The invariant polarisation-tensor field for spin-1 particles in storage rings

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Abstract. We complement the concept of the invariant spin field (ISF) in storage rings by defining the invariant polarisation–tensor field (ITF) for spin-1 particles and we suggest how to calculate it by stroboscopic averaging or directly from the invariant spin field. The ITF the ISF are used to construct equilibrium spin density–matrix fields, and thereby offer a clean framework for describing equilibrium ensembles of spin-1 particles such as deuterons in storage rings.

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INTRODUCTION

Spin motion for particles moving in electric and magnetic fields is governed by the T–BMT equation [1]. This describes the rate of precession of the rest–frame, pure–state, spin expectation value $\vec{S}$ ("the spin") of a particle. In circular particle accelerators and storage rings we take the distance around the ring as the independent variable [2]. Then, at the position s along the design orbit and the point u in the 6–dimensional phase space, we write the T–BMT equation as $d\vec{S}/ds = \vec{\Omega}(u;s) \times \vec{S}$ where the vector $\vec{\Omega}(u;s)$ describing the precession axis and the rate of precession, depends on the electric and magnetic fields in the laboratory, and on the reference energy of the ring. The motion of a particle is governed by the Lorentz force [1]. Thus both the motion of the particle and the motion of the spin expectation value can be treated classically. Nevertheless, we still need to look at the quantum mechanics. For this we exploit the spin density matrix.

The spin state of an ensemble of spin-1/2 fermions is completely specified by means of the 2×2 density matrix $\rho = \frac{1}{2}\{I + \vec{P} \cdot \vec{\sigma}\}$ where $\vec{P}$ is the vector polarisation, I is the unit matrix, the

$$\sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

are the Pauli spin matrices [3] and $\vec{\sigma}$ is the corresponding matrix-valued 3-vector. The vector polarisation is $\langle \vec{\sigma} \rangle = \vec{P} = Tr\,(\rho\vec{\sigma})$ where $\langle\rangle$ denotes a mixed-state expectation value.

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Spin-1 particles such as deuterons need a 3×3 density matrix and we write it as [3]

$$\rho = \frac{1}{3}\left\{ I + \frac{3}{2}\vec{P}\cdot\vec{\mathfrak{J}} + \sqrt{\frac{3}{2}}\sum_{i,j} T_{ij}\,(\mathfrak{J}_i\mathfrak{J}_j + \mathfrak{J}_j\mathfrak{J}_i)\right\} ,$$

where the three matrices $\mathfrak{J}$

$$\mathfrak{J}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},\ \mathfrak{J}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ \mathfrak{J}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

representing the normalised spin operators $\hat{s}_i \ \ (i=1,2,3)$, are the analogues for spin-1 of the Pauli matrices and where $\vec{\mathfrak{J}}$ is the corresponding matrix–valued 3–vector. The rank–2, 3×3, real, symmetric, traceless, Cartesian tensor T has just five independent components. With the three components of the vector polarisation there is a total of eight parameters for defining ρ as required for a Hermitian matrix with a definite trace (unity). The degree of tensor polarisation $\mathfrak{T} \equiv \sqrt{\sum_{i,j} T_{ij}^2} = \sqrt{Tr\,(T^2)}$ is at most unity [3].

In accelerator physics, we define a density matrix for each point u and s. Then at each u and s we write the density matrix for spin-1/2 fermions in terms of the local vector polarisation $\vec{P}_{\rm loc}(u;s)$. For spin-1 particles we need the local polarisation tensor $T_{\rm loc}(u;s)$ too. If the vector polarisation for spin-1/2 fermions is periodic from turn to turn, the density matrix can be described in terms of the vector $\hat{n}$ of the Invariant Spin Field (ISF) and the value of the equilibrium vector polarisation. Is there an analogous object for the polarisation tensor? This paper answers this in the affirmative by describing the new concepts of the Invariant Tensor Field (ITF) and the equilibrium tensor polarisation. Together with the ISF and the equilibrium vector polarisation they provide a complete description of the equilibrium spin state of a beam of spin-1 particles such as deuterons in a storage ring.

This paper provides just a brief summary. More information can be found in [4].

