# 1 Introduction

This work, which is a sequel to [BEH], is based largely on mathematical concepts and ideas in the PhD Thesis [He2] of the first author (KH), where methods from the theory of dynamical systems and topology are exploited to distil some essential features of spin motion in storage rings.

In storage-ring physics there are two main approaches for dealing with the independent variable in the equations of motion (EOM), namely use of the flow formalism or the map formalism. In the flow formalism the EOM is an ordinary differential equation (ODE), whence the independent variable is the continuous variable $\theta \in \mathbb{R}$ describing the distance around the ring. In the map formalism the independent variable in the EOM is the discrete variable $n \in \mathbb{Z}$ labelling the turn number where $\mathbb{Z}$ denotes the set of integers. In dynamical-systems theory it is common practice to refer to the independent variable in the EOM such as $\theta$, the "time" and that is the convention that we will use here. Thus the two approaches are based on continuous time and discrete time and in the following we will refer to these as the continuous-time and discrete-time formalisms. In [BEH] we used the continuous-time formalism. Here, the emphasis is on use of discrete-time.

In [BEH] we undertook an extensive study of the concept of spin tune in storage rings based on the Thomas–Bargmann–Michel–Telegdi (T–BMT) equation [Ja] of spin precession. For this we assumed that the orbital motion was independent of the spin, and was described by an integrable Hamiltonian system in action-angle variables, $J, \phi$. We further assumed that the electric and magnetic fields were smooth (of class $C^1$, i.e., continuously differentiable) both in $\phi$ and $\theta$. Thus the T–BMT equation became a linear system of ODEs for the spin motion with smooth coefficients depending quasiperiodically on $\theta$. This quasiperiodic structure led us to a generalization of the Floquet theorem and a new approach to the spin tune. Since physical electric and magnetic fields are smooth, this is a perfectly reasonable approach. On the other hand, practical numerical spin–orbit tracking simulations are usually

Principal Bundles, i.e. mathematical structures familiar from Yang-Mills theories & particle physics carried out with fields which cut off sharply at the ends of magnets and/or with thin-lens approximations. Thus in [BEH] our formalism involved smoothness in the time variable $\theta$ although numerical calculations cited there in Sec. X had been obtained using hard-edged and thin-lens fields. However, hard-edged and thin-lens ring elements fit naturally into the discrete-time formalism. In particular, for this, we merely require that the fields are continuous (i.e., of class $C^0$) in the orbital phases and we allow jump discontinuities in $\theta$. Of course, this still allows study of systems with fields smooth in $\theta$ and/or the orbital phases. The way that the discrete-time formalism derives from the continuous-time formalism is explained in Section 2.1.

Although accelerator physicists tend to concentrate on studying spin motion in real storage rings, many of the issues surrounding the invariant spin field and spin tune depend just on the *structure* of the equations of spin-orbit motion and can be treated in isolation from the original physical system. This, together with the use of discrete time, is the strategy to be adopted here and it clears the way for the focus on purely mathematical matters and in particular for the exploitation of methods from the theory of dynamical systems and theorems from cocycle theory and the topology of bundles. It is also an easy exercise to translate the machinery of the present work to the continuous-time formalism and this would give results which substantially go beyond those in [BEH].

This work is designed so that it can be read independently of [BEH]. However, we wish to avoid repeating the copious contextual material contained in [BEH]. We therefore invite the reader to consult the Introduction and the Summary and Conclusion in [BEH] in order to acquire a better appreciation of the context. In this work, as in [BEH], the orbital motion is integrable and we allow the number of angle variables, $d$, to be arbitrary (but $\geq 1$) although for spin motion in storage rings, the case $d = 3$ is the most important. We use the symbols $\phi = (\phi_1, ..., \phi_d)^t$, $J = (J_1, ..., J_d)^t$ and $\omega(J) = (\omega_1(J), ..., \omega_d(J))^t$ respectively the lists of orbital angles, orbital actions and orbital tunes where with continuous time $d\phi/d\theta = \omega(J)$ . Note that $^t$ denotes the transpose. In the continuous-time formalism, the T–BMT equation is written as $d\mathrm{S}/d\theta = \Omega(\theta, J, \phi(\theta)) \times \mathrm{S}$ where the vector $\mathrm{S}$ is the spin expectation value ("the spin") in the rest frame of a particle and $\Omega$ is the precession vector obtained as indicated in [BEH] from the electric and magnetic fields on the particle trajectory. Generally, if $\phi$ appears as an independent variable in a function, the function will be $2\pi$–periodic in each $\phi_1, ..., \phi_d$. In that case we say for brevity that the function is $2\pi$–periodic in $\phi$. For the purposes of this work it will suffice to confine ourselves to motion on a single torus. Thus the actions $J$ are just parameters.

The work is structured as follows. In Section 2.1 we discuss the continuous-time formalism which will motivate, in Section 2.2, the discrete-time concept of the spin-orbit torus. Here we also see for the first time the utility of so-called cocycles. In Section 2.1 we define the proper spin-orbit tori which are those spin-orbit tori which can be derived from the continuous-time formalism. In Section 2.2 we define the set $\mathcal{SOT}$ of spin-orbit tori and the notion of spin transfer matrix of a spin-orbit torus and show that spin transfer matrices are cocycles. From Chapter 3 onwards we focus on spin-orbit tori, i.e., we only exploit the discrete-time formalism. The transformation theory of spin-orbit tori is introduced in Chapter 3 and it partitions the set of spin-orbit tori into equivalence classes. While this transformation rule is natural and well known in the polarized beam community, it here gains further meaning by becoming a transformation rule of cocycles linking it thus with the concept of cohomologous cocycles and, in Section ??, with the concept of $SO(3)$ gauge transformation. Moreover, since spin tunes describe constant rates of precession in appropriate reference frames, special spin-orbit tori are needed which can be reached by transforming from the original spin-orbit tori to such frames. This is handled in Chapter 4 and leads to the definition of the subsets $\mathcal{ACB}, \mathcal{CB}$ of $\mathcal{SOT}$ and of spin tunes and spin-orbit resonances of first kind. Moreover at the end of Chapter 4, by using the transformation theory of spin-orbit tori, the concept of $H$ normal form of spin-orbit tori is defined for every subgroup $H$ of $SO(3)$. In Chapter 5 we define spin and polarization fields and these lead to the definition of the invariant spin field (ISF). Chapter 6 covers Theorem 6.1 which addresses the issue of how many ISF's exist. Note that Theorem 6.1 is one of several results for which we don't have space to prove them but whose proofs can be found in [He2]. We then derive from Theorem 6.1 a practically important estimation formula for the maximum polarization of a bunch. In Chapter 7, the fact that the spin equation of motion contains a quasiperiodic function leads to the exploitation of quasiperiodicity and the definition of spin tunes of second kind. Then spin-orbit tori can be classified as being well-tuned or ill-tuned as in [BEH]. Chapter 8 takes as an example, a simple well known model and analyzes it to demonstrate that ISF's not always exist since here the occurrence of an ISF depends crucially on a certain parameter for that model. Then, in Chapter 9, we employ the theory developed up to that point to take a broader view of the spin-orbit tori by using the concept of the principal bundle [Sc, NS, Na1, Na2, Hus1, Hus2, tDi2, Ma]. It turns out by Corollary 9.10 in Section ?? that the existence of an ISF is a property of a certain principal bundle, $\lambda_d$, and this result is accomplished by Theorem 9.12 in Section ?? which connects the invariant reductions of $\lambda_d$, which are certain principla bundles introduced by Definition ?? in Section ??, with the quotient structural equations, introduced in Section ??. It also turns out by Corollary 9.10 that the presence of a spin-orbit resonance of first kind is an analogous property of the principal bundle $\lambda_d$. Thus the existence of an ISF and the presence of a spin-orbit resonance of first kind describe the same mathematical situations which only differ by the subgroup $H$ of $SO(3)$ of the corresponding invariant $H$ reduction of $\lambda_d$ (in fact $H = SO(2)$ for the ISF and $H = \{I_{3\times3}\}$ for the spin-orbit resonance of first kind where $I_{3\times3}$ denotes the $3 \times 3$ unit matrix). Section ?? resp. Section ?? also point out a close relationship between the $H$ normal forms and the quotient structural equations resp. the invariant $H$ reductions of $\lambda_d$. The principal bundle $\lambda_d$ also has a practical aspect since we outline, at the end of Section ??, recipes for numerically solving the structural equations. In Section ?? we consider the automorphism group of $\lambda_d$ to give on the one hand further insight into the invariant reduction machinery of Section ?? and to show on the other hand that the transformations of spin-orbit tori introduced in Chapter 3 are $SO(3)$ gauge transformations of $\lambda_d$. With Chapter 10 we put our work into context. In particular we point out how the present work can be generalized from continuous to Borel measurable functions. In Chapter 11 we give a brief summary and outlook. At the end of the outlook we mention Hill's equation which is covered in Appendix ?? and we mention Chao's 6D matrix formalism. Several mathematical definitions can be found in Appendix A.

