_0^{\infty} dJ_{II} \int_0^{\infty} dJ_{III} \cdot \frac{e^{-\left[J_I/\hat{J}_I + J_{II}/\hat{J}_{II} + J_{III}/\hat{J}_{III} \right]}}{\hat{J}_I \cdot \hat{J}_{II} \cdot \hat{J}_{III}}
$$

$$
\begin{aligned}
& \times \frac{1}{(2\pi)^3} \int_0^{2\pi} d\Phi_I \int_0^{2\pi} d\Phi_{II} \int_0^{2\pi} d\Phi_{III} \\
& \times \left\{ \left(\frac{\partial \vec{n}}{\partial \hat{\alpha}}\right)^2 \cdot F_1(s)^2 + 2 \left(\frac{\partial \vec{n}}{\partial \hat{\alpha}} \cdot \frac{\partial \vec{n}}{\partial \hat{\beta}}\right) \cdot F_1(s)\, F_2(s) + \left(\frac{\partial \vec{n}}{\partial \hat{\beta}}\right)^2 \cdot F_2(s)^2 \right\} \\
= \quad & R(s+L) \; . \qquad (8.21)
\end{aligned}
$$

Taking into account the 1st order relations:

$$
\begin{pmatrix} \hat{\alpha}^{(1)}(s) \\ \hat{\beta}^{(1)}(s) \end{pmatrix} \quad = \quad \sum_\mu \hat{A}_\mu \cdot \hat{\vec{w}}_\mu\,(s) \; ; \qquad (8.22)
$$

$$
\begin{aligned}
\begin{pmatrix} F_1^{(1)}(s) \\ F_1^{(1)}(s) \end{pmatrix} \quad & = \quad \sum_\mu \hat{\vec{w}}_\mu\,(s) \cdot \hat{V}_\mu \\
& = \quad \sum_\mu \vec{w}_\mu\,(s) \cdot V_\mu \\
& = \quad i \cdot \sum_{k=I,II,III} \{\vec{w}_k \cdot v_{k5}^* + compl.conj.\} \\
& = \quad -2 \cdot \Im m \sum_{k=I,II,III} \vec{w}_k \cdot i\, v_{k5}^* \qquad (8.23)
\end{aligned}
$$

(see eqns. (6.11) and (8.15)) the depolarisation rate ${\tau_D}^{-1}$ reads as:

$$
\tau_D^{-1} = 2 \cdot \frac{c}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot \omega(\tilde{s}) \cdot \sum_{\mu=1}^{2} \left(\Im m \sum_{k=I,II,III} [v_{k5}^*(\tilde{s}) \cdot w_{k\mu}(\tilde{s})] \right)^2 \; . \qquad (8.24)
$$

This result has already been derived in a different manner by A. Chao [15]) (see also Refs. [7, 12]), and serves as the basis of various schemes for maximising the polarisation [16, 28, 29, 30, 31, 32, 33].

In the same way higher orders of τ_D may be obtained by using eqns. (6.26) and (8.15).

Remark:

Note that since

$$
\int_0^{2\pi} d\Phi_k \cdot (A_k)^\mu \, (A_{-k})^\nu \quad = \quad 0 \quad \text{for} \quad \mu + \nu \neq 0
$$

(with A_k given by eqn. (4.24)), many terms of the integral (8.21) vanish. Only terms containing even balanced products of A_k's such as $A_k A_{-k}$ remain [18].

9 Summary

We have investigated a number of aspects of spin dynamics in storage rings on the basis of the T–BMT equation.

The orbital motion was described in terms of the fully coupled 6–dimensional formalism with the canonical variables $x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma = (1/\beta_0^2)\cdot\eta$.

Orbital action–angle variables in linear approximation were introduced via a canonical transformation taking into account all kinds of coupling (synchro–betatron coupling and coupling of the betatron oscillations by skew quadrupoles and solenoids).

In addition to the wellknown orbital variables $x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma$ of the fully coupled 6–dimensional formalism we defined the canonical spin variables α and β which were used to determine the $\vec{n}$–axis by the method of forced solution.

In order to describe spin motion, various coordinate systems were introduced, characterised by the triad $(\vec{e}_x,\ \vec{e}_z,\ \vec{e}_s)$, and the dreibeins $(\vec{n}_0,\ \vec{m},\ \vec{l})$ $(\vec{n}_1,\ \vec{n}_2,\ \vec{n})$ and $(\hat{\vec{n}}_1,\ \hat{\vec{n}}_2,\ \vec{n})$.

With respect to the dreibein $(\hat{\vec{n}}_1,\ \hat{\vec{n}}_2,\ \vec{n})$ we were then able to define action–angle variables of spin motion on an arbitrary particle path characterised by the orbital variables $J_k,\ \Phi_k$ and to calculate the spin tune as a function of J_k.