THE INVARIANT SPIN AND TENSOR FIELDS

Since the T–BMT equation is linear in $\vec{S}$ and since the particles at (u,s) all see the same $\vec{\Omega}(u;s)$, $\vec{P}_{\rm loc}(u(s);s)$ obeys the T–BMT equation along trajectories. Furthermore, $P_{\rm loc} \equiv |\vec{P}_{\rm loc}|$ is constant along a trajectory. For a storage ring at fixed energy, $\vec{\Omega}$ is 1–turn periodic in s at a fixed position in phase space u so that $\vec{\Omega}(u;s) = \vec{\Omega}(u;s+C)$ where C is the circumference. This opens the possibility that the polarisation is the same from turn to turn in the sense that $\vec{P}_{\rm loc}(u;s+C) = \vec{P}_{\rm loc}(u;s)$. We denote such a $\vec{P}_{\rm loc}$ by $\vec{P}_{\rm eq}$ since, if the phase space density has the same periodicity, the polarisation of the whole beam is 1-turn periodic, i.e. in equilibrium. Since $\vec{P}_{\rm eq}$ also obeys the T–BMT equation we then have $\vec{P}_{\rm eq}(M(u;s+C,s);s+C) = \vec{P}_{\rm eq}(M(u;s+C,s);s) = R(u;s+C,s)\vec{P}_{\rm eq}(u;s)$. where $M(u;s+C,s)$ is the new position in phase space after one turn starting at u and s, $R(u;s+C,s)$ is the corresponding 3×3 spin transfer matrix representing the solution to the T–BMT equation for one turn from s to $s+C$ and where here, $\vec{P}_{\rm eq}$ is represented by a column matrix of its components.

These relations motivate the introduction of a vector *field* $\hat{n}(u;s)$ of unit length, obeying similar constraints, namely $\hat{n}(M(u;s+C,s);s+C) = \hat{n}(M(u;s+C,s);s) = R(u;s+C,s)\hat{n}(u;s)$ where $\hat{n}$ is represented by a column matrix of its components. Since the vector field $\hat{n}(u;s)$ is invariant from turn to turn and independent of the real state of a beam it is called the invariant spin field (ISF) [5, 6]. We define $\hat{n}_0(s) \equiv \hat{n}(0;s)$.

We assume that u can be parametrised in terms of three pairs of action–angle variables $(J_i, \phi_i,\ i=1,2,3)$ which we abbreviate by (J,ϕ). The actions J are constants of motion. Thus the orbital phase space is partitioned into disjoint tori characterised by unique sets J. We also assume that the orbital motion is nonresonant so that, in time, a trajectory could cover its torus. Then from the definition of $\vec{P}_{\rm eq},\ P_{\rm eq} \equiv |\vec{P}_{\rm eq}|$ must be the same at all points ϕ on a torus but can depend on the J. Away from spin-orbit resonances and apart from the sign, $\hat{n}(J,\phi;s)$ is a unique function of ϕ and s, 2π–periodic in ϕ [5]. So we may write $\vec{P}_{\rm eq}(J,\phi;s) = P_{\rm eq}(J)\,\hat{n}(J,\phi;s)$, bearing in mind that $|\hat{n}|=1$. If we also require that the particles on a torus are distributed uniformly in ϕ, we find that the polarisation for the torus, defined as the average $P_{\rm eq}(J)/(2\pi)^3\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\hat{n}(J,\phi;s)d\phi$, is invariant from turn to turn, i.e., in equilibrium. The key aspects of the ISF are summarised in [5]. The existence of a 1–turn periodic $\vec{P}_{\rm eq}$ and the corresponding ISF implies the existence of an equilibrium spin density–matrix field (ESDeMF), $\rho^{\rm eq}$, obeying the periodicity condition $\rho^{\rm eq}(u;s+C) = \rho^{\rm eq}(u;s)$ and which we can write in obvious notation as:

$$\rho^{\rm eq}_{P_{\rm eq}(J)}(J,\phi;s) = \frac{1}{2}\{I + P_{\rm eq}(J)\,\hat{n}(J,\phi;s)\cdot\vec{\sigma}\}\ .$$

We now extend the concept of invariance to the polarisation tensor by defining the Invariant Tensor Field $T^{\rm I}$ by the periodicity condition $T^{\rm I}(M(u;s+C,s);s+C) = T^{\rm I}(M(u;s+C,s);s) = R(u;s+C,s)\ T^{\rm I}(u;s)R^{\rm T}(u;s+C,s)$ where we have used the transformation rule [7] of a Cartesian tensor. We normalise so that $\sqrt{Tr\,([T^{\rm I}]^2)} = 1$.