# 2 Spin-orbit tori

## 2.1 Deriving a discrete-time dynamical system from the continuous time formalism: proper spin-orbit tori

We begin our study by deriving our discrete-time dynamical system from a continous-time initial value problem (IVP) which takes the form

$$\frac{d\phi}{d\theta} = \omega \, , \qquad \phi(\theta_0) = \phi_0 \in \mathbb{R}^d \, , \tag{2.1}$$

$$\frac{dS}{d\theta} = \mathcal{A}(\theta,\phi)S \, , \qquad S(\theta_0) = S_0 \in \mathbb{R}^3 \, , \tag{2.2}$$

where $\omega \in \mathbb{R}^d$ and where the function $\mathcal{A} : \mathbb{R}^{d+1} \to \mathbb{R}^{3\times3}$ is continuous in $\phi$ and piecewise continuous in $\theta$. More precisely, $\mathcal{A}$ is either continuous or a finite number of $\theta$ values $\theta_1, ..., \theta_N$ exist such that $\mathcal{A}$ is continuous on $(\mathbb{R} \setminus \{\theta_1, ..., \theta_N\}) \times \mathbb{R}^d$ and such that $\mathcal{A}(\theta_1; \cdot), ..., \mathcal{A}(\theta_N; \cdot)$ are continuous. Moreover we assume that $\mathcal{A}$ is $2\pi$-periodic in each of its $d+1$ arguments and that it is skew–symmetric, i.e., $\mathcal{A}^t(\theta,\phi) = -\mathcal{A}(\theta,\phi)$. We denote the set of pairs $(\omega, \mathcal{A})$, where $\omega \in \mathbb{R}^d$ and where $\mathcal{A}$ satisfies the above conditions, by $\widetilde{\mathcal{SOT}}(d,\omega)$.

As is clear from the Introduction, the above IVP and the assumptions on $\mathcal{A}$ are motivated by our underlying interest in spin-orbit motion in storage rings. In the application to spin motion in storage rings, $S$ is a column vector of components of the spin $\mathbf{S}$ and $\mathcal{A}(\theta,\phi) \equiv \mathcal{A}_J(\theta,\phi)$ represents the rotation rate vector $\Omega(\theta, J, \phi)$ of the T-BMT equation [BEH]. Here $J, \phi$ are the action-angle variables of an integrable orbital motion. In this work we are only occasionally interested in the $J$-dependence. So we suppress the $J$ unless we need it since most definitions in this work do not involve the $J$. The acronym $\mathcal{SOT}$ stands for spin-orbit torus, reflecting the fact that orbital motion, $\phi(\theta) = \phi_0 + (\theta - \theta_0)\omega$ can be represented on a torus $\mathbb{T}^d$. The set $\widetilde{\mathcal{SOT}}(d,\omega)$ includes standard spin-orbit motion but need not, and is therefore more general, in keeping with our wish to investigate the properties of any system defined by (2.1, 2.2).

The system (2.1, 2.2) has a unique solution for each $(\omega, \mathcal{A})$ in $\widetilde{\mathcal{SOT}}(d,\omega)$ (see Remark 1 below). However, because $\mathcal{A}(\cdot;\phi)$ is $2\pi$-periodic it is convenient to study the behavior of a solution in terms of a Poincaré map (PM) [AP, HK2]. For this we introduce the general solution $\varphi$ of the IVP (2.1, 2.2).

Let $x = (\phi, S)^t$ and write (2.1, 2.2) as

$$\frac{dx}{d\theta} = f(x,\theta) \, , \qquad x(\theta_0) = x_0 \in \mathbb{R}^{d+3} \, , \tag{2.3}$$

a map
≈ principal
soln matrix

then the general solution $\varphi = \varphi(\theta, \theta_0; x_0)$ is defined uniquely by

$$\frac{\partial}{\partial\theta}\varphi(\theta,\theta_0;x) = f(\varphi(\theta,\theta_0;x),\theta) \, , \qquad \varphi(\theta_0,\theta_0;x) = x \, . \tag{2.4}$$

The two basic properties of $\varphi$ are:

$$\text{(P1) (Flow property)} \qquad \varphi(\theta,\theta_0;x) = \varphi(\theta,\theta_1;\varphi(\theta_1,\theta_0;x)) \, . \tag{2.5}$$

$$\text{(P2) (Periodicity property)} \qquad \varphi(\theta + 2\pi, \theta_0 + 2\pi; x) = \varphi(\theta, \theta_0; x)) \,. \tag{2.6}$$

Both (P1) and (P2) are proven by noting that the left and right hand sides satisfy the ODE in (2.4) and are equal for one $\theta$ value, namely $\theta = \theta_1$ in (2.5) and $\theta = \theta_0$ in (2.6). The proof of (P2) makes use of the fact that $f(x, \theta + 2\pi) = f(x, \theta)$. Because of (P2) the solutions of (2.3) can be studied in terms of the sequences

$$x_n = \varphi(\theta_0 + 2\pi n, \theta_0; x_0) \,, \quad (n \in \mathbb{Z}) \,, \tag{2.7}$$

as parametrized by $x_0$. Because of (P2) the $x_n$ are given by iteration as follows

$$x_{n+1} = \varphi(\theta_0 + 2\pi, \theta_0; x_n) \,, \quad n = ..., -2, -1, 0, 1, 2, ... \tag{2.8}$$

This can be written as

$$x_{n+1} = \mathcal{P}^{\theta_0}_{\omega,\mathcal{A}}(x_n) \,, \tag{2.9}$$

where

$$\mathcal{P}^{\theta_0}_{\omega,\mathcal{A}}(x) := \varphi(\theta_0 + 2\pi, \theta_0; x) \,, \tag{2.10}$$

is the PM. Eq. (2.9) is an example of an interated map. Let $f = \mathcal{P}^{\theta_0}_{\omega,\mathcal{A}}$ and define $f^n$ recursively by $f_0 = id_{\mathbb{R}^{d+3}}$ and $f^{n+1} = f \circ f^n$ where $\circ$ denotes the composition of functions. Clearly $f^{m+n} = f^n \circ f^m = f^m \circ f^n$ and $x_n = f^n(x_0)$. With this we have constructed the required discrete-time dynamical system.

So far we have made no use of the special structure of (2.1, 2.2). This implies a simple structure of the PM and which is of fundamental importance for all that follows. In fact the rhs of the ODE in (2.1) is independent of $S$ and (2.2) is a linear ODE for $S$. This is sometimes referred to as a skew-product dynamical system (see, e.g., [Wi]).

For more details on this structure see the remarks in Section 3.2 on skew products. Solving (2.1) gives

$$\phi(\theta) = \phi_0 + (\theta - \theta_0)\omega \,. \tag{2.11}$$

We thus introduce the IVP

$$\frac{dS}{d\theta} = \hat{\mathcal{A}}(\theta; P)S \,, \qquad S(\theta_0) = S_0 \in \mathbb{R}^3 \,, \tag{2.12}$$

where $P = (P_1, P_2)$ and where the function $\hat{\mathcal{A}} : \mathbb{R}^{d+2} \to \mathbb{R}^{3\times3}$ is defined by

$$\hat{\mathcal{A}}(\theta; P) := \mathcal{A}(\theta, P_2 + (\theta - P_1)\omega) \,. \qquad * \tag{2.13}$$

Clearly (2.12) is identical with (2.2) if $P = (\theta_0, \phi_0)$. We have introduced the notation $P, P_1, P_2$ because $\theta_0$ and $\phi_0$ are parameters in $\hat{\mathcal{A}}$ and we wish to separate the two dependencies on $\theta_0$ in (2.1),(2.2). The Principal Solution Matrix $\Phi$ for (2.12) is the analogue of $\varphi$ for (2.3) and is defined by the IVP

$$\frac{\partial}{\partial\theta}\Phi(\theta, \theta_0; P) = \hat{\mathcal{A}}(\theta; P)\Phi(\theta, \theta_0; P) \,, \quad \Phi(\theta_0, \theta_0; P) = I_{3\times3} \,. \tag{2.14}$$

*
$P_3$ has 3 bits
$P_{\parallel}$ has 1 bit
$\theta$ has 1 bit

Thus the general solution of (2.12) is

$$S(\theta) = \Phi(\theta, \theta_0; P)S_0 \,, \tag{2.15}$$

and we have the following representation for $\varphi$ namely

$$\varphi(\theta, \theta_0; \phi_0, S_0) = \begin{pmatrix} \phi_0 + (\theta - \theta_0)\omega \\ \Phi(\theta, \theta_0; \theta_0, \phi_0)S_0 \end{pmatrix} . \tag{2.16}$$

From (P1) we obtain

(P1′) (Flow property) $\Phi(\theta, \theta_0; \theta_0, \phi) = \Phi(\theta, \theta_1; \theta_1, \phi + (\theta_1 - \theta_0)\omega)\Phi(\theta_1, \theta_0; \theta_0, \phi)$ , (2.17)

and from (P2)

(P2′) (First periodicity property) $\Phi(\theta + 2\pi, \theta_0 + 2\pi; P_1 + 2\pi, P_2) = \Phi(\theta, \theta_0; P_1, P_2)$ . (2.18)

In addition, from (2.14)

(P3′) (Second periodicity property) $\Phi(\theta, \theta_1; P_1, \phi)$ is $2\pi-$ periodic in $\phi$ . (2.19)

We have now obtained the main result of this section, namely the following representation for the PM

$$\mathcal{P}^{\theta_0}_{\omega,\mathcal{A}}(\phi, S) = \begin{pmatrix} \phi + 2\pi\omega \\ A^{\theta_0}_{\omega,\mathcal{A}}(\phi)S \end{pmatrix} , \tag{2.20}$$

where

$$A^{\theta_0}_{\omega,\mathcal{A}}(\phi) := \Phi(\theta_0 + 2\pi, \theta_0; \theta_0, \phi) \,. \tag{2.21}$$

Just (P2')? From (P3′) the function $A^{\theta_0}_{\omega,\mathcal{A}}$ is $2\pi$-periodic. Also it is $SO(3)$-valued since the values of $\mathcal{A}$ are real skew–symmetric $3 \times 3$ matrices where $SO(3)$ is the set of real $3 \times 3$–matrices $R$ for which $R^tR = I$ and $\det(R) = 1$.

