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Bog

Itemize the Appendix

The 3rd paper should also go to PRSTAB!

Am I physicist? forget Husemoller! Sect 7.3 problem

Spin motion in storage rings: basic theory and exploiting tools from bundle theory

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K. Heinemann[b], D. P. Barber[a], J. A. Ellison[b] and Mathias Vogt[a]

[a] Deutsches Elektronen–Synchrotron, DESY, 22607 Hamburg, Germany

[b] Department of Mathematics and Statistics, The University of New Mexico, Albuquerque, New Mexico 87131, U.S.A.

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Abstract

We return to our study [BEH] of spin motion in storage rings by expanding our toolset. After basic chapters on spin-orbit motion and polarization fields we present a rather modern method, used in Dynamical-Systems theory, Differential Geometry and other fields, allowing us to clarify and extend the current theory of spin-orbit motion and polarization fields. A central part of this is what we call the "Method of Association" which provides a well defined infinite collection of "allowed" time evolution laws which of course include the two cases, denoted by $L_{\omega,A}, \tilde{L}_{\omega,A}$, of spin-orbit motion and polarization fields respectively. Most importantly we extract from this collection an infinite subset of time evolution laws denoted by $\acute{L}_{H,\omega,A}$ which give us new insights into old notions, such as the invariant spin field, the invariant frame field and the spin-orbit resonance. Underlying the ~~Method~~ Technique of Association is a principal bundle which coordinates the dynamics as well as the transformation behavior of the theory. The time evolution laws $\acute{L}_{H,\omega,A}$ are especially natural and well understood since they originate in the reduction theory of the underlying principal bundle. A unifying byproduct of the reduction theory is the insight that invariant spin fields and spin-orbit resonances are essentially the same phenomenona. The Method of Association is very general and versatile and it therefore has potential for other parts of Accelerator Physics, e.g., nonlinear beam dynamics. For readers not well versed in bundle theory we believe it is easier and more direct to develop the ideas first in an elementary way. Only in the final chapter do we put the work into the context of principal bundles. Our use of principal bundles comes from Dynamical-Systems and Ergodic Theory as a part of the so-called Zimmer program developed in the 1980s and it differs from Yang-Mills theory so that a brief comparison is made as well.

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Contents

1 Introduction

This work, which is a sequel to [BEH], is based largely on mathematical concepts and ideas in the PhD Thesis [He2] of the first author (KH), where a method from the theory of dynamical systems is exploited to distil some essential features of spin motion in storage rings. As to be seen in Chapters 7 and ?? this method clarifies and considerably extends the current theory of [BEH].

In [BEH] we undertook an extensive study of the concept of spin tune in storage rings based on the Thomas–Bargmann–Michel–Telegdi (T–BMT) equation [Ja] of spin precession. For this we assumed that the orbital motion was independent of the spin, and was described by an integrable Hamiltonian system in action-angle variables, J, ϕ. We further assumed that the electric and magnetic fields were smooth (of class C^1, i.e., continuously differentiable) both in ϕ and θ. Thus the T–BMT equation became a linear system of ordinary differential equations (ODE) for the spin-orbit motion with smooth coefficients depending quasiperiodically on θ. This quasiperiodic structure led us to a generalization of the Floquet theorem and a new approach to the spin tune.

Although accelerator physicists tend to concentrate on studying spin-orbit motion in real storage rings, many of the issues surrounding the spin tune and the so-called invariant spin field depend just on the *structure* of the equations of spin-orbit motion and can be treated in isolation from the original physical system. This is the strategy to be adopted here and it clears the way for the focus on purely mathematical matters and in particular for the exploitation of methods from the theory of dynamical systems and the theory of principal bundles.

The history of principal bundles began in the 1930's motivated by the necessities of Algebraic Topology. The notion of the principal bundle then subsequently penetrated other areas of Mathematics as for example Differential Geometry and Dynamical-Systems and Ergodic Theory. With the advent in 1954 of the gauge theories of the Yang-Mills type, principal bundles gradually began to play a role in Particle Physics and since the 1970s they are indispensible tools, in particular for investigations of quantization and renormalization. [CB, DV, Na2, NS, Sc]. Our use of principal bundle theory is that from Dynamical-Systems and Ergodic Theory loosely characterized as a part of the so-called Zimmer program. For details and similarities with applications of the Yang-Mills type, see Chapter ??.

time formalism. Moreover in Section 2.1 we introduce the torus $\mathbb{T}^d$. In Section 2.2 we define the set $\mathcal{SOT}$ of spin-orbit tori and the notion of spin transfer matrix of a spin-orbit torus. In Section 2.3 we introduce the important notion of left group action. The transformation theory of spin-orbit tori, which partitions the set of spin-orbit tori into equivalence classes, is introduced in Chapter 3 and will turn out in Chapter ?? to come from $SO(3)$ gauge transformations of a principal bundle. In Section 3.3 we introduce the important notion of right group action. Moreover, since spin tunes describe constant rates of precession in appropriate reference frames, special spin-orbit tori are needed which can be reached by transforming from the original spin-orbit tori to such frames. This is handled in Chapter 4 and leads to the definition of the subsets $\mathcal{ACB}, \mathcal{CB}$ of $\mathcal{SOT}$ and of spin tunes and spin-orbit resonances of first kind. Furthermore at the end of Chapter 4, by using the transformation theory of spin-orbit tori, the concept of H normal form of spin-orbit tori is defined for every subgroup H of $SO(3)$. In the special case $H = SO(2)$ this leads us to the definition of the invariant frame field (IFF). In Chapter 5 we define spin and polarization fields and these lead to the definition of the invariant spin field (ISF). Chapter 6 covers Theorem 6.1 which addresses the issue of how many ISF's exist. Note that Theorem 6.1 is one of several results for which we don't have space to prove them but whose proofs can be found in [He2]. We then derive from Theorem 6.1 a practically important estimation formula for the maximum polarization of a bunch. Then, in Chapter 7 we generalize and revisit the studies of the previous chapters by using what we call the "Technique of Association" (ToA). This method provides a well defined infinite collection of "allowed" time evolution laws which include the two laws of Chapters 2 and 5 and the new evolution laws $\acute{L}_{H,\omega,A}$. In fact the latter gives us in Section 7.4 new insights into SORs, IFFs and ISFs. While the methods of Chapter 7 are as elementary as the ones in the previous chapters, in Chapter ?? we will use the more subtle tools of bundle theory allowing us further insight into Chapter 7. In particular we will identify in Chapter ?? a principal bundle, λ_d, which underlies the ToA and therefore is of fundamental importance for the theory of spin-orbit tori. Of special importance in Chapter ?? is the reduction theory of λ_d since it gives, via the time evolution laws $\acute{L}_{H,\omega,A}$, further insights into the results of Section 7.4 about the notions of invariant spin field, IFF and spin-orbit resonance. Moreover the transformation rules of spin-orbit motion and polarization fields of Chapters 2 and 5 turn out to originate in a $SO(3)$ gauge transformation rule of λ_d. In Appendix A we introduce the basic analytic notions like continuous and identifying functions, homeomorphisms and lifts.

2 Spin-orbit tori

2.1 Deriving the discrete-time spin-orbit motion from the continuous time spin-orbit motion

We begin our study by deriving our discrete-time ~~dynamical~~ spin-orbit motion from a continous-time initial value problem (IVP) which takes the form

$$\frac{d\phi}{d\theta} = \omega \, , \qquad \phi(0) = \phi_0 \in \mathbb{R}^d \, , \tag{2.1}$$

Note that $\Phi_{\omega,\mathcal{A}}$ is $SO(3)$-valued, since the values of $\mathcal{A}$ are real skew–symmetric 3×3 matrices, where $SO(3)$ is the set of real 3×3–matrices R for which $R^t R = I$ and $\det(R) = 1$. By adding the parameter vector ϕ_0 in Cronin's proof, and using the fact that $\mathcal{A}(\theta;\phi)$ is continuous in ϕ, we conclude from (2.7) that $\Phi_{\omega,\mathcal{A}} \in \mathcal{C}(\mathbb{R}^{d+1}, SO(3))$ where $\mathcal{C}(\mathbb{R}^{d+1}, SO(3))$ is the set of continuous functions from $\mathbb{R}^{d+1}$ into $SO(3)$ (see also Appendix A). Furthermore $\Phi_{\omega,\mathcal{A}}(\theta,\phi)$ is 2π-periodic in the components of ϕ. Using (2.3) and (2.6), the solution of the IVP (2.1, 2.2) can now be written as

$$\begin{pmatrix} \phi(\theta) \\ S(\theta) \end{pmatrix} = \varphi(\theta;\phi_0,S_0)\ , \tag{2.8}$$

where the function $\varphi \in \mathcal{C}(\mathbb{R}^{d+4},\mathbb{R}^{d+3})$, is defined by

$$\varphi(\theta,\phi,S) := \begin{pmatrix} \phi + \omega\theta \\ \Phi_{\omega,\mathcal{A}}(\theta,\phi)S \end{pmatrix} . \tag{2.9}$$

The PM on $\mathbb{R}^{d+3}$ is defined by

Starting at $\theta = 0$

$$\varphi(2\pi,\phi,S) = \begin{pmatrix} \phi + 2\pi\omega \\ \Phi_{\omega,\mathcal{A}}(2\pi;\phi)S \end{pmatrix} . \tag{2.10}$$

With (2.10) the Poincaré map $\varphi(2\pi,\cdot)$ is determined by the parameters ω and $\Phi_{\omega,\mathcal{A}}(2\pi;\cdot)$. Since ϕ is an angular variable and since we will have to prove many analytic properties down the road it is very convenient to replace $\mathbb{R}^d$ by the torus $\mathbb{T}^d$. Have replaced cont. time with ~~a 1 turn map~~ discrete time.

Space

Remark:

(1) Following Appendix A we define $\mathbb{T}^d$ by

$$\mathbb{T}^d := \{[\phi]_d : \phi \in \mathbb{R}^d\}\ , \tag{2.11}$$

where $[\phi]_d \subset \mathbb{R}^d$ is defined by

$$[\phi]_d := \{\phi + 2\pi n : n \in \mathbb{Z}^d\}\ , \tag{2.12}$$

Really? this is the opposite. Surely

with $\phi \in \mathbb{R}^d$. So ~~Note that~~ $[\phi]_d$ is actually ~~is also known as~~ $\phi \bmod 2\pi$. Since we have to deal with continuous functions which have the domain $\mathbb{T}^d$ we need a topology on $\mathbb{T}^d$. To equip $\mathbb{T}^d$ with its natural topology we define the surjection $\pi_d : \mathbb{R}^d \to \mathbb{T}^d$ by

$$\pi_d(\phi) := [\phi]_d = \{\phi + 2\pi n : n \in \mathbb{Z}^d\}\ , \tag{2.13}$$

whence a subset M of $\mathbb{T}^d$ is open iff $\pi_d^{-1}(M)$ is open in $\mathbb{R}^d$. By Appendix A, the natural topology of $\mathbb{T}^d$ is the unique topology for which the function π_d is identifying. Of course an identifying function is continuous (but not vice versa).

As pointed out in Appendix A, the above method of equipping $\mathbb{T}^d$ with its natural topology is a special case of a general method applied repeatedly ~~time and again~~ in this work. In fact this method is based on the notion of right group space (see Definition 3.5 in Section 3.3). □

does a physicist need this?

where $\mathcal{P}_\omega \in \mathcal{C}(\mathbb{T}^d, \mathbb{T}^d)$ is defined by

$$\mathcal{P}_\omega([\phi]_d) := [\phi + 2\pi\omega]_d \, . \tag{2.20}$$

We call $\mathcal{P}_{\omega,A}$ the "1-turn map of (ω, A)". Note that the continuity of $\mathcal{P}_\omega$ can be shown by the method of Remark 2 above whence $\mathcal{P}_{\omega,A}$ is continuous too.

Of course, in the special case where the spin-orbit torus (ω, A) belongs to $\mathcal{SOT}_{cont}$ the 1-turn map of (ω, A) carries the data of the PM, i.e., $\mathcal{P}_{\omega,A}(z, S) = \mathcal{P}_{\omega,A_{\omega,A}}(z, S)$.