The dreibein $(\hat{\vec{n}}_1,\ \hat{\vec{n}}_2,\ \vec{n})$ is a single valued function of the orbital coordinates $(\Phi_k,\ J_k,\ s)$.

Then we presented a way to derive the general formula of the characteristic spin depolarisation time τ_D for the diffusion of spins resulting from stochastic orbit motion and in linear approximation obtained the usual result.

The equations presented in this report can serve to develop an 8–dimensional tracking program for the combined spin–orbit system, taking into account nonlinear spin motion.

Finally we remark that, starting from the variables $x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma\ \alpha,\ \beta$ and using analytical techniques as described in Refs. [9, 10, 34] one can also develop an 8–dimensional dispersion formalism.

In this paper we have neglected the Stern–Gerlach forces [1]. We shall incorporate these in another report.

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Appendix A: The Hamiltonian for Spin Motion

A.1 The Spin Hamiltonian

Spin motion can be described in terms of a spin Hamiltonian, namely:

$$\frac{d}{dt}\,\alpha \;=\; +\frac{\partial \mathcal{H}_{spin}}{\partial \beta}\;; \tag{A.1a}$$

$$\frac{d}{dt}\,\beta \;=\; -\frac{\partial \mathcal{H}_{spin}}{\partial \alpha}\;. \tag{A.1b}$$

where $\mathcal{H}_{spin}$ reads as [1] :

$$\mathcal{H}_{spin}(\alpha,\, \beta;\; t) \;=\; \sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}\cdot[\Omega_1\cdot\alpha+\Omega_2\cdot\beta]+\left[1-\frac{1}{2}\left(\alpha^2+\beta^2\right)\right]\cdot\Omega_3\;. \tag{A.2}$$

Then:

$$\begin{aligned}
\frac{d}{dt}\,\alpha \;&=\; +\frac{\partial \mathcal{H}_{spin}}{\partial \beta} \\
&=\; +\frac{-\beta}{4\sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}}\cdot[\Omega_1\cdot\alpha+\Omega_2\cdot\beta]+\sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}\cdot\Omega_2-\beta\cdot\Omega_3; \quad \text{(A.3a)} \\
\frac{d}{dt}\,\beta \;&=\; -\frac{\partial \mathcal{H}_{spin}}{\partial \alpha} \\
&=\; -\frac{-\alpha}{4\sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}}\cdot[\Omega_1\cdot\alpha+\Omega_2\cdot\beta]-\sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}\cdot\Omega_1+\alpha\cdot\Omega_3. \quad \text{(A.3b)}
\end{aligned}$$

The components ξ_1, ξ_2, ξ_3 of the spin vector $\vec{\xi}$

$$\vec{\xi} \;=\; \xi_1\cdot\vec{e}_1+\xi_2\cdot\vec{e}_2+\xi_3\cdot\vec{e}_3 \tag{A.4}$$

are given by:

$$\xi_1 \;=\; \alpha\cdot\sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}\,; \tag{A.5a}$$

$$\xi_2 \;=\; \beta\cdot\sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}\,; \tag{A.5b}$$

$$\xi_3 \;=\; 1-\frac{1}{2}\left(\alpha^2+\beta^2\right)\;; \tag{A.5c}$$

$$\Longrightarrow\quad \mathcal{H}_{spin} \;=\; \Omega_1\cdot\xi_1+\Omega_2\cdot\xi_2+\Omega_3\cdot\xi_3\;. \tag{A.6}$$

A.2 The Equations of Spin Motion

In terms of the components ξ_i we have:

