7.4 Applying the ToA to the left $SO(3)$-spaces $(SO(3)/H, l_H)$

We now go beyond the familiar situation of Chapters 2-6 where the underlying left $SO(3)$-space is $(\mathbb{R}^3, l_{spin})$ as we have just shown. For this we introduce the quotient set $(SO(3)/H$ defined in (7.49) and consider those left $SO(3)$-spaces $(SO(3)/H, l_H)$ which are different from $(\mathbb{R}^3, l_{spin})$ and which, when used for the ToA, will give us the opportunity to learn more about spin-orbit tori via the $\mathbb{Z}$ actions $L_d[l_H] \odot \mathcal{H}_{\omega,A}$ and $\tilde{L}_d[l_H] \odot \mathcal{H}_{\omega,A}$. We will proceed as follows.

In Section 7.4.1, we consider an arbitrary closed subgroup, H, of $SO(3)$, i.e., a subgroup which is, at the same time, a closed subset of $SO(3)$, i.e., $\bar{H} = H$ (see Appendix A). Then with the quotient set $(SO(3)/H$ of (7.49) we introduce the left $SO(3)$-spaces $(SO(3)/H, l_H)$. With $(SO(3)/H, l_H)$ the ToA leads us to the $\mathbb{Z}$ actions $L_d[l_H] \odot \mathcal{H}_{\omega,A}$ and $\tilde{L}_d[l_H] \odot \mathcal{H}_{\omega,A}$.

In Section 7.4.2 the fixed points of $\tilde{L}_d[l_H] \odot \mathcal{H}_{\omega,A}$ are related with so-called structural equations. In Section 7.4.4 these equations give us new insights into the nature of spin-orbit resonances of the first kind via our First SOR Theorem. In Section 7.4.5 they give new insights into the nature of ISFs via the First ISF Theorem. Every g in $\mathcal{C}(\mathbb{T}^d, SO(3)/H)$ defines a unique subset, $E_H(g)$ of E_d, to be introduced in Section 7.4.7 and we further characterize the $E_H(g)$, in Section 7.4.8, via the First Reduction Theorem. In Section 7.4.9 we study the cross sections of the functions $p_d|E_H(g)$ and thereby obtain new insights into the nature of IFFs. In Sections 7.4.10 and 7.4.11 we revisit the First SOR and ISF Theorems in terms of the $E_H(g)$. The fundamental importance of the left $SO(3)$-spaces $(SO(3)/H, l_H)$ will become clear in Section 7.5.

7.4.1 Defining the left $SO(3)$-spaces $(SO(3)/H, l_H)$

We now apply the ToA to the left $SO(3)$-spaces $(SO(3)/H, l_H)$ where, by Appendix A.2, $SO(3)/H$ is given by

$$SO(3)/H = SO(3)/R_{SO(3),H} = \{rH : r \in SO(3)\}\ , \quad rH = \{rh : h \in H\}\ , \tag{7.49}$$

and where, as in Appendix A.2, the left $SO(3)$ action $l_H := L_{SO(3),H}$ is the function $l_H : SO(3) \times SO(3)/H \to SO(3)/H$ given, for $r, r' \in SO(3)$, by

$$l_H(r'; rH) := L_{SO(3),H}(r'; rH) = (r'r)H\ . \tag{7.50}$$

To start with the dynamical part of the ToA we have to consider the $\mathbb{Z}$ actions $L_d[l_H] \odot \mathcal{H}_{\omega,A}$ and $\tilde{L}_d[l_H] \odot \mathcal{H}_{\omega,A}$. We define the functions $L_{H,\omega,A} : \mathbb{Z} \times \mathbb{T}^d \times SO(3)/H \to \mathbb{T}^d \times SO(3)/H$ and $\tilde{L}_{H,\omega,A} : \mathbb{Z} \times \mathcal{C}(\mathbb{T}^d, SO(3)/H) \to \mathcal{C}(\mathbb{T}^d, SO(3)/H)$ by

$$L_{H,\omega,A} := L_d[l_H] \odot \mathcal{H}_{\omega,A}\ , \quad \tilde{L}_{H,\omega,A} := \tilde{L}_d[l_H] \odot \mathcal{H}_{\omega,A}\ , \tag{7.51}$$

However, in this work we do not study $L_{H,\omega,A}$, but concentrate instead on $\tilde{L}_{H,\omega,A}$, the entity of major interest. For $n \in \mathbb{Z}$, $f \in \mathcal{C}(\mathbb{T}^d, SO(3)/H)$ and by (7.32)and (7.51), we have

$$\tilde{L}_{H,\omega,A}(n; g) = (\tilde{L}_d[l_H] \odot \mathcal{H}_{\omega,A})(n; g) = l_H\Big(\Psi_{\omega,A}(n, \cdot); g\Big) \circ L_\omega(-n; \cdot)\ . \tag{7.52}$$

$$\tilde{L}_d[l]\Big(\hat{a}_d(j_2,k_2);\tilde{L}_d[l]\Big(\hat{a}_d(j_1,k_1);f\Big)\Big) = \tilde{L}_d[l]\Big(\hat{a}_d(j_2,k_2);id_{\mathbb{T}^d},l(k_1;f)\circ j_1^{-1}\Big)$$
$$= l\Big(k_2;l(k_1;f)\circ j_1^{-1}\Big)\circ j_2^{-1}\Big)$$
$$= l\Big(k_2\circ j_2^{-1};l\Big(k_1\circ j_1^{-1}\circ j_2^{-1};f\circ j_1^{-1}\circ j_2^{-1}\Big)\Big)$$
$$= l\Big((k_2\circ j_2^{-1})(k_1\circ j_1^{-1}\circ j_2^{-1});f\circ j_1^{-1}\circ j_2^{-1}\Big)\,, \tag{7.42}$$
$$\tilde{L}_d[l]\Big(\hat{a}_d(j_2,k_2)\Diamond\hat{a}_d(j_1,k_1);f\Big) = \tilde{L}_d[l]\Big(\hat{a}_d\Big(j_2\circ j_1,(k_2\circ j_1)k_1\Big);f\Big)$$
$$= l\Big((k_2\circ j_1)k_1;f\Big)\circ j_1^{-1}\circ j_2^{-1}$$
$$= l\Big((k_2\circ j_2^{-1})(k_1\circ j_1^{-1}\circ j_2^{-1});f\circ j_1^{-1}\circ j_2^{-1}\Big)\,, \tag{7.43}$$

so that indeed $\tilde{L}_d[l]$ is a left $\mathfrak{A}_d$ action on $\mathcal{C}(\mathbb{T}^d,E)$. Note that in the third equality of (7.41) and in the fourth equality of (7.42) we used the fact that l is a left $SO(3)$ action on E. □

While in this chapter the ToA is merely factory of group actions, it will become clear in Chapter ??, via the principal bundle λ_d, that the ToA is deeply rooted in principal-bundle theory.

7.3.2 Applying the ToA to the left $SO(3)$-space $(\mathbb{R}^3, l_{spin})$: Recovering $L_{\omega,A}$

In this section we arrive at the already-announced left $SO(3)$-space $(\mathbb{R}^3, l_{spin})$ by applying the ToA to the case of the spin-orbit motion introduced in Section 2.2. In particular we show that a left $SO(3)$-space (E,l) exists such that, for all (ω,A) in $\mathcal{SOT}(d)$, $L_{\omega,A}$ is of the form $L_d[l]\odot\mathcal{H}_{\omega,A}$ and we identify the l. We also show that the transformation rule (3.6) is a special case of (7.36).

Since, for every (ω,A) in $\mathcal{SOT}(d)$, $L_{\omega,A}$ is a left $\mathbb{Z}$ action on $\mathbb{T}^d\times\mathbb{R}^3$ we need $E=\mathbb{R}^3$. Moreover by inspection of $L_{\omega,A}$ in (2.28) and using (7.31) we are easily led to choose $l=l_{spin}$ where the function $l_{spin}:SO(3)\times\mathbb{R}^3\to\mathbb{R}^3$ is defined by

$$l_{spin}(r,S) := rS\,. \tag{7.44}$$

Clearly $(\mathbb{R}^3, l_{spin})$ is a left $SO(3)$-space and we see, by (7.31) and (7.44), that

$$(L_d[l_{spin}]\odot\mathcal{H}_{\omega,A})(n;z,S) = \Big(L_\omega(n;z),l_{spin}\Big(\Psi_{\omega,A}(n;z);S\Big)\Big) = \Big(L_\omega(n;z),\Psi_{\omega,A}(n;z)S\Big)\,,$$

whence, by (2.28),

$$L_d[l_{spin}]\odot\mathcal{H}_{\omega,A} = L_{\omega,A}\,. \tag{7.45}$$

$\chi = \Psi_{\omega,A}$, giving us

$$(L_d[l] \odot \mathcal{H}_{\omega,A})(n; z, x) = \Bigg(L_\omega(n; z), l\Big(\Psi_{\omega,A}(n, z); x \Big) \Bigg) , \tag{7.31}$$

$$(\tilde{L}_d[l] \odot \mathcal{H}_{\omega,A})(n; f) = l\Big(\Psi_{\omega,A}(n, \cdot); f \Big) \circ L_\omega(-n; \cdot) , \tag{7.32}$$

where $z \in \mathbb{T}^d$, $x \in E$, $f \in \mathcal{C}(\mathbb{T}^d, E), n \in \mathbb{Z}$ and $(\omega, A) \in \mathcal{SOT}(d)$

Of course $L_d[l] \odot \mathcal{H}_{\omega,A}$ and $\tilde{L}_d[l] \odot \mathcal{H}_{\omega,A}$ are $\mathbb{Z}$-actions. Recalling Definition 2.4 and using the fact that $\mathbb{Z}$ is equipped with the discrete topology and that l is continuous it is an easy exercise to even show, by (7.31), that $(\mathbb{T}^d \times E, L_d[l] \odot \mathcal{H}_{\omega,A})$ is a $\mathbb{Z}$-space.

END NEW

Remark:

(3) For every left $SO(3)$-space (E, l) and every spin-orbit torus (ω, A) in $\mathcal{SOT}(d)$ the $\mathbb{Z}$-action $L_d[l] \odot \mathcal{H}_{\omega,A}$ can be viewed as describing particle motion on the "phase space" E. Moreover the $\mathbb{Z}$-action $\tilde{L}_d[l] \odot \mathcal{H}_{\omega,A}$ can be viewed as describing field motion on the same phase space E. This view is further corroborated by using (7.31) and (7.32) since they imply that

$$(L_d[l] \odot \mathcal{H}_{\omega,A})(n; z, f(z)) = \Bigg(L_\omega(n; z), \Big((\tilde{L}_d[l] \odot \mathcal{H}_{\omega,A})(n; f) \Big) (L_\omega(n; z)) \Bigg) , \tag{7.33}$$

which can be interpreted as the statement that the "characteristic curves" of the field motion $\tilde{L}_d[l] \odot \mathcal{H}_{\omega,A}$ are trajectories of the particle motion $L_d[l] \odot \mathcal{H}_{\omega,A}$. In particular in the special case of the left $SO(3)$-space $(\mathbb{R}^3, l_{spin})$ the characteristic curves of the polarization fields will be spin-orbit trajectories. □

We now take a closer look at how to use $\mathcal{H}_d^{trans}$ in the ToA. So let (E, l) be a left $SO(3)$-space. Then the left $\mathcal{C}(\mathbb{T}^d, SO(3))$ action $L_d[l] \odot \mathcal{H}_d^{trans}$ on $\mathbb{T}^d \times E$ and the left $\mathcal{C}(\mathbb{T}^d, SO(3))$ action $\tilde{L}_d[l] \odot \mathcal{H}_d^{trans}$ on $\mathcal{C}(\mathbb{T}^d, E)$ satisfy

$$\begin{aligned}(L_d[l] \odot \mathcal{H}_d^{trans})(T; z, x) &= L_d[l](\mathcal{H}_d^{trans}(T); z, x) = L_d[l]\Big(\hat{a}_d(id_{\mathbb{T}^d}, T); z, x \Big) \\ &= \Bigg(z, l\Big(T(z); x \Big) \Bigg) ,\end{aligned} \tag{7.34}$$

$$(\tilde{L}_d[l] \odot \mathcal{H}_d^{trans})(T; f) = \tilde{L}_d[l](\mathcal{H}_d^{trans}(T); f) = \tilde{L}_d[l]\Big(\hat{a}_d(id_{\mathbb{T}^d}, T); f \Big) = l(T; f) , \tag{7.35}$$

where $z \in \mathbb{T}^d$, $x \in E$, $f \in \mathcal{C}(\mathbb{T}^d, E)$ and $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and where we used (7.12), (7.24), (7.25), (7.26) and (7.27).