We find it convenient to work in the more general setting of $\mathcal{SOT}(d,\omega)$ and (2.19) than with the special setting of $\mathcal{SOT}_{cont}(d,\omega)$. However the main physical interest is in a small subset of $\mathcal{SOT}_{cont}(d,\omega)$. There is a natural mathematical question: given (ω, A) in $\mathcal{SOT}(d,\omega)$, does it belong to $\mathcal{SOT}_{cont}(d,\omega)$? This is an analogue of the following question from beam dynamics: given a symplectic map, can it be generated as the one-turn map of a Hamiltonian system? We do not deal with this question. However as is shown in Section 7.2 of [He2] by using simple arguments from Homotopy Theory and by defining $A \in \mathcal{C}(\mathbb{T}, SO(3))$ by

$$A([\phi]_1) := \begin{pmatrix} \cos m\phi & -\sin m\phi & 0 \\ \sin m\phi & \cos m\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{2.21}$$

one can show, for every real ω, that $(\omega, A) \in \mathcal{SOT} \setminus \mathcal{SOT}_{cont}$, iff m is odd. The continuity of A in (2.21) can be shown by the method of Remark 2 above.

It is an easy exercise to show that for $(\omega, A), (\omega', A') \in \mathcal{SOT}(d)$

$$\mathcal{P}_{\omega',A'} \circ \mathcal{P}_{\omega,A} = \mathcal{P}_{\omega+\omega',A''} \, , \tag{2.22}$$

where $A'' \in \mathcal{C}(\mathbb{T}^d, SO(3))$ is defined by

$$A'' := (A' \circ \mathcal{P}_\omega)A \, , \tag{2.23}$$

whence

$$\mathcal{P}_{\omega,A} = \mathcal{P}_{\omega,A_{d,0}} \circ \mathcal{P}_{0,A} \, , \tag{2.24}$$

so that the inverse, $\mathcal{P}^{-1}_{\omega,A}$, of $\mathcal{P}_{\omega,A}$ is given by

$$\mathcal{P}^{-1}_{\omega,A} = \mathcal{P}_{0,A^t} \circ \mathcal{P}_{-\omega,A_{d,0}} \, , \tag{2.25}$$

where $A_{d,0} \in \mathcal{C}(\mathbb{T}^d, SO(3))$ is defined by $A_{d,0}(z) := I_{3\times 3}$. Thus $\mathcal{P}^{-1}_{\omega,A} \in \mathcal{C}(\mathbb{T}^d \times \mathbb{R}^3, \mathbb{T}^d \times \mathbb{R}^3)$ whence $\mathcal{P}_{\omega,A} \in Homeo(\mathbb{T}^d \times \mathbb{R}^3)$. As can be understood from Appendix A, $Homeo(\mathbb{T}^d \times \mathbb{R}^3)$ denotes the set of homeomorphisms on $\mathbb{T}^d \times \mathbb{R}^3$. Note also that since $\mathcal{P}_{-\omega}$ is the inverse of $\mathcal{P}_\omega$ we conclude that $\mathcal{P}_\omega \in Homeo(\mathbb{T}^d)$.

Clearly $\mathcal{P}^2_{\omega,A}(z, S) = (\mathcal{P}_{\omega,A} \circ \mathcal{P}_{\omega,A})(z, S) = \mathcal{P}_{\omega,A}(\mathcal{P}_\omega(z), A(z)S)$ $= (\mathcal{P}_\omega(\mathcal{P}_\omega(z)), A(\mathcal{P}_\omega(z))A(z)S) = (\mathcal{P}^2_\omega(z), A(\mathcal{P}_\omega(z))A(z)S)$ and the iterates of the 1-turn map $\mathcal{P}_{\omega,A}$ must have the form

$$\mathcal{P}^n_{\omega,A}(z, S) = \begin{pmatrix} L_\omega(n; z) \\ \Psi_{\omega,A}(n; z)S \end{pmatrix} , \tag{2.26}$$

and $(\mathbb{T}^d \times \mathbb{R}^3, L_{\omega,A})$ are examples of the important concept of a $\mathbb{Z}$ space which we define in the next section in a more general setting; a setting which will be needed later.

We now introduce some more terminology. We call the n-th iterate $\mathcal{P}^n_{\omega,A} = L_{\omega,A}(n;\cdot)$ the "n-turn map of (ω, A)" and we call $\Psi_{\omega,A}$ the "spin transfer matrix" of (ω, A). Using the standard topology on $\mathbb{T}^d$ (see Appendix A) and the standard topology on $\mathbb{Z}$ (for the latter see also Section 2.3) a function on $\mathbb{Z} \times \mathbb{T}^d$ is continuous if it is continuous in the second argument. Since $\Psi_{\omega,A}(n;\cdot)$ is continuous, every spin transfer matrix is a continuous function. We call $\Psi_{\omega,A}(n;\cdot)$ the "n-turn spin transfer matrix" of (ω, A). Clearly

$$\Psi_{\omega,A}(1; z) = A(z) , \tag{2.35}$$

which justifies calling A the 1-turn spin transfer matrix.

For $n \in \mathbb{Z}, z_0 \in \mathbb{T}^d, S_0 \in \mathbb{R}^3$ we define the function $S : \mathbb{Z} \to \mathbb{R}^3$ by

$$S(n) := \Psi_{\omega,A}(n; z_0)S_0 , \tag{2.36}$$

and call S a "spin trajectory of (ω, A) over z_0". It follows from (2.33), (2.36) that

$$S(n+1) = A(L_\omega(n; z_0))S(n) , \qquad S(0) = S_0 \in \mathbb{R}^3 . \tag{2.37}$$

Note that (2.37) is a linear, non-autonomous IVP for S.

Clearly the behavior of the spin trajectories in (2.37) depends on the values of A reached by the orbital ~~dynamics~~ $L_\omega(n; z_0)$ in its argument which in turn depends on the value of ω. Since the domain of A is the torus, for some values of ω the argument will remain in a confined subset of the torus and for other values it can cover the torus densely. To be more precise we define resonance. We say $\chi \in \mathbb{R}^n$ is *resonant* if there exists a non-zero integer vector $k \in \mathbb{Z}^n$ such that $k \cdot \chi = 0$ and nonresonant if not resonant. If $(1, \omega)$ is nonresonant then the argument of A covers the torus densely and since A is continuous all values of A affect the spin trajectory. Whereas if $(1, \omega)$ is resonant the values of A are only sampled by its values on a sub-torus. The spin-orbit torus (ω, A) is said to be "off orbital resonance" if $(1, \omega)$ is nonresonant and "on orbital resonance" otherwise. Thus spin trajectories may exhibit significantly different qualitative behaviors on and off orbital resonance.

Let $\phi_0 \in \mathbb{R}^d$ and $(\omega, \mathcal{A}) \in \widetilde{SOT}(d, \omega)$ and let S solve the EOM (2.4) $dS/d\theta = \mathcal{A}(\theta, \phi_0 + \omega\theta)S$. Defining the function $\hat{S} : \mathbb{Z} \to \mathbb{R}^3$ by $\hat{S}(n) := S(2\pi n)$ we observe that $\hat{S}$ is a spin trajectory over $z_0 = [\phi_0]_d$ of $(\omega, A_{\omega,\mathcal{A}})$. Moreover

$$\Psi_{\omega,A_{\omega,\mathcal{A}}}(n; [\phi_0]_d) = \Phi_{\omega,\mathcal{A}}(2\pi n; \phi_0) . \tag{2.38}$$

2.3 Left group actions and cocycles

With Section 2.2 complete, it is appropriate to introduce some standard definitions which will be crucial for the remainder of this work.

Definition 2.1 *(Group)*
A "group" is a pair $(G, *)$ where G is a set and $*$ is a binary operation on G, i.e., a function $* : G \times G \to G$ such that

(G1) (Associativity) $\qquad \forall_{g_1,g_2,g_3 \in G}\ (g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$, Any left G set can be loosely called a dynamical system or a transformation group. Moreover any left $\mathbb{Z}$ set can be loosely called a discrete time dynamical system and any left $\mathbb{R}$ set can be loosely called a continuous time dynamical system.

It is clear by (2.32) that L_ω is a left $\mathbb{Z}$ action on $\mathbb{T}^d$ whence $(\mathbb{T}^d, L_\omega)$ is a left $\mathbb{Z}$ set. Since the group $\mathbb{Z}$ is Abelian, L_ω is a $\mathbb{Z}$ action on $\mathbb{T}^d$ and $(\mathbb{T}^d, L_\omega)$ is a $\mathbb{Z}$ set. Analogously it follows from (2.30) that $L_{\omega,A}$ is a left $\mathbb{Z}$ action (and a $\mathbb{Z}$ action) on $\mathbb{T}^d \times \mathbb{R}^3$ and that $(\mathbb{T}^d \times \mathbb{R}^3, L_{\omega,A})$ is a left $\mathbb{Z}$ set (and a $\mathbb{Z}$ set). Clearly the $\mathbb{Z}$ set $(\mathbb{T}^d \times \mathbb{R}^3, L_{\omega,A})$ carries valuable information about (ω, A). Further below we will see more $\mathbb{Z}$ sets carrying valuable information about (ω, A). In particular in Chapter 7 we will meet a factory for those $\mathbb{Z}$ sets called the ToA. Note that if L is $\mathbb{Z}$ action on E then

$$Fix(L) = \bigcap_{n\in\mathbb{Z}} \{x \in E : L(n;x) = x\} = \{x \in E : L(1;x) = x\} . \qquad (2.41)$$

END NEW

There are many books which cover group actions. Two books, dedicated exclusively to group actions, are [tDi1, Ka].

In this work we are often interested in left G sets where G and E have a topology and L is continuous. This is formalized in the following definition.

NEW

Definition 2.4 *(Left G space)*
Let E be a topological space where E is nonempty, G be a topological group, and let L be a left G action on E with L being continuous where $G \times E$ carries the product topology.

DO PHYSICITS EVER NEED SETS E WHICH ARE NONEMPTY(????????????)

Then the pair (E, L) is called a "left G space". Note that each $L(g;\cdot)$ is a homeomorphism. Recalling from Definition (2.3) the notation E/L and p_L, we equip E/L with its natural topology, i.e., a subset M of E/L is open iff $p_L^{-1}(M)$ is open in E. Thus the function p_L is onto E/L and identifying and one calls E/L an "orbit space". Also each orbit is equipped with the relative topology from E. In the important subcase when the topology of G is discrete (e.g., when $G = \mathbb{Z}$) the condition that L is continuous is equivalent to $L(g;\cdot)$ being continuous for all $g \in G$. The definition of "fixed point" is the same as for left G sets.

If (E, L) is a left G space and if the group G is Abelian then we also call (E, L) a "G space". □

Where is G in the notation?

Since, by (2.27), $L_\omega(n;\cdot)$ is continuous it is clear that the $\mathbb{Z}$ set $(\mathbb{T}^d, L_\omega)$ is a left $\mathbb{Z}$ space (and therefore a $\mathbb{Z}$ space). Recalling that $\Psi_{\omega,A}$ is continuous, it is equally clear by (2.28) that $L_{\omega,A}(n;\cdot)$ is continuous whence the $\mathbb{Z}$ set $(\mathbb{T}^d \times \mathbb{R}^3, L_{\omega,A})$ is a left $\mathbb{Z}$ space (and therefore a $\mathbb{Z}$ space).

END NEW

The spin transfer matrix is an example of a cocycle and (2.33) is the cocycle condition. Cocycles are central to our work and we now provide a definition.

Definition 2.6 *(Cocycle)*
Let G, K ~~be topological groups~~ and let (E, L) be a left G space. Then a function $f \in C(G \times E, K)$ is called a "K cocycle over the left G space (E, L)" if, for $g, g' \in G, x \in E$,

$$f(gg';x) = f(g; L(g';x))f(g';x) . \qquad (2.42)$$

Notation: $f \to K$, $k \in K$

Why not an "f cocycle"?

This gives a partition of $\mathcal{SOT}(d,\omega)$ as we formalize in the next two definitions.