$$
\begin{aligned}
\frac{d}{dt}\,\xi_3 &= -\alpha\cdot\frac{d}{ds}\,\alpha-\beta\cdot\frac{d}{ds}\,\beta \\
&= -\alpha\sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}\cdot\Omega_2+\beta\sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}\cdot\Omega_1 \\
&= \Omega_1\cdot\xi_2-\Omega_2\cdot\xi_1\ ; \qquad\qquad \text{(A.7a)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\,\xi_1 &= \sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}\cdot\frac{d}{ds}\,\alpha+\frac{\alpha}{4\sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}}\cdot\left[-\alpha\cdot\frac{d}{ds}\,\alpha-\beta\cdot\frac{d}{ds}\,\beta\right] \\
&= -\frac{\beta}{4}\cdot[\Omega_1\cdot\alpha+\Omega_2\cdot\beta]+\left[1-\frac{1}{4}\left(\alpha^2+\beta^2\right)\right]\cdot\Omega_2-\beta\sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}\cdot\Omega_3 \\
&\quad+\frac{\alpha}{4\sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}}\cdot[\Omega_1\cdot\xi_2-\Omega_2\cdot\xi_1] \\
&= -\frac{\beta}{4}\cdot[\Omega_1\cdot\alpha+\Omega_2\cdot\beta]+\left[1-\frac{1}{4}\left(\alpha^2+\beta^2\right)\right]\cdot\Omega_2-\xi_2\cdot\Omega_3 \\
&\quad+\frac{\alpha}{4}\cdot[\Omega_1\cdot\beta-\Omega_2\cdot\alpha] \\
&= -\frac{1}{4}\left(\alpha^2+\beta^2\right)\cdot\Omega_2+\left[1-\frac{1}{4}\left(\alpha^2+\beta^2\right)\right]\cdot\Omega_2-\xi_2\cdot\Omega_3 \\
&= \left[1-\frac{1}{2}\left(\alpha^2+\beta^2\right)\right]\cdot\Omega_2-\xi_2\cdot\Omega_3 \\
&= \Omega_2\cdot\xi_3-\Omega_3\cdot\xi_2\ ; \qquad\qquad \text{(A.7b)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\,\xi_2 &= \sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}\cdot\frac{d}{ds}\,\beta+\frac{\beta}{4\sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}}\cdot\left[-\alpha\cdot\frac{d}{ds}\,\alpha-\beta\cdot\frac{d}{ds}\,\beta\right] \\
&= +\frac{\alpha}{4}\cdot[\Omega_1\cdot\alpha+\Omega_2\cdot\beta]-\left[1-\frac{1}{4}\left(\alpha^2+\beta^2\right)\right]\cdot\Omega_1+\alpha\sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}\cdot\Omega_3 \\
&\quad+\frac{\beta}{4\sqrt{1-\frac{1}{4}\,(\alpha^2+\beta^2)}}\cdot[\Omega_1\cdot\xi_2-\Omega_2\cdot\xi_1] \\
&= +\frac{\alpha}{4}\cdot[\Omega_1\cdot\alpha+\Omega_2\cdot\beta]-\left[1-\frac{1}{4}\left(\alpha^2+\beta^2\right)\right]\cdot\Omega_1+\xi_1\cdot\Omega_3 \\
&\quad+\frac{\beta}{4}\cdot[\Omega_1\cdot\beta-\Omega_2\cdot\alpha] \\
&= +\frac{1}{4}\left(\alpha^2+\beta^2\right)\cdot\Omega_1-\left[1-\frac{1}{4}\left(\alpha^2+\beta^2\right)\right]\cdot\Omega_2+\xi_1\cdot\Omega_3
\end{aligned}
$$

$$= -\left[1-\frac{1}{2}\left(\alpha^2+\beta^2\right)\right]\cdot\Omega_1+\xi_1\cdot\Omega_3$$

$$= \Omega_3\cdot\xi_1-\Omega_1\cdot\xi_3\ ; \tag{A.7c}$$

$$\Longrightarrow\quad \frac{d}{dt}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = \begin{pmatrix}\Omega_1\\ \Omega_2\\ \Omega_3\end{pmatrix}\times\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}\ . \tag{A.8}$$

The last formula is identical with the T–BMT equation (2.7a) or (2.10).

Appendix B: The Periodic Spin Frame $(\vec{n}_0, \vec{m}, \vec{l})$ along the Closed Orbit

In order to investigate spin motion along the closed orbit, we write eqn. (4.2b) [3] in the form:

$$\frac{d}{ds}\ \vec{\xi}_0(s) = \underline{\Omega}^{(0)}(s)\cdot\vec{\xi}_0(s) \tag{B.1}$$

where we have set

$$\vec{\xi}_0 = \begin{pmatrix}\xi_{0s}\\ \xi_{0x}\\ \xi_{0z}\end{pmatrix} \tag{B.2a}$$

and

$$\underline{\Omega}^{(0)}(s) = \begin{pmatrix} 0 & -\Omega_z^{(0)} & \Omega_x^{(0)}\\ \Omega_z^{(0)} & 0 & -\Omega_s^{(0)}\\ -\Omega_x^0 & \Omega_s^{(0)} & 0\end{pmatrix}\ . \tag{B.2b}$$

The transfer matrix $\underline{M}_{(spin)}(s, s_0)$ for the spin motion defined by

$$\vec{\xi}_0(s) = \underline{M}_{(spin)}(s, s_0)\cdot\vec{\xi}_0(s_0)$$

satisfies the relationships:

$$\underline{M}^T_{(spin)}(s, s_0)\cdot\underline{M}_{(spin)}(s, s_0) = \underline{1}\ ; \tag{B.3a}$$

$$\det\ [\underline{M}_{(spin)}(s, s_0)] = 1 \tag{B.3b}$$

[3] This equation can be solved by methods as described in Appendix C, using $\vec{\Omega}^{(0)}$ instead of $\hat{\vec{\Omega}}$.