**Remark:**

(1) The backward iteration of (2.7) using (2.16) gives

$$x_{-1} = \varphi(\theta_0 - 2\pi, \theta_0; x_0) = \begin{pmatrix} \phi_0 - 2\pi\omega \\ \Phi(\phi_0 - 2\pi, \theta_0; \theta_0, \phi_0)S_0 \end{pmatrix} . \tag{2.22}$$

From (2.17) with $\theta = \theta_0$ and $\theta_1 = \theta_0 - 2\pi\omega$ we obtain

$$I_{3\times3} = \Phi(\theta_0, \theta_0 - 2\pi; \theta_0 - 2\pi, \phi_0 - 2\pi\omega)\Phi(\theta_0 - 2\pi, \theta_0; \theta_0, \phi_0) \,. \tag{2.23}$$

Using the periodicity we obtain

$$\Phi(\theta_0 - 2\pi, \theta_0; \theta_0, \phi_0) = \Phi^t(\theta_0 + 2\pi, \theta_0; \theta_0, \phi_0 - 2\pi\omega) = (A^{\theta_0}_{\omega,\mathcal{A}})^t(\phi)(\phi_0 - 2\pi\omega) \tag{2.24}$$

Thus the inverse of the PM is

$$(\mathcal{P}^{\theta_0}_{\omega,\mathcal{A}})^{-1}(\phi, S) = \begin{pmatrix} \phi - 2\pi\omega \\ (A^{\theta_0}_{\omega,\mathcal{A}})^t(\phi)(\phi_0 - 2\pi\omega)S \end{pmatrix} . \tag{2.25}$$

(2) Since $\hat{\mathcal{A}}(\theta; P)$ is piecewise continuous in $\theta$ it can be shown that (2.12) has a unique solution $S$ such that $S(\theta_0) = S_0$ in the sense that

$$S(\theta) = S_0 + \int_{\theta_0}^{\theta} \hat{\mathcal{A}}(t; P) S(t) dt \,. \tag{2.26}$$

It follows that $S(\theta)$ is continuous in $\theta$. The proof in Cronin [Cr] which assumes only that $\hat{\mathcal{A}}(\cdot; P_1, P_2)$ is Riemann integrable, doesn't include the parameters $P_1, P_2$ but they are easily added. By (2.13) and (2.26) we have

$$\Phi_{\omega,\mathcal{A}}(\theta, \theta_0; P) = I_{3\times3} + \int_{\theta_0}^{\theta} \hat{\mathcal{A}}(t; P)\Phi(t, \theta_0; P) dt$$

$$= I_{3\times3} + \int_{\theta_0}^{\theta} \mathcal{A}(t, P_2 + (t - P_1)\omega)\Phi(t, \theta_0; P) dt \,. \tag{2.27}$$

By adding the parameter vector $P$ in Cronin's proof, and using the fact that $\hat{\mathcal{A}}(\theta; P)$ is continuous in $P$, we conclude from (2.27) that $\Phi \in \mathcal{C}(\mathbb{R}^{d+3}, \mathbb{R}^{3\times3})$ where $\mathcal{C}(\mathbb{R}^{d+3}, \mathbb{R}^{3\times3})$ is the set of continuous functions from $\mathbb{R}^{d+3}$ into $\mathbb{R}^{3\times3}$ (see also Appendix A). □

With (2.20) the Poincaré map $\mathcal{P}^{\theta_0}_{\omega,\mathcal{A}}$ is determined by the parameters $\omega$ and $A^{\theta_0}_{\omega,\mathcal{A}}$. With this in mind we define, for $\omega \in \mathbb{R}^d$, Union over all $\theta_0$

$$\mathcal{SOT}_{prop}(d, \omega) := \bigcup_{\theta_0 \in \mathbb{R}} \{(\omega, A^{\theta_0}_{\omega,\mathcal{A}}) : (\omega, \mathcal{A}) \in \widetilde{\mathcal{SOT}}(d, \omega)\}$$

$$= \{(\omega, A^{0}_{\omega,\mathcal{A}}) : (\omega, \mathcal{A}) \in \widetilde{\mathcal{SOT}}(d, \omega)\} \,, \quad \text{— just } \theta_0 = 0 \tag{2.28}$$

and denote the union of the $\mathcal{SOT}_{prop}(d, \omega)$ over $d$ and $\omega$ by $\mathcal{SOT}_{prop}$ and call every element of $\mathcal{SOT}_{prop}$ a "proper spin-orbit torus". Note that the second equality in (2.28) follows from the simple fact that every $\mathcal{A}$ can be modified into another $\mathcal{A}$ by shifting the argument $\theta$ by a fixed amount. ?

## 2.2 Introducing the set $\mathcal{SOT}$ of spin-orbit tori

We now generalize the notion of "proper spin-orbit torus". In particular, we define, for every $\omega \in \mathbb{R}^d$,

$$\mathcal{SOT}(d, \omega) := \{(\omega, A) : A \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))\} \,, \tag{2.29}$$

where $A$ need not be derivable from (2.1, 2.2). We denote the union of the $\mathcal{SOT}(d, \omega)$ over $d$ and $\omega$ by $\mathcal{SOT}$ and call every pair $(\omega, A)$ in $\mathcal{SOT}$ a "spin-orbit torus". Since the function $A^{\theta_0}_{\omega,\mathcal{A}}$ belongs to $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ we see from (2.28) and (2.29) that $\mathcal{C}_{per}$: App A ?

$$\mathcal{SOT}_{prop}(d, \omega) \subset \mathcal{SOT}(d, \omega) \,, \qquad \mathcal{SOT}_{prop} \subset \mathcal{SOT} \,. \tag{2.30}$$

We call $\omega$ the "orbital tune vector" of a spin-orbit torus $(\omega, A)$ and $A$ its "1-turn spin transfer matrix" of our discrete-time system. Next, motivated by (2.20) we define, for every $(\omega, A)$ in $\mathcal{SOT}(d, \omega)$, the function $\mathcal{P}_{\omega,A} \in Homeo(\mathbb{R}^{d+3})$ by

$$\mathcal{P}_{\omega,A}(\phi, S) := \begin{pmatrix} \phi + 2\pi\omega \\ A(\phi)S \end{pmatrix} \,, \tag{2.31}$$

and call it the "1-turn map of $(\omega, A)$". As can be understood from Appendix A, $Homeo(\mathbb{R}^{d+3})$ denotes the set of homeomorphisms on $\mathbb{R}^{d+3}$. Of course, in the special case where the spin-orbit torus $(\omega, A)$ is proper, the 1-turn map of $(\omega, A)$ is the PM.

We find it convenient to work in the more general setting of $\mathcal{SOT}(d,\omega)$ and (2.31) than with the special setting of $\mathcal{SOT}_{prop}(d,\omega)$ and (2.20). However the main physical interest is in a small subset of $\mathcal{SOT}_{prop}(d,\omega)$. There is a natural mathematical question: given $(\omega, A)$ in $\mathcal{SOT}(d,\omega)$, does it belong to $\mathcal{SOT}_{prop}(d,\omega)$? This is an analogue of the following question from beam dynamics: given a symplectic map, can it be generated as the one-turn map of a Hamiltonian system? We do not deal with this question. However there are many spin-orbit tori which are not proper as is shown in Section 7.2 of [He2] by using simple arguments from Homotopy Theory. For example, for $A \in \mathcal{C}_{per}(\mathbb{R}, SO(3))$ defined by $A(\phi) := \begin{pmatrix} \cos m\phi & -\sin m\phi & 0 \\ \sin m\phi & \cos m\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$, one can show, for every real $\omega$, that the spin-orbit torus

? $(\omega, A)$ is proper, i.e., $(\omega, A) \in \mathcal{SOT} \setminus \mathcal{SOT}_{prop}$, iff $m$ is even.

The iterates of the 1-turn map $\mathcal{P}_{\omega,A}$ must have the form

$$\mathcal{P}^n_{\omega,A}(\phi, S) = \begin{pmatrix} L_\omega(n;\phi) \\ \Psi_{\omega,A}(n;\phi)S \end{pmatrix} , \tag{2.32}$$

as we now argue. With this notation, we impose the convention that $\mathcal{P}^0_{\omega,A}$ is the identity function on $\mathbb{R}^{d+3}$ and that for $n$ negative, $\mathcal{P}^n_{\omega,A}$ is the $|n|$-th iterate of the inverse $\mathcal{P}^{-1}_{\omega,A}$. Clearly

$$\mathcal{P}^{-1}_{\omega,A}(\phi, S) = \begin{pmatrix} \phi - 2\pi\omega \\ A^t(\phi - 2\pi\omega)S \end{pmatrix} . \tag{2.33}$$