Definition 3.1 *(Transformation rule of spin-orbit tori)*
Let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d,\omega)$ then a T in $\mathcal{C}(\mathbb{T}^d, SO(3))$ is called a "transfer field from (ω, A) to (ω, A')" iff (3.1) holds. We also say that "(ω, A') is the transform of (ω, A) under T". We denote the collection of all transfer fields from (ω, A) to (ω, A') by $\mathcal{TF}_{d,\omega}(A, A')$. □

Clearly $\mathcal{TF}_{d,\omega}(A, A') \neq \emptyset$ iff (ω, A') is a transform of (ω, A). To obtain an equivalence relation on $\mathcal{SOT}(d,\omega)$ we note, by (2.22), that

$$
\begin{aligned}
\mathcal{P}_{0,A_{d,0}} &= id_{\mathbb{T}^d \times \mathbb{R}^3} \, , \\
\mathcal{P}_{0,T}\mathcal{P}_{0,T'} &= \mathcal{P}_{0,TT'} \, , \\
\mathcal{P}_{0,T}^{-1} &= \mathcal{P}_{0,T^t} \, ,
\end{aligned}
\tag{3.3}
$$

where $T, T' \in \mathcal{C}(\mathbb{T}^d, SO(3))$. This leads us to:

Definition 3.2 *Let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d,\omega)$. Then we write $(\omega, A) \sim_{d,\omega} (\omega, A')$ and say that $(\omega, A), (\omega, A')$ are "similar" iff (ω, A') is a transform of (ω, A), i.e., a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists which satisfies (3.1). Note that, by (3.1), (3.3) it is easy to show that $\sim_{d,\omega}$ is an equivalence relation on $\mathcal{SOT}(d,\omega)$. We denote the equivalence class of (ω, A) by $\overline{(\omega, A)}$.* □

Since $\sim_{d,\omega}$ is an equivalence relation on $\mathcal{SOT}(d,\omega)$, its equivalence classes give a partition of $\mathcal{SOT}(d,\omega)$. Each (ω, A) in a given equivalence class gives rise to similar dynamics since many properties are shared as explained in Remarks 1 and 2 below. This is important for spin-orbit tori since it suggests that, given an (ω, A) in $\mathcal{SOT}(d,\omega)$, one should look for the "simple" elements of $\overline{(\omega, A)}$. See Chapter 4) too.

Remarks:

(1) Two spin-orbit tori which are similar share many important properties, e.g., the existence or nonexistence of an ISF as in Chapter 5. We will see other properties shared by similar spin-orbit tori below.

(2) We observe by (2.37) and (3.2) that if $T \in \mathcal{TF}_{d,\omega}(A, A')$ and if S is a spin-orbit trajectory of (ω, A) over, say z_0, then S', defined by

$$
S'(n) := T^t(L_\omega(n; z_0))S(n) \, , \tag{3.4}
$$

is a spin trajectory of (ω, A') over z_0.

(3) It is important to note that the transformation rule outlined in (3.4) is no stranger to the polarized beam community. In fact when researchers deal with the topics of spin tune, spin frequency, spin resonances, resonance strengths etc. then they often appeal more or less directly to the above transformation rule. However our aim here is to deal with rather fundamental aspects, and their physical implications, which are usually either not addressed or not addressed in such an explicit form. □

the $\mathbb{Z}$ space $(\mathbb{T}^d \times \mathbb{R}^3, L_{\omega,A})$ is a so-called "skew product" over the $\mathbb{Z}$ space $(\mathbb{T}^d, L_\omega)$. Note that this skew product structure has its origin in the fact that the first component of $L_{\omega,A}$ in (2.28) is independent of S and that the second component of $L_{\omega,A}$ is linear in S. It is easy to see that $\Psi_{\omega,A}$ being a cocycle is equivalent to $(\mathbb{T}^d \times \mathbb{R}^3, L_{\omega,A})$ being a skew product. For more details on skew products and their relation to cocycles see, e.g., [HK1]. $\square$

It follows from (2.33),(3.2) and by induction in n that, if $T \in \mathcal{TF}_{d,\omega}(A, A')$, then the spin transfer matrices of $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$ are related by

$$\Psi_{\omega,A'}(n; z) = T^t(L_\omega(n; z))\Psi_{\omega,A}(n; z)T(z) \ . \tag{3.9}$$

Recall from Section 2.3 that $\Psi_{\omega,A}$ and $\Psi_{\omega,A'}$ are cocycles. Then (3.9) implies that the cocycles $\Psi_{\omega,A}$ and $\Psi_{\omega,A'}$ are "cohomologous" For this notion, see, e.g., [HK1, He2, KR, Zil]. Eq. (3.9) once again shows that our transformation rule of spin-orbit tori is very natural.

3.3 Right group actions and topological G-maps of right G spaces

The transformation rule of Definition 3.1 for spin-orbit tori can be formalized in terms of the function $R_{d,\omega} : \mathcal{C}(\mathbb{T}^d, SO(3)) \times \mathcal{SOT}(d, \omega) \to \mathcal{SOT}(d, \omega)$ defined by

$$R_{d,\omega}(T; \omega, A) := \Big(\omega, (T^t \circ \mathcal{P}_\omega)AT\Big) \ , \tag{3.10}$$

i.e., $R_{d,\omega}(T; \omega, A) = (\omega, A')$ where $A' \in \mathcal{C}(\mathbb{T}^d, SO(3))$ is given by (3.2). According to the following definition, $R_{d,\omega}$ is a right $\mathcal{C}(\mathbb{T}^d, SO(3))$ action on $\mathcal{SOT}(d, \omega)$.

NEW

Really needed ?

Definition 3.4 *(Right G action, right G set)*
Let G be a group with identity element e_G and let E be a nonempty set. Then a function $R : G \times E \to E$ is called a "right G action on E" if, for $g_1, g_2 \in G, x \in E$,

what about G

$$R(e_G; x) = x \tag{3.11}$$
$$R(g_1g_2; x) = R(g_2; R(g_1; x)) \ . \tag{3.12}$$

If R is a right G action on E then the pair (E, R) is called a "right G set". Since $R(e_G; x) = R(g; L(g^{-1}; x))$ each $R(g; \cdot)$ is a bijection, i.e., $R(g; \cdot) \in \mathrm{Bij}(E)$. If (E, R) is a right G set and $x \in E$ then the set $\{R(g; x) : g \in G\}$ is called the "orbit of x under R". We denote the set of orbits by E/R and define the function $p_R : E \to E/R$ by $p_R(x) := \{R(g; x) : g \in G\}$. Thus $p_R(x)$ the orbit of x under R. Note that the orbits form a partition of E. If the orbit of x only consists of x itself then x is called a "fixed point" of (E, R) and of R and the set of fixed points of (E, R) is denoted by $Fix(R)$. Note that as in Definition 2.3, this notion of "orbit" is different from the physical notion of "orbital motion".

One calls R "free" if, for all $x \in E$, the function $R(\cdot; x)$ is one-one (and then it is a bijection from G onto the orbit of x). For the definition of one-one, see Appendix A.

A right G set (E, R) is called "trivial" if every $R(g, \cdot)$ is the identity function on E and one then calls (E, R) the "trivial right G set over E".

mention G

being continuous for all $g \in G$. The definition of "fixed point" is the same as for right G sets.

A right G space (E, R) is called "trivial" if it is trivial as a right G set, i.e., if every $R(g, \cdot)$ is the identity function on E. One then calls (E, R) the "trivial right G space over E".

If (E, R) is a right G space and if the group G is Abelian then we call (E, R) a "G space". Thus if G is an Abelian group the notions right G space, left G space, and G space are synonymous. □

Very important right G spaces are considered in Chapter 7 and in the context of principal bundles in ??. In fact principal G-bundles are right G spaces with some extra structure.

The following definition is the right analogue to Definition 3.3.

Definition 3.6 *(G-maps of right G sets, topological G-maps of right G spaces)*
a) Consider ~~Let G be a group~~ and let $(E, R), (E', R')$ be right G sets and let $f : E \to E'$ ~~be a function.~~ Consider the function
If f satisfies, for $g \in G, x \in E$,

$$f(R(g; x)) = R'(g; f(x)) , \tag{3.14}$$

then f is called a "G-map from (E, R) to (E', R')".
b) Let G be a topological group. Let $(E, R), (E', R')$ be right G spaces and let $f \in C(E, E')$. If f satisfies (3.7) then f is called a "topological G-map from (E, R) to (E', R')". □

If f is a G-map from the right G set (E, R) to the right G set (E', R') and if f is a bijection onto E' then f is a G-map from (E', R') to (E, R) and in this situation the G sets (E', R') and (E, R) said to be are called "isomorphic" and thus are effectively the same. We then also say that R' and R are "isomorphic". Analogously, when f is a topological G-map from the right G space (E, R) to the right G space (E', R') and if f is a homeomorphism onto E' then (E', R') and (E, R) are called "isomorphic" and thus are effectively the same.

END NEW

Remark:

(6) We here give a simple example of a G-map of right group sets.

Let $\omega \in \mathbb{R}^d$ and let the function $f : \mathcal{SOT}(d, \omega) \to \mathcal{SOT}(d, -\omega)$ be defined by

$$f(\omega, A) := \Big(-\omega, A^t \circ \mathcal{P}_{-\omega}\Big) . \tag{3.15}$$

Then, by (3.10),

$$f\Big(R_{d,\omega}(T; \omega, A)\Big) = f\Big(\omega, (T^t \circ \mathcal{P}_\omega)AT\Big) = \Big(-\omega, (T^t \circ \mathcal{P}_{-\omega})(A \circ \mathcal{P}_{-\omega})T\Big) ,$$

$$R_{d,-\omega}(T; f(\omega, A)) = R_{d,-\omega}\Big(T; -\omega, A^t \circ \mathcal{P}_{-\omega}\Big)$$

$$= \Big(-\omega, (T^t \circ \mathcal{P}_{-\omega})(A^t \circ \mathcal{P}_{-\omega})T\Big) ,$$

the group H is. Arguably the simplest elements in every $\mathcal{SOT}(d,\omega)$ are those (ω, A) with constant A. We therefore introduce the sets

$$\begin{aligned} \mathcal{SOT}_H^{const}(d,\omega) &:= \{(\omega, A) \in \mathcal{SOT}_H(d,\omega) : A(z) \text{ is independent of } z\}\,, \\ \mathcal{SOT}_H^{const} &:= \bigcup_{d\in\mathbb{N},\omega\in\mathbb{R}^d} \mathcal{SOT}_H^{const}(d,\omega)\,, \end{aligned} \tag{4.1b}$$

The most extreme case is, of course, when H is the trivial group. These sets will now be condidered in Sections 4.1 and 4.2.

means? i.e. $\{I_{3\times 3}\}$

4.1 The subsets $\mathcal{ACB}$ and $\mathcal{CB}$ of $\mathcal{SOT}$

As mentioned above, it seems sensible to look for the simplest elements, of $\overline{(\omega, A)}$, for (ω, A) in $\mathcal{SOT}_{SO(3)}^{const}(d,\omega)$. So we collect these into the set $\bigcup_{(\omega,A)\in\mathcal{SOT}_{SO(3)}^{const}(d,\omega)} \overline{(\omega, A)}$.

An obvious subset of $\mathcal{SOT}_{SO(3)}^{const}$ is $\mathcal{SOT}_{SO(2)}^{const}$ where the subgroup $SO(2)$ of $SO(3)$ is defined by

Not yet H

$$\begin{aligned} SO(2) &:= \left\{ \begin{pmatrix} \cos(x) & -\sin(x) & 0 \\ \sin(x) & \cos(x) & 0 \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \\ &= \{\exp(x\mathcal{J}) : x \in \mathbb{R}\} = \{\exp(x\mathcal{J}) : x \in [0, 2\pi)\}\,, \end{aligned} \tag{4.2}$$

with

$$\mathcal{J} := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{4.3}$$

In fact if $r \in SO(3)$ then there exist $\nu \in [0,1)$ and $W \in SO(3)$ such that

$$r = W^t \exp(2\pi\nu\mathcal{J})W\,, \tag{4.4}$$

and ν is uniquely determined by r. See, e.g., Lemma 2.1 of [BEH]. Thus we define, for every $\nu \in [0,1)$ and $d \in \mathbb{N}$, the constant-valued function $A_{d,\nu} \in \mathcal{C}(\mathbb{T}^d, SO(2))$ by

$$A_{d,\nu}(z) := \exp(2\pi\nu\mathcal{J}) = \begin{pmatrix} \cos(2\pi\nu) & -\sin(2\pi\nu) & 0 \\ \sin(2\pi\nu) & \cos(2\pi\nu) & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \tag{4.5}$$

so that, by (4.1b) and (4.2),

$$\mathcal{SOT}_{SO(2)}^{const}(d,\omega) = \{(\omega, A_{d,\nu}) : \nu \in [0,1)\}\,. \tag{4.6}$$

It follows from (4.1b), (4.4), (4.5) and Definition 3.2, that for every (ω, A) in $\mathcal{SOT}_{SO(3)}^{const}(d,\omega)$, a $\nu \in [0,1)$ exists such that (ω, A) is similar to $(\omega, A_{d,\nu})$ whence, by (4.6),

$$\bigcup_{(\omega,A)\in\mathcal{SOT}_{SO(3)}^{const}(d,\omega)} \overline{(\omega, A)} = \bigcup_{(\omega,A)\in\mathcal{SOT}_{SO(2)}^{const}(d,\omega)} \overline{(\omega, A)} = \bigcup_{\nu\in[0,1)} \overline{(\omega, A_{d,\nu})}\,. \tag{4.7}$$

which ?