since (using eqn. (B.1))

$$\frac{d}{ds}\,\underline{M}_{(spin)}(s,s_0) = \underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s,s_0)\ ;$$

$$\underline{M}_{(spin)}(s_0,s_0) = \underline{1}$$

and therefore (with $[\underline{\Omega}^{(0)}]^T = -\underline{\Omega}^{(0)}$)

$$\begin{aligned}
\frac{d}{ds}\left[\underline{M}^T_{(spin)}(s,s_0)\cdot\underline{M}_{(spin)}(s,s_0)\right] &= \left[\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s,s_0)\right]^T\cdot\underline{M}_{(spin)}(s,s_0) \\
&\quad +\underline{M}^T_{(spin)}(s,s_0)\cdot\left[\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s,s_0)\right] \\
&= -\underline{M}_{(spin)}(s,s_0)^T\cdot\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s,s_0) \\
&\quad +\underline{M}^T_{(spin)}(s,s_0)\cdot\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s,s_0) \\
&= \underline{0}\ ; \\
\det\ M_{(spin)}(s,s_0) &= \det\ M_{(spin)}(s_0,s_0) = 1\ ,
\end{aligned}$$

i.e. $\underline{M}_{(spin)}(s,s_0)$ is an orthogonal matrix with determinant 1.

Let us now consider the eigenvalue problem for the revolution matrix $\underline{M}_{(spin)}(s_0+L,s_0)$ with the eigenvalues α_μ and eigenvectors $\vec{r}_\mu(s_0)$:

$$\underline{M}_{(spin)}(s_0+L,s_0)\ \vec{r}_\mu(s_0) = \alpha_\mu\cdot\vec{r}_\mu(s_0)\ ; \tag{B.4}$$

$$(\mu = 1,\ 2,\ 3)$$

using the notation

$$\vec{r}_\mu \ \hat{=}\ \begin{pmatrix} \vec{r}_\mu\cdot\vec{e}_s \\ \vec{r}_\mu\cdot\vec{e}_x \\ \vec{r}_\mu\cdot\vec{e}_z \end{pmatrix} .$$

Because of (B.3a,b) we can write [35]:

$$\begin{aligned}
\alpha_1 &= 1\ ; \\
\alpha_2 &= e^{i\cdot 2\pi\cdot Q_{spin}}\ ; \\
\alpha_3 &= e^{-i\cdot 2\pi\cdot Q_{spin}}\ ;
\end{aligned} \tag{B.5}$$

$$(Q_{spin} = \text{real number})$$

and

$$\vec{r}_1(s_0) = \vec{n}_0(s_0) \; ; \tag{B.6a}$$
$$\vec{r}_2(s_0) = \vec{m}_0(s_0) + i \cdot \vec{l}_0(s_0) \; ; \tag{B.6b}$$
$$\vec{r}_3(s_0) = \vec{m}_0(s_0) - i \cdot \vec{l}_0(s_0) \; ; \tag{B.6c}$$
$$(\vec{n}_0, \vec{m}_0, \vec{l}_0 = \text{real vectors}) \; .$$

If we require that

$$\vec{r}_1^{\,+} \cdot \vec{r}_1 = 1 \; ; \tag{B.7a}$$
$$\vec{r}_2^{\,+} \cdot \vec{r}_2 \equiv \vec{r}_3^{\,+} \cdot \vec{r}_3 = 2 \; ; \tag{B.7b}$$
$$\text{(normalising conditions)}$$

we find, using eqn. (B.3a) [35]:

$$|\vec{n}_0(s_0)| = |\vec{m}_0(s_0)| = |\vec{l}_0(s_0)| = 1 \; ; \tag{B.8a}$$

$$\vec{n}_0(s_0) \perp \vec{m}_0(s_0) \perp \vec{l}_0(s_0) \; . \tag{B.8b}$$

Thus the vectors $\vec{n}_0(s_0)$, $\vec{m}_0(s_0)$ and $\vec{l}_0(s_0)$ form an orthogonal system of unit vectors. Choosing the direction of $\vec{n}_0(s_0)$ such that

$$\vec{n}_0(s_0) = \vec{m}_0(s_0) \times \vec{l}_0(s_0) \tag{B.8c}$$

these vectors form a righthanded coordinate system.

In this way we have found a coordinate frame for the position $s = s_0$.