We now see that every left $SO(3)$-space (E, l) leads to transformation rules since the general transformation rule (7.14) which is an equality of group homomorphisms becomes, where $g \in G$, $z \in \mathbb{T}^d, x \in E$ and $f \in \mathcal{C}(\mathbb{T}^d, E)$.

Remark:

(1) We expect that $L_d[l] \odot \mathcal{H}$ and $\tilde{L}_d[l] \odot \mathcal{H}$ to be left G actions and it is easy to show that they are. In fact this follows from the following simple lemma.

Let G, G' be groups, let $\psi : G \to G'$ be a group homomorphism and let (X, L) be a left G' set. We then define the function $(L \odot \psi) : G \times X \to X$ by

$$(L \odot \psi)(g, x) = L(\psi(g), x) \,. \tag{7.28}$$

Since ψ is a group homomorphism and (X, L) is a left G' set, it is an easy exercise to show by (7.28) that $(X, L \odot \psi)$ is a left G-set.

Clearly this lemma provides us with group actions by combining the group actions $L_d[l]$ and $\tilde{L}_d[l]$ with group homomorphisms. In fact in the special case where $G' = \mathfrak{A}_d$, $\psi = \mathcal{H}$, $X = \mathbb{T}^d \times E$ and $L = L_d[l]$ or $L = \tilde{L}_d[l]$ we find that $L_d[l] \odot \mathcal{H}$ and $\tilde{L}_d[l] \odot \mathcal{H}$ are left G actions, as was to be shown. □

Let $\mathcal{L}$ denote the class of all those l for which an E exists such that (E, l) is a left $SO(3)$-space. We thus define the ToA as the method which gives us the group actions $L_d[l] \odot \mathcal{H}$ and $\tilde{L}_d[l] \odot \mathcal{H}$ where l varies over $\mathcal{L}$ and where $\mathcal{H}$ is any group homomorphism into $\mathfrak{A}_d$ provided by the cocycle theorem, Theorem 7.1. The name "Technique of Association" reflects its origin in bundle theory as we will explain in Section ?? where we will tie the ToA with the notion of "associated bundle". Clearly the ToA is very general. So some words of clarification are in order.

The ToA can be viewed as a machine fabricating group actions by turning the two independent "knobs" $\mathcal{H}$ and (E, l). Since the knobs are independent, care is needed when one is looking for group actions $L_d[l] \odot \mathcal{H}$ and $\tilde{L}_d[l] \odot \mathcal{H}$ which give useful information about spin-orbit tori.

We first discuss the knob $\mathcal{H}$. While $\mathcal{H}$ can be any group homomorphism into $\mathfrak{A}_d$ provided by Theorem 7.1, the only $\mathcal{H}$ we are interested in this work are the $\mathcal{H}_{\omega,A}$ and the $\mathcal{H}_d^{trans}$. See however, Remark 2 below. Thus the adjustments for the first knob are clear from the start. Clearly our application of the ToA has a dynamical aspect via $\mathcal{H}_{\omega,A}$ and a transformational aspect via $\mathcal{H}_d^{trans}$.

Remark:

(2) While the only group homomorphisms $\mathcal{H}$ that interest us in this work are the $\mathcal{H}_{\omega,A}$ and the $\mathcal{H}_d^{trans}$, other group homomorphisms provided by Theorem 7.1 would be of interest in further studies as well. In particular in so-called "rigidity" studies one supplements each $\mathcal{H}_{\omega,A}$ by other group homomorphisms, say $\mathcal{H}'_{\omega,A}$, which extend $\mathcal{H}_{\omega,A}$ (so that $\mathbb{Z}$ is a subgroup of the domain of $\mathcal{H}'_{\omega,A}$) or which restrict $\mathcal{H}_{\omega,A}$ (so that the domain of $\mathcal{H}'_{\omega,A}$ is a subgroup of $\mathbb{Z}$). □

The second knob, the left $SO(3)$-space (E, l), determines the geometrical situation underlying $L_d[l] \odot \mathcal{H}$ and $\tilde{L}_d[l] \odot \mathcal{H}$. Thus the adjustments for the second knob will change as

Theorem 7.1 *(Cocycle Theorem) Let $\chi \in COC(\mathbb{T}^d, L, SO(3))$ where $(\mathbb{T}^d, L)$ is a left G-space and G is a topological group (recall Definition 2.6). Then the function $\mathcal{H}[\chi] : G \to \mathfrak{A}_d$, defined by*

$$\mathcal{H}[\chi](g) := \hat{a}_d\Big(L(g,\cdot), \chi(g,\cdot)\Big) , \tag{7.15}$$

where $g \in G$, is a group homomorphism from G to $\mathfrak{A}_d$.

Proof of Theorem 7.1: If $g, g' \in G$ then from (7.15) we find that

$$\mathcal{H}[\chi](g'g) = \hat{a}_d\Big(L(g'g,\cdot), \chi(g'g,\cdot)\Big) = \hat{a}_d\Big(L(g',\cdot) \circ L(g,\cdot), \chi(g'g,\cdot)\Big)$$
$$= \hat{a}_d\Big(L(g',\cdot) \circ L(g,\cdot), \chi(g'; L(g;\cdot))\chi(g;\cdot)\Big) , \tag{7.16}$$
$$\mathcal{H}[\chi](g')\diamond\mathcal{H}[\chi](g) = \hat{a}_d\Big(L(g',\cdot), \chi(g',\cdot)\Big)\diamond\hat{a}_d\Big(L(g,\cdot), \chi(g,\cdot)\Big) , \tag{7.17}$$

where in the second equality of (7.16) we used the fact that L is a left G-action and where in the third equality of (7.16) we used the cocycle condition (2.42) of χ. It follows from (7.6) and (7.17) that

$$\mathcal{H}[\chi](g')\diamond\mathcal{H}[\chi](g) = \hat{a}_d\Big(L(g',\cdot) \circ L(g,\cdot), (\chi(g';\cdot) \circ L(g;\cdot))\chi(g;\cdot)\Big) , \tag{7.18}$$

whence, by (7.16), $\mathcal{H}[\chi](g'g) = \mathcal{H}[\chi](g')\diamond\mathcal{H}[\chi](g)$ which implies, by Definition 2.1, that $\mathcal{H}[\chi]$ is a group homomorphism. □

We now show that $\mathcal{H}_{\omega,A}$ and $\mathcal{H}_d^{trans}$ belong to this set of group homomorphisms.

We start with $\mathcal{H}_{\omega,A}$. So let $(\omega, A) \in \mathcal{SOT}(d,\omega)$. In fact inspection of (7.10) easily leads us to the choice $\chi = \Psi_{\omega,A}$. Thus using (7.10) and (7.15) and recalling from Section 2.3 that $\Psi_{\omega,A} \in COC(\mathbb{T}^d, L_\omega, SO(3))$, we obtain

$$\mathcal{H}[\Psi_{\omega,A}](n) = \hat{a}_d\Big(L_\omega(n,\cdot), \Psi_{\omega,A}(n,\cdot)\Big) = \mathcal{H}_{\omega,A}(n) , \tag{7.19}$$

whence $\mathcal{H}_{\omega,A}$ indeed belongs to the group homomorphisms provided by Theorem 7.1.

To do the same for $\mathcal{H}_d^{trans}$, i.e., to identify $\mathcal{H}_d^{trans}$ as an $\mathcal{H}[\chi]$, we obviously need χ to belong to $COC(\mathbb{T}^d, L_d^{trans}, SO(3))$ where $(L_d^{trans}, SO(3))$ is an appropriately chosen left $\mathcal{C}(\mathbb{T}^d, SO(3))$-space. In fact inspection of (7.12) and (7.15) easily leads us to the following definitions. We first define the function $L_d^{trans} : \mathcal{C}(\mathbb{T}^d, SO(3)) \times \mathbb{T}^d \to \mathbb{T}^d$ by

$$L_d^{trans}(T; z) := z , \tag{7.20}$$

and we see that L_d^{trans} is a left $\mathcal{C}(\mathbb{T}^d, SO(3))$-action on $\mathbb{T}^d$. Clearly, regardless of the topology chosen on $\mathcal{C}(\mathbb{T}^d, SO(3))$, the function L_d^{trans} is continuous whence, by Definition 2.4, $(\mathbb{T}^d, L_d^{trans})$ is a left $\mathcal{C}(\mathbb{T}^d, SO(3))$-space (recall from Definition 2.6 that the topology on $\mathcal{C}(\mathbb{T}^d, SO(3)) \times \mathbb{T}^d$ is the product topology). We now define the function χ_d^{trans} : $\mathcal{C}(\mathbb{T}^d, SO(3)) \times \mathbb{T}^d \to SO(3)$ by

$$\chi_d^{trans}(T; z) := T(z) , \tag{7.21}$$

As always $\circ$ denotes composition of functions. To see that the rhs of (7.5) is in $\mathfrak{A}_d$ we note that $\check{a}_d(j,k)$ maps (z,r) to $(z',r') = (j(z), k(z)r)$ and $\check{a}_d(j',k')$ maps (z',r') to $(j'(z'), k'(z')r') = (j'(j(z)), k'(j(z))k(z)r)$. Thus $\check{a}_d(j',k') \circ \check{a}_d(j,k)$ maps (z,r) to $(j'(j(z)), k'(j(z))k(z)r)$ and we obtain for $j, j' \in Homeo(\mathbb{T}^d, \mathbb{T}^d)$ and $k, k' \in C(\mathbb{T}^d, SO(3))$,

$$\hat{a}_d(j',k') \diamond \hat{a}_d(j,k) = \hat{a}_d\Big(j' \circ j, (k' \circ j)k \Big) \ , \tag{7.6}$$

which is in $\mathfrak{A}_d$. Using (7.6) and noting that $(id_{E_d}, id_{\mathbb{T}^d})$ is the identity element of the group it is straightforward to check the group axioms in Definition 2.1. In particular $\hat{a}_d(j^{-1}, k^t \circ j^{-1})$ is the inverse of $\hat{a}_d(j,k)$ whence $\check{a}_d(j^{-1}, k^t \circ j^{-1})$ is the inverse of $\check{a}_d(j,k)$ so that $\check{a}_d(j,k) \in Homeo(E_d)$.