Recalling our earlier remark, Definition 4.1 tells us that $\mathcal{ACB}$ contains the most important spin-orbit tori in applications. However as explained in Section 7.6 in [He2] and Chapter ?? of the current work, it is easy to show that spin-orbit tori exist which are not in $\mathcal{ACB}$. (???? give details about why. E.g what about prop? ??????) In fact the current work is by no means limited to the spin-orbit tori in $\mathcal{ACB}$ (see especially Chapter 7).

???????IMPROVE????????

Figures 1 and 2 provide symbolic illustrations of these ideas. Thus $\mathcal{SOT}^{const}_{SO(2)}(d,\omega)$ is denoted by a circle whereby each point on the circle ~~is~~ of the form $(\omega, A_{d,\nu})$ and belongs to the equivalence class $\overline{(\omega, A_{d,\nu})} = \mathcal{ACB}_\nu(d,\omega)$, ~~shown~~ by a curve with multiple intersections on the circle. If (ω, A) is on orbital resonance the "curve" $\mathcal{ACB}_\nu(d,\omega)$ intersects the circle ~~on~~ at a finite number of ~~times~~ place corresponding to the orbital resonance multiplicity (see [DEH]) whereas if (ω, A) is off orbital resonance the "curve" $\mathcal{ACB}_\nu(d,\omega)$ intersects the circle at a dense and countably infinite set. (THIS PARA NEEDS FIXING WHEN THE FIGS ARE READY?????????)

represent a s-o torus

The left is redundant

Explain better Point & Sect 4.2

The ⊙ covers $\nu = 0 \to 1$

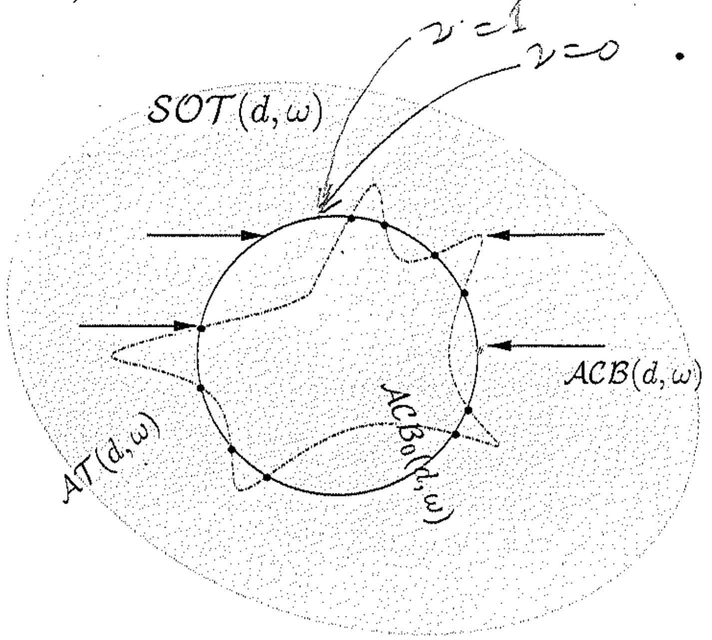

Figure 1: A symbolic representation of the relations between the sets $\mathcal{SOT}(d,\omega)$, $\mathcal{AT}(d,\omega)$, $\mathcal{ACB}(d,\omega)$ and $\mathcal{ACB}_0(d,\omega)$ defined in the text. The pink area represents a part of $\mathcal{SOT}(d,\omega)$ and the red, blue and green locii represent $\mathcal{AT}(d,\omega)$, $\mathcal{ACB}_0(d,\omega)$ and $\mathcal{ACB}(d,\omega)$ respectively. The $\mathcal{ACB}_0(d,\omega)$ crosses the $\mathcal{AT}(d,\omega)$ at.......

Note that every n-turn spin transfer matrix of an (ω, A) in $\mathcal{SOT}^{const}_{SO(3)}(d,\omega)$ is constant valued since, by (2.34), it satisfies

$$\Psi_{\omega,A}(n;\phi) = A^n \,. \tag{4.17}$$

Thus for every spin-orbit torus in $\mathcal{SOT}^{const}_{SO(3)}$ every n-turn spin transfer matrix is constant valued so that, by Definition 4.1, every spin-orbit torus in $\mathcal{ACB}$ is similar to a spin-orbit torus for which every n-turn spin transfer matrix is constant valued. This motivates our

?

(2) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$. The elements of $\mathcal{TF}^{const}_{SO(2)}(\omega, A)$ are the discrete-time analogues of the so-called uniform invariant frame fields introduced in [BEH]. Thus we abbreviate

$$\mathcal{UIFF}(\omega, A) := \mathcal{TF}^{const}_{SO(2)}(\omega, A) , \tag{4.20}$$

and we call every element of $\mathcal{UIFF}(\omega, A)$ a "uniform IFF of (ω, A)". By the remarks after (4.19), $\mathcal{UIFF}(\omega, A)$ is nonempty iff $(\omega, A) \in \mathcal{ACB}(d,\omega)$. The basis for the analogy is explained in Section 4.3. □

One equivalence class is especially important, namely $\mathcal{ACB}_0(d,\omega)$. As we shall see this is central to the discussion of spin-orbit resonance, whence the following definition is needed.

Definition 4.2 *($\mathcal{CB}(d,\omega), \mathcal{CB}$)*
We denote the set of those spin-orbit tori in $\mathcal{SOT}(d,\omega)$ which are similar to the trivial spin-orbit torus $(\omega, A_{d,0})$ by $\mathcal{CB}(d,\omega)$, i.e., by recalling Definition 4.1:

$$\begin{aligned}\mathcal{CB}(d,\omega) &:= \mathcal{ACB}_0(d,\omega) = \overline{(\omega, A_{d,0})} \\ &= \{R_{d,\omega}(T;\omega, A_{d,0}) : T \in \mathcal{C}(\mathbb{T}^d, SO(3))\} ,\end{aligned} \tag{4.21}$$

and we denote their union by $\mathcal{CB}$, i.e.,

$$\mathcal{CB} := \bigcup_{d\in\mathbb{N},\omega\in\mathbb{R}^d} \mathcal{CB}(d,\omega) . \tag{4.22}$$

Note that, by Definition 4.1, $\mathcal{CB} = \mathcal{ACB}_0$. The acronym $\mathcal{CB}$ comes from the language of cocycles and will be explained further below. □

Let $(\omega, A) \in \mathcal{CB}(d,\omega)$, thus by Definition 4.2 and (4.18) $\mathcal{TF}^{const}_{G_0}(\omega, A)$ is nonempty and every transfer field, T, in $\mathcal{TF}^{const}_{G_0}(\omega, A)$ satisfies

$$(T^t \circ \mathcal{P}_\omega)AT = A_{d,0} , \tag{4.23}$$

so that, by (2.34) and (4.5),

$$\Psi_{\omega,A}(n;z) = T(L_\omega(n;z))T^t(z) . \tag{4.24}$$

Eq. (4.24) motivates the acronym $\mathcal{CB}$ in Definition 4.2 since the spin transfer matrix $\Psi_{\omega,A}$ in (4.24) belongs to that class of cocycles which are called "coboundaries" (see [HK1, HK2]).

Remark:

(3) Definition 4.2 gives us another property shared by similar spin-orbit tori. In particular, it implies that if (ω, A) belongs to $\mathcal{CB}$ then every spin-orbit torus in $\overline{(\omega, A)}$ belongs to $\mathcal{CB}$. □

Recall that spin-orbit tori exist which are not in $\mathcal{ACB}$ (and therefore not in $\mathcal{CB}$). In fact the problem of deciding whether a given spin-orbit torus is in $\mathcal{ACB}$ or in $\mathcal{CB}$, is fruitful both theoretically and practically. In particular the techniques of Chapter 7 are meant to head us into the study of this general problem.(IMPROVE????????)

?

4.3, that $\nu' \in \Xi_1(\omega, A)$. A complete proof of (4.26), uses the tool of quasiperiodic functions, see Chapter 7.

An important implication of (4.26) is the following. If $\nu, \mu \in [0,1), \omega \in \mathbb{R}^d$ then, since by Definition 4.1, $\mathcal{ACB}_\mu(d,\omega)$ and $\mathcal{ACB}_\nu(d,\omega)$ are equivalence classes of $\sim_{d,\omega}$ they are either equal or disjoint. Moreover, it follows from (4.5), (4.26) and Definition 4.3 that if $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and $\nu \in \Xi_1(\omega, A)$ then only those $(\omega, A_{d,\mu})$ are similar to (ω, A) for which $\varepsilon \in \{1,-1\}, m \in \mathbb{Z}^d, n \in \mathbb{Z}$ exist such that $\mu = \varepsilon\nu + m\cdot\omega + n$. Thus, by Definition 4.1, for arbitrary $\nu, \mu \in [0,1)$ we have $\mathcal{ACB}_\mu(d,\omega) = \mathcal{ACB}_\nu(d,\omega)$ iff $\varepsilon \in \{1,-1\}, m \in \mathbb{Z}^d, n \in \mathbb{Z}$ exist such that $\mu = \varepsilon\nu + m\cdot\omega + n$.

IMPROVE

This is illustrated in Fig. 2 by points intersecting the curve with the circle. (IMPROVE??????)

WE MUST DECIDE ON THE NUMERING CONVENTION FOR REMARKS.

Remark:

(4) If one considers a family $(\omega(J), A_J)_{J\in\Lambda}$ of spin-orbit tori (see Chapters 5 and 6) and if every $(\omega(J), A_J)$ has a spin tune of the first kind, say ν_J, then ν_J is called an amplitude dependent spin tune (ADST). □

We need another definition.

Definition 4.4 *(Spin-orbit resonance of the first kind)*
Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$. We say that (ω, A) is on spin-orbit resonance of the first kind if $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and if for every $\nu \in \Xi_1(\omega, A)$ we can find $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ such that

$$\nu = m\cdot\omega + n\,. \tag{4.27}$$

We say that (ω, A) is "off spin-orbit resonance of first kind" iff $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and if (ω, A) is not on spin-orbit resonance of the first kind. Note that a spin-orbit torus which has no spin tunes of first kind is neither on nor off spin-orbit resonance of the first kind. □

Remarks:

(5) By (4.26) and Definition 4.4 an $(\omega, A) \in \mathcal{ACB}(d,\omega)$ is on spin-orbit resonance of the first kind iff (4.27) holds for some $m \in \mathbb{Z}^d, n \in \mathbb{Z}, \nu \in \Xi_1(\omega, A)$. Thus a single spin tune ν of (ω, A) of the first kind is sufficient to determine if (ω, A) is on spin-orbit resonance of the first kind. Note that, by (4.26) and Definition 4.4, a spin-orbit torus $(\omega, A) \in \mathcal{ACB}(d,\omega)$ is on spin-orbit resonance of the first kind iff $0 \in \Xi_1(\omega, A)$. Thus, by Definitions 4.2, a spin-orbit torus $(\omega, A) \in \mathcal{ACB}(d,\omega)$ is on spin-orbit resonance of the first kind iff $(\omega, A) \in \mathcal{CB}(d,\omega)$. Thus a spin-orbit torus is on a spin-orbit resonance of the first kind iff it is in $\mathcal{CB}$.