An orthogonal system of unit vectors at an arbitrary position s can be defined by applying the transfer matrix $\underline{M}_{(spin)}(s, s_0)$ to the vectors $\vec{n}_0(s_0)$, $\vec{m}_0(s_0)$ and $\vec{l}_0(s_0)$:

$$\vec{n}_0(s) = \underline{M}_{(spin)}(s, s_0) \; \vec{n}_0(s_0) \; ; \tag{B.9a}$$
$$\vec{m}_0(s) = \underline{M}_{(spin)}(s, s_0) \; \vec{m}_0(s_0) \; ; \tag{B.9b}$$
$$\vec{l}_0(s) = \underline{M}_{(spin)}(s, s_0) \; \vec{l}_0(s_0) \; . \tag{B.9c}$$

Because of eqn. (B.3a,b) the orthogonality relations remain unchanged:

$$\vec{n}_0(s) = \vec{m}_0(s) \times \vec{l}_0(s) \tag{B.10a}$$
$$\vec{m}_0(s) \perp \vec{l}_0(s) \; ; \tag{B.10b}$$
$$|\vec{n}_0(s)| = |\vec{m}_0(s)| = |\vec{l}_0(s)| = 1 \; . \tag{B.10c}$$

The coordinate frame defined by $\vec{n}_0(s)$, $\vec{m}_0(s)$ and $\vec{l}_0(s)$ is not yet appropriate for a description of the spin motion, because it does not transform into itself after one revolution of the particles:

$$\begin{aligned} \vec{m}_0(s_0 + L) + i\,\vec{l}_0(s_0 + L) &= \underline{M}_{(spin)}(s_0 + L, s_0) \left[\vec{m}_0(s_0) + i\vec{l}_0(s_0)\right] \\ &= e^{i \cdot 2\pi \cdot Q_{spin}} \cdot \left[\vec{m}_0(s_0) + i\vec{l}_0(s_0)\right] \\ &\neq \vec{m}_0(s_0) + i\,\vec{l}_0(s_0) \end{aligned}$$

$$(\text{if } \; Q_{spin} \neq \text{integer}).$$

i.e. although $\vec{n}_0(s)$ is periodic by eqns. (B.5), (B.6a), $\vec{m}_0(s)$ and $\vec{l}_0(s)$ are not periodic.

But by introducing a phase function $\psi(s)$ and using another orthogonal matrix $\underline{\hat{D}}(s)$:

$$\underline{\hat{D}}(s) = \begin{pmatrix} \cos[\psi_{spin}(s)] & \sin[\psi_{spin}(s)] \\ -\sin[\psi_{spin}(s)] & \cos[\psi_{spin}(s)] \end{pmatrix} \tag{B.11}$$

with

$$\underline{\hat{D}}^T(s) \cdot \underline{\hat{D}}(s) \;=\; \underline{1} \; ; \tag{B.12a}$$

$$\det\,[\underline{\hat{D}}(s)] = 1 \tag{B.12b}$$

we can construct a periodic orthogonal system of unit vectors from $\vec{n}_0(s)$, $\vec{m}_0(s)$ and $\vec{l}_0(s)$. Namely, if we put [36]:

$$\begin{pmatrix} \vec{m}(s) \\ \vec{l}(s) \end{pmatrix} = \underline{\hat{D}}(s) \begin{pmatrix} \vec{m}_0(s) \\ \vec{l}_0(s) \end{pmatrix}$$

$$\begin{aligned} \Longrightarrow \qquad \vec{m}(s) + i\,\vec{l}(s) \;&=\; e^{-i\,\cdot\,\psi_{spin}(s)} \cdot \left[\vec{m}_0(s) + i\,\vec{l}_0(s)\right] \\ &\neq\; \vec{m}_0(s_0) + i\,\vec{l}_0(s_0) \end{aligned} \tag{B.13}$$

we find, using eqns. (B.12a, b) :

$$\vec{n}_0(s) \;=\; \vec{m}(s) \times \vec{l}(s) \; ; \tag{B.14a}$$

$$\vec{m}(s) \;\perp\; \vec{l}(s) \; ; \tag{B.14b}$$

$$|\vec{n}_0(s)| \;=\; |\vec{m}(s)| \;=\; |\vec{l}(s)| \;=\; 1 \; . \tag{B.14c}$$

Since

$$\begin{aligned} \vec{m}(s+L) + i \cdot \vec{l}(s+L) \;&=\; e^{-i\,\cdot\,\psi_{spin}(s+L)} \cdot \left[\vec{m}_0(s+L) + i \cdot \vec{l}_0(s+L)\right] \\ &=\; e^{-i\,\cdot\,\psi_{spin}(s+L)} \cdot e^{i\,\cdot\,2\pi\,Q_{spin}} \cdot \left[\vec{m}_0(s) + i \cdot \vec{l}_0(s)\right] \\ &=\; e^{-i\,\cdot\,\psi_{spin}(s+L)} \cdot e^{i\,\cdot\,2\pi\,Q_{spin}} \cdot e^{+i\,\cdot\,\psi(s)} \cdot \left\{\vec{m}(s) + i \cdot \vec{l}(s)\right\} \\ &=\; e^{-i\,\cdot\,[\psi_{spin}(s+L) - \psi_{spin}(s)]} \cdot e^{i\,\cdot\,2\pi\,Q_{spin}} \cdot \left\{\vec{m}(s) + i \cdot \vec{l}(s)\right\} \end{aligned}$$