To introduce dynamics let $(\omega, A) \in \mathcal{SOT}(d)$ and we consider the function $\check{\mathcal{P}}_{\omega,A} : E_d \to E_d$ defined by

$$\check{\mathcal{P}}_{\omega,A}(z,r) := \begin{pmatrix} \mathcal{P}_\omega(z) \\ A(z)r \end{pmatrix} \ , \tag{7.7}$$

where $z \in \mathbb{T}^d$ and $r \in SO(3)$. By (7.3) and (7.7) and since $\mathcal{P}_\omega \in Homeo(\mathbb{T}^d)$,

$$(\check{\mathcal{P}}_{\omega,A}, \mathcal{P}_\omega) = \hat{a}_d(\mathcal{P}_\omega, A) \ , \tag{7.8}$$

so that $(\check{\mathcal{P}}_{\omega,A}, \mathcal{P}_\omega) \in \mathfrak{A}_d$ and we thereby see how $\mathfrak{A}_d$ provides a group structure for handling familiar objects. Since $(\mathfrak{A}_d, \diamond)$ is a group, we can see that for $n \in \mathbb{Z}$ the n-th power of $(\check{\mathcal{P}}_{\omega,A}, \mathcal{P}_\omega)$ belongs to $\mathfrak{A}_d$ whence we define the function $\mathcal{H}_{\omega,A} : \mathbb{Z} \to \mathfrak{A}_d$ by

$$\mathcal{H}_{\omega,A}(n) := (\check{\mathcal{P}}_{\omega,A}, \mathcal{P}_\omega)^n = (\check{\mathcal{P}}^n_{\omega,A}, \mathcal{P}^n_\omega) = (\check{\mathcal{P}}^n_{\omega,A}, \mathcal{P}_{n\omega}) \ . \tag{7.9}$$

Note that, for negative n, $(\check{\mathcal{P}}_{\omega,A}, \mathcal{P}_\omega)^n$ is the $|n|$-th iterate of the inverse $(\check{\mathcal{P}}_{\omega,A}, \mathcal{P}_\omega)^{-1}$. Note also that, by (2.27), (2.34), (7.6), (7.8) and (7.9),

$$\mathcal{H}_{\omega,A}(n) = \hat{a}_d\Big(L_\omega(n,\cdot), \Psi_{\omega,A}(n,\cdot) \Big) \ . \tag{7.10}$$

Equations (7.9) and (7.10) show how both he dynamical data of the spin-orbit motion in (2.28) and the polarization fields in (5.7) enter $\mathcal{H}_{\omega,A}$. By (7.5) and (7.9) we see that

$$\begin{aligned} \mathcal{H}_{\omega,A}(n+m) &= \Big(\check{\mathcal{P}}^{n+m}_{\omega,A}, \mathcal{P}^{n+m}_\omega \Big) = \Big(\check{\mathcal{P}}^n_{\omega,A} \circ \check{\mathcal{P}}^m_{\omega,A}, \mathcal{P}^n_\omega \circ \mathcal{P}^m_\omega \Big) \\ &= \Big(\check{\mathcal{P}}^n_{\omega,A}, \mathcal{P}^n_\omega \Big) \diamond \Big(\check{\mathcal{P}}^m_{\omega,A}, \mathcal{P}^m \Big) = \mathcal{H}_{\omega,A}(n) \diamond \mathcal{H}_{\omega,A}(m) \ , \end{aligned} \tag{7.11}$$

whence according to Definition 2.1, $\mathcal{H}_{\omega,A}$ is a group homomorphism from $(\mathbb{Z}, +)$ into $(\mathfrak{A}_d, \diamond)$ so that the range of $\mathcal{H}_{\omega,A}$ is a subgroup of $(\mathfrak{A}_d, \diamond)$. It is easy to see, although it is of no importance for us, that the range of $\mathcal{H}_{\omega,A}$ is rather simple since, by the isomorphism theorem of group theory, it is an Abelian group which is either finite or isomorphic to $(\mathbb{Z}, +)$.

7 Introducing the ToA. Applying the ToA to the left $SO(3)$-spaces $(SO(3)/H, l_H)$ to investigate SORs, IFFs and ISFs

7.1 Preliminaries

In this chapter we revisit and generalize the studies of the previous chapters by using our already-announced Technique of Association (ToA). In Chapters 2-6 we introduced the pair of $\mathbb{Z}$ actions $L_{\omega,A}$ and $\tilde{L}_{\omega,A}$. We will now show how to arrive at these and others via the ToA. In particular, we show how the ToA equips every spin-orbit torus with a pair of $\mathbb{Z}$ actions for each left $SO(3)$-space and thereby provides an infinite, but well defined class of $\mathbb{Z}$ actions to be used for every spin-orbit torus. Moreover, every such pair of $\mathbb{Z}$ actions contains one $\mathbb{Z}$ action describing a particle motion on some "phase space" and one $\mathbb{Z}$ action of motion on a field defined on that same phase space. In the case of $L_{\omega,A}$ and $\tilde{L}_{\omega,A}$ the underlying left $SO(3)$-space is called $(\mathbb{R}^3, l_{spin})$ and it will be defined in Section 7.3.

Then in Section 7.4 we go beyond the left $SO(3)$-space $(\mathbb{R}^3, l_{spin})$ by introducing an infinite family of left $SO(3)$-spaces, $(SO(3)/H, l_H)$, where H is a subgroup of $SO(3)$ and where H is assumed to be a closed subset of $SO(3)$. Then for each spin-orbit torus (ω, A) and each $(SO(3)/H, l_H)$, the ToA provides a pair of $\mathbb{Z}$ actions which we denote by $L_{H,\omega,A}$ and $\tilde{L}_{H,\omega,A}$ respectively. In fact, as we shall see in Section 7.4, the $\mathbb{Z}$ actions $\tilde{L}_{H,\omega,A}$ give us new insights into SORs, IFFs and ISFs via elegant existence criteria which lead to new avenues for studying spin-orbit tori. In Section 7.5 we show that in some sense the left $SO(3)$-spaces $(SO(3)/H, l_H)$ contain all the data one will ever extract from the ToA.

The tools of this chapter are as elementary as the ones in the previous chapters. We postpone the use of the more subtle tools of bundle theory to Chapter ?? (?????) but then we will gain further insight into the constructions of this chapter via the principal bundle λ_d. In fact, as we shall see, the ToA will turn out to be a technique having its origin in the bundles associated with λ_d.

We will proceed as follows. In Section 7.2 we will introduce the group $(\mathfrak{A}_d, \diamond)$ and the group homomorphisms $\mathcal{H}_{\omega,A}$ and $\mathcal{H}_d^{trans}$ into this group. In Section 7.3 the group homomorphism $\mathcal{H}_{\omega,A}$ will turn out to be the crucial tool by which the ToA provides the pair of $\mathbb{Z}$ actions for (ω, A) from every given left $SO(3)$-space. Moreover $\mathcal{H}_d^{trans}$ will be presented as the tool which provides the transformation rules for each pair of $\mathbb{Z}$ actions. In Section 7.4 the ToA will provide us with the $\mathbb{Z}$ actions $L_{H,\omega,A}$ and $\tilde{L}_{H,\omega,A}$ and demonstrate the impact of $\tilde{L}_{H,\omega,A}$ on the SOR, IFF and ISF. In Section 7.5 we show that the left $SO(3)$-spaces $(SO(3)/H, l_H)$ are fundamental in the sense that left $SO(3)$-spaces (E, l) can be "decomposed" into left $SO(3)$-spaces of the form $(SO(3)/H, l_H)$. So when applying the ToA to a left $SO(3)$-space (E, l) one can use the machinery of Section 7.4 on the individual components.

(7) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$. It is clear, by (5.10), that (5.22) maps the set $\mathcal{PF}(\omega, A)$ of polarization fields of (ω, A) bijectively onto the set $\mathcal{PF}(\omega, A')$. Moreover it is clear, by (5.13), that (5.22) maps $\mathcal{ISF}(\omega, A)$ bijectively onto $\mathcal{ISF}(\omega, A')$. In particular two similar spin-orbit tori have the same number of ISFs. Thus we arrived at another property shared by similar spin-orbit tori.

(8) Clearly the transformation rule (5.23) of the polarization field and the transformation rule (3.4) of spin trajectories are very similar. Unsurprisingly, one can even show the following. Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and let $\mathcal{S}$ be a polarization field of (ω, A). Also, $(\omega, A') := R_{d,\omega}(T;\omega, A)$ where of course $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and let $\mathcal{S}'$ be the polarization field of (ω, A') which is the transform of $\mathcal{S}$ as in (5.22). Clearly by Definition 5.1, if we pick $z \in \mathbb{T}^d$ then the function S_z defined by $S_z(n) := \mathcal{S}(n, \mathcal{P}^n_\omega(z))$ is a spin trajectory of (ω, A) over z and the function S'_z defined by $S'_z(n) := \mathcal{S}'(n, \mathcal{P}^n_\omega(z))$ is a spin trajectory of (ω, A') over z. The point here is that S'_z is the transform of S_z via (3.4). □

5.3 Polarization

We now tie together the concepts of polarization field and polarization. Thus consider a family $(\omega(J), A_J)_{J\in\Lambda}$ of spin-orbit tori where $(\omega(J), A_J) \in \mathcal{SOT}(d, \omega(J))$ and Λ is the set of action values.

We note (see also [BH, BV]) that for every $J \in \Lambda$, we have a so-called "local polarization", say $\mathcal{S}_{loc,J}$ which by definition is a polarization field of $(\omega(J), A_J)$ satisfying

$$|\mathcal{S}_{loc,J}| \leq 1 \, . \tag{5.24}$$

The associated bunch polarization is then given by

$$P(n) = \left| \int_\Lambda dJ \rho_{eq}(J) \int_{[0,2\pi]^d} d\phi \mathcal{S}_{loc,J}(n, [\phi]_d) \right| , \tag{5.25}$$

where $\rho_{eq} = \rho_{eq}(J)$ is the equilibrium orbital phase space density. In the so-called "spin equilibrium" the polarization fields $\mathcal{S}_{loc,J}$ are, by the definition of the spin equilibrium, invariant. Thus the bunch polarization for the combined beam equilibrium and spin equilibrium reads as

$$P(n) = P(0) = \left| \int_\Lambda dJ \rho_{eq}(J) \int_{[0,2\pi]^d} d\phi \mathcal{S}_{loc,J}(0, [\phi]_d) \right| , \tag{5.26}$$

whence

$$P(0) \leq \int_\Lambda dJ \rho_{eq}(J) \left| \int_{[0,2\pi]^d} d\phi \mathcal{S}_{loc,J}(0, [\phi]_d) \right| . \tag{5.27}$$

Note that we assume that the function ρ_{eq} is regular enough to ensure that the integrals in (5.25), (5.26) and (5.27) are meaningful. Then under the assumption that every $(\omega(J), A_J)$ has an ISF and since $|\mathcal{S}_{loc,J}| \leq 1$, with (5.27) we have

$$\mathcal{P}(0) \leq \mathcal{P}_{max}(0) \, , \tag{5.28}$$

If a spin-orbit torus (ω, A) has an ISF $\mathcal{S}$ then $-\mathcal{S}$ is also an ISF of (ω, A). So since $\mathcal{S} \neq -\mathcal{S}$, if (ω, A) has a finite number of ISF's, then this number is even. The important subcase where (ω, A) has exactly two ISF's is dealt with in Chapter 6. It is known [BV] that spin-orbit tori exist which are on orbital resonance and which have no ISF. Note that the spin-orbit torus of this example is on orbital resonance. There are some indications, mainly from numerical computations on ISF's, that practically relevant spin-orbit tori which have no ISF are "rare". Thus we state the following conjecture, which we call the "ISF-conjecture": If a spin-orbit torus (ω, A) is off orbital resonance, then it has an ISF. The ISF-conjecture is, at least to our knowledge, unsettled. The existence problem of the ISF is important both theoretically and practically and Chapter 7 presents a new framework for discussing it.

We now make some remarks on the relationship between ISFs and IFFs.

MATHIAS: explain why the ISF is tricky/special – continuity everywhere etc.

Remarks:

(3) Let $(\omega, A) \in \mathcal{CB}_{SO(2)}(d, \omega)$. Then (ω, A) has an ISF. To show that, we recall from Section 4.3 that (ω, A) has an IFF, say T, whence $N \in \mathbb{Z}^d$ and $h \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ exist such that (4.35) holds. This implies that

$$A([\phi]_d)T([\phi]_d) = T([\phi + 2\pi\omega]_d)\exp(\mathcal{J}[N \cdot \phi + 2\pi h([\phi]_d)]) \,. \tag{5.16}$$

Then by multiplying (5.16) from the right by $(0,0,1)^t$ and by using (4.3) we have,

$$\begin{aligned} A([\phi]_d)T([\phi]_d)(0,0,1)^t &= T([\phi + 2\pi\omega]_d)\exp(\mathcal{J}[N \cdot \phi + 2\pi h([\phi]_d)])(0,0,1)^t \\ &= T([\phi + 2\pi\omega]_d)(0,0,1)^t \,. \end{aligned} \tag{5.17}$$

Next, with $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ defined by $f(z) := T(z)(0,0,1)^t$, we see that $|f(z)| = |T(z)(0,0,1)^t| = |(0,0,1)^t| = 1$ so that $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{S}^2)$. Then from (5.17) $f \circ \mathcal{P}_\omega = Af$. So f satisfies the ISF criterion for (ω, A) whence (ω, A) indeed has an ISF. We have thereby shown that the third column of every IFF is the generator of an ISF.