(6) As stated at the beginning of this chapter, the phenomenon of spin-orbit resonance can lead to instability of spin motion. For example it can lead to a large spread of the ISF, to be introduced in Chapter 5, and that can lead to unacceptably low equilibrium polarization. Furthermore, in Chapter 6 spin-orbit resonances of the first kind lead

$$= \left\{ T \in \mathcal{C}(\mathbb{T}^d, SO(3)) : T^t(\mathcal{P}_\omega(z))A(z)T(z) \in H \right\} . \tag{4.30}$$

Thus $(\omega, A) \in \mathcal{CB}_H(d,\omega)$ *iff* $\mathcal{TF}_H(\omega, A)$ *is nonempty.* □

We now make some remarks on Definition 4.5.

Remarks:

(8) Let (ω, A) be in $\mathcal{SOT}(d,\omega)$. It follows from Definitions 4.1, 4.2, 4.5 and (4.8) that (ω, A) has a G_0 normal form iff $(\omega, A) \in \mathcal{CB}(d,\omega)$. Therefore, by Definition 4.5,

$$\mathcal{CB}_{G_0}(d,\omega) = \mathcal{CB}(d,\omega) , \qquad \mathcal{CB}_{G_0} = \mathcal{CB} . \tag{4.31}$$

Thus the sets $\mathcal{CB}_H(d,\omega)$ are generalizations of the set $\mathcal{CB}(d,\omega)$ and the sets $\mathcal{CB}_H$ are generalizations of the set $\mathcal{CB}$. This circumstance motivates the acronym $\mathcal{CB}_H$.

(9) Definition 4.5 gives us another property shared by similar spin-orbit tori since it implies that if (ω, A) belongs to $\mathcal{CB}_H$ then every spin-orbit torus in $\overline{(\omega, A)}$ belongs to $\mathcal{CB}_H$.

(10) In Chapter 7 we link H normal forms with the exploitation of structural equations (see Proposition 7.3) and therefore connect them with the principal bundle λ_d. One practical aspect of this link is that the structural equations are a tool for finding a transfer field in $\mathcal{TF}_H(\omega, A)$.

(11) Let (ω, A) be in $\mathcal{SOT}(d,\omega)$ and H, H' be subgroups of $SO(3)$ such that $H' \subset H$. Then $\mathcal{CB}_{H'}(d,\omega) \subset \mathcal{CB}_H(d,\omega)$ and $\mathcal{CB}_{H'} \subset \mathcal{CB}_H$.

(12) Let (ω, A) be in $\mathcal{SOT}_H(d,\omega)$ where H is a subgroup of $SO(3)$. Recall that we equip H with the relative topology from $SO(3)$. Then, by Definition 2.6 and (2.34) the spin transfer matrix $\Psi_{\omega,A}$ of (ω, A) belongs to $COC(\mathbb{T}^d, L_\omega, H)$.

(13) We now discuss briefly the method of "rational approximation" to find an H normal form of a $(\omega, A) \in \mathcal{CB}_H(d,\omega)$. If $\omega \in \mathbb{Q}^d$ and if we choose a nonzero integer j such that $j\omega \in \mathbb{Z}^d$ then the criterion (4.29) simplifies to the following property for $T(z)$:

$$T^t(z)\Psi(j;z)T(z) \in H . \tag{4.32}$$

If ω is not in $\mathbb{Q}^d$ then one try (??????can) do a "rational approximation". This entails replacing ω by an element ω in $\mathbb{Q}^d$, choosing a nonzero integer j such that $j\omega \in \mathbb{Z}^d$ and then solving (4.32).. (IS THIS STABLE NEAR A S-O RESONANCE, IN PARTICULAR NEAR SNAKE RESONANCES??????) □

We now discuss the important case $H = SO(2)$. For this one can show, e.g., as in Appendix C in [He2], that if $g \in \mathcal{C}(\mathbb{T}^d, SO(2))$ it easily follows from (??) that a constant $N \in \mathbb{Z}^d$ and a $h \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ exist such that

$$g([\phi]_d) = \exp(\mathcal{J}[N \cdot \phi + 2\pi h([\phi]_d)]) . \tag{4.33}$$

?

5 Polarization fields and invariant spin fields

5.1 Generalities

High precision measurements of the spin-dependent properties of the interactions of colliding protons, deuterons, electrons and positrons in storage rings depend on the equilibrium spin polarization being maximized. This, in turn, is facilitated by an understanding of the meaning of the term equilibrium, both as it applies to the value of the polarization and to its direction at each point in phase space. We will return to these matters in Section 5.3 but continue now with a definition and an exploration of the effects of maps.

Suppose therefore that $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and that, at time $n = 0$, a spin vector S_{z_0} has been assigned to every point $z_0 = [\phi_0]_d \in \mathbb{T}^d$ of the "orbital torus" and consider its evolution according to (2.37). Let $S_{z_0}(n)$ denote the spin trajectory over z_0 and for the initial value $S_0 = S_{z_0}$. We define the field $\mathcal{S} = \mathcal{S}(n,z)$ by $\mathcal{S}(n, \mathcal{P}^n_\omega(z)) = S_z(n)$ where n, z vary over $\mathbb{Z}, \mathbb{T}^d$ respectively. Clearly $\mathcal{S}(n,\cdot)$ is the distribution of spins which started at $n = 0$ with the assignment S_{z_0} and evolved under the dynamics of (2.37). Since (2.37) gives us $S_z(n+1) = A(\mathcal{P}^n_\omega(z))S_z(n)$, we have

$$\mathcal{S}(n+1, z) = A\Big(\mathcal{P}_{-\omega}(z)\Big)\mathcal{S}\Big(n, \mathcal{P}_{-\omega}(z)\Big) . \tag{5.1}$$

Definition 5.1 *(Polarization field, invariant spin field)*
Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$. We call a function $\mathcal{S} \in \mathcal{C}(\mathbb{Z}\times\mathbb{T}^d, \mathbb{R}^3)$ a "polarization field of (ω, A)", if it satisfies the evolution equation (5.1). Clearly $\mathcal{S}(n,\cdot) \in \mathcal{C}(\mathbb{T}^d,\mathbb{R}^3)$ and we call $\mathcal{S}(0,\cdot)$ the "generator of $\mathcal{S}$".

A polarization field $\mathcal{S}$ is called "invariant" if it is time-independent, i.e., if $\mathcal{S}(n,\cdot) = \mathcal{S}(0,\cdot)$. We call a polarization field $\mathcal{S}$ a "spin field" if $|\mathcal{S}| = 1$. Note that a polarization field $\mathcal{S}$ is a spin field iff $|\mathcal{S}(0,\cdot)| = 1$. If an invariant polarization field is a spin field we call it an "invariant spin field (ISF)". □

Remark:

(1) According to Definition 5.1, every polarization field $\mathcal{S}$ fulfills two different conditions: the "dynamical" condition (5.1) and the "regularity" condition that $\mathcal{S}(0,\cdot)$ is continuous. However, in contrast to the dynamical condition, the regularity condition is a matter of choice. While in this work, and in [He2], we choose continuity as the regularity property of the functions A and $\mathcal{S}(0,\cdot)$ etc., this property is ~~a matter of choice and~~ can basically vary between the extremes "Borel measurable" and "of class C^∞".
□

We define the function $\tilde{\mathcal{P}}_{\omega,A} : \mathcal{C}(\mathbb{T}^d,\mathbb{R}^3) \to \mathcal{C}(\mathbb{T}^d,\mathbb{R}^3)$ by

note the $(-\omega)$?
So $\tilde{\mathcal{P}}_{\omega,A}$ transports from the past
– not into the future.

$$\tilde{\mathcal{P}}_{\omega,A}(f) := (A\circ\mathcal{P}_{-\omega})(f\circ\mathcal{P}_{-\omega}) , \tag{5.2}$$

useful

i.e., $(\tilde{\mathcal{P}}_{\omega,A}(f))([\phi]_d) := A([\phi - 2\pi\omega]_d)f([\phi-2\pi\omega]_d)$ where $f \in \mathcal{C}(\mathbb{T}^d,\mathbb{R}^3)$. It is an easy exercise to show that for $(\omega, A), (\omega', A') \in \mathcal{SOT}(d)$

$$\tilde{\mathcal{P}}_{\omega',A'} \circ \tilde{\mathcal{P}}_{\omega,A} = \tilde{\mathcal{P}}_{\omega+\omega',A''} , \tag{5.3}$$

Therefore the linearity of (5.1) has the following implication. Since $\mathbb{R}^3$ has a natural structure of a real vector space, the set $\mathcal{C}(\mathbb{T}^d,\mathbb{R}^3)$ is a real linear space in a natural way. Moreover, by (5.7), we have for for $n \in \mathbb{Z}, x, x' \in \mathbb{R}, f, f' \in \mathcal{C}(\mathbb{T}^d,\mathbb{R}^3)$,

$$\tilde{L}_{\omega,A}(n; xf + x'f') = x\tilde{L}_{\omega,A}(n;f) + x'\tilde{L}_{\omega,A}(n;f') , \tag{5.9}$$

whence every $\tilde{L}_{\omega,A}(n;\cdot)$ is a linear function. It thus (then ?????) follows from the above that $\tilde{L}_{\omega,A}(n;\cdot) \in GL(\mathcal{C}(\mathbb{T}^d,\mathbb{R}^3))$ and that the function $\tilde{L}^{hom}_{\omega,A}$ defined above is a representation of the group $\mathbb{Z}$ on the linear space $\mathcal{C}(\mathbb{T}^d,\mathbb{R}^3)$. Of ~~course~~ we also have $\tilde{P}_{\omega,A} \in GL(\mathcal{C}(\mathbb{T}^d,\mathbb{R}^3))$. where? □

Denoting the set of polarization fields of (ω, A) by $\mathcal{PF}(\omega, A)$ and referring to (5.8) and Definition 5.1 we have

$$\mathcal{PF}(\omega,A) = \left\{ \mathcal{S} \in \mathcal{C}(\mathbb{Z}\times\mathbb{T}^d,\mathbb{R}^3) : (\forall n \in \mathbb{Z})\ \mathcal{S}(n,\cdot) = \tilde{L}_{\omega,A}(n;\mathcal{S}(0,\cdot)) \right\} . \tag{5.10}$$

Moreover, denoting the set of invariant spin fields of (ω, A) by $\mathcal{ISF}(\omega, A)$ and referring to (5.10) and Definition 5.1, we have

$$\mathcal{ISF}(\omega,A) = \{\mathcal{S} \in \mathcal{PF}(\omega,A) \cap \mathcal{C}(\mathbb{Z}\times\mathbb{T}^d,\mathbb{S}^2) : (\forall n \in \mathbb{Z})\ \mathcal{S}(0,\cdot) = \tilde{L}_{\omega,A}(n;\mathcal{S}(0,\cdot))\} , \tag{5.11}$$

where the 2-sphere is defined by $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ and is equipped with the relative topology from $\mathbb{R}^3$ (i.e., a subset of $\mathbb{S}^2$ is open iff it is the intersection of $\mathbb{S}^2$ with an open subset of $\mathbb{R}^3$).

It follows from (5.2) and Definition 5.1 that any polarization field $\mathcal{S}$ of (ω, A) is invariant iff

for defn of $\tilde{P}_{\omega,A}$

$$\underbrace{\mathcal{S}(0,\cdot)}_{=\mathcal{S}(1,\cdot)} = \tilde{\mathcal{P}}_{\omega,A}(\mathcal{S}(0,\cdot)) . \tag{5.12}$$

Transforms from $\varphi - 2\pi\tilde{\nu}$ up to φ

So from (5.11),(5.12) we have

Needed?

$$\mathcal{ISF}(\omega,A) = \{\mathcal{S} \in \mathcal{PF}(\omega,A) \cap \mathcal{C}(\mathbb{Z}\times\mathbb{T}^d,\mathbb{S}^2) : \mathcal{S}(0,\cdot) = \tilde{\mathcal{P}}_{\omega,A}(\mathcal{S}(0,\cdot))\} . \tag{5.13}$$

Since $\mathcal{S}(0,\cdot)$ is the generator of $\mathcal{S}$, (5.13) delivers the following criterion, which we call the "ISF criterion":

- (ISF criterion) A spin-orbit torus (ω, A) in $\mathcal{SOT}(d,\omega)$ has an ISF iff a function $f \in \mathcal{C}(\mathbb{T}^d,\mathbb{S}^2)$ exists which satisfies the functional equation

a field – also no mention of "time"

$$f \circ \mathcal{P}_\omega = Af . \quad (\forall \omega) \tag{5.14}$$

Note that if a function $f \in \mathcal{C}(\mathbb{T}^d,\mathbb{S}^2)$ satisfies the ISF criterion of (ω, A) then f is the generator of an ISF $\mathcal{S}$, i.e.,

$$\mathcal{S}(n,\cdot) = \tilde{L}_{\omega,A}(n;f) . \tag{5.15}$$

Clearly the generator of every ISF satisfies the ISF criterion. The ISF criterion (5.12) will play a major role in Chapter 7.