it follows, that the condition of periodicity for $\vec{n}_0$, $\vec{m}$ and $\vec{l}$:

$$\left(\vec{n}_0,\; \vec{m},\; \vec{l}\right)_{s\,=\,s_0+L} \;=\; \left(\vec{n}_0,\; \vec{m},\; \vec{l}\right)_{s\,=\,s_0} \tag{B.15}$$

can indeed be fulfilled if the phase function $\psi(s)$ satisfies the following relationship:

$$\psi_{spin}(s+L) - \psi_{spin}(s) = 2\pi \cdot Q_{spin} \ ; \qquad \text{(B.16a)}$$

$$(Q_{spin} = \text{spin tune}).$$

For instance we can choose:

$$\psi_{spin}(s) = 2\pi \cdot Q_{spin} \cdot \frac{s}{L} \ . \qquad \text{(B.16b)}$$

In this frame, spins on the closed orbit precess uniformly with respect to $\vec{m}$ and $\vec{l}$.

Note that the spin tune Q_{spin} can be separated into an arbitrary integer part κ and a fractional part $\tilde{Q}_{spin}$:

$$\begin{aligned} Q_{spin} &= \kappa + \tilde{Q}_{spin} \ ; \\ 0 &\leq \tilde{Q}_{spin} < 1 \ . \end{aligned}$$

Taking the derivatives of $\vec{m}(s)$ and $\vec{l}(s)$ with respect to s, and taking into account eqns. (B.13), (B.9), and (4.2b) we get

$$\frac{d}{ds}\begin{pmatrix} \vec{m}\cdot\vec{e}_s \\ \vec{m}\cdot\vec{e}_x \\ \vec{m}\cdot\vec{e}_z \end{pmatrix} = \underline{\Omega}^{(0)}(s) \begin{pmatrix} \vec{m}\cdot\vec{e}_s \\ \vec{m}\cdot\vec{e}_x \\ \vec{m}\cdot\vec{e}_z \end{pmatrix} + \psi\,'(s)\cdot \begin{pmatrix} \vec{l}\cdot\vec{e}_s \\ \vec{l}\cdot\vec{e}_x \\ \vec{l}\cdot\vec{e}_z \end{pmatrix} \ ; \qquad \text{(B.17a)}$$

$$\frac{d}{ds}\begin{pmatrix} \vec{l}\cdot\vec{e}_s \\ \vec{l}\cdot\vec{e}_x \\ \vec{l}\cdot\vec{e}_z \end{pmatrix} = \underline{\Omega}^{(0)}(s) \begin{pmatrix} \vec{l}\cdot\vec{e}_s \\ \vec{l}\cdot\vec{e}_x \\ \vec{l}\cdot\vec{e}_z \end{pmatrix} - \psi\,'(s)\cdot \begin{pmatrix} \vec{m}\cdot\vec{e}_s \\ \vec{m}\cdot\vec{e}_x \\ \vec{m}\cdot\vec{e}_z \end{pmatrix} \qquad \text{(B.17b)}$$

and $\vec{n}_0(s)$ satisfies (see (B.9a))

$$\frac{d}{ds}\begin{pmatrix} \vec{n}_0\cdot\vec{e}_s \\ \vec{n}_0\cdot\vec{e}_x \\ \vec{n}_0\cdot\vec{e}_z \end{pmatrix} = \underline{\Omega}^{(0)}(s) \begin{pmatrix} \vec{n}_0\cdot\vec{e}_s \\ \vec{n}_0\cdot\vec{e}_x \\ \vec{n}_0\cdot\vec{e}_z \end{pmatrix} \ . \qquad \text{(B.17c)}$$

Finally, the vectors

$$\begin{aligned} \vec{r}_1(s) &= \vec{n}_0(s) \equiv \underline{M}_{(spin)}(s,s_0)\,\vec{r}_1(s_0) \ ; && \text{(B.18a)} \\ \vec{r}_2(s) &= \vec{m}_0(s) + i\cdot\vec{l}_0(s) \equiv \underline{M}_{(spin)}(s,s_0)\,\vec{r}_2(s_0) \ ; && \text{(B.18b)} \\ \vec{r}_3(s) &= \vec{m}_0(s) - i\cdot\vec{l}_0(s) \equiv \underline{M}_{(spin)}(s,s_0)\,\vec{r}_3(s_0) && \text{(B.18c)} \end{aligned}$$

are eigenvectors of the revolution matrix $\underline{M}_{(spin)}$ with the same eigenvalues as in (B.5):

$$\underline{M}_{(spin)}(s+L,s)\,\vec{r}_\mu(s) = \alpha_\mu\cdot\vec{r}_\mu(s) \ . \qquad \text{(B.19)}$$

Thus, the eigenvalues α_μ and the quantity Q_{spin} defined by eqn. (B.5) are independent of the chosen initial position s_0.