(4) Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$. We now prove the converse of Remark 3. Thus let (ω, A) have an ISF so that, by the ISF criterion, a $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{S}^2)$ exists such that $f \circ \mathcal{P}_\omega = Af$. Let $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and let the third column of T be f. Then by the ISF criterion

$$T([\phi + 2\pi\omega]_d)(0,0,1)^t = f([\phi + 2\pi\omega]_d) = A([\phi]_d)f([\phi]_d) \,,$$

whence $(0,0,1)^t = T^t([\phi + 2\pi\omega]_d)A([\phi]_d)T([\phi]_d)(0,0,1)^t$. This implies, by (4.3), that $T^t([\phi + 2\pi\omega]_d)A([\phi]_d)T([\phi]_d)$ so that by Definition 4.5, $T \in \mathcal{TF}_{SO(2)}(\omega, A)$ whence T is an IFF of (ω, A). This raises the natural question about the conditions under which a spin-orbit torus with ISF has an IFF. Of course by Remark 3 we know that there are many situations where such a T exists. However, as shown in [He2], it can happen that (ω, A) has an ISF but no IFF. Thus the following question arises: Under which conditions on a $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{S}^2)$ does a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exist such that $f = T(0,0,1)^t$? In Chapter 7 of [He2] this question was studied by using simple arguments from Homotopy Theory. In Chapter 7 of the current work we will consider this question from another point of view.

where $A'' \in \mathcal{C}(\mathbb{T}^d, SO(3))$ is defined by (2.23), i.e., $A'' := (A' \circ \mathcal{P}_\omega)A$, whence

$$\tilde{\mathcal{P}}_{\omega,A} = \tilde{\mathcal{P}}_{\omega,A_{d,0}} \circ \tilde{\mathcal{P}}_{0,A} \, , \tag{5.4}$$

so that the inverse, $\tilde{\mathcal{P}}^{-1}_{\omega,A}$, of $\tilde{\mathcal{P}}_{\omega,A}$ is given by

$$\tilde{\mathcal{P}}^{-1}_{\omega,A} = \tilde{\mathcal{P}}_{0,A^t} \circ \tilde{\mathcal{P}}_{-\omega,A_{d,0}} \, . \tag{5.5}$$

Thus $\tilde{\mathcal{P}}_{\omega,A}$ is a bijection whence the function $\tilde{L}_{\omega,A} : \mathbb{Z} \times \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3) \to \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$, defined by

$$\tilde{L}_{\omega,A}(n; \cdot) := \tilde{\mathcal{P}}^n_{\omega,A} \, , \tag{5.6}$$

is a $\mathbb{Z}$-action on $\mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ where $\tilde{\mathcal{P}}^n_{\omega,A}$ denotes the n-th iteration of $\tilde{\mathcal{P}}_{\omega,A}$. Clearly $(\mathcal{C}(\mathbb{T}^d, \mathbb{R}^3), \tilde{L}_{\omega,A})$ is a $\mathbb{Z}$-set. Note that, by (2.34), (5.2) and (5.6),

$$\tilde{L}_{\omega,A}(n; f) = \Big(\Psi_{\omega,A}(n; \cdot) f \Big) \circ L_\omega(-n; \cdot) \, . \tag{5.7}$$

Of course, with (5.2) the evolution equation (5.1) can be written as $\mathcal{S}(n+1, \cdot) = \tilde{\mathcal{P}}_{\omega,A}(\mathcal{S}(n, \cdot))$ whence, by (5.6), for every polarization field $\mathcal{S}$

$$\mathcal{S}(n, \cdot) = \tilde{L}_{\omega,A}(n; \mathcal{S}(0, \cdot)) \, . \tag{5.8}$$

Remark:

(2) Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$. Comparing (5.1) and (2.37) we see they are both linear systems. However (5.1) is more complex in that it depends on two independent variables but it is simpler in that it is autonomous. Accordingly, the transformation rule (5.23) of the polarization field is autonomous while the transformation rule (3.4) of the spin trajectories is nonautonomous.

Before we take a closer look at the linearity of (5.1) we make some general comments on linearity. Let G be a group and (E, L) be a left G set. Also let E be a vector space and let every $L(g, \cdot)$ be linear where, of course, $g \in G$. Recalling Definition 2.3, the $L(g, \cdot)$ are bijections whence, since they are linear, they are automorphisms of the vector space E, i.e., $L(g, \cdot) \in GL(E)$ where $GL(E)$ denotes the set of automorphisms of the vector space E. One can cast the data of L into the function $L^{hom} : G \to GL(E)$ defined by $L^{hom}(g) := L(g, \cdot)$. The key point here is that, since $L(g, \cdot) \in GL(E)$, the function L^{hom} is a homomorphism from the group $\mathbb{Z}$ into the group $GL(E)$ where the group multiplication in $GL(E)$ is understood to be the composition of functions. In other words L^{hom} is a so-called "representation" of the group $\mathbb{Z}$ on the linear space E. Thus the notion of group representation emerges as a specialization of the notion of left group action. In fact the first contact of a physicist with group actions typically occurs via group representations. For the definition of group and group homomorphism, see Definition 2.1.

Now let $(\omega, A) \in \mathcal{SOT}_{SO(2)}(d,\omega)$ so that, by (4.1b), $A \in \mathcal{C}(\mathbb{T}^d, SO(2))$. Then, by (4.33), $N \in \mathbb{Z}^d$ and $a \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ exist such that

$$A([\phi]_d) = \exp(\mathcal{J}[N \cdot \phi + 2\pi a([\phi]_d)]) \,. \tag{4.34}$$

The elements of $\mathcal{TF}_{SO(2)}(\omega, A)$ are the discrete-time analogues of the invariant frame field (IFF) described in the continuous-time formalism, e.g., in [BEH]. This can be seen as follows.

Let $(\omega, A) \in \mathcal{CB}_{SO(2)}(d,\omega)$ and let us pick a $T \in \mathcal{TF}_{SO(2)}(\omega, A)$. Then, by (3.10), (4.1b), (4.34), (4.37), $N \in \mathbb{Z}^d$ and $h \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ exist such that (4.38???????)

$$T^t([\phi + 2\pi\omega]_d)A([\phi]_d)T([\phi]_d) = \exp(\mathcal{J}[N \cdot \phi + 2\pi h([\phi]_d)]) \,. \tag{4.35}$$

Of course the rhs of (4.35) is the 1-turn spin transfer matrix of $R_{d,\omega}(T;\omega, A)$. If S is a spin trajectory of (ω, A) over $z_0 = [\phi_0]_d$, then by our transformation rule (3.4) we can transform S into a spin trajectory $S'(n) = T^t(L_\omega(n; z_0))S(n)$ of $R_{d,\omega}(T;\omega, A)$ and we see by (2.37) and (4.35) that S' obeys the simple EOM:

$$S'(n+1) = \exp\Big(\mathcal{J}[N \cdot (\phi_0 + 2\pi\omega n) + 2\pi h(L_\omega(n; z_0))]\Big)S'(n) \,. \tag{4.36}$$

We now define

$$\begin{aligned} \mathcal{IFF}(\omega, A) &:= \mathcal{TF}_{SO(2)}(\omega, A) \\ &= \{T \in \mathcal{C}(\mathbb{T}^d, SO(3)) : R_{d,\omega}(T;\omega, A) \in \mathcal{SOT}_{SO(2)}(d,\omega)\} \,. \end{aligned} \tag{4.37}$$

We call every element of $\mathcal{IFF}(\omega, A)$ an "IFF of (ω, A)". Clearly, by Definition 4.5, $\mathcal{IFF}(\omega, A)$ is nonempty iff $(\omega, A) \in \mathcal{CB}_{SO(2)}(d,\omega)$.

(MENTION QUAIPERIODICITY??????????)

For the case when T is chosen so that the argument of the exponential is independent of ϕ, $T(\phi)$ is analogous to the uniform IFF of the continuous-time formalism [BEH]. In that case we can write the argument as $2\pi\nu$ where ν is the ADST. Of course since, by (4.18) and (4.30), $\mathcal{TF}^{const}_{SO(2)}(\omega, A) \subset \mathcal{TF}_{SO(2)}(\omega, A)$ we have, by (4.20) and (4.37),

$$\mathcal{UIFF}(\omega, A) \subset \mathcal{IFF}(\omega, A) \,. \tag{4.38}$$

It is noteworthy that the constant N in (4.35) carries interesting information about A. For example for (ω, A) to be proper it is necessary that all d components of N are even integers. This is shown in Section 7.2 of [He2] by using simple arguments from Homotopy Theory. We will return to (4.35) later on.

Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$. By Remark 2 in Section 4.1 the set $\mathcal{UIFF}(\omega, A)$ is nonempty so that, by (4.38), $\mathcal{IFF}(\omega, A)$ is nonempty. Then we have

$$\mathcal{ACB}(d,\omega) \subset \mathcal{CB}_{SO(2)}(d,\omega) \,. \tag{4.39}$$

Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$. We will now briefly discuss how IFFs are important from a practical point of view. In fact in the computer code SPRINT [EPAC98, BHV98, Ho, Vo, BHV00, BEH00] one computes a spin tune ν of the first kind in two steps. By (4.39) $\mathcal{IFF}(\omega, A)$ is nonempty. Then in the first step of SPRINT one computes an IFF of (ω, A), say T. By (4.35), an $N \in \mathbb{Z}^d$ and a $h \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ exist such that (4.35) holds. In the second step of SPRINT one computes ν by doing some averaging (NEEDS FIXING ?????) Fourier Analysis of h. For more remarks on the first step see Remark 6 in Chapter 5.

us to the Uniqueness Theorem for the ISF [Yo1, DK73]. A rigorous definition, as in Definition 4.4, is therefore very relevant for understanding real spin motion.

See [?, ?, ?, ?, ?] for formalisms and calculations which have demonstrated the potential for a large spread of the ISF near spin-orbit resonances. For detailed further comments see Section X in [BEH].

c) SPECIAL STRUCTURE IN CHAPTER 9,

Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and let us perturb $\mathcal{P}_{\omega,A}(z,S)$ into $\mathcal{P}_{\omega,A}(z,S) + \varepsilon \begin{pmatrix} 2\pi a \\ B(z)S \end{pmatrix}$. Then on the basis of the above notion of spin-orbit resonance of the first kind, we will have motions far from leading order resonances (FLOR) and near to leading order resonances (NLOR), where $a \in \mathbb{R}^d$, $B(z) \in \mathbb{R}^{3\times 3}$. For example, $\nu - m\cdot\omega - n$ will appear as a small divisor in the analysis. (IMPROVE?????) □

4.3 *H* normal forms and the subsets $\mathcal{CB}_H$ of $\mathcal{SOT}$

Recall again that each spin-orbit torus shares many properties with all similar ones so that in order to study these properties of (ω, A) one should look for the simple elements of $\overline{(\omega, A)}$. In Sections 4.1 and 4.2 we studied this issue for when these simple elements belong to $\mathcal{SOT}^{const}_{SO(3)}(d,\omega)$. Of course, then $\overline{(\omega, A)}$ even contains spin-orbit tori from $\mathcal{SOT}^{const}_{SO(2)}(d,\omega) \subset \mathcal{SOT}_{SO(2)}(d,\omega)$.

Thus it is a natural to look into the more general situation when $\overline{(\omega, A)} \cap \mathcal{SOT}_{SO(2)}(d,\omega)$ is nonempty or the even more general case when $\overline{(\omega, A)} \cap \mathcal{SOT}_H(d,\omega)$ is nonempty. Note that $\mathcal{SOT}^{const}_{SO(2)}(d,\omega)$ is a proper subset of $\mathcal{SOT}_{SO(2)}(d,\omega)$. So this point of view is indeed a generalization of the one in Sections 4.1,4.2. (IMPROVE TEXT?????)

Thus in this section we discuss those (ω, A) for which $\overline{(\omega, A)}$ contains elements in $\mathcal{SOT}_H(d,\omega)$ where H is a subgroup of $SO(3)$ with special emphasis on the case $H = SO(2)$. This leads to the concept of "H normal form" given by the following definition.