From the first component of (2.31) we conclude that the function $L_\omega : \mathbb{Z}\times\mathbb{R}^d \to \mathbb{R}^d$ in (2.32) is given by

$$L_\omega(n;\phi) := \phi + 2\pi n\omega . \tag{2.34}$$

From the second component of (2.31) we conclude that the function $\Psi_{\omega,A}(n;\phi)$ is $SO(3)$-valued, i.e., $\Psi_{\omega,A} : \mathbb{Z}\times\mathbb{R}^d \to SO(3)$. We define the function $L_{\omega,A} : \mathbb{Z}\times\mathbb{R}^{d+3} \to \mathbb{R}^{d+3}$ for $n \in \mathbb{Z}, \phi \in \mathbb{R}^d, S \in \mathbb{R}^3$ by

$$L_{\omega,A}(n;\phi,S) := \mathcal{P}^n_{\omega,A}(\phi, S) = \begin{pmatrix} L_\omega(n;\phi) \\ \Psi_{\omega,A}(n;\phi)S \end{pmatrix} , \tag{2.35}$$

and since

$$\mathcal{P}^{m+n}_{\omega,A} = \mathcal{P}^n_{\omega,A} \circ \mathcal{P}^m_{\omega,A} , \tag{2.36}$$

in the paragraph

as after (2.10), we have

$$L_{\omega,A}(n+m,\phi,S) = L_{\omega,A}(n; L_{\omega,A}(m;\phi,S)) , \tag{2.37}$$

which is analogous to (2.5). This gives, by (2.35),

$$\begin{pmatrix} L_\omega(n+m;\phi) \\ \Psi_{\omega,A}(n+m;\phi)S \end{pmatrix} = L_{\omega,A}(n+m,\phi,S) = L_{\omega,A}(n;L_{\omega,A}(m;\phi,S))$$
$$= L_{\omega,A}\left(n;\begin{pmatrix} L_\omega(m;\phi) \\ \Psi_{\omega,A}(m;\phi)S \end{pmatrix}\right) = \begin{pmatrix} L_\omega(n;L_\omega(m;\phi)) \\ \Psi_{\omega,A}(n;L_\omega(m;\phi))\Psi_{\omega,A}(m;\phi)S \end{pmatrix}\Bigg) , \tag{2.38}$$

which implies

$$L_\omega(n+m,\phi) = L_\omega(n;L_\omega(m;\phi)) , \tag{2.39}$$

Just use 2.34!

and

$$\Psi_{\omega,A}(n+m;\phi) = \Psi_{\omega,A}(n;L_\omega(m;\phi))\Psi_{\omega,A}(m;\phi) . \tag{2.40}$$

Move this up

Note that (2.40) is analogous to (2.17). As a consequence of (2.40) we get

$$\begin{aligned}
&\Psi_{\omega,A}(0;\phi) = I_{3\times 3} , \\
&\Psi_{\omega,A}(n;\phi) = A(\phi+2\pi(n-1)\omega)\cdots A(\phi+2\pi\omega)A(\phi) , \qquad (n=1,2,...) , \\
&\Psi_{\omega,A}(n;\phi) = A^t(\phi+2\pi n\omega)\cdots A^t(\phi-4\pi\omega)A^t(\phi-2\pi\omega) , \quad (n=-1,-2,...) ,
\end{aligned} \tag{2.41}$$

where we also used (2.34). It is easy to obtain (2.40) directly by iteration of (2.31), i.e., without using $L_{\omega,A}$, however the procedure here is more pedagogical since the pairs $(\mathbb{R}^d, L_\omega)$ and $(\mathbb{R}^{d+3}, L_{\omega,A})$ are examples of the important concept of a $\mathbb{Z}$ space which we define in the next section in a more general setting; a setting which will be needed later.

Said it ? be

We now introduce some more terminology. We call the $n$-th iterate $\mathcal{P}^n_{\omega,A} = L_{\omega,A}(n;\cdot)$ the "$n$-turn map of $(\omega,A)$" and we call $\Psi_{\omega,A}$ the "spin transfer matrix" of $(\omega,A)$. Using the standard topology on $\mathbb{R}^d$ and the standard topology on $\mathbb{Z}$ (for the latter see also Section 2.3) a function on $\mathbb{Z}\times\mathbb{R}^d$ is continuous if it is continuous in the second argument. Since $\Psi_{\omega,A}(n;\cdot)$ is continuous, every spin transfer matrix is a continuous function. We call $\Psi_{\omega,A}(n;\cdot)$ the "$n$-turn spin transfer matrix" of $(\omega,A)$. Clearly

$$\Psi_{\omega,A}(1;\phi) = A(\phi) , \tag{2.42}$$

which justifies calling $A$ the 1-turn spin transfer matrix.

For $n\in\mathbb{Z}, \phi_0\in\mathbb{R}^d, S_0\in\mathbb{R}^3$ we define the function $S:\mathbb{Z}\to\mathbb{R}^3$ by

$$S(n) := \Psi_{\omega,A}(n;\phi_0)S_0 , \tag{2.43}$$

and call $S$ a "spin trajectory of $(\omega,A)$ over $\phi_0$". It follows from (2.40), And (2.43) that

$$S(n+1) = A(\phi_0+2\pi n\omega)S(n) , \qquad S(0) = S_0\in\mathbb{R}^3 . \tag{2.44}$$

Note that (2.44) is a linear, non-autonomous IVP for $S$. Note also that $A(\phi_0+2\pi n\omega)$ is an $\omega$-quasiperiodic function of $n$ as defined in Chapter 7. This quasiperiodicity has a strong influence on the spin trajectories and it is therefore important for spin-orbit tori. For more details, see Chapter 7.

Clearly the behavior of the spin trajectories in (2.44) depend on the values of $A$ reached by the orbital dynamics $\phi_0 + 2\pi n\omega$ in its argument which in turn depends on the value of $\omega$. Since $A$ is periodic its argument can be viewed on the torus and for some values of $\omega$ the argument will remain in a confined subset of the torus and for other values it can cover the torus densely. To be more precise we define resonance. We say $\chi \in \mathbb{R}^n$ is *resonant* if there exists a non-zero integer vector $k \in \mathbb{Z}^n$ such that $k \cdot \chi = 0$ and nonresonant if not resonant. If $(1, \omega)$ is nonresonant then the argument of $A$ covers the torus densely and since $A$ is continuous all values of $A$ affect the spin trajectory. Whereas if $(1, \omega)$ is resonant the values of $A$ are only sampled by its values on a sub-torus. The spin-orbit torus $(\omega, A)$ is said to be "off orbital resonance" if $(1, \omega)$ is nonresonant and "on orbital resonance" otherwise. Thus spin trajectories may exhibit significantly different qualitative behaviors on and off orbital resonance.

Let $\phi_0 \in \mathbb{R}^d$ and $(\omega, \mathcal{A}) \in \widetilde{SOT}(d, \omega)$ and let $S$ solve the EOM (2.12) in the special case where $P_1 = \theta_0, P_2 = \phi_0$, i.e., $dS/d\theta = \mathcal{A}(\theta, \phi_0 + (\theta - \theta_0)\omega)S$. Defining the function $\hat{S} : \mathbb{Z} \to \mathbb{R}^3$ by $\hat{S}(n) := S(2\pi n)$ we observe that $\hat{S}$ is a spin trajectory over $\phi_0$ of the proper spin-orbit torus $(\omega, A^{\theta_0}_{\omega,\mathcal{A}})$.

## 2.3 Left group actions and cocycles

With Section 2.2 complete, it is appropriate to introduce some standard definitions which will be crucial for the remainder of this work.

**Definition 2.1** *(Group)*
*A "group" is a pair $(G, *)$ where $G$ is a set and $*$ is a binary operation on $G$, i.e., a function: $G \times G \to G$ such that*

| | |
|---|---|
| (G1) (Associativity) | $\forall_{g_1,g_2,g_3 \in G}\ (g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$ , |
| (G2) (Identity element $e_G$) | $\exists_{e_G \in G}\ \forall_{g \in G}\ e_G = e_G * g = g * e_G$ , |
| (G3) (Inverse elements) | $\forall_{g_1 \in G}\ \exists_{g_2 \in G}\ e_g = g_1 * g_2 = g_2 * g_1$ , |

*A group $(G, *)$ is called "Abelian" if*

| | |
|---|---|
| (G4) (Commutativity) | $\forall_{g_1,g_2 \in G}\ g_1 * g_2 = g_2 * g_1$ . |

□

Note that we always abbreviate $(G, *)$ as $G$ when the operation $*$ is clear from the context and that we often write $g_1 * g_2$ as $g_1 g_2$ when the operation $*$ is clear from the context. If $G$ is Abelian then $*$ is often replaced with $+$. The inverse element of a $g \in G$ is denoted by $g^{-1}$. Important examples of groups in Section 2.2 are $(\mathbb{Z}, +)$, $(SO(3), *)$ (where the binary operation is matrix multiplication) and $(\mathbb{R}, +)$. We abbreviate them by $\mathbb{Z}, SO(3)$ and $\mathbb{R}$ and note that $\mathbb{Z}$ and $\mathbb{R}$ are Abelian.

**Definition 2.2** *(Topological group)*
*A "topological group" is a group $(G, *)$ where $G$ is a topological space, where the binary operation $*$ is continuous and where the function $g \mapsto g^{-1}$ on $G$ is continuous, too.* □

Note that the above mentioned groups $\mathbb{Z}$ and $SO(3)$ in Section 2.2 are topological as we consider them to be equipped with their standard topologies. Thus the topology of $\mathbb{Z}$ is discrete, i.e., every subset of $\mathbb{Z}$ is open and $SO(3)$ is equipped with the relative topology from $\mathbb{R}^{3\times 3}$.