$\hat{n}(s, M(z)) = R_{3\times3}\, \hat{n}(s, z)$
But $R_{3\times3}\, \hat{n}(s,z)$ is also $\hat{n}(s+C, M(z))$
So $\hat{n}(s+C, M(z)) = \hat{n}(s, M(z)) \Rightarrow$ invariance

(5) Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$ so that, by (4.39), $(\omega, A) \in \mathcal{CB}_{SO(2)}(d,\omega)$. Then (ω, A) has an IFF whence, by Remark 3, (ω, A) has an ISF.

(6) Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$. Recall that we briefly discussed at the end of Section 4.3 the computer code SPRINT that computes a spin tune of first kind in two steps and that in the first step an IFF of (ω, A) is computed, say T. Note that by Remark 5 an IFF exists. The first step of SPRINT consists of two substeps. In the first substep a generator f of an ISF is computed by using the fact that f satisfies the ISF criterion. This makes it possible to apply the technique of stroboscopic averaging to compute f. The second substep is a simple orthonormalization procedure to obtain a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ whose third column is f. Thus, by Remark 4 above, T is an IFF (needs work???????). □

5.2 Transforming polarization fields

We now derive a natural rule for transforming polarization fields according to our transformation rule for a spin-orbit torus from Chapter 3. Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and let $(\omega, A') := R_{d,\omega}(T;\omega, A)$ where, of course, $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Then from (5.3) and for $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ we obtain

$$\tilde{\mathcal{P}}_{0,T^t} = \tilde{\mathcal{P}}_{0,T}^{-1} , \qquad \tilde{\mathcal{P}}_{0,T}^{-1} \circ \tilde{\mathcal{P}}_{\omega,A} \circ \tilde{\mathcal{P}}_{0,T} = \tilde{\mathcal{P}}_{\omega,A'} , \tag{5.18}$$

whence, by (5.6),

$$\tilde{L}_{\omega,A'}(n;\cdot) = \tilde{\mathcal{P}}^n_{\omega,A'} = \tilde{\mathcal{P}}_{0,T}^{-1} \circ \tilde{\mathcal{P}}^n_{\omega,A} \circ \tilde{\mathcal{P}}_{0,T} = \tilde{\mathcal{P}}_{0,T}^{-1} \circ \tilde{L}_{\omega,A}(n,\cdot) \circ \tilde{\mathcal{P}}_{0,T} . \tag{5.19}$$

Thus, recalling Definition 3.3, $\tilde{\mathcal{P}}_{0,T}^{-1}$ is a topological $\mathbb{Z}$-map from the $\mathbb{Z}$ set $(\mathcal{C}(\mathbb{T}^d, \mathbb{R}^3), \tilde{L}_{\omega,A})$ to the $\mathbb{Z}$ set $(\mathcal{C}(\mathbb{T}^d, \mathbb{R}^3), \tilde{L}_{\omega,A'})$. Since $\tilde{\mathcal{P}}_{0,T}^{-1}$ is a bijection both $\mathbb{Z}$ sets are isomorphic as $\mathbb{Z}$ sets.

With (5.19) it is simple to transform polarization fields. In fact if f is the generator of a polarization field $\mathcal{S}$ of (ω, A) then we can relate it to the polarization field $\mathcal{S}'$ of (ω, A') whose generator is $\tilde{\mathcal{P}}_{0,T}^{-1}(f) = T^t f$. Thus, by (5.8),

$$\mathcal{S}(n,\cdot) = \tilde{L}_{\omega,A}(n; f) , \quad \mathcal{S}'(n,\cdot) = \tilde{L}_{\omega,A}(n; T^t f) , \tag{5.20}$$

whence, by (5.6),(5.18),

$$\begin{aligned}\mathcal{S}'(n,\cdot) &= \tilde{L}_{\omega,A'}(n; \mathcal{S}'(0,\cdot)) = \tilde{\mathcal{P}}^n_{\omega,A'}(\mathcal{S}'(0,\cdot)) = (\tilde{\mathcal{P}}_{0,T}^{-1} \circ \tilde{\mathcal{P}}^n_{\omega,A} \circ \tilde{\mathcal{P}}_{0,T})(\mathcal{S}'(0,\cdot)) \\ &= (\tilde{\mathcal{P}}_{0,T}^{-1} \circ \tilde{\mathcal{P}}^n_{\omega,A} \circ \tilde{\mathcal{P}}_{0,T} \circ \tilde{\mathcal{P}}_{0,T}^{-1})(f) = \tilde{\mathcal{P}}_{0,T}^{-1}(\tilde{\mathcal{P}}^n_{\omega,A}(f) \\ &= \tilde{\mathcal{P}}_{0,T}^{-1}(\tilde{L}_{\omega,A}(n, f) = \tilde{\mathcal{P}}_{0,T}^{-1}(\mathcal{S}(n,\cdot)) ,\end{aligned} \tag{5.21}$$

i.e.,

$$\mathcal{S}'(n,\cdot) = \tilde{\mathcal{P}}_{0,T}^{-1}(\mathcal{S}(n,\cdot)) . \tag{5.22}$$

We conclude from (5.22) that if $\mathcal{S}$ is a polarization field of (ω, A) then $\mathcal{S}'$, defined by the lhs of (5.22), is a polarization field of (ω, A'). Thus with (5.22) we have a natural transformation rule of polarization fields. Note that, by (5.2) and (5.18), one can write (5.22) as

$$\mathcal{S}'(n,z) = T^t(z)\mathcal{S}(n,z) . \tag{5.23}$$

Remarks:

where

$$\mathcal{P}_{max}(0) := \sup\Big(\Big\{\int_\Lambda dJ\rho_{eq}(J)\Big|\int_{[0,2\pi]^d} d\phi \mathcal{S}_J(0,[\phi]_d)\Big| : \mathcal{S}_J \in \mathcal{ISF}(\omega(J),A_J)\Big\}\Big) \quad (5.29)$$

Thus the ISFs provide an upper bound for the polarization and this is why they are so important in practice. In Chapter 6 we will see how the Uniqueness Theorem leads to simplification of the rhs of (5.29).

6 Uniqueness of invariant spin fields

We saw with formula (5.28) in Chapter 5 that the invariant spin fields play a role in the estimation of the polarization of the bunch. We now reconsider (5.28) using the fact that the invariant spin fields also play a role for spin tunes of the first kind.

Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$. Then, by Remark 5 in Chapter 5, (ω, A) has an ISF. Since, as shown in Chapter 4, every spin-orbit torus in $\mathcal{ACB}$ has spin tunes of first kind and, as just shown, it has an ISF it is, for every given $(\omega, A) \in \mathcal{ACB}(d,\omega)$, natural to ask about the impact of the set $\Xi_1(\omega, A)$ on $\mathcal{ISF}(\omega, A)$. In fact, if (ω, A) is off orbital resonance, this question is partially answered by the following theorem whose proof can be found in Section F.10 of [He2].

Theorem 6.1 *(The Uniqueness Theorem) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ (ACB: MATHIAS' COMMENT????????????) be off orbital resonance, i.e., let $(1,\omega)$ be nonresonant. Also, let (ω, A) be off spin-orbit resonance of first kind. Then (ω, A) has an ISF, say $\mathcal{S}$, and $\mathcal{S}$ and $-\mathcal{S}$ are the only ISF's of (ω, A).* □

Note that the claim of Theorem 6.1 that (ω, A) has an ISF is trivial since, from Remark 5 in Chapter 5, we know that every $(\omega, A) \in \mathcal{ACB}(d,\omega)$ has an ISF. Thus the meat of the claim of Theorem 6.1 is that (ω, A) has two ISF's. Recall also from Chapter 5 that the set of ISF's of a spin-orbit torus is either infinite or contains an even number of elements.

The Uniqueness Theorem allows us to simplify the formula for the maximum polarization (5.29). For that purpose we assume that the proper spin-orbit tori $(\omega(J), A_J)$ in (5.29) satisfy the assumptions of Theorem 6.1 for almost every J, i.e., we assume that a set $M \subset \Lambda$ exists which has Lebesgue measure zero and such that, for every $J \in (\Lambda \setminus M)$, the spin-orbit torus $(\omega(J), A_J)$ is off orbital resonance and off spin-orbit resonance of first kind. Thus, by Theorem 6.1 and for every $J \in \Lambda \setminus M)$, the spin-orbit torus $(\omega(J), A_J)$ has an ISF, say $\mathcal{S}_J$, and $\mathcal{ISF}(\omega(J), A_J)$ just has the two elements $\mathcal{S}_J, -\mathcal{S}_J$. Thus (5.29) simplifies to

$$\mathcal{P}_{max}(0) = \int_\Lambda dJ\rho_{eq}(J)\Big|\int_{[0,2\pi]^d} d\phi \mathcal{S}_J(0,[\phi]_d)\Big| , \quad (6.1)$$

where we assume that the functional dependences of $\rho_{eq}(J)$ and $\mathcal{S}_J(0,[\phi]_d)$ on J are regular enough to ensure that the integrals ~~in (6.1)~~ are meaningful.

MATHIAS: mention P_lim? Cite earlier work? Mention distance to resonance?

7.2 The basic group homomorphisms $\mathcal{H}_{\omega,A}$ and $\mathcal{H}_d^{trans}$

In Chapters 2-4 we studied the single particle spin-orbit motion defined in terms of the $\mathbb{Z}$ space $(\mathbb{T}^d \times \mathbb{R}^3, L_{\omega,A})$ with the $\mathbb{Z}$ action $L_{\omega,A} : \mathbb{Z} \times \mathbb{T}^d \times \mathbb{R}^3 \to \mathbb{T}^d \times \mathbb{R}^3$ defined by (2.28), i.e.,

$$L_{\omega,A}(n; z, S) = \begin{pmatrix} L_\omega(n; z) \\ \Psi_{\omega,A}(n; z)S \end{pmatrix} ,$$

where the $\mathbb{Z}$ action $L_\omega : \mathbb{Z} \times \mathbb{T}^d \to \mathbb{T}^d$ is defined by (2.27), i.e.,

$$L_\omega(n; z) = \mathcal{P}_\omega^n(z) = \mathcal{P}_{n\omega}(z) .$$

In Chapter 5 we studied the motion of the polarization field in terms of the $\mathbb{Z}$ action $\tilde{L}_{\omega,A} : \mathbb{Z} \times \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3) \to \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ given by (5.7), i.e.,

$$\tilde{L}_{\omega,A}(n; f) = \Big(\Psi_{\omega,A}(n; \cdot)f\Big) \circ L_\omega(-n; \cdot) = \Psi_{\omega,A}(n; L_\omega(-n; \cdot))f(L_\omega(-n; \cdot)) .$$

Since we do not need (EXPLAIN??????)a topology on $\mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$, we consider $(\mathcal{C}(\mathbb{T}^d, \mathbb{R}^3), \tilde{L}_{\omega,A})$ just as a $\mathbb{Z}$ set.