Definition 4.5 *(H normal form, $\mathcal{CB}_H(d,\omega), \mathcal{CB}_H$)*
Let H be a subgroup of $SO(3)$ and let (ω, A) be in $\mathcal{SOT}(d,\omega)$. Then we call a (ω, A') in $\mathcal{SOT}_H(d,\omega)$ an "H normal form of (ω, A)" if $(\omega, A) \sim (\omega, A')$, i.e., $(\omega, A') \in \overline{(\omega, A)}$. We denote by $\mathcal{CB}_H(d,\omega)$ the set of all spin-orbit tori in $\mathcal{SOT}(d,\omega)$ which have an H normal form, i.e.,

$$\mathcal{CB}_H(d,\omega) := \bigcup_{(\omega,A)\in\mathcal{SOT}_H(d,\omega)} \overline{(\omega, A)} \,. \tag{4.28}$$

Thus $(\omega, A) \in \mathcal{CB}_H(d,\omega)$ iff $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that

$$T^t(\mathcal{P}_\omega(z))A(z)T(z) \in H \,, \tag{4.29}$$

holds for every $z \in \mathbb{T}^d$. We denote, for fixed H, the union of all $\mathcal{CB}_H(d,\omega)$ by $\mathcal{CB}_H$. The acronym $\mathcal{CB}_H$ will be explained further below. We also define

$$\mathcal{TF}_H(\omega, A) := \Big\{ T \in \mathcal{C}(\mathbb{T}^d, SO(3)) : R_{d,\omega}(T;\omega, A) \in \mathcal{SOT}_H(d,\omega) \Big\} \,.$$

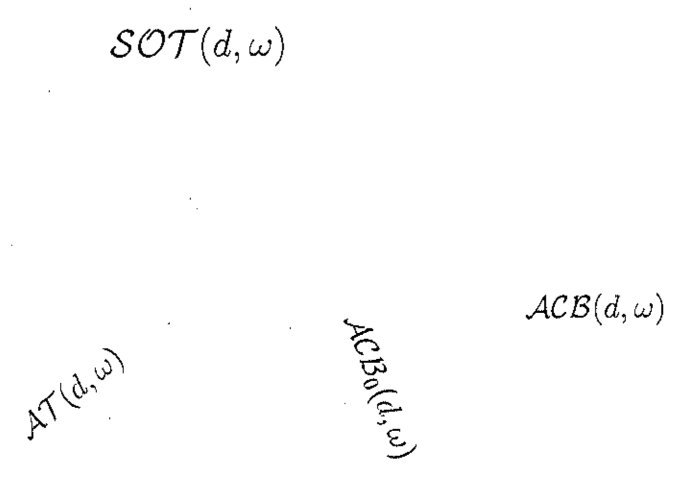

Figure 3: A symbolic representation of the relations between the sets $\mathcal{ACB}$ etc........

4.2 Spin tunes and spin-orbit resonances of the first kind

We now come to the definition of spin tune. A $\nu \in [0,1)$ is said to be a spin tune for $(\omega, A) \in \mathcal{SOT}(d,\omega)$ if $(\omega, A) \sim_{d,\omega} (\omega, \exp(2\pi\nu\mathcal{J}))$. More formally we have the definition

Definition 4.3 *(Spin tune of the first kind)*
If $(\omega, A) \in \mathcal{SOT}(d,\omega)$ we define the set

$$\Xi_1(\omega, A) := \{\nu \in [0,1) : (\omega, A) \in \mathcal{ACB}_\nu\} = \{\nu \in [0,1) : (\omega, A) \in \overline{(\omega, A_{d,\nu})}\} . \qquad (4.25)$$

We call the elements of $\Xi_1(\omega, A)$ the spin tunes of the first kind of (ω, A). □

It follows from Definitions 4.1, 4.3 that, for every $(\omega, A) \in \mathcal{SOT}(d,\omega)$, the set $\Xi_1(\omega, A)$ is nonempty iff $(\omega, A) \in \mathcal{ACB}(d,\omega)$. Note also, by Definitions 4.2, 4.3, that 0 is a spin tune of the first kind of a spin-orbit torus iff that spin-orbit torus is in $\mathcal{CB}$. Most importantly, for every $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and every $\nu \in \Xi_1(\omega, A)$ we have the relation

$$\Xi_1(\omega, A) = [0,1) \cap \left\{ \varepsilon\nu + m \cdot \omega + n : \varepsilon \in \{1,-1\}, m \in \mathbb{Z}^d, n \in \mathbb{Z} \right\} . \qquad (4.26)$$

In fact the inclusion $\Xi_1(\omega, A) \supset [0,1) \cap \left\{ \nu + m \cdot \omega + n : m \in \mathbb{Z}^d, n \in \mathbb{Z} \right\}$ is very easily demonstrated as follows.

So let $\nu \in \Xi_1(\omega, A)$. Then, by Definition 4.3, $(\omega, A) \in \overline{(\omega, A_{d,\nu})}$ whence recalling Definition 3.1, the function $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$, defined by $T([\phi]_d) := \exp(-\mathcal{J} m \cdot \phi)$ with $m \in \mathbb{Z}^d$, belongs to $\mathcal{TF}_{d,\omega}(A_{d,\nu}, A_{d,\nu'})$ where $\nu' \in [0,1)$ is defined by $\nu' := \nu + m \cdot \omega \bmod 1$. Thus, by Definition 3.2, $\overline{(\omega, A_{d,\nu})} = \overline{(\omega, A_{d,\nu'})}$ so that $(\omega, A) \in \overline{(\omega, A_{d,\nu'})}\}$ which implies, by Definition

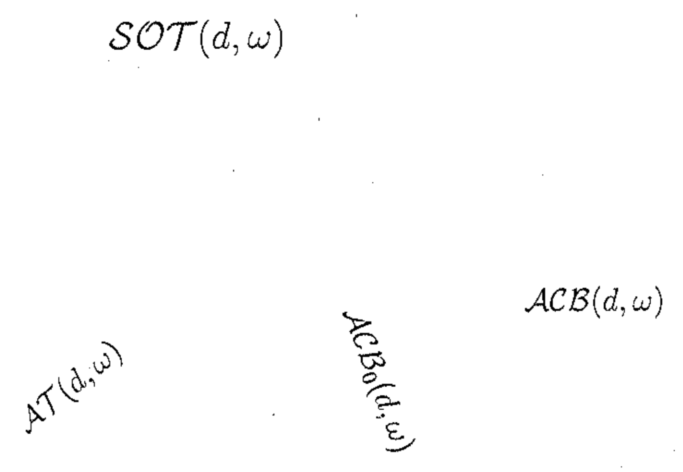

Figure 2: A symbolic representation of the relations between the sets $\mathcal{SOT}(d,\omega)$, $\mathcal{AT}(d,\omega)$, $\mathcal{ACB}(d,\omega)$ and $\mathcal{ACB}_0(d,\omega)$ defined in the text. The pink area represents a part of $\mathcal{SOT}(d,\omega)$ and the red, blue and green locii represent $\mathcal{AT}(d,\omega)$, $\mathcal{ACB}_0(d,\omega)$ and $\mathcal{ACB}(d,\omega)$ respectively. The $\mathcal{ACB}_0(d,\omega)$ crosses the $\mathcal{AT}(d,\omega)$ at.......

acronym $\mathcal{ACB}$ in Definition 4.1 since the spin transfer matrices of the spin-orbit tori in $\mathcal{ACB}$ are so-called "almost coboundaries" (see, e.g., [KR]).

Remark:

(1) Definition 4.1 gives us another property shared by similar spin-orbit tori since it implies that if (ω, A) belongs to $\mathcal{ACB}$ then every spin-orbit torus in $\overline{(\omega, A)}$ belongs to $\mathcal{ACB}$. □

We now formalize the transfer fields associated with $\mathcal{SOT}_H^{const}(d,\omega)$. Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$. Then we define

$$\mathcal{TF}_H^{const}(\omega, A) := \{T \in \mathcal{C}(\mathbb{T}, SO(3)) : R_{d,\omega}(T; \omega, A) \in \mathcal{SOT}_H^{const}(d,\omega)\} , \quad (4.18)$$

i.e., $\mathcal{TF}_H^{const}(\omega, A)$ is the set of all transfer fields from (ω, A) to spin-orbit tori in $\mathcal{SOT}_H^{const}(d,\omega)$. In the special case $H = SO(2)$, (4.18) gives us

$$\mathcal{TF}_{SO(2)}^{const}(\omega, A) = \bigcup_{\nu \in [0,1)} \{T \in \mathcal{TF}_{d,\omega}(A, A_{d,\nu})\} . \quad (4.19)$$

By Definition 4.1 it is clear that a spin-orbit torus $(\omega, A) \in \mathcal{SOT}(d,\omega)$ belongs to $\mathcal{ACB}(d,\omega)$ iff $\mathcal{TF}_{SO(2)}^{const}(\omega, A)$ is nonempty, i.e., iff $\mathcal{TF}_{SO(3)}^{const}(\omega, A)$ is nonempty. Note also that the arguments that led to (4.4) imply that if $(\omega, A) \in \mathcal{SOT}_{SO(3)}^{const}(d,\omega)$ then $\mathcal{TF}_{SO(2)}^{const}(\omega, A)$ contains a transfer field which is constant valued.

Remark:

The set in (4.7) contains the most important spin-orbit tori in applications. For the subgroup

$$G_\nu := \{\exp(2\pi n\nu\mathcal{J}) : n \in \mathbb{Z}\}\,, \tag{4.8}$$

of $SO(2)$ we see that $A_{d,\nu} \in G_\nu$ and that

$$\begin{aligned} \mathcal{SOT}^{const}_{SO(2)}(d,\omega) &= \bigcup_{\nu\in[0,1)} \mathcal{SOT}^{const}_{G_\nu}(d,\omega)\,, \\ \mathcal{SOT}^{const}_{SO(2)} &= \bigcup_{\nu\in[0,1)} \mathcal{SOT}^{const}_{G_\nu}\,. \end{aligned} \tag{4.9}$$

Of course, with (4.8), the trivial subgroup of $SO(3)$ is G_0, i.e.,

$$G_0 = \{I_{3\times 3}\}\,, \tag{4.10}$$

and by (4.1b), (4.5) and(4.8) we have

$$\mathcal{SOT}_{G_0} = \mathcal{SOT}^{const}_{G_0} = \{(\omega, A_{d,0}) : d \in \mathbb{N}, \omega \in \mathbb{R}^d\}\,, \tag{4.11}$$

where, by (4.5), $A_{d,0}$ is the $I_{3\times3}$ valued function on $\mathbb{R}^d$. Note that all spin-orbit tori in every $\mathcal{SOT}^{const}_{SO(3)}(d,\omega)$ are proper, i.e.,

$$\mathcal{SOT}^{const}_{SO(3)}(d,\omega) \subset \mathcal{SOT}_{cont}(d,\omega)\,. \tag{4.12}$$

In fact if $(\omega, A) \in \mathcal{SOT}^{const}_{SO(3)}(d,\omega)$ then a skew-symmetric matrix $\mathcal{A}$ exists in $\mathbb{R}^{3\times3}$ such that $A = \exp(2\pi\mathcal{A})$ and we see from Section 2.1 that

$$A^{\theta_0}_{\omega,\mathcal{A}} = A\,, \tag{4.13}$$

for every $\theta_0 \in \mathbb{R}$. We now formalize these ideas into a definition.