**Definition 2.3** *(Left G action, left G set)*
*Let $G$ be a group with identity element $e_G$ and let $E$ be a set. Then a function $L : G\times E \to E$ is called a "left $G$ action on $E$" if, for $g_1, g_2 \in G, x \in E$,* es stating with $G$ & $E$ and getting $E$

$$L(e_G; x) = x \tag{2.45}$$
$$L(g_1 g_2; x) = L(g_1; L(g_2; x)) \,. \tag{2.46}$$

*If $L$ is a left $G$ action on $E$ then the pair $(E, L)$ is called a "left $G$ set". Since $L(e_G; x) = L(g; L(g^{-1}; x))$ each $L(g;\cdot)$ is a bijection onto $E$, i.e., $L(g;\cdot) \in \mathrm{Bij}(E)$. If $(E, L)$ is a left $G$ set and $x \in E$ then the set $\{L(g;x) : g \in G\}$ is called the "orbit of $x$ under $L$". Note that this notion of "orbit" is different from the physical notion of "orbital motion". If the orbit of $x$ only consists of $x$ itself then $x$ is called a "fixed point" of $(E, L)$.*

*If $G$ is an Abelian group then every left $G$ action $L$ is also called a "$G$ action" and every left $G$ set is also called "$G$ set". Any left $G$ set can be loosely called a dynamical system or a transformation group. Moreover any left $\mathbb{Z}$ set can be loosely called a discrete time dynamical system and any left $\mathbb{R}$ set can be loosely called a continuous time dynamical system.* □

It is clear by (2.39) that $L_\omega$ is a left $\mathbb{Z}$ action on $\mathbb{R}^d$ whence $(\mathbb{R}^d, L_\omega)$ is a left $\mathbb{Z}$ set. Since the group $\mathbb{Z}$ is Abelian, $L_\omega$ is a $\mathbb{Z}$ action on $\mathbb{R}^d$ and $(\mathbb{R}^d, L_\omega)$ is a $\mathbb{Z}$ set. Analogously it follows from (2.37) that $L_{\omega,A}$ is a left $\mathbb{Z}$ action (and a $\mathbb{Z}$ action) on $\mathbb{R}^{d+3}$ and that $(\mathbb{R}^{d+3}, L_{\omega,A})$ is a left $\mathbb{Z}$ set (and a $\mathbb{Z}$ set). There are many books which cover group actions. Two books, dedicated exclusively to group actions, are [tDi1, Ka].

In this work we are only interested ~~in~~ left $G$ sets which have additional structure as formalized in the following definition. [illegible]

**Definition 2.4** *(Left G space)*
*Let $E$ be a topological space, $G$ be a topological group, and let $L$ be a left $G$ action on $E$ with $L$ being continuous. Then the pair $(E, L)$ is called a "left $G$ space". Note that each $L(g;\cdot)$ is a homeomorphism onto $E$. In the important subcase when the topology of $G$ is discrete (e.g., when $G = \mathbb{Z}$) the condition that $L$ is continuous is equivalent to $L(g;\cdot)$ being continuous for all $g \in G$. The definition of "fixed point" is the ~~same as for left G sets.~~* defined ? to be ?

*If $(E, L)$ is a left $G$ space and if the group $G$ is Abelian then we also call $(E, L)$ a "$G$ space".* □

Since, by (2.34), $L_\omega(n;\cdot)$ is continuous it is clear that the $\mathbb{Z}$ set $(\mathbb{R}^d, L_\omega)$ is a left $\mathbb{Z}$ space (and therefore a $\mathbb{Z}$ space). It is equally clear by (2.35) that $L_{\omega,A}(n;\cdot)$ is continuous whence the $\mathbb{Z}$ set $(\mathbb{R}^{d+3}, L_{\omega,A})$ is a left $\mathbb{Z}$ space (and therefore a $\mathbb{Z}$ space).

**Remark:**

(1) The following argument demonstrates how natural the concept of left group action is. ~~Let $G$ be a group and let $E$ be a set.~~ Let $L : G \times E \to E$ be a function such that
and E be a group and set respectively
— or better

every $L(g;\cdot)$ is a bijection on $E$. Thus this condition is fulfilled when $L$ is a left $G$ action. Then one can express the action (coin the data?????) of $L$ with (into?????) the function $L^{hom} : G \to \mathrm{Bij}(E)$ defined by $L^{hom}(g) := L(g,\cdot)$. Under composition $\mathrm{Bij}(E)$ is a group. The key point here is that $L$ is a left $G$ action iff $L^{hom}$ is a group homomorphism, i.e., a homomorphism from the group $G$ into the group $\mathrm{Bij}(E)$. This fact follows immediately from (2.45) and (2.46) and it motivates the notation $L^{hom}$.

Analogously, let $G$ be a topological group and let $(E, L)$ be a left $G$ space. Since each $L(g;\cdot)$ is a homeomorphism onto $E$, the function $L^{hom}$ defined above, is a homomorphism from the group $\mathbb{Z}$ into the group $Homeo(E)$ where the group multiplication in $Homeo(E)$ is understood to be the composition of functions. □

The spin transfer matrix is an example of a cocycle and (2.40) is the cocycle condition. Cocycles are central to our work and we now provide a definition.

**Definition 2.5** *(Cocycle, periodic cocycle)*
*Let $G, K$ be topological groups and let $(E, L)$ be a left $G$ space. Then a function $f \in \mathcal{C}(G \times E, K)$ is called a "$K$ cocycle over the left $G$ space $(E, L)$" if, for $g, g' \in G, x \in E$,*

Where is K here?

$$f(gg';x) = f(g; L(g';x))f(g';x)\,. \tag{2.47}$$

$f \equiv K$?, $G = \mathbb{Z}$

*We denote the collection of all $K$ cocycles over $(E, L)$ by $COC(E, L, K)$. We deal mainly with the case $E = \mathbb{R}^d$. So let $(\mathbb{R}^d, L)$ be a left $G$ space. Then a $f \in COC(\mathbb{R}^d, L, K)$ is called a "periodic $K$ cocycle over $(\mathbb{R}^d, L)$" if every $f(g;\cdot)$ is $2\pi$-periodic. We denote the set of all periodic cocycles in $COC(\mathbb{R}^d, L, K)$ by $COC_{per}(\mathbb{R}^d, L, K)$. For literature on cocycles, see, e.g., [HK1, KR, Zi1].* □

Since $(\mathbb{R}^d, L_\omega)$ is a left $\mathbb{Z}$ space and $SO(3)$ is a topological group, the set $COC(\mathbb{R}^d, L_\omega, SO(3))$ is well defined. It follows from (2.40)

???????...that, for every $(\omega, A) \in \mathcal{SOT}(d,\omega)$, $\Psi_{\omega,A} \in COC(\mathbb{R}^d, L_\omega, SO(3))$. Of course, since $\Psi_{\omega,A}(n;\cdot) \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$, we see that $\Psi_{\omega,A} \in COC_{per}(\mathbb{R}^d, L_\omega, SO(3))$. Conversely, every $\Psi$ in $COC_{per}(\mathbb{R}^d, L_\omega, SO(3))$ is the spin transfer matrix of a spin-orbit torus since, by defining $A(\cdot) := \Psi(1;\cdot)$, we have $\Psi_{\omega,A} = \Psi$ so that $\Psi$ is the spin transfer matrix of $(\omega, A)$. We thus ~~arrive at~~ conclude that

$$COC_{per}(\mathbb{R}^d, L_\omega, SO(3)) = \{\Psi_{\omega,A} : (\omega, A) \in \mathcal{SOT}(d,\omega)\}\,. \tag{2.48}$$

Clearly cocycles are important for spin-orbit tori.

# 3 Transforming spin-orbit tori

## 3.1 Conjugacies and the transformation rule of spin-orbit tori

We now consider the basic structure of $\mathcal{SOT}(d,\omega)$. The dynamics of each element of $\mathcal{SOT}(d,\omega)$ is given by its 1-turn map $\mathcal{P}_{\omega,A}$ and we are interested in those $(\omega, A)$ which have similar dynamics. This is made precise by the notion of conjugacy. Recall that

$\mathcal{P}_{\omega,A} \in Homeo(\mathbb{R}^{d+3})$. Two functions $f, g \in Homeo(\mathbb{R}^n)$ are said to be "conjugate" if a $t \in Homeo(\mathbb{R}^n)$ exists such that $g = t^{-1} \circ f \circ t$. We denote this similarity by $f \sim g$. To see the effect on the dynamics we note that $y_n := g^n(y_0) = (t^{-1} \circ f \circ t)^n(y_0) = (t^{-1} \circ f^n \circ t)(y_0)$ whence $t(y_n) = f^n(t(y_0))$. So $x_n = t(y_n)$ where $x_n := f^n(x_0)$ and many properties of $x_0, x_1, \ldots$ and $y_0, y_1, \ldots$ are similar, e.g., existence of fixed points or periodic solutions. Furthermore $f \sim g$ defines an equivalence relation on $Homeo(\mathbb{R}^n)$ whose equivalence classes form a partition of $Homeo(\mathbb{R}^n)$.

We now formalize this in the context of $\mathcal{SOT}(d,\omega)$. By (2.31), $\mathcal{P}_{\omega,A}(\phi, S) = \begin{pmatrix} \phi + 2\pi\omega \\ A(\phi)S \end{pmatrix}$ and we will focus on its spin component, i.e., the $A$. Thus we consider the transformations $\mathcal{P}_{0,T}$ defined by $\mathcal{P}_{0,T}(\phi) = \begin{pmatrix} \phi \\ T(\phi) \end{pmatrix}$.

$x_0 = t(y_0)$

$\mathcal{P}_{0,T}(\phi)$ has no $S$.

Clearly if $(\omega, A), (\omega, A') \in \mathcal{SOT}(d,\omega)$ then, by (2.31), the equality

$$\mathcal{P}_{0,T}^{-1} \circ \mathcal{P}_{\omega,A} \circ \mathcal{P}_{0,T} = \mathcal{P}_{\omega,A'} \tag{3.1}$$

holds iff

$$A'(\phi) = T^t(\phi + 2\pi\omega)A(\phi)T(\phi) \ . \tag{3.2}$$

This gives a partition of $\mathcal{SOT}(d,\omega)$ as we formalize in the next two definitions.