Then we can construct an ESDeMF on a torus J for spin-1 particles too:

$$\rho^{\rm eq}_{[P_{\rm eq},\mathfrak{T}_{\rm eq}](J)}(J,\phi;s) = \frac{1}{3}\left\{I + \frac{3}{2}P_{\rm eq}(J)\,\hat{n}\cdot\vec{\mathfrak{J}} + \sqrt{\frac{3}{2}}\mathfrak{T}_{\rm eq}(J)\sum_{i,j}T^{\rm I}_{ij}\ (\mathfrak{J}_i\mathfrak{J}_j + \mathfrak{J}_j\mathfrak{J}_i)\right\}\ ,$$

where $\mathfrak{T}_{\rm eq}(J)$ is the degree of equilibrium tensor polarisation on the torus. The ISF and ITF provide "scaffolding" on which to "hang" equilibrium spin distributions for spin-1 particles and the ESDeMF then depends on just two scalar parameters, $P_{\rm eq}(J)$ and $\mathfrak{T}_{\rm eq}(J)$.

The most general, *model–independent* way, to construct the ISF is by so–called stroboscopic averaging [8, 9, 10, 11]. This just requires a spin–orbit tracking code such as SPRINT [10] or EpsSLICK [12] which delivers 3×3 spin transport matrices along particle trajectories. As explained in [8, 9, 10, 11], with stroboscopic averaging the ISF, $\hat{n}(u_0;s_0)$, at the starting positions u_0 and $s=s_0$ can be found in terms of multi–turn spin transfer matrices by taking the average

$$\vec{f}_N(u_0;s_0) \equiv \frac{1}{N+1}\sum_{k=0}^{N} R(u(s_0-kC);s_0,s_0-kC))\hat{n}_0(s_0)\ ,$$

for very large N and normalising this to unity: $\hat{n}(u_0;s_0) = \vec{f}_N(u_0;s_0)/|\vec{f}_N(u_0;s_0)|$. In this expression we have used notation similar to that in [8, eq.22] and chosen the "seed" spin field to be $\hat{n}_0$, although more general choices could be used [11].

The ITF can also be found by stroboscopic averaging:

$$g_N(u_0;s_0) \equiv \frac{1}{N+1}\sum_{k=0}^{N} R(u(s_0-kC);s_0,s_0-kC)\; T(s_0)\; R^{\mathrm{T}}(u(s_0-kC);s_0,s_0-kC)$$

where N is very large and $T(s_0)$ is a fixed 3×3 symmetric matrix with zero trace. The ITF is obtained as $T^{\mathrm{I}}(u_0;s_0) \equiv g_N/\sqrt{Tr(g_N^2)}$.

Given the above periodicity condition defining the ITF it is tempting to try to express the ITF in terms of the ISF. An ansatz in terms of $\hat{n}$ fulfilling this condition is:

$$T^{\mathrm{I}} = \pm\sqrt{\frac{3}{2}}\left\{\hat{n}\hat{n}^{\mathrm{T}} - \frac{1}{3}I\right\} .$$

Moreover, using the relation $T_{ij} = \frac{1}{2}\sqrt{\frac{3}{2}}\left\{\langle \hat{s}_i\hat{s}_j + \hat{s}_j\hat{s}_i\rangle - \frac{4}{3}\delta_{ij}\right\}$ [3], we find that this form for T^{I} is precisely that required if $\rho^{\mathrm{eq}}(J,\phi;s)$ is to be able to represent an ensemble of spins in the eigenstate of $\hat{n}\cdot\vec{\mathfrak{J}}$ whose eigenvalue is +1, -1 or 0. This is also the form obtained numerically by stroboscopic averaging. If this ansatz is unique the equilibrium spin-1 state on a torus J is then completely defined just by the vector field $\hat{n}(J,\phi;s)$ and the scalars $P_{\mathrm{eq}}(J)$ and $\mathfrak{T}_{\mathrm{eq}}(J)$:

$$\rho^{\mathrm{eq}} \;=\; \left\{\frac{I}{3} + \frac{1}{2}P_{\mathrm{eq}}\,\hat{n}\cdot\vec{\mathfrak{J}} + \mathfrak{T}_{\mathrm{eq}}\sum_{i,j}(\hat{n}_i\hat{n}_j - \frac{1}{3}\delta_{ij})\frac{(\mathfrak{J}_i\mathfrak{J}_j + \mathfrak{J}_j\mathfrak{J}_i)}{2}\right\} .$$

For example, for the eigenvalues ± 1 of $\hat{n}\cdot\vec{\mathfrak{J}}$ we have $P_{\mathrm{eq}} = \pm 1$ and $\mathfrak{T}_{\mathrm{eq}} = 1/2$. If the eigenvalue is 0, $P_{\mathrm{eq}} = 0$ and $\mathfrak{T}_{\mathrm{eq}} = -1$.

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