Definition 4.1 *($\mathcal{ACB}_\nu(d,\omega), \mathcal{ACB}_\nu, \mathcal{ACB}(d,\omega), \mathcal{ACB}$)*
For $\nu \in [0,1), \omega \in \mathbb{R}^d$ we denote the set of those spin-orbit tori in $\mathcal{SOT}(d,\omega)$ which are similar to $(\omega, \exp(2\pi\nu\mathcal{J}))$ by $\mathcal{ACB}_\nu(d,\omega)$, i.e.,

$$\mathcal{ACB}_\nu(d,\omega) := \overline{(\omega, A_{d,\nu})} = \{R_{d,\omega}(T;\omega,A_{d,\nu}) : T \in \mathcal{C}(\mathbb{T}^d, SO(3))\}\,, \tag{4.14}$$

and we denote, for fixed ν, their union by $\mathcal{ACB}_\nu$, i.e.,

$$\mathcal{ACB}_\nu := \bigcup_{d\in\mathbb{N},\omega\in\mathbb{R}^d} \mathcal{ACB}_\nu(d,\omega)\,. \tag{4.15}$$

We also define

$$\begin{aligned} \mathcal{ACB}(d,\omega) &:= \bigcup_{\nu\in[0,1)} \mathcal{ACB}_\nu(d,\omega) = \bigcup_{\nu\in[0,1)} \overline{(\omega, A_{d,\nu})} \\ &= \bigcup_{(\omega,A)\in\mathcal{SOT}^{const}_{SO(3)}(d,\omega)} \overline{(\omega,A)} = \bigcup_{(\omega,A)\in\mathcal{SOT}^{const}_{SO(2)}(d,\omega)} \overline{(\omega,A)}\,, \end{aligned} \tag{4.16}$$

where in the third and fourth equalities we used (4.7). We abbreviate their union by $\mathcal{ACB}$, i.e., $\mathcal{ACB} := \bigcup_{d\in\mathbb{N},\omega\in\mathbb{R}^d} \mathcal{ACB}(d,\omega)$. □

whence

$$f\Big(R_{d,\omega}(T;\omega,A)\Big) = R_{d,-\omega}\Big(T; f(\omega,A)\Big),$$

so that, by Definition 3.6, f is a $\mathcal{C}(\mathbb{T}^d, SO(3))$-map from $(\mathcal{SOT}(d,\omega), R_{d,\omega})$ to $(\mathcal{SOT}(d,-\omega), R_{d,-\omega})$.

The function f has an obvious interpretation in terms of time reversal as follows. Using the EOM (2.37), we see that if S is a spin trajectory of (ω, A) over z_0 then the "time inverted" function S', defined by $S'(n) := S(-n)$, is a spin trajectory over z_0 of the spin-orbit torus $f(\omega, A)$. Note also that f is a bijection onto $\mathcal{SOT}(d,-\omega)$ since f is its own inverse, that is, $f \circ f$ is the identity function on $\mathcal{SOT}(d,\omega)$. Thus recalling Definition 3.6, the right $\mathcal{C}(\mathbb{T}^d, SO(3))$ sets $(\mathcal{SOT}(d,\omega), R_{d,\omega})$ and $(\mathcal{SOT}(d,-\omega), R_{d,-\omega})$ are isomorphic right G sets.

(7) Let G be a topological group and let (E, R) be a right G space. Then, by Definitions 3.5 and 3.6, p_R is a topological G-map from (E, R) to the trivial right G space over E/R. □

4 Spin tunes and spin-orbit resonances of first kind and H normal forms

WE NO LONGER HAVE SPIN TUNES OF THE SECOND KIND. SO WE SHOULD MENTION THEM SOMEHOW (????????????)

One important motivation for the transformation rule of Definition 3.1 is that, under certain circumstances, a spin-orbit torus can be transformed into a simpler one. In fact as mentioned in Chapter 3 each spin-orbit torus shares many properties with all similar ones so that in order to study these properties of (ω, A) one should look for the simple elements of $\overline{(\omega, A)}$. This is the subject of this section and it will enable us to associate extra tunes, namely spin tunes, with our spin-orbit tori. As in other dynamical systems, tunes can lead to the recognition of resonances and consequent instabilities. Here, spin tunes will lead to recognition of spin-orbit resonances. In the case of real spin motion, where spins are subject to the electric and magnetic fields on synchro-betatron trajectories, the definition of spin-orbit resonance allows us predict at which orbital tunes spin motion might be particularly unstable. The definition of spin tune is also associated with the concept of normal forms for spin. Here, we will go beyond the usual definition of normal form for spin [Yo2] to take a broader view by introducing H normal forms where H is a subgroup of $SO(3)$.

We thus define, for an arbitrary subgroup H of $SO(3)$,

$$\mathcal{SOT}_H(d,\omega) := \{(\omega, A) \in \mathcal{SOT}(d,\omega) : A \in \mathcal{C}(\mathbb{T}^d, H)\},$$
$$\mathcal{SOT}_H := \bigcup_{d\in\mathbb{N},\omega\in\mathbb{R}^d} \mathcal{SOT}_H(d,\omega), \tag{4.1a}$$

where, as always in this work, the topology of a subgroup H of $SO(3)$ is the relative topology from $SO(3)$. Clearly the sets in (4.1a) give us spin-orbit tori which are the simpler the smaller

rephrase?

If G is an Abelian group then every right G action R is also called a "G action" and every right G set is also called "G set". Thus if G is an Abelian group then the notions right G action, left G action, and G action are synonymous and the notions right G set, left G set, and G set are synonymous. □

END NEW

Remark:

(5) The notions of left and right are dual. In fact if R is a right G action on E then the function $L : G \times E \to E$ defined by $L(g;x) := R(g^{-1};x)$ is a left G action on E. Moreover if L is a left G action on E then the function $R : G \times E \to E$ defined by $R(g;x) := L(g^{-1};x)$ is a right G action on E. □

To show that $R_{d,\omega}$ is a right action, note that $(G,*) = (\mathcal{C}(\mathbb{T}^d, SO(3)), *)$ is a group with identity element e_G where $*$ denotes pointwise multiplication. In particular, $T * T'$ is defined by $(T * T')(z) := T(z)T'(z)$ where e_G is the constant $I_{3\times 3}$ valued function, i.e., $e_G = A_{d,0}$ where $A_{d,0} \in \mathcal{C}(\mathbb{T}^d, SO(3))$ is defined by $A_{d,0}(z) = I_{3\times 3}$. Using Definition 3.1 and (3.10) we obtain

$$\begin{aligned} R_{d,\omega}(e_G;\omega,A) &= \Big(\omega, A_{d,0}^t A A_{d,0}\Big) = (\omega, A)\,, \\ R_{d,\omega}\Big(T'; R_{d,\omega}(T;\omega,A)\Big) &= R_{d,\omega}\Big(T';\omega,(T^t \circ \mathcal{P}_\omega)AT\Big) \\ &= \Big(\omega, ((T')^t \circ \mathcal{P}_\omega)(T^t \circ \mathcal{P}_\omega)ATT'\Big) = R_{d,\omega}(TT';\omega,A)\,, \end{aligned} \tag{3.13}$$

whence $R_{d,\omega}$ is indeed a right $\mathcal{C}(\mathbb{T}^d, SO(3))$ action on $\mathcal{SOT}(d,\omega)$ so that $(\mathcal{SOT}(d,\omega), R_{d,\omega})$ is a right $\mathcal{C}(\mathbb{T}^d, SO(3))$ set. The orbits of the group action $R_{d,\omega}$ are the equivalence classes of $\sim_{d,\omega}$. Note also that the group $\mathcal{C}(\mathbb{T}^d, SO(3))$ is not Abelian since the group $SO(3)$ is not Abelian so that the right action $R_{d,\omega}$ is not a left action.

While $R_{d,\omega}$ is of course important in this work since it determines the transformation rule, the fact that $R_{d,\omega}$ is a group action does not play a major role in this paper. In fact we introduced the right group action $R_{d,\omega}$ just to prepare the reader for important right actions in Chapter 7. These right group actions have additional structure as formalized in the following definition.

NEW

Definition 3.5 *(Right G space)*
Let E be a topological space where E is nonempty, G be a topological group, and let R be a right G action on E with R being continuous where $G \times E$ carries the product topology. Then the pair (E,R) is called a "right G space". Note that each $R(g;\cdot)$ is a homeomorphism. Recalling from Definition (3.4) the notation E/R and p_R, we equip E/R with its natural topology, i.e., a subset M of E/R is open iff $p_R^{-1}(M)$ is open in E. Thus the function p_R is onto E/R and identifying and one calls E/R an "orbit space". Also each orbit is equipped with the relative topology from E. In the important subcase when the topology of G is discrete (e.g., when $G = \mathbb{Z}$) the condition that R is continuous is equivalent to $R(g;\cdot)$

3.2 Topological G-maps of left G spaces

We now look at how the left $\mathbb{Z}$ spaces $(\mathbb{T}^d \times \mathbb{R}^3, L_{\omega,A})$, defined in Section 2.3, are related for similar spin-orbit tori. From (3.1)

$$\mathcal{P}_{0,T}^{-1} \circ \mathcal{P}_{\omega,A}^{n} \circ \mathcal{P}_{0,T} = \mathcal{P}_{\omega,A'}^{n} \,. \tag{3.5}$$

Therefore, by (2.28), $L_{\omega,A'}(n;\cdot) = \mathcal{P}_{0,T}^{-1} \circ L_{\omega,A}(n;\cdot) \circ \mathcal{P}_{0,T}$, so that

$$\mathcal{P}_{0,T}^{-1} \circ L_{\omega,A}(n;\cdot) = L_{\omega,A'}(n;\cdot) \circ \mathcal{P}_{0,T}^{-1} \,. \tag{3.6}$$

Thus according to the following definition, $\mathcal{P}_{0,T}$ and $\mathcal{P}_{0,T}^{-1}$ are topological $\mathbb{Z}$-maps.
NEW

Definition 3.3 *(G-maps of left G sets, topological G-maps of left G spaces)*
a) Let G be a group and let $(E, L), (E', L')$ be left G sets and consider the function $f : E \to E'$. If for $g \in G, x \in E$, f satisfies

$$f(L(g; x)) = L'(g; f(x)) \,, \tag{3.7}$$

then f is called a "G-map from (E, L) to (E', L')".
b) Let G be a topological group. Let $(E, L), (E', L')$ be left G spaces and let $f \in C(E, E')$. If f satisfies (3.7) then f is called a "topological G-map from (E, L) to (E', L')". □

If f is a G-map from the left G set (E, L) to the left G set (E', L') and if f is a bijection onto E', then f^{-1} is a G-map from (E', L') to (E, L) and (E', L') and (E, L) are called "isomorphic" and thus are effectively the same. We then also say that L' and L are "isomorphic".

Analogously when f is a topological G-map from the left G space (E, L) to the left G space (E', L') and if f is a homeomorphism onto E' then (E', L') and (E, L) are called "isomorphic" and are effectively the same.

Parallel behavior

In our special case it follows from (3.6) and Definitions 3.1 and 3.3 that if T is a transfer field from (ω, A) to (ω, A') then the $\mathcal{P}_{0,T}^{-1}$ is a topological $\mathbb{Z}$-map from the left $\mathbb{Z}$ space $(\mathbb{T}^d \times \mathbb{R}^3, L_{\omega,A})$ to the left $\mathbb{Z}$ space $(\mathbb{T}^d \times \mathbb{R}^3, L_{\omega,A'})$ and, since $\mathcal{P}_{0,T}^{-1} \in Homeo(\mathbb{T}^d \times \mathbb{R}^3)$, $\mathcal{P}_{0,T}$ is a topological $\mathbb{Z}$-map from $(\mathbb{T}^d \times \mathbb{R}^3, L_{\omega,A'})$ to $(\mathbb{T}^d \times \mathbb{R}^3, L_{\omega,A})$.
END NEW

Remark:

(4) Eqn. (2.28) provides another example of a topological $\mathbb{Z}$-map in our context. Thus let $f \in C(\mathbb{T}^d \times \mathbb{R}^3, \mathbb{T}^d)$ be defined by $f(z, S) := z$. Then, by (2.28), $f\Big(L_{\omega,A}(n; z, S)\Big) = L_\omega(n; z) = L_\omega(n; f(z, S))$, i.e.,

$$f \circ L_{\omega,A}(n;\cdot) = L_\omega(n;\cdot) \circ f \,. \tag{3.8}$$

So we see by (3.8) and Definition 3.3 that f is a topological $\mathbb{Z}$-map from the $\mathbb{Z}$ space $(\mathbb{T}^d \times \mathbb{R}^3, L_{\omega,A})$ to the $\mathbb{Z}$ space $(\mathbb{T}^d, L_\omega)$. Since $\mathbb{T}^d \times \mathbb{R}^3$ is the cartesian product of $\mathbb{T}^d$ and $\mathbb{R}^3$ and f is the projection onto $\mathbb{T}^d$, the fact that f is a topological $\mathbb{Z}$-map means that

MATHIAS: where is K in this definition? That's why the examples are quoted.