Finally, we remark that $(\vec{n}_0,\, \vec{m}_0,\, \vec{l}_0)$ are all T–BMT solutions whereas $(\vec{m},\, \vec{l})$ in general are not T–BMT solutions.

Appendix C: Solution of the Equations of Spin Motion

C.1 Thin Lens Approximation

1) Using the variables α and β.

The solution of the canonical spin equation (5.14a, b) can approximated by the form:

$$\begin{aligned} \alpha(s+\Delta s) &= \alpha(s) + \Delta s \cdot \frac{\partial \mathcal{H}_{spin}}{\partial \beta} \; ; \\ \beta(s+\Delta s) &= \beta(s) - \Delta s \cdot \frac{\partial \mathcal{H}_{spin}}{\partial \alpha} \; . \end{aligned}$$

Expanding the spin Hamiltonian (5.16) into a power series of α and β, we obtain various orders of approximation. The computer program "SLIM" [15] works with the linear order of spin motion. In general this solution is not symplectic, i.e. the spin transformation is not orthogonal.

2) Using the variables ξ_n, ξ_m and ξ_l.

For point–like fields, the matrix $\underline{\hat{\Omega}}(s)$ appearing in (5.11) is given by

$$\underline{\hat{\Omega}}(s) = \underline{P}(s_0) \cdot \delta(s - s_0) \; . \tag{C.1}$$

Writing $\underline{P}(s_0)$ in the form:

$$\underline{P}(s_0) = \begin{pmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{pmatrix} \tag{C.2}$$

the solution of eqns. (5.11), (B.1) and (B.2) leads to the transfer matrix:

$$\underline{M}_{spin}(s_0+0, s_0-0) =$$

$$\begin{pmatrix} \hat{P}_1^2(1-\cos P)+\cos P & \hat{P}_1\hat{P}_2(1-\cos P)-\hat{P}_3\sin P & \hat{P}_1\hat{P}_3(1-\cos P)+\hat{P}_2\sin P \\ \hat{P}_2\hat{P}_1(1-\cos P)+\hat{P}_3\sin P & \hat{P}_2^2(1-\cos P)+\cos P & \hat{P}_2\hat{P}_3(1-\cos P)-\hat{P}_1\sin P \\ \hat{P}_3\hat{P}_1(1-\cos P)-\hat{P}_2\sin P & \hat{P}_3\hat{P}_2(1-\cos P)+\hat{P}_1\sin P & \hat{P}_3^2(1-\cos P)+\cos P \end{pmatrix} \tag{C.3}$$

with

$$\hat{P}_\nu = \frac{1}{P} \cdot P_\nu \; ; \quad (\nu = 1,2,3) \; ; \tag{C.4a}$$

$$P = \sqrt{P_1^2 + P_2^2 + P_3^2} \; . \tag{C.4b}$$

If ξ_n, ξ_m, ξ_l are known, the canonical variables α and β can be obtained from eqn. (5.13). In this way, eqn. (B.3) represents a symplectic integration method of the canonical spin equations (5.14a, b) on the basis of a thin lens approximation, i.e. the spin transformation is orthogonal.

C.2 Integration by Lie Series

A system of differential equations of the form:

$$\frac{d}{ds} z_i(s) = \vartheta_i (z_1, z_2, \cdots z_n) ;$$
$$(i = 1, 2, \cdots n)$$

with

$$\frac{\partial}{\partial s} \vartheta_i = 0 ,$$

(no explicit s dependence) whereby the terms $\vartheta_i (z_1, z_2, \cdots z_n)$ represent analytical functions, can be solved by Lie series [37]:

$$z_i(s) = e^{(s-s_0) D} \hat{z}_i$$

with

$$D = \vartheta_1 (\hat{z}_1, \hat{z}_2, \cdots \hat{z}_n) \frac{\partial}{\partial \hat{z}_1} + \vartheta_2 (\hat{z}_1, \hat{z}_2, \cdots \hat{z}_n) \frac{\partial}{\partial \hat{z}_2} + \cdots + \vartheta_n (\hat{z}_1, \hat{z}_2, \cdots \hat{z}_n) \frac{\partial}{\partial \hat{z}_n}$$

and

$$z_i(s_0) \equiv \hat{z}_i .$$

Applying this result to the canonical spin equations (5.14a, b):

$$z_1 \equiv \alpha ; \quad z_2 \equiv \beta ;$$

$$\vartheta_1 = +\frac{\partial}{\partial \beta} \mathcal{H}_{spin}(\alpha, \beta, x, p_x, z, p_z, \sigma, p_\sigma) ;$$
$$\vartheta_2 = -\frac{\partial}{\partial \alpha} \mathcal{H}_{spin}(\alpha, \beta, x, p_x, z, p_z, \sigma, p_\sigma)$$