To prepare ourselves for the ToA in Section 7.3 we now introduce the three objects, namely the group $\mathfrak{A}_d$ and the functions $\mathcal{H}_{\omega,A}$ and $\mathcal{H}_d^{trans}$ where $(\omega, A) \in \mathcal{SOT}(d)$ and where $\mathcal{H}_{\omega,A} : \mathbb{Z} \to \mathfrak{A}_d$ carries the dynamics and $\mathcal{H}_d^{trans} : \mathcal{C}(\mathbb{T}^d, SO(3)) \to \mathfrak{A}_d$ carries the transformations. The range of both functions lies in $\mathfrak{A}_d$ which we now define. Let $j \in Homeo(\mathbb{T}^d), k \in \mathcal{C}(\mathbb{T}^d, SO(3))$, and let

$$E_d := \mathbb{T}^d \times SO(3) , \tag{7.1}$$

and consider $\check{a}_d(j, k) \in \mathcal{C}(E_d, E_d)$ is defined by

$$(\check{a}_d(j, k))(z, r) := \begin{pmatrix} j(z) \\ k(z)r \end{pmatrix} , \tag{7.2}$$

with $z \in \mathbb{T}^d, r \in SO(3)$. With this we define the pair

$$\hat{a}_d(j, k) := (\check{a}_d(j, k), j) , \tag{7.3}$$

MATHIAS: we need a pair in order to describe orbital and spin motion separately
Note that E_d is equipped with its natural topology. We then define $\mathfrak{A}_d$ by

$$\mathfrak{A}_d := \Big\{\hat{a}_d(j, k) : j \in Homeo(\mathbb{T}^d), k \in \mathcal{C}(\mathbb{T}^d, SO(3))\Big\} . \tag{7.4}$$

This forms a group $(\mathfrak{A}_d, \diamond)$ with $(id_{E_d}, id_{\mathbb{T}^d})$ being the identity element of the group and where, for $j, j' \in Homeo(\mathbb{T}^d)$ and $k, k' \in \mathcal{C}(\mathbb{T}^d, SO(3))$,

$$\hat{a}_d(j', k')\diamond\hat{a}_d(j, k) := \Big(\check{a}_d(j', k') \circ \check{a}_d(j, k), j' \circ j\Big) . \tag{7.5}$$

If $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ then $\hat{a}_d(\mathcal{P}_0, T) \in \mathfrak{A}_d$ whence we define, in analogy to $\mathcal{H}_{\omega,A}$, the function $\mathcal{H}_d^{trans} : \mathcal{C}(\mathbb{T}^d, SO(3)) \to \mathfrak{A}_d$ by

$$\mathcal{H}_d^{trans}(T) := \hat{a}_d(\mathcal{P}_0, T) = \hat{a}_d(id_{\mathbb{T}^d}, T) = (\check{\mathcal{P}}_{0,T}, \mathcal{P}_0) = (\check{\mathcal{P}}_{0,T}, id_{\mathbb{T}^d}) \; . \tag{7.12}$$

By (7.7),

$$\check{\mathcal{P}}_{0,T} \circ \check{\mathcal{P}}_{0,T'} = \check{\mathcal{P}}_{0,TT'} \; , \tag{7.13}$$

where $T, T' \in \mathcal{C}(\mathbb{T}^d, SO(3))$ whence, by (7.12),

$$\mathcal{H}_d^{trans}(T'T) = \hat{a}_d(id_{\mathbb{T}^d}, T'T) = \hat{a}_d(id_{\mathbb{T}^d}, T') \diamond \hat{a}_d(id_{\mathbb{T}^d}, T) = \mathcal{H}_d^{trans}(T') \diamond \mathcal{H}_d^{trans}(T) \; ,$$

so that $\mathcal{H}_d^{trans}$ is also a group homomorphism from $\mathcal{C}(\mathbb{T}^d, SO(3))$ into $\mathfrak{A}_d$. The group structure of $\mathcal{C}(\mathbb{T}^d, SO(3))$ is defined in Section 3.3.

As already mentioned, the $\mathcal{H}_{\omega,A}$ will be exploited in Section 7.3 as the tools which implement (GENERATE????????) the various $\mathbb{Z}$ actions via the ToA. To accomplish that it is crucial that the $\mathcal{H}_{\omega,A}$ are group homomorphisms, as we shall see later.

We now derive the general transformation formula (7.14). With (7.6), (7.10) and (7.12) we obtain

$$\begin{aligned}
&\mathcal{H}_d^{trans}(T^t) \diamond \mathcal{H}_{\omega,A}(n) \diamond \mathcal{H}_d^{trans}(T) = \hat{a}_d(id_{\mathbb{T}^d}, T^t) \diamond \hat{a}_d\Big(L_\omega(n,\cdot), \Psi_{\omega,A}(n,\cdot) \Big) \diamond \hat{a}_d(id_{\mathbb{T}^d}, T) \\
&= \hat{a}_d\Big(L_\omega(n,\cdot), (T^t \circ L_\omega(n,\cdot)) \Psi_{\omega,A}(n,\cdot) \Big) \diamond \hat{a}_d(id_{\mathbb{T}^d}, T) \\
&= \hat{a}_d\Big(L_\omega(n,\cdot), T^t \circ L_\omega(n,\cdot)) \Psi_{\omega,A}(n,\cdot) T \Big) \; ,
\end{aligned}$$

so that, using (3.2), we obtain

$$\mathcal{H}_d^{trans}(T^t) \diamond \mathcal{H}_{\omega,A}(n) \diamond \mathcal{H}_{trans,d}(T) = \hat{a}_d\Big(L_\omega(n,\cdot), \Psi_{\omega,A'}(n,\cdot) \Big) \; ,$$

which implies, by (7.10),

$$\mathcal{H}_d^{trans}(T^t) \diamond \mathcal{H}_{\omega,A}(n) \diamond \mathcal{H}_d^{trans}(T) = \mathcal{H}_{\omega,A'}(n) \; . \tag{7.14}$$

The fact that $\mathcal{H}_d^{trans}$ is a group homomorphism is as crucial for us as the fact that $\mathcal{H}_{\omega,A}$ is a group homomorphism. In fact the homomorphism involving $\mathcal{H}_d^{trans}$ allows us to apply (7.14) to the ToA and create a transformation rule for every $\mathbb{Z}$ action provided by the ToA.

NEW

Following these two examples demonstrating group homomorphisms and recognizing that they rely on the cocycle structure implicit in the functions $\hat{a}_d$, it is tempting to search for other examples. This is mediated by the following theorem which, as we shall see in 7.3.1, also defines the set of all group homomorphisms which are eligible for the ToA. The first equality in (7.19) below, provides a concrete example of the definition (7.15) in the theorem.

and observe, by (7.20), that it satisfies

$$\chi_d^{trans}(TT';z) = T(z)T'(z) = \chi_d^{trans}(T; L_d^{trans}(T';z))\chi_d^{trans}(T';z) \ , \tag{7.22}$$

whence the cocycle condition (2.42) is satisfied by χ_d^{trans}. To show that χ_d^{trans} is continuous we equip $\mathcal{C}(\mathbb{T}^d, SO(3))$ with its natural topology, i.e., the compact-open topology and observe by (7.21) that χ_d^{trans} indeed is continuous (see [tDi2, Section 2.4]). We thus conclude that χ_d^{trans} belongs to $COC(\mathbb{T}^d, L_d^{trans}, SO(3))$ whence $\mathcal{H}[\chi_d^{trans}]$ is a well defined group homomorphism from $\mathcal{C}(\mathbb{T}^d, SO(3))$ to $\mathfrak{A}_d$. Finally, by using (7.12), (7.15), (7.20) and (7.21), we compute

$$\mathcal{H}[\chi_d^{trans}](T) = \hat{a}_d\Big(L_d^{trans}(T,\cdot), \chi_d^{trans}(T,\cdot)\Big) = \hat{a}_d\Big(id_{\mathbb{T}^d}, \chi_d^{trans}(T,\cdot)\Big) = \hat{a}_d\Big(id_{\mathbb{T}^d}, T\Big)$$
$$= \mathcal{H}_d^{trans}(T) \ , \tag{7.23}$$

whence $\mathcal{H}_d^{trans}$ indeed belongs to the group homomorphisms provided by Theorem 7.1.
END NEW

7.3 The ToA. Generalization of the spin-orbit motion and polarization field dynamics of Chapters 2-6

7.3.1 Defining the ToA

We can now build innumerable group actions to study spin-orbit tori, in particular innumerable $\mathbb{Z}$-actions based on $\mathcal{H}_{\omega,A}$ to study (ω, A).

Consider an arbitrary left $SO(3)$-space, (E, l), and the functions $L_d[l] : \mathfrak{A}_d \times \mathbb{T}^d \times E \to \mathbb{T}^d \times E$ and $\tilde{L}_d[l] : \mathfrak{A}_d \times \mathcal{C}(\mathbb{T}^d, E) \to \mathcal{C}(\mathbb{T}^d, E)$, defined by

$$L_d[l]\Big(\hat{a}_d(j,k); z, x\Big) := \Big(j(z), l(k(z);x)\Big) \ , \tag{7.24}$$

$$\tilde{L}_d[l]\Big(\hat{a}_d(j,k); f\Big) := l(k;f)\circ j^{-1} \ , \tag{7.25}$$

where $j \in Homeo(\mathbb{T}^d)$, $k \in \mathcal{C}(\mathbb{T}^d, SO(3))$, $z \in \mathbb{T}^d$, $x \in E$ and $f \in \mathcal{C}(\mathbb{T}^d, E)$. As shown in Remarks 4 and 5 below, the functions in (7.24) and (7.25) are left $\mathfrak{A}_d$ actions on E and on $\mathcal{C}(\mathbb{T}^d, E)$ respectively. So for example, we could have ....THEN PUT IN SOME STUFF WITH THE L_ω, $\Psi_{\omega,A}$ etc TO ILLUSTRATE WHAT (7.24) and (7.25) MEAN PHYSICALLY (???????) 6

The only applications of (7.24) and (7.25) in this work arise by combining the group actions $L_d[l]$ and $\tilde{L}_d[l]$ with the group homomorphisms given by the cocycle theorem, Theorem 7.1, in particular with $\mathcal{H}_{\omega,A}$ and $\mathcal{H}_d^{trans}$. In fact if G is a group and if $\mathcal{H} : G \to \mathfrak{A}_d$ is a group homomorphism then we define the function $L_d[l]\odot\mathcal{H} : G \times \mathbb{T}^d \times E \to \mathbb{T}^d \times E$ by

$$(L_d[l]\odot\mathcal{H})(g,z,x) := L_d[l](\mathcal{H}(g),z,x) \ , \tag{7.26}$$

and the function $\tilde{L}_d[l]\odot\mathcal{H} : G \times \mathcal{C}(\mathbb{T}^d, E) \to \mathcal{C}(\mathbb{T}^d, E)$ by

$$(\tilde{L}_d[l]\odot\mathcal{H})(g,f) := \tilde{L}_d[l](\mathcal{H}(g),f) \ , \tag{7.27}$$

Trans example?

we progress through this chapter. Two obvious aspects of the ToA are therefore its use to identify those (E, l) which are already "in use" and to find "new" left $SO(3)$-spaces.

For example in Sections 7.3.2 and 7.3.3 we will show that the familiar $\mathbb{Z}$ actions from Chapters 2-6 arise from the ToA via the left $SO(3)$-space, $(\mathbb{R}^3, l_{spin})$, defined in Section 7.3.2. In fact using the choices $\mathcal{H} = \mathcal{H}_{\omega,A}$ and $(E, l) = (\mathbb{R}^3, l_{spin})$ we will find that $L_{\omega,A} = L_d[l_{spin}] \odot \mathcal{H}_{\omega,A}$ and that $\tilde{L}_{\omega,A} = \tilde{L}_d[l_{spin}] \odot \mathcal{H}_{\omega,A}$. Thus $(\mathbb{R}^3, l_{spin})$ is already "in use".

BEGIN TENSOR

A second example of a left $SO(3)$-space which is in use is (E_{tensor}, l_{tensor}) where $E_{tensor} := \{M \in \mathbb{R}^{3\times 3} : M^t = M, Tr(M) = 0\}$ and where the function $l_{tensor} : SO(3) \times E_{tensor} \to E_{tensor}$ is defined by $l_{tensor}(r; M) := rMr^t$ with $r \in SO(3), M \in E_{tensor}$ and with Tr denoting the trace. In fact it is an easy exercise to show that (E_{tensor}, l_{tensor}) is a left $SO(3)$-space where E_{tensor} is equipped with the relative topology from $\mathbb{R}^{3\times 3}$. This left $SO(3)$-space is used for studies of polarized beams of spin-1 particles like deuterons. For example one finds that the fixed points of the $\mathbb{Z}$-action $\tilde{L}_d[l_{tensor}] \odot \mathcal{H}_{\omega,A}$ are the "generators" (=initial values) of the so-called "invariant polarisation-tensor fields" of (ω, A) [BV2].