We denote the collection of all K cocycles over (E, L) by $COC(E, L, K)$. Note that $G \times E$ carries the product topology. For literature on cocycles, see, e.g., [HK1, KR, Zi1]. □

The reader will easily appreciate the similarity between the structures of (2.33) and (2.42) and the correspondence between the functions $\Psi_{\omega,A} \in \mathcal{C}(\mathbb{Z} \times \mathbb{T}^d, SO(3))$ and $f \in \mathcal{C}(G \times E, K)$. Since $(\mathbb{T}^d, L_\omega)$ is a left $\mathbb{Z}$ space and $SO(3)$ is a topological group, the set $COC(\mathbb{T}^d, L_\omega, SO(3))$ is well defined. In fact since $\Psi_{\omega,A} \in \mathcal{C}(\mathbb{Z} \times \mathbb{T}^d, SO(3))$ it follows from (2.33) that, for every $(\omega, A) \in \mathcal{SOT}(d, \omega)$, $\Psi_{\omega,A} \in COC(\mathbb{T}^d, L_\omega, SO(3))$. Conversely, every Ψ in $COC(\mathbb{T}^d, L_\omega, SO(3))$ is the spin transfer matrix of a spin-orbit torus since, by defining $A := \Psi(1; \cdot)$, we have $\Psi_{\omega,A} = \Psi$ so that Ψ is the spin transfer matrix of (ω, A). We thus arrive at

$$COC(\mathbb{T}^d, L_\omega, SO(3)) = \{\Psi_{\omega,A} : (\omega, A) \in \mathcal{SOT}(d, \omega)\} . \tag{2.43}$$

Clearly the cocycles $\Psi_{\omega,A}$ are important for spin-orbit motion of spin-orbit tori. Further below when we will see more $\mathbb{Z}$ sets carrying valuable information about (ω, A) that all of them carry $\Psi_{\omega,A}$ in an explicit way (see Chapter 7).

THIS SENTENCE NEEDS WORK ON THE SYNTAX. WHY NOT MENTION CH.7 AT THE START (???????)

3 Transforming spin-orbit tori

3.1 Conjugacies and the transformation rule of spin-orbit tori

We now consider the basic structure of $\mathcal{SOT}(d, \omega)$. The dynamics of each element of $\mathcal{SOT}(d, \omega)$ is given by its 1-turn map $\mathcal{P}_{\omega,A}$ and we are interested in those (ω, A) which have similar dynamics. This is made precise by the notion of conjugacy. Recall that $\mathcal{P}_{\omega,A} \in Homeo(\mathbb{T}^d \times \mathbb{R}^3)$. Two functions $f, g \in Homeo(\mathbb{T}^d \times \mathbb{R}^3)$ are said to be "conjugate" if a $t \in Homeo(\mathbb{T}^d \times \mathbb{R}^3)$ exists such that $g = t^{-1} \circ f \circ t$. We denote this similarity by $f \sim g$. To see the effect on the dynamics we note that $y_n := g^n(y_0) = (t^{-1} \circ f \circ t)^n(y_0) = (t^{-1} \circ f^n \circ t)(y_0)$ whence $t(y_n) = f^n(t(y_0))$. So $x_n = t(y_n)$ where $x_n := f^n(x_0)$ and many properties of $x_0, x_1, \ldots$ and $y_0, y_1, \ldots$ are similar, e.g., existence of fixed points or periodic solutions. Furthermore $f \sim g$ defines an equivalence relation on $Homeo(\mathbb{T}^d \times \mathbb{R}^3)$ whose equivalence classes form a partition of $Homeo(\mathbb{T}^d \times \mathbb{R}^3)$.

We now formalize this in the context of $\mathcal{SOT}(d, \omega)$. By (2.19), $\mathcal{P}_{\omega,A}(z, S) = \begin{pmatrix} \mathcal{P}_\omega(z) \\ A(z)S \end{pmatrix}$ and we will focus on its spin component, i.e., the A. Thus we consider the transformations $\mathcal{P}_{0,T}$ defined by $\mathcal{P}_{0,T}(z, S) = \begin{pmatrix} z \\ T(z)S \end{pmatrix}$.

Clearly if $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$ then, by (2.19), the equality

$$\mathcal{P}_{0,T}^{-1} \circ \mathcal{P}_{\omega,A} \circ \mathcal{P}_{0,T} = \mathcal{P}_{\omega,A'} \tag{3.1}$$

holds iff

$$A'(z) = T^t(\mathcal{P}_\omega(z))A(z)T(z) . \tag{3.2}$$

(G2) (Identity element e_G) $\exists_{e_G \in G} \forall_{g \in G}\ e_G = e_G * g = g * e_G$,

(G3) (Inverse elements) $\forall_{g_1 \in G} \exists_{g_2 \in G}\ e_g = g_1 * g_2 = g_2 * g_1$,

Note that $*(g_1, g_2)$ is abbreviated by $g_1 * g_2$. (WHERE DOES THIS COME FROM??????) Note that we always abbreviate $(G, *)$ as G when the operation $*$ is clear from the context and that we often write $g_1 * g_2$ as $g_1 g_2$ when the operation $*$ is clear from the context. The inverse element of a $g \in G$ is denoted by g^{-1}. Two subgroups G', G'' of a group G are called "conjugate" if $g \in G$ exists such that $G'' = \{gg'g^{-1} : g' \in G'\}$.

A group $(G, *)$ is called "Abelian" if

(G4) (Commutativity) $\forall_{g_1, g_2 \in G}\ g_1 * g_2 = g_2 * g_1$.

If $(G, *)$ and $(G', **)$ are groups then a function $f : G \to G'$ is called a group homomorphism from $(G, *)$ to $(G', **)$ if $f(g_1 * g_2) = f(g_1) ** f(g_2)$. If f is even a bijection then it is called a group isomorphism from $(G, *)$ to $(G', **)$ and the groups $(G, *), (G', **)$ are called "isomorphic". Clearly conjugate subgroups are isomorphic. □

If G is Abelian then $*$ is often replaced with $+$. Important examples of groups in Section 2.2 are $(\mathbb{Z}, +)$, $(SO(3), *)$ (where the binary operation is matrix multiplication) and $(\mathbb{R}, +)$. We abbreviate them by $\mathbb{Z}, SO(3)$ and $\mathbb{R}$ and note that $\mathbb{Z}$ and $\mathbb{R}$ are Abelian.

Definition 2.2 *(Topological group)*
A "topological group" is a group $(G, *)$ where G is a topological space, where the binary operation $*$ is continuous and where the function $g \mapsto g^{-1}$ on G is also continuous. □

The above mentioned groups $\mathbb{Z}$ and $SO(3)$ in Section 2.2 are topological as we consider them to be equipped with their standard topologies. Thus the topology of $\mathbb{Z}$ is discrete, i.e., every subset of $\mathbb{Z}$ is open and $SO(3)$ is equipped with the relative topology from $\mathbb{R}^{3\times3}$.

NEW

Definition 2.3 *(Left G action, left G set)*
Let G be a group with identity element e_G and let E be a nonempty set. Then a function $L : G \times E \to E$ is called a "left G action on E" if, for $g_1, g_2 \in G, x \in E$,

$$L(e_G; x) = x \tag{2.39}$$

$$L(g_1 g_2; x) = L(g_1; L(g_2; x)) . \tag{2.40}$$

MATHIAS: expand the symbol (E, L) to include G?

If L is a left G action on E then the pair (E, L) is called a "left G set". Since $L(e_G; x) = L(g; L(g^{-1}; x))$ each $L(g; \cdot)$ is a bijection, i.e., $L(g; \cdot) \in \mathrm{Bij}(E)$. If (E, L) is a left G set and $x \in E$ then the set $\{L(g; x) : g \in G\}$ is called the "orbit of x under L". We denote the set of orbits by E/L and define the function $p_L : E \to E/L$ by $p_L(x) := \{L(g; x) : g \in G\}$. Thus $p_L(x)$ the orbit of x under L. Note that the orbits form a partition of E. If the orbit of x only consists of x itself then x is called a "fixed point" of (E, L) and of L and the set of fixed points of (E, L) is denoted by $Fix(L)$. Note that this notion of "orbit" is different from the physical notion of "orbital motion". For the definition of $L(g; \cdot)$ and bijection, see Appendix A. If G is an Abelian group then every left G action L is also called a "G action" and every left G set is also called "G set". □

as we now argue. With this notation, we impose the convention that $\mathcal{P}^0_{\omega,A}$ is the identity function on $\mathbb{T}^d \times \mathbb{R}^3$ and that for n negative, $\mathcal{P}^n_{\omega,A}$ is the $|n|$-th iterate of the inverse $\mathcal{P}^{-1}_{\omega,A}$. From the first component of (2.19) we conclude that the function $L_\omega : \mathbb{Z} \times \mathbb{T}^d \to \mathbb{T}^d$ in (2.26) is given by

$$L_\omega(n; z) := \mathcal{P}^n_\omega(z) = \mathcal{P}_{n\omega}(z) \ . \tag{2.27}$$

From the second component of (2.19) we conclude that the function $\Psi_{\omega,A}$ is $SO(3)$-valued, i.e., $\Psi_{\omega,A} : \mathbb{Z} \times \mathbb{T}^d \to SO(3)$. We define the function $L_{\omega,A} : \mathbb{Z} \times \mathbb{T}^d \times \mathbb{R}^3 \to \mathbb{T}^d \times \mathbb{R}^3$ for $n \in \mathbb{Z}, z \in \mathbb{T}^d, S \in \mathbb{R}^3$ by

$$L_{\omega,A}(n; z, S) := \mathcal{P}^n_{\omega,A}(z, S) = \begin{pmatrix} L_\omega(n; z) \\ \Psi_{\omega,A}(n; z)S \end{pmatrix} \ . \tag{2.28}$$

By the definition of composition

$$\mathcal{P}^{m+n}_{\omega,A} = \mathcal{P}^n_{\omega,A} \circ \mathcal{P}^m_{\omega,A} \ , \tag{2.29}$$

and we have

$$L_{\omega,A}(n + m, z, S) = L_{\omega,A}(n; L_{\omega,A}(m; z, S)) \ . \tag{2.30}$$

This gives, by (2.28),

$$\begin{pmatrix} L_\omega(n + m; z) \\ \Psi_{\omega,A}(n + m; z)S \end{pmatrix} = L_{\omega,A}(n + m, z, S) = L_{\omega,A}(n; L_{\omega,A}(m; z, S))$$
$$= L_{\omega,A}\left(n; \begin{pmatrix} L_\omega(m; z) \\ \Psi_{\omega,A}(m; z)S \end{pmatrix}\right) = \left(\begin{matrix} L_\omega(n; L_\omega(m; z)) \\ \Psi_{\omega,A}(n; L_\omega(m; z))\Psi_{\omega,A}(m; z)S \end{matrix}\right)\Bigg) \ , \tag{2.31}$$

which implies

$$L_\omega(n + m, z) = L_\omega(n; L_\omega(m; z)) \ , \tag{2.32}$$

MATHIAS: mention cocycles here already?
and

$$\Psi_{\omega,A}(n + m; z) = \Psi_{\omega,A}(n; L_\omega(m; z))\Psi_{\omega,A}(m; z) \ . \tag{2.33}$$

As a consequence of (2.33) we get

$$\begin{aligned} &\Psi_{\omega,A}(0; z) = I_{3\times 3} \ , \\ &\Psi_{\omega,A}(n; z) = A(L_\omega(n - 1; z)) \cdots A(L_\omega(1; z))A(z) \ , \qquad (n = 1, 2, \ldots) \ , \\ &\Psi_{\omega,A}(n; z) = A^t(L_\omega(n; z)) \cdots A^t(L_\omega(-1; z)) \ , \quad (n = -1, -2, \ldots) \ , \end{aligned} \tag{2.34}$$

where we also used (2.27). It is easy to obtain (2.33) directly by iteration of (2.19), i.e., without using $L_{\omega,A}$, however the procedure here is more pedagogical since the pairs $(\mathbb{T}^d, L_\omega)$

With Remark 1 we define $A_{\omega,\mathcal{A}} \in \mathcal{C}(\mathbb{T}^d, SO(3))$ by

$$A_{\omega,\mathcal{A}}([\phi]_d) := \Phi_{\omega,\mathcal{A}}(2\pi;\phi) , \tag{2.14}$$

and we ~~abbreviate~~ define

$$\mathcal{SOT}_{cont}(d,\omega) := \{(\omega, A_{\omega,\mathcal{A}}) : (\omega,\mathcal{A}) \in \widetilde{\mathcal{SOT}}(d,\omega)\} , \tag{2.15}$$

where the suffix "cont" indicates that the elements of $\mathcal{SOT}_{cont}(d,\omega)$ come from a continuous time treatment. Moreover we define the function $\mathcal{P}_{\omega,\mathcal{A}} \in \mathcal{C}(\mathbb{T}^d \times \mathbb{R}^3, \mathbb{T}^d \times \mathbb{R}^3)$ by

$$\mathcal{P}_{\omega,\mathcal{A}}([\phi]_d, S) := \begin{pmatrix} [\phi + 2\pi\omega]_d \\ A_{\omega,\mathcal{A}}(\phi) S \end{pmatrix} . \tag{2.16}$$

Note that $\mathcal{P}_{\omega,\mathcal{A}}$ is the PM on $\mathbb{T}^d \times \mathbb{R}^3$.