associated with the equations for orbital motion:

$$z_3 \equiv x; \quad z_4 \equiv p_x ;$$
$$z_3 \equiv x; \quad z_4 \equiv p_x ;$$
$$z_5 \equiv z; \quad z_6 \equiv p_z ;$$
$$z_7 \equiv \sigma; \quad z_8 \equiv p_\sigma ;$$

$$\vartheta_3 = +\frac{\partial}{\partial p_x} \mathcal{H}_{orb}(x, p_x, z, p_z, \sigma, p_\sigma) ;$$
$$\vartheta_4 = -\frac{\partial}{\partial x} \mathcal{H}_{orb}(x, p_x, z, p_z, \sigma, p_\sigma) ;$$

$$
\begin{aligned}
\vartheta_5 &= +\frac{\partial}{\partial p_z}\,\mathcal{H}_{orb}(x,p_x,z,p_z,\sigma,p_\sigma)\ ; \\
\vartheta_6 &= -\frac{\partial}{\partial z}\,\mathcal{H}_{orb}(x,p_x,z,p_z,\sigma,p_\sigma)\ ; \\
\vartheta_7 &= +\frac{\partial}{\partial p_\sigma}\,\mathcal{H}_{orb}(x,p_x,z,p_z,\sigma,p_\sigma)\ ; \\
\vartheta_8 &= -\frac{\partial}{\partial \sigma}\,\mathcal{H}_{orb}(x,p_x,z,p_z,\sigma,p_\sigma)
\end{aligned}
$$

we obtain:

$$\alpha(s) = e^{(s-s_0)\,D}\,\hat{\alpha}\ ; \tag{C.5a}$$

$$\beta(s) = e^{(s-s_0)\,D}\,\hat{\beta}\ ; \tag{C.5b}$$

$$x(s) = e^{(s-s_0)\,D}\,\hat{x}\ ; \tag{C.5c}$$

$$p_x(s) = e^{(s-s_0)\,D}\,\hat{p_x}\ ; \tag{C.5d}$$

$$z(s) = e^{(s-s_0)\,D}\,\hat{z}\ ; \tag{C.5e}$$

$$p_x(s) = e^{(s-s_0)\,D}\,\hat{p_z}\ ; \tag{C.5f}$$

$$\sigma(s) = e^{(s-s_0)\,D}\,\hat{\sigma}\ ; \tag{C.5g}$$

$$p_x(\sigma) = e^{(s-s_0)\,D}\,\hat{p_\sigma} \tag{C.5h}$$

with

$$
\begin{aligned}
D = &\left[\frac{\partial}{\partial\hat{\beta}}\,\mathcal{H}_{spin}\right]\frac{\partial}{\partial\hat{\alpha}} - \left[\frac{\partial}{\partial\hat{\alpha}}\,\mathcal{H}_{spin}\right]\frac{\partial}{\partial\hat{\beta}} \\
&+ \left[\frac{\partial}{\partial\hat{p_x}}\,\mathcal{H}_{orb}\right]\frac{\partial}{\partial\hat{x}} - \left[\frac{\partial}{\partial\hat{x}}\,\mathcal{H}_{orb}\right]\frac{\partial}{\partial\hat{p_x}} \\
&+ \left[\frac{\partial}{\partial\hat{p_z}}\,\mathcal{H}_{orb}\right]\frac{\partial}{\partial\hat{z}} - \left[\frac{\partial}{\partial\hat{z}}\,\mathcal{H}_{orb}\right]\frac{\partial}{\partial\hat{p_z}} \\
&+ \left[\frac{\partial}{\partial\hat{p_\sigma}}\,\mathcal{H}_{orb}\right]\frac{\partial}{\partial\hat{\sigma}} - \left[\frac{\partial}{\partial\hat{\sigma}}\,\mathcal{H}_{orb}\right]\frac{\partial}{\partial\hat{p_\sigma}}
\end{aligned}
\tag{C.6}
$$

and

$$\hat{\alpha} \equiv \alpha(s_0)\ ;\quad \hat{\beta} \equiv \beta(s_0)\ ;$$

$$
\begin{aligned}
\hat{x} &\equiv x(s_0)\ ; \quad \hat{p_x} \equiv p_x(s_0)\ ; \\
\hat{z} &\equiv z(s_0)\ ; \quad \hat{p_z} \equiv p_z(s_0)\ ; \\
\hat{\sigma} &\equiv \sigma(s_0)\ ; \quad \hat{p_\sigma} \equiv p_\sigma(s_0)\ .
\end{aligned} \tag{C.7}
$$

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