END TENSOR

When we come to Section 7.4 we will go beyond the above two examples by introducing the "new" left $SO(3)$-spaces, $(SO(3)/H, l_H)$ and apply them to the ToA.

NEW

Perhaps surprisingly, in Section 7.5 we will see how the left $SO(3)$-spaces $(SO(3)/H, l_H)$ play a fundamental role in the ToA by allowing to "decompose" every left $SO(3)$-space (E, l), e.g., $(\mathbb{R}^3, l_{spin})$, (E_{tensor}, l_{tensor}) into left $SO(3)$-spaces of the form $(SO(3)/H, l_H)$ and thereby effectively reducing each problem to the machinery of Section 7.4.

In the meantime, we now take a closer look at how to use the ToA. Of course a group homomorphism eligible for the ToA is obtained, via Theorem 7.1, by letting G be a topological group, letting $(\mathbb{T}^d, L)$ be a left G-space and picking a $\chi \in COC(\mathbb{T}^d, L, SO(3))$ giving us the group homomorphism $\mathcal{H}[\chi]$ defined by (7.15). Thus if (E, l) is a left $SO(3)$-space then the left G-action $L_d[l] \odot \mathcal{H}[\chi]$ on $\mathbb{T}^d \times E$ and the left G-action $\tilde{L}_d[l] \odot \mathcal{H}[\chi]$ on $\mathcal{C}(\mathbb{T}^d, E)$ are given by

$$
\begin{aligned}
(L_d[l] \odot \mathcal{H}[\chi])(g; z, x) &= L_d[l](\mathcal{H}[\chi](g); z, x) = L_d[l]\Big(\hat{a}_d\Big(L(g,\cdot), \chi(g,\cdot)\Big); z, x\Big) \\
&= \Big(L(g; z), l\Big(\chi(g, z); x\Big)\Big) , \qquad (7.29)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{L}_d[l] \odot \mathcal{H}[\chi])(g; f) &= \tilde{L}_d[l](\mathcal{H}[\chi](g); f) = \tilde{L}_d[l]\Big(\hat{a}_d\Big(L(g,\cdot), \chi(g,\cdot)\Big); f\Big) \\
&= l\Big(\chi(g,\cdot); f\Big) \circ L(g^{-1}; \cdot) , \qquad (7.30)
\end{aligned}
$$

where $z \in \mathbb{T}^d$, $x \in E$, $f \in \mathcal{C}(\mathbb{T}^d, E)$ and $g \in G$ and where we used (7.15),(7.24),(7.25), (7.26) and (7.27). Of course since we are focused in this work on the group homomorphisms $\mathcal{H}_{\omega,A}$ and $\mathcal{H}_d^{trans}$ we are interested in the special case of (7.29), (7.30) where $L = L_\omega$ and via (7.26) and (7.27), an equality of $\mathbb{Z}$ actions. Let $n \in \mathbb{Z}$ and $T \in C(\mathbb{T}^d, SO(3))$ and define $(\omega, A') := R_{d,\omega}(T; \omega, A)$. Then, by (7.14) and (7.26),

$$
\begin{aligned}
(L_d[l] \odot \mathcal{H}_{\omega,A'})(n,\cdot) &= \left(L_d[l] \odot \left(\mathcal{H}_d^{trans}(T^t) \diamond \mathcal{H}_{\omega,A}(n) \diamond \mathcal{H}_d^{trans}(T), \cdot \right) \right)(n,\cdot) \\
&= L_d[l] \left(\mathcal{H}_{d,tran}(T^t) \diamond \mathcal{H}_{\omega,A}(n) \diamond \mathcal{H}_d^{trans}(T), \cdot \right) \\
&= (L_d[l] \odot \mathcal{H}_d^{trans})(T^t,\cdot) \circ (L_d[l] \odot \mathcal{H}_{\omega,A})(n,\cdot) \circ (L_d[l] \odot \mathcal{H}_d^{trans})(T,\cdot) ,
\end{aligned}
$$

so that we obtain the following equality of $\mathbb{Z}$ actions:

$$
(L_d[l] \odot \mathcal{H}_{\omega,A'})(n,\cdot) = (L_d[l] \odot \mathcal{H}_d^{trans})(T^t,\cdot) \circ (L_d[l] \odot \mathcal{H}_{\omega,A})(n,\cdot) \circ (L_d[l] \odot \mathcal{H}_d^{trans})(T,\cdot) . \tag{7.36}
$$

Analogously

$$
(\tilde{L}_d[l] \odot \mathcal{H}_{\omega,A'})(n,\cdot) = (\tilde{L}_d[l] \odot \mathcal{H}_d^{trans})(T^t,\cdot) \circ (\tilde{L}_d[l] \odot \mathcal{H}_{\omega,A})(n,\cdot) \circ (\tilde{L}_d[l] \odot \mathcal{H}_d^{trans})(T,\cdot) . \tag{7.37}
$$

Remarks (left actions):

(4) Let (E, l) be a left $SO(3)$-space. To show that $L_d[l]$ is a left $\mathfrak{A}_d$ action on $\mathbb{T}^d \times E$ we compute, for $j, j' \in Homeo(\mathbb{T}^d), k, k' \in C(\mathbb{T}^d, SO(3)), z \in \mathbb{T}^d$ and $x \in E$ and by using (7.3), (7.6) and (7.24),

$$
L_d[l](id_{E_d}, id_{\mathbb{T}^d}; z, x) = L_d[l]\left(\hat{a}_d(id_{\mathbb{T}^d}, A_{d,0}); z, x \right) = \left(id_{\mathbb{T}^d}(z), l(A_{d,0}; x) \right) = (z, x) , \tag{7.38}
$$

$$
\begin{aligned}
L_d[l]\left(\hat{a}_d(j',k'); L_d[l]\left(\hat{a}_d(j,k); z, x \right) \right) &= L_d[l]\left(\hat{a}_d(j',k'); j(z), l(k(z); x) \right) \\
&= \left(j'(j(z)), l\left(k'(z); l(k(j(z)); x) \right) \right) = \left(j'(j(z)), l\left(k'(z)k(j(z)); x \right) \right) ,
\end{aligned} \tag{7.39}
$$

$$
\begin{aligned}
L_d[l]\left(\hat{a}_d(j',k') \diamond \hat{a}_d(j,k), z, x \right) &= L_d[l]\left(\hat{a}_d\left(j' \circ j, (k' \circ j)k \right), z, x \right) \\
&= \left(j'(j(z)), l\left(k'(z)k(j(z)); x \right) \right) ,
\end{aligned} \tag{7.40}
$$

so that indeed $L_d[l]$ is a left $\mathfrak{A}_d$ action on $\mathbb{T}^d \times E$. Note that in the third equalities of (7.38) and (7.39) we used the fact that l is a left $SO(3)$ action on E.

(5) Let (E, l) be a left $SO(3)$-space (E, l). To show that $\tilde{L}_d[l]$ is a left $\mathfrak{A}_d$ action on $C(\mathbb{T}^d, E)$ we compute, for $j_1, j_2 \in Homeo(\mathbb{T}^d), k_1, k_2 \in C(\mathbb{T}^d, SO(3))$ and $f \in C(\mathbb{T}^d, E)$ and by using (7.3), (7.6) and (7.25),

$$
\tilde{L}_d[l](id_{E_d}, id_{\mathbb{T}^d}; f) = \tilde{L}_d[l]\left(\hat{a}_d(id_{\mathbb{T}^d}, A_{d,0}); f \right) = l(A_{d,0}; f) = f , \tag{7.41}
$$

With (7.45) we have completed the first successful run of the dynamical part of the ToA. One can even show that $(\mathbb{R}^3, l_{spin})$ is the only left $SO(3)$-space (E, l) which gives (7.45) for all (ω, A) in $\mathcal{SOT}(d)$. In fact it is an easy exercise to show that for every $r \in SO(3)$ there exists an (ω, A) in $\mathcal{SOT}(d)$ and $n \in \mathbb{Z}, z \in \mathbb{T}^d$ such that $r = \Psi_{\omega,A}(n; z)$ whence l_{spin} is uniquely determined by (7.45).

We now come to the transformational part of the ToA by showing that (3.6) is a special case of (7.36). Recalling Section 7.3.1, $L_d[l_{spin}] \odot \mathcal{H}_d^{trans}$ is a left $\mathcal{C}(\mathbb{T}^d, SO(3))$ action on $\mathbb{T}^d \times \mathbb{R}^3$ and to write it explicitly we use (2.19), (7.34) and (7.44) to obtain

$$(L_d[l_{spin}] \odot \mathcal{H}_d^{trans})(T; z, S) = \Big(z, l_{spin}(T(z); S)\Big) = (z, T(z)S) = \mathcal{P}_{0,T}(z, S) \ . \quad (7.46)$$

By defining $(\omega, A') := R_{d,\omega}(T; \omega, A)$ it follows from (7.45) and (7.46) that in the current case (7.36) reads as

$$L_{\omega,A'}(n; \cdot) = \mathcal{P}_{0,T}^{-1} \circ L_{\omega,A}(n; \cdot) \circ \mathcal{P}_{0,T} \ ,$$

which confirms that the transformation rule (3.6) is a special case of (7.36).

7.3.3 Applying the ToA to the left $SO(3)$-space $(\mathbb{R}^3, l_{spin})$: Recovering $\tilde{L}_{\omega,A}$

In this section we apply the ToA to the case of the polarization fields introduced in Chapter 5 by showing that a left $SO(3)$-space (E, l) exists such that, for all (ω, A) in $\mathcal{SOT}(d)$, $\tilde{L}_{\omega,A}$ is of the form $\tilde{L}_d[l] \odot \mathcal{H}_{\omega,A}$. We will also show that the transformation rule (5.19) is a special case of (7.37).

Since, for every (ω, A) in $\mathcal{SOT}(d)$, $\tilde{L}_{\omega,A}$ is a left $\mathbb{Z}$ action on $\mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ we need $E = \mathbb{R}^3$. Moreover by inspection of $\tilde{L}_{\omega,A}$ in (5.7) and using (7.32) we are easily led to choose the $l = l_{spin}$defined in 7.44. Next, we observe, by (7.32) and (7.44), that

$$(\tilde{L}_d[l_{spin}] \odot \mathcal{H}_{\omega,A})(n; f) = l_{spin}\Big(\Psi_{\omega,A}(n, \cdot); f\Big) \circ L_\omega(-n; \cdot) = \Big(\Psi_{\omega,A}(n, \cdot) f\Big) \circ L_\omega(-n; \cdot) \ ,$$

whence, by (5.7),

$$\tilde{L}_d[l_{spin}] \odot \mathcal{H}_{\omega,A} = \tilde{L}_{\omega,A} \ . \quad (7.47)$$

One can even show that $(\mathbb{R}^3, l_{spin})$ is the only left $SO(3)$-space (E, l) which gives (7.47) for all (ω, A) in $\mathcal{SOT}(d)$. In fact recalling the remarks after (7.45) we know that for every $r \in SO(3)$ there exists an (ω, A) in $\mathcal{SOT}(d)$ and $n \in \mathbb{Z}, z \in \mathbb{T}^d$ such that $r = \Psi_{\omega,A}(n; z)$ so that l_{spin} is uniquely determined by (7.47).

We now come to the transformational part of the ToA by showing that (5.19) is a special case of (7.36). Recalling Section 7.3.1, $\tilde{L}_d[l_{spin}] \odot \mathcal{H}_d^{trans}$ is a left $\mathcal{C}(\mathbb{T}^d, SO(3))$ action on $\mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ and to write it explicitly we use (5.2), (7.35) and (7.44) to obtain

$$(\tilde{L}_d[l_{spin}] \odot \mathcal{H}_d^{trans})(T; f) = l_{spin}(T; f) = Tf = \tilde{\mathcal{P}}_{0,T}(f) \ . \quad (7.48)$$

By defining $(\omega, A') := R_{d,\omega}(T; \omega, A)$ it follows from (7.47) and (7.48) that in the current case (7.37) as

$$\tilde{L}_{\omega,A'}(n; \cdot) = \tilde{\mathcal{P}}_{0,T}^{-1} \circ \tilde{L}_{\omega,A}(n; \cdot) \circ \tilde{\mathcal{P}}_{0,T} \ ,$$

which confirms that the transformation rule (5.19) is a special case of (7.37).