**Definition 3.1** *(Transformation rule of spin-orbit tori)*
*Let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d,\omega)$ then a $T$ in $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ is called a "transfer field from $(\omega, A)$ to $(\omega, A')$" iff (3.1) holds. We also say that "$(\omega, A')$ is the transform of $(\omega, A)$ under $T$". We denote the collection of all transfer fields from $(\omega, A)$ to $(\omega, A')$ by $\mathcal{TF}_{d,\omega}(A, A')$.* □

Clearly $\mathcal{TF}_{d,\omega}(A, A') \neq \emptyset$ iff $(\omega, A')$ is a transform of $(\omega, A)$. To obtain an equivalence relation on $\mathcal{SOT}(d,\omega)$ we note, by (2.31), that

$$\begin{aligned} &\mathcal{P}_{0,I_{3\times3}} = id_{\mathbb{R}^{d+3}} \ , \\ &\mathcal{P}_{0,T}\mathcal{P}_{0,T'} = \mathcal{P}_{0,TT'} \ , \\ &\mathcal{P}_{0,T}^{-1} = \mathcal{P}_{0,T^t} \ , \end{aligned} \tag{3.3}$$

where $T, T' \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. This leads us to:

**Definition 3.2** *Let $(\omega, A), (\omega, A') \in \mathcal{SOT}(d,\omega)$. Then we write $(\omega, A) \sim_{d,\omega} (\omega, A')$ and say that $(\omega, A), (\omega, A')$ are "similar" iff $(\omega, A')$ is a transform of $(\omega, A)$, i.e., a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists which satisfies (3.1). Note that, by (3.1),(3.3) it is easy to show that $\sim_{d,\omega}$ is an equivalence relation on $\mathcal{SOT}(d,\omega)$. We denote the equivalence class of $(\omega, A)$ by $\overline{(\omega, A)}$.* □

Since $\sim_{d,\omega}$ is an equivalence relation on $\mathcal{SOT}(d,\omega)$, its equivalence classes give a partition of $\mathcal{SOT}(d,\omega)$. Each $(\omega, A)$ in a given equivalence class gives rise to similar dynamics since many properties are shared as explained in Remarks 1 and 2 below. This is important for spin-orbit tori since it suggests that, given an $(\omega, A)$ in $\mathcal{SOT}(d,\omega)$, one should look for the "simple" elements of $\overline{(\omega, A)}$. See Chapter 4) too.

**Remarks:**

(1) Two spin-orbit tori which are similar share many important properties, e.g., the existence or nonexistence of an ISF as in Chapter 5. We will see other properties shared by similar spin-orbit tori below.

(2) We observe by (2.44) and (3.2) that if $T \in \mathcal{TF}_{d,\omega}(A, A')$ and if $S$ is a spin-orbit trajectory of $(\omega, A)$ over, say $\phi_0$, then $S'$, defined by

$$S'(n) := T^t(\phi_0 + 2\pi\omega n)S(n) , \tag{3.4}$$

is a spin trajectory of $(\omega, A')$ over $\phi_0$. We will define quasiperodicity in Chapter 7. It will follow that $T^t(\phi_0 + 2\pi\omega n)$ is quasiperiodic and if $S(n)$ is quasiperiodic then so is $T^t(\phi_0 + 2\pi\omega n)S(n)$ and thus $S'(n)$. This gives us another property shared by similar spin-orbit tori since it implies that if all spin trajectories of $(\omega, A)$ are quasiperiodic then, for every spin-orbit torus in $\overline{(\omega, A)}$, all of its spin trajectories are quasiperiodic.

killed in short version

(3) It is important to note that the transformation rule outlined in (3.4) is no stranger to the polarized beam community. In fact when researchers deal with the topics of spin tune, spin frequency, spin resonances, resonance strengths etc. then they often appeal more or less directly to the above transformation rule. However our aim here is to deal with rather fundamental aspects, and their physical implications, which are usually either not addressed or not addressed in such an explicit form. □

## 3.2 Topological $G$ maps of left $G$ spaces

We now look at how the left $\mathbb{Z}$ spaces $(\mathbb{R}^{d+3}, L_{\omega,A})$, defined in Section 2.3, are related for similar spin-orbit tori. From (3.1)

$$\mathcal{P}_{0,T}^{-1} \circ \mathcal{P}_{\omega,A}^{n} \circ \mathcal{P}_{0,T} = \mathcal{P}_{\omega,A'}^{n} . \tag{3.5}$$

$P^{-1}_{0T}$ gives a $P_{0T}$ of computation

Therefore, by (2.35), $L_{\omega,A'}(n;\cdot) = \mathcal{P}_{0,T}^{-1} \circ L_{\omega,A}(n;\cdot) \circ \mathcal{P}_{0,T}$, so that

$$\mathcal{P}_{0,T}^{-1} \circ L_{\omega,A}(n;\cdot) = L_{\omega,A'}(n;\cdot) \circ \mathcal{P}_{0,T}^{-1} . \tag{3.6}$$

Thus according to the following definition, $\mathcal{P}_{0,T}$ and $\mathcal{P}_{0,T}^{-1}$ are $\mathbb{Z}$ maps. strange name – $P_{0T}$, $P^{-1}_{0T}$ don't shift anything.

**Definition 3.3** *($G$ maps of left $G$ sets, topological $G$ maps of left $G$ spaces)*
*a) Let $G$ be a group and let $(E, L), (E', L')$ be left $G$ sets and consider the function $f : E \to E'$. If for $g \in G, x \in E$, $f$ satisfies*

$$f(L(g;x)) = L'(g; f(x)) , \tag{3.7}$$

*then $f$ is called a "$G$ map from $(E, L)$ to $(E', L')$".*
Fullstop
*b) Let $G$ be a topological group. Let $(E, L), (E', L')$ be left $G$ spaces and let $f \in C(E, E')$. If $f$ satisfies (3.7) then $f$ is called a "topological $G$ map from $(E, L)$ to $(E', L')$".* □

If $f$ is a $G$ map from the left $G$ set $(E, L)$ to the left $G$ set $(E', L')$ and if $f$ is a bijection onto $E'$, then $f^{-1}$ is a $G$ map from $(E', L')$ to $(E, L)$ and $(E', L')$ and $(E, L)$ are called "isomorphic" and thus are essentially the same. Analogously, when $f$ is a topological $G$ map

$$(Y, L) \xrightarrow{f} (X', L')$$

$$f \circ L = L' \circ f$$

$$f(g_1 \circ g_2) = f(g_1) \times f(g_2)$$

from the left $G$ space $(E, L)$ to the left $G$ space $(E', L')$ and if $f$ is a homeomorphism onto $E'$ then $(E', L')$ and $(E, L)$ are called "isomorphic" and are essentially the same. ?

In our special case it follows from (3.6) and Definitions 3.1 and 3.3 that if $T$ is a transfer field from $(\omega, A)$ to $(\omega, A')$ then the $\mathcal{P}_{0,T}^{-1}$ is a topological $\mathbb{Z}$ map from the left $\mathbb{Z}$ space $(\mathbb{R}^{d+3}, L_{\omega,A})$ to the left $\mathbb{Z}$ space $(\mathbb{R}^{d+3}, L_{\omega,A'})$ and, since $\mathcal{P}_{0,T}^{-1} \in Homeo(\mathbb{R}^{d+3})$, $\mathcal{P}_{0,T}$ is a topological $\mathbb{Z}$ map from $(\mathbb{R}^{d+3}, L_{\omega,A'})$ to $(\mathbb{R}^{d+3}, L_{\omega,A})$.

**Remark:**

(1) Eqn. (2.35) provides another example of a topological $\mathbb{Z}$ map in our context. Thus let $f \in \mathcal{C}(\mathbb{R}^{d+3}, \mathbb{R}^d)$ be defined by $f(\phi, S) := \phi$. Then, by (2.35),
$f\Big(L_{\omega,A}(n; \phi, S)\Big) = L_\omega(n; \phi) = L_\omega(n; f(\phi, S))$, i.e.,

$$f \circ L_{\omega,A}(n; \cdot) = L_\omega(n; \cdot) \circ f \; . \tag{3.8}$$

So we see by (3.8) and Definition 3.3 that $f$ is a topological $\mathbb{Z}$ map from the $\mathbb{Z}$ space $(\mathbb{R}^{d+3}, L_{\omega,A})$ to the $\mathbb{Z}$ space $(\mathbb{R}^d, L_\omega)$. Since $\mathbb{R}^{d+3}$ is the cartesian product of $\mathbb{R}^d$ and $\mathbb{R}^3$ and $f$ is the projection onto $\mathbb{R}^d$, the fact that $f$ is a topological $\mathbb{Z}$ map means that the $\mathbb{Z}$ space $(\mathbb{R}^{d+3}, L_{\omega,A})$ is a so-called "skew product" over the $\mathbb{Z}$ space $(\mathbb{R}^d, L_\omega)$. Note that this skew product structure has its origin in the fact that the first component of $L_{\omega,A}$ in (2.35) is independent of $S$ and that the second component of $L_{\omega,A}$ is linear in $S$. It is easy to see that $\Psi_{\omega,A}$ being a cocycle is equivalent to $(\mathbb{R}^{d+3}, L_{\omega,A})$ being a skew product. For more details on skew products and their relation to cocycles see, e.g., [HK1]. □

It follows from (2.40),(3.2) and by induction in $n$ that, if $T \in \mathcal{TF}_{d,\omega}(A, A')$, then the spin transfer matrices of $(\omega, A), (\omega, A') \in \mathcal{SOT}(d, \omega)$ are related by

$$\Psi_{\omega,A'}(n; \phi) = T^t(\phi + 2\pi n\omega)\Psi_{\omega,A}(n; \phi)T(\phi) \; . \tag{3.9}$$

Recall from Section 2.3 that $\Psi_{\omega,A}$ and $\Psi_{\omega,A'}$ are cocycles. Then (3.9) implies that the cocycles $\Psi_{\omega,A}$ and $\Psi_{\omega,A'}$ are "cohomologous" For this notion, see, e.g., [HK1, He2, KR, Zi1]. Eq. (3.9) once again shows that our transformation rule of spin-orbit tori is very natural.

## 3.3 Right group actions and topological $G$ maps of right $G$ spaces

The transformation rule of Definition 3.1 for spin-orbit tori can also be formalized in terms of the function $R_{d,\omega} : \mathcal{C}_{per}(\mathbb{R}^d, SO(3)) \times \mathcal{SOT}(d, \omega) \to \mathcal{SOT}(d, \omega)$ defined by

$$R_{d,\omega}(T; \omega, A) := \Big(\omega, T^t(\cdot + 2\pi\omega)AT\Big) \; , \tag{3.10}$$

i.e., $R_{d,\omega}(T; \omega, A) = (\omega, A')$ where $A' \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ is given by (3.2). According to the following definition, $R_{d,\omega}$ is a right $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ action on $\mathcal{SOT}(d, \omega)$.

**Definition 3.4** *(Right G action, right G set)*
*Let $G$ be a group with identity element $e_G$ and let $E$ be a set. Then a function $R : G \times E \to E$ is called a "right $G$ action on $E$" if, for $g_1, g_2 \in G, x \in E$,*

Transitive, partition

$$R(e_G; x) = x \tag{3.11}$$
$$R(g_1 g_2; x) = R(g_2; R(g_1; x)) \,. \tag{3.12}$$

*If $R$ is a right $G$ action on $E$ then the pair $(E, R)$ is called a "right $G$ set". Note that each $R(g; \cdot)$ is a permutation of $E$. If $(E, R)$ is a right $G$ set and $x \in E$ then the image of $R(\cdot; x)$, i.e., the set $R(G \times \{x\}) = \{R(g; x) : g \in G\}$, is called the "orbit of $x$ under $R$". We denote the collection of orbits by $E/R$ and define the function $p_R : E \to E/R$ by $p_R(x) := R(G \times \{x\})$. Note that $p_R$ maps every $x$ to its orbit and that the orbits form a partition of $E$. ~~Note that~~ Again as in Definition 2.3, this notion of "orbit" is different from the physical notion of "orbital motion".*

Again

*A right $G$ set $(E, R)$ is called "trivial" if every $R(g, \cdot)$ is the identity function on $E$ and one then calls $(E, R)$ the "trivial right $G$ set over $E$".*

*If $G$ is an Abelian group then every right $G$ action $R$ is also called a "$G$ action" and every right $G$ set is also called "$G$ set". Thus if $G$ is an Abelian group then the notions right $G$ action, left $G$ action and $G$ action are synonymous and the notions right $G$ set, left $G$ set, and $G$ set are synonymous. Any right $G$ set can be loosely called a transformation group.* □

Not yet defined? See 2.30.

**Remark:**

(1) The notions of left and right are dual. In fact if $R$ is a right $G$ action on $E$ then the function $L : G \times E \to E$ defined by $L(g; x) := R(g^{-1}; x)$ is a left $G$ action on $E$. Moreover if $L$ is a left $G$ action on $E$ then the function $R : G \times E \to E$ defined by $R(g; x) := L(g^{-1}; x)$ is a right $G$ action on $E$. □

6

To show that $R_{d,\omega}$ is a right action, note that $(G, *) = (\mathcal{C}_{per}(\mathbb{R}^d, SO(3)), *)$ is a group with identity element $e_G$ where $*$ denotes pointwise multiplication. In particular, $T * T'$ is defined by $(T * T')(\phi) := T(\phi)T'(\phi)$ where $e_G$ is the constant $I_{3\times3}$ valued function. Using Definition 3.1 and (3.10) we obtain

$$R_{d,\omega}(e_G; \omega, A) = \Big(\omega, I^t_{3\times3} A I_{3\times3}\Big) = (\omega, A) \,,$$
$$R_{d,\omega}\Big(T'; R_{d,\omega}(T; \omega, A)\Big) = R_{d,\omega}\Big(T'; \omega, T^t(\cdot + 2\pi\omega) A T\Big)$$
$$= \Big(\omega, (T')^t(\cdot + 2\pi\omega) T^t(\cdot + 2\pi\omega) A T T'\Big) = R_{d,\omega}(TT'; \omega, A) \,, \tag{3.13}$$

whence $R_{d,\omega}$ is indeed a right $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ action on $\mathcal{SOT}(d, \omega)$ so that $(\mathcal{SOT}(d, \omega), R_{d,\omega})$ is a right $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ set. The orbits of the group action $R_{d,\omega}$ are the equivalence classes of $\sim_{d,\omega}$. Note also that the group $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ is not Abelian since the group $SO(3)$ is not Abelian so that the right action $R_{d,\omega}$ is not a left action.

While $R_{d,\omega}$ is of course important in this work since it determines the transformation rule, the fact that $R_{d,\omega}$ is a group action does not play a major role in this paper. In fact we introduced the right group action $R_{d,\omega}$ just to prepare the reader for important right actions in Chapter 9. These right group actions have additional structure as formalized in the following definition.

**Definition 3.5** *(Right G space)*
*Let $E$ be a topological space, $G$ be a topological group, and let $R$ be a right $G$ action on $E$ with $R$ being continuous. Then the pair $(E, R)$ is called a "right $G$ space". Note that each $R(g;\cdot)$ is a homeomorphism onto $E$. Recalling from Definition (3.4) the notation $E/R$ and $p_R$, we equip $E/R$ with its natural topology, i.e., a subset $\mathcal{O}$ of $E/R$ is open iff $p_R^{-1}(\mathcal{O})$ is open in $E$. Thus the function $p_R$ is onto $E/R$ and identifying and one calls $E/R$ an "orbit space". Also each orbit is equipped with the relative topology from $E$. A right $G$ space $(E, R)$ is called "trivial" if it is trivial as a right $G$ set, i.e., if every $R(g,\cdot)$ is the identity function on $E$. One then calls $(E, R)$ the "trivial right $G$ space over $E$".*

*If $(E, R)$ is a right $G$ space and if the group $G$ is Abelian then we call $(E, R)$ a "$G$ space". Thus if $G$ is an Abelian group the notions right $G$ space, left $G$ space, and $G$ space are synonymous.* □

Very important right $G$ spaces are considered in Chapter 9 in the context of principal $G$ bundles. In fact principal $G$ bundles are right $G$ spaces with some extra structure. Note that in Chapter 9 $G$ will be a subgroup of $SO(3)$.

The following definition is the right analogue to Definition 3.3.

**Definition 3.6** *(G maps of right G sets, topological G maps of right G spaces)*
*a) Let $G$ be a group and let $(E, R), (E', R')$ be right $G$ sets and let $f : E \to E'$ be a function. If $f$ satisfies, for $g \in G, x \in E$,*

$$f(R(g;x)) = R'(g;f(x)) \,, \tag{3.14}$$

*then $f$ is called a "$G$ map from $(E, R)$ to $(E', R')$".*
*b) Let $G$ be a topological group. Let $(E, R), (E', R')$ be right $G$ spaces and let $f \in \mathcal{C}(E, E')$. If $f$ satisfies (3.7) then $f$ is called a "topological $G$ map from $(E, R)$ to $(E', R')$".* □

If $f$ is a $G$ map from the right $G$ set $(E, R)$ to the right $G$ set $(E', R')$ and if $f$ is a bijection onto $E'$ then $f$ is a $G$ map from $(E', R')$ to $(E, R)$ and in this situation the $G$ sets $(E', R')$ and $(E, R)$ are called "isomorphic" and are indistinguishable as right $G$ sets. Analogously when $f$ is a topological $G$ map from the right $G$ space $(E, R)$ to the right $G$ space $(E', R')$ and if $f$ is a homeomorphism onto $E'$ then $(E', R')$ and $(E, R)$ are called "isomorphic" and are ~~indistinguishable as right $G$ spaces~~. ?

**Remark:**

(2) We here give a simple example of a $G$ map of right ~~group~~ $G$ sets.

Let $\omega \in \mathbb{R}^d$ and let the function $f : SOT(d, \omega) \to SOT(d, -\omega)$ be defined by

$$f(\omega, A) := \Big(-\omega, A^t(\cdot - 2\pi\omega)\Big) \,. \tag{3.15}$$

Then, by (3.10),

$$f\Big(R_{d,\omega}(T;\omega,A)\Big) = f\Big(\omega, T^t(\cdot + 2\pi\omega)AT\Big)$$
$$= \Big(-\omega, T^t(\cdot - 2\pi\omega)A(\cdot - 2\pi\omega)T\Big) ,$$
$$R_{d,-\omega}(T; f(\omega,A)) = R_{d,-\omega}\Big(T; -\omega, A^t(\cdot - 2\pi\omega)\Big)$$
$$= \Big(-\omega, T^t(\cdot - 2\pi\omega)A^t(\cdot - 2\pi\omega)T\Big) ,$$

whence

$$f\Big(R_{d,\omega}(T;\omega,A)\Big) = R_{d,-\omega}\Big(T; f(\omega,A)\Big) ,$$

What happened to the (3.7) and Spin $(\omega, A)$?

so that, by Definition 3.6, $f$ is a $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ map from $(\mathcal{SOT}(d,\omega), R_{d,\omega})$ to $(\mathcal{SOT}(d,-\omega), R_{d,-\omega})$.

The function $f$ has an obvious interpretation in terms of time reversal as follows. Using the EOM (2.44), we see that if $S$ is a spin trajectory of $(\omega, A)$ over $\phi_0$ then the "time inverted" function $S'$, defined by $S'(n) := S(-n)$, is a spin trajectory over $\phi_0$ of the spin-orbit torus $f(\omega, A)$. Note also that $f$ is a bijection onto $\mathcal{SOT}(d,-\omega)$ since $f$ is its own inverse, that is, $f \circ f$ is the identity function on $\mathcal{SOT}(d,\omega)$. Thus recalling a remark after Definition 3.6, one can say that the right $\mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ sets $(\mathcal{SOT}(d,\omega), R_{d,\omega})$ and $(\mathcal{SOT}(d,-\omega), R_{d,-\omega})$ are isomorphic right $G$ sets.

?

(3) Let $G$ be a topological group and let $(E, R)$ be a right $G$ space. Then, by Definitions 3.5 and 3.6, $p_R$ is a topological $G$ map from $(E, R)$ to the trivial right $G$ space over $E/R$. □

# 4 Spin tunes and spin-orbit resonances of first kind and $H$ normal forms

One important motivation for the transformation rule of Definition 3.1 is that, under certain circumstances, a spin-orbit torus can be transformed into a simpler one. In fact as mentioned in Chapter 3 each spin-orbit torus shares many properties with all similar ones so that in order to study these properties of $(\omega, A)$ one should look for the simple elements of $\overline{(\omega, A)}$. This is the subject of this section and it will enable us to associate extra tunes, namely spin tunes, with our spin-orbit tori. As in other dynamical systems, tunes can lead to the recognition of resonances and consequent instabilities. Here, spin tunes will lead to recognition of spin-orbit resonances. In the case of real spin motion, where spins are subject to the electric and magnetic fields on synchro-betatron trajectories, the definition of spin-orbit resonance allows us predict at which orbital tunes spin motion might be particularly unstable. The definition of spin tune is also associated with the concept of normal forms for spin. Here, we will go beyond the usual definition of normal form for spin [?] to take a broader view by introducing $H$ normal forms where $H$ is a subgroup of $SO(3)$.

We thus define, for an arbitrary subgroup $H$ of $SO(3)$,

$$\begin{aligned} \mathcal{SOT}_H(d,\omega) &:= \{(\omega, A) \in \mathcal{SOT}(d,\omega) : A \in \mathcal{C}_{per}(\mathbb{R}^d, H)\} \,, \\ \mathcal{SOT}_H &:= \bigcup_{d\in\mathbb{N},\omega\in\mathbb{R}^d} \mathcal{SOT}_H(d,\omega) \,, \end{aligned} \tag{4.1a}$$

where, as always in this work, the topology of a subgroup $H$ of $SO(3)$ is the relative topology from $SO(3)$. Clearly the sets in (4.1a) give us spin-orbit tori which are the simpler the smaller the group $H$ is. Arguably the simplest elements in every $\mathcal{SOT}(d,\omega)$ are those $(\omega, A)$ with constant $A$. We therefore introduce the sets

$$\begin{aligned} \mathcal{SOT}_H^{const}(d,\omega) &:= \{(\omega, A) \in \mathcal{SOT}_H(d,\omega) : A(\phi) \text{ is independent of } \phi\} \,, \\ \mathcal{SOT}_H^{const} &:= \bigcup_{d\in\mathbb{N},\omega\in\mathbb{R}^d} \mathcal{SOT}_H^{const}(d,\omega) \,, \end{aligned} \tag{4.1b}$$

The most extreme case is, of course, when $H$ is the trivial group. These sets will now be condidered in Sections 4.1 and 4.2.

## 4.1 The subsets $\mathcal{ACB}$ and $\mathcal{CB}$ of $\mathcal{SOT}$

As mentioned above, it seems sensible to look for the simplest elements, of $\overline{(\omega, A)}$, for $(\omega, A)$ in $\mathcal{SOT}_{SO(3)}^{const}(d,\omega)$. So we collect these into the set $\bigcup_{(\omega,A)\in\mathcal{SOT}_{SO(3)}^{const}(d,\omega)} \overline{(\omega, A)}$.

An obvious subset of $\mathcal{SOT}_{SO(3)}^{const}$ is $\mathcal{SOT}_{SO(2)}^{const}$ where the subgroup $SO(2)$ of $SO(3)$ is defined by

$$\begin{aligned} SO(2) &:= \left\{ \begin{pmatrix} \cos(x) & -\sin(x) & 0 \\ \sin(x) & \cos(x) & 0 \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \\ &= \{\exp(x\mathcal{J}) : x \in \mathbb{R}\} = \{\exp(x\mathcal{J}) : x \in [0, 2\pi)\} \,, \end{aligned} \tag{4.2}$$

with

$$\mathcal{J} := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.3}$$

In fact if $r \in SO(3)$ then there exist $\nu \in [0, 1)$ and $W \in SO(3)$ such that

$$r = W^t \exp(2\pi\nu\mathcal{J})W \,, \tag{4.4}$$

and $\nu$ is uniquely determined by $r$. See, e.g., Lemma 2.1 of [BEH]. Thus we define, for every $\nu \in [0, 1)$ and $d \in \mathbb{N}$, the constant-valued function $A_{d,\nu} \in \mathcal{C}_{per}(\mathbb{R}^d, SO(2))$ by

$$A_{d,\nu}(\phi) := \exp(2\pi\nu\mathcal{J}) = \begin{pmatrix} \cos(2\pi\nu) & -\sin(2\pi\nu) & 0 \\ \sin(2\pi\nu) & \cos(2\pi\nu) & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{4.5}$$

so that, by (4.1b) and (4.2),

$$\mathcal{SOT}^{const}_{SO(2)}(d,\omega) = \{(\omega, A_{d,\nu}) : \nu \in [0,1)\} . \tag{4.6}$$

It follows from (4.1b), (4.4), (4.5) and Definition 3.2, that for every $(\omega, A)$ in $\mathcal{SOT}^{const}_{SO(3)}(d,\omega)$, a $\nu \in [0,1)$ exists such that $(\omega, A)$ is similar to $(\omega, A_{d,\nu})$ whence, by (4.6),

$$\bigcup_{(\omega,A)\in\mathcal{SOT}^{const}_{SO(3)}(d,\omega)} \overline{(\omega, A)} = \bigcup_{(\omega,A)\in\mathcal{SOT}^{const}_{SO(2)}(d,\omega)} \overline{(\omega, A)} = \bigcup_{\nu\in[0,1)} \overline{(\omega, A_{d,\nu})} . \tag{4.7}$$

The set in (4.7) contains the most important spin-orbit tori in applications. For the subgroup

$$G_\nu := \{\exp(2\pi n\nu\mathcal{J}) : n \in \mathbb{Z}\} , \tag{4.8}$$

of $SO(2)$ we see that $A_{d,\nu} \in G_\nu$ and that

$$\begin{aligned} \mathcal{SOT}^{const}_{SO(2)}(d,\omega) &= \bigcup_{\nu\in[0,1)} \mathcal{SOT}^{const}_{G_\nu}(d,\omega) , \\ \mathcal{SOT}^{const}_{SO(2)} &= \bigcup_{\nu\in[0,1)} \mathcal{SOT}^{const}_{G_\nu} . \end{aligned} \tag{4.9}$$

Of course, with (4.8), the trivial subgroup of $SO(3)$ is $G_0$, i.e.,

$$G_0 = \{I_{3\times3}\} , \tag{4.10}$$

and by (4.1b), (4.5) and(4.8) we have

$$\mathcal{SOT}_{G_0} = \mathcal{SOT}^{const}_{G_0} = \{(\omega, A_{d,0}) : d \in \mathbb{N}, \omega \in \mathbb{R}^d\} , \tag{4.11}$$

where, by (4.5), $A_{d,0}$ ~~is the~~ $I_{3\times3}$ valued function on $\mathbb{R}^d$. Note that all spin-orbit tori in every $\mathcal{SOT}^{const}_{SO(3)}(d,\omega)$ are proper, i.e.,

$$\mathcal{SOT}^{const}_{SO(3)}(d,\omega) \subset \mathcal{SOT}_{prop}(d,\omega) . \tag{4.12}$$

In fact if $(\omega, A) \in \mathcal{SOT}^{const}_{SO(3)}(d,\omega)$ then a skew-symmetric matrix $\mathcal{A}$ exists in $\mathbb{R}^{3\times3}$ such that $A = \exp(2\pi\mathcal{A})$ and we see from Section 2.1 that

$$A^{\theta_0}_{\omega,A} = A , \tag{4.13}$$

for every $\theta_0 \in \mathbb{R}$. We now formalize these ideas into a definition.

**Definition 4.1** *($\mathcal{ACB}_\nu(d,\omega), \mathcal{ACB}_\nu, \mathcal{ACB}(d,\omega), \mathcal{ACB}$)*
*For $\nu \in [0,1), \omega \in \mathbb{R}^d$ we denote the set of those spin-orbit tori in $\mathcal{SOT}(d,\omega)$ which are similar to $(\omega, \exp(2\pi\nu\mathcal{J}))$ by $\mathcal{ACB}_\nu(d,\omega)$, i.e.,*

$$\mathcal{ACB}_\nu(d,\omega) := \overline{(\omega, A_{d,\nu})} = \{R_{d,\omega}(T;\omega, A_{d,\nu}) : T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))\} , \tag{4.14}$$

Orb J.?

*and we denote, for fixed $\nu$, their union by $\mathcal{ACB}_\nu$, i.e.,*

$$\mathcal{ACB}_\nu := \bigcup_{d