Remark:

(2) It is worthwile to show how one proves the continuity of $A_{\omega,\mathcal{A}}$ since we will use this method time and again. First of all we note that $A_{\omega,\mathcal{A}}$ is well defined since, if $[\phi]_d = [\phi']_d$ then $(\phi - \phi')/2\pi \in \mathbb{Z}^d$ whence $A_{\omega,\mathcal{A}}([\phi]_d) = \Phi_{\omega,\mathcal{A}}(2\pi;\phi) = \Phi_{\omega,\mathcal{A}}(2\pi;\phi') = A_{\omega,\mathcal{A}}([\phi']_d)$ where in the second equality we used the fact that $\Phi_{\omega,\mathcal{A}}(2\pi;\phi)$ is 2π-periodic in the components of ϕ. To show that $A_{\omega,\mathcal{A}}$ is continuous we recall from Remark 1 above that π_d is identifying. Since π_d is identifying and $\Phi_{\omega,\mathcal{A}}(2\pi;\cdot)$ is continuous and since, by (2.14), $A_{\omega,\mathcal{A}} \circ \pi_d = \Phi_{\omega,\mathcal{A}}(2\pi;\cdot)$ we conclude that $A_{\omega,\mathcal{A}}$ is continuous, i.e., $A_{\omega,\mathcal{A}} \in \mathcal{C}(\mathbb{T}^d, SO(3))$. As always $\circ$ denotes composition of functions (see Appendix A).

The continuity of $\mathcal{P}_{\omega,\mathcal{A}}$ can be shown by the same method. □

2.2 Introducing the set $\mathcal{SOT}$ of spin-orbit tori

We now generalize $\mathcal{SOT}_{cont}(d,\omega)$ by defining, for every $\omega \in \mathbb{R}^d$,

$$\mathcal{SOT}(d,\omega) := \{(\omega, A) : A \in \mathcal{C}(\mathbb{T}^d, SO(3))\} , \tag{2.17}$$

where A need not be derivable from (2.1, 2.2) via (2.14) as we will see below. We denote the union of the $\mathcal{SOT}(d,\omega)$ over ω by $\mathcal{SOT}(d)$ and the union of the $\mathcal{SOT}(d,\omega)$ over d and ω by $\mathcal{SOT}$ and call every pair (ω, A) in $\mathcal{SOT}$ a "spin-orbit torus". Since the function $A_{\omega,\mathcal{A}}$ belongs to $\mathcal{C}(\mathbb{T}^d, SO(3))$ we see from (2.15) and (2.17) that

$$\mathcal{SOT}_{cont}(d,\omega) \subset \mathcal{SOT}(d,\omega) . \tag{2.18}$$

We call ω the "orbital tune vector" of a spin-orbit torus (ω, A) and A its "1-turn spin transfer matrix". Next, motivated by (2.10) we define, for every (ω, A) in $\mathcal{SOT}(d,\omega)$, the function $\mathcal{P}_{\omega,A} : \mathbb{T}^d \times \mathbb{R}^3 \to \mathbb{T}^d \times \mathbb{R}^3$ by

$$\mathcal{P}_{\omega,A}(z, S) := \begin{pmatrix} \mathcal{P}_\omega(z) \\ A(z) S \end{pmatrix} , \tag{2.19}$$

$$\frac{dS}{d\theta} = \mathcal{A}(\theta,\phi)S\,, \qquad S(0) = S_0 \in \mathbb{R}^3\,, \tag{2.2}$$

where $\omega \in \mathbb{R}^d$ and where the function $\mathcal{A} : \mathbb{R}^{d+1} \to \mathbb{R}^{3\times3}$ is continuous in ϕ and piecewise continuous in θ. More precisely $\mathcal{A}$ is either continuous or a finite number of θ values $\theta_1, ..., \theta_N$ exist such that $\mathcal{A}$ is continuous on $(\mathbb{R}\setminus\{\theta_1,...,\theta_N\})\times\mathbb{R}^d$ and such that $\mathcal{A}(\theta_1;\cdot), ..., \mathcal{A}(\theta_N;\cdot)$ are continuous. For the $\cdot$ notion see Appendix A. Moreover we assume that $\mathcal{A}$ is 2π-periodic in each of its $d+1$ arguments and that it is skew–symmetric, i.e., $\mathcal{A}^t(\theta,\phi) = -\mathcal{A}(\theta,\phi)$. We denote the set of pairs $(\omega,\mathcal{A})$, where $\omega\in\mathbb{R}^d$ and where $\mathcal{A}$ satisfies the above conditions, by $\widetilde{\mathcal{SOT}}(d,\omega)$.

As is clear from the Introduction, the above IVP and the assumptions on $\mathcal{A}$ are motivated by our underlying interest in spin-orbit motion in storage rings. In the application to spin motion in storage rings, S is a column vector of components of the spin S and $\mathcal{A}(\theta,\phi) \equiv \mathcal{A}_J(\theta,\phi)$ represents the rotation rate vector $\Omega(\theta,J,\phi)$ of the T-BMT equation [BEH]. Here J,ϕ are the action-angle variables of an integrable orbital motion. In this work we are only occasionally interested in the J-dependence. So we suppress the J unless we need it since most definitions in this work do not involve the J. The acronym $\mathcal{SOT}$ stands for spin-orbit torus, reflecting the fact that orbital motion, $\phi(\theta) = \phi_0 + \omega\theta$ can be represented on a torus $\mathbb{T}^d$. In fact from Section 2.2 onwards the orbital motion will be on $\mathbb{T}^d$. The set $\widetilde{\mathcal{SOT}}(d,\omega)$ includes standard spin-orbit motion but need not, and is therefore more general, in keeping with our wish to investigate the properties of any system defined by (2.1, 2.2).

Since the system (2.1, 2.2) is periodic in θ it is convenient to study the behavior of solutions in terms of the Poincaré map (PM) [AP, HK2]. We now derive a convenient representation for the PM. Solving (2.1) gives

Starting at $\theta=0$

$$\phi(\theta) = \phi_0 + \omega\theta\,, \tag{2.3}$$

whence (2.2) reads as

$$\frac{dS}{d\theta} = \mathcal{A}(\theta,\phi_0+\omega\theta)S\,, \qquad S(0) = S_0 \in \mathbb{R}^3\,. \tag{2.4}$$

Since $\mathcal{A}(\theta;\phi)$ is piecewise continuous in θ it can be shown [Cr] that the IVP (2.4) has a unique solution S in the sense that

$$S(\theta) = S_0 + \int_0^\theta \mathcal{A}(t,\phi_0+\omega t)S(t)dt\,. \tag{2.5}$$

It follows that $S(\theta)$ is continuous in θ. The proof in Cronin [Cr] doesn't include the parameter ϕ_0 but it is easily added.

Since the EOM (2.4) is linear in S the general solution of (2.5) can be written

$$S(\theta) = \Phi_{\omega,\mathcal{A}}(\theta;\phi_0)S_0\,, \tag{2.6}$$

where the function $\Phi_{\omega,\mathcal{A}} : \mathbb{R}\times\mathbb{R}^d \to SO(3)$ satisfies, due to (2.5),

$$\Phi_{\omega,\mathcal{A}}(\theta;\phi_0) = I_{3\times3} + \int_0^\theta \mathcal{A}(t,\phi_0+\omega t)\Phi_{\omega,\mathcal{A}}(\theta;\phi_0)dt\,. \tag{2.7}$$

In storage-ring physics there are two main approaches for dealing with the independent variable in the equations of motion (EOM), namely use of the flow formalism or the map formalism. In the flow formalism the EOM is an ODE, whence the independent variable is the continuous variable $\theta \in \mathbb{R}$ describing the distance around the ring. In the map formalism the independent variable in the EOM is the discrete variable $n \in \mathbb{Z}$ labelling the turn number where $\mathbb{Z}$ denotes the set of integers. In Dynamical-Systems theory it is common practice to refer to the independent variable in the EOM such as θ, the "time" and that is the convention that we will use here. Thus the two approaches are based on continuous time and discrete time and in the following we will refer to these as the continuous-time and discrete-time formalisms. In [BEH] we used the continuous-time formalism. Here, the emphasis is on use of discrete-time.

Physical electric and magnetic fields are smooth. So the assumption of smoothness adopted in [BEH] appears to be perfectly reasonable. On the other hand, practical numerical spin–orbit tracking simulations are usually carried out with fields which cut off sharply at the ends of magnets and/or with thin-lens approximations. Thus in [BEH] our formalism involved smoothness in the time variable θ although numerical calculations cited there in Sec. X had been obtained using hard-edged and thin-lens fields. However, hard-edged and thin-lens ring elements fit naturally into the discrete-time formalism. In particular, for this, we merely require that the fields are continuous (i.e., of class C^0) in the orbital phases and we allow jump discontinuities in θ. Of course, this still allows study of systems with fields smooth in θ and/or the orbital phases. The way that the discrete-time formalism derives from the continuous-time formalism is explained in Section 2.1. It is also an easy exercise to translate the machinery of the present work to the continuous-time formalism and this would give results which substantially go beyond those in [BEH].

This work is designed so that it can be read independently of [BEH]. However, we wish to avoid repeating the copious contextual material contained in [BEH]. We therefore invite the reader to consult the Introduction and the Summary and Conclusion in [BEH] in order to acquire a better appreciation of the context. In this work, as in [BEH], the orbital motion is integrable and we allow the number of angle variables, d, to be arbitrary (but ≥ 1) although for spin-orbit motion in storage rings, the case $d = 3$ is the most important. We use the symbols $\phi = (\phi_1, ..., \phi_d)^t$, $J = (J_1, ..., J_d)^t$ and $\omega(J) = (\omega_1(J), ..., \omega_d(J))^t$ respectively the lists of orbital angles, orbital actions and orbital tunes where with continuous time $d\phi/d\theta = \omega(J)$. Note that ^{t} denotes the transpose. In the continuous-time formalism, the T–BMT equation is written as $dS/d\theta = \Omega(\theta, J, \phi(\theta)) \times S$ where the vector S is the spin expectation value ("the spin") in the rest frame of a particle and Ω is the precession vector obtained as indicated in [BEH] from the electric and magnetic fields on the particle trajectory. Note that $\phi \in \mathbb{R}^d$ but as it is common and convenient we replace $\mathbb{R}^d$ by the torus $\mathbb{T}^d$ and thereby map ϕ to ϕ mod 2π in $\mathbb{T}^d$. For the purposes of this work we don't need to consider the whole (J, ϕ) phase space since it will suffice to confine ourselves to a fixed J-value whence to spin-orbit motion on a single torus. Thus the actions J are just parameters. However it is likely that our work can be easily generalized to arbitrary orbital motion if one maintains our condition that the orbital motion is unaffected by the spin motion.

The work is structured as follows. In Section 2.1 we discuss the continuous-time formalism which will motivate, in Section 2.2, the discrete-time concept of the spin-orbit torus. In Section 2.1 we also define those spin-orbit tori which can be derived from the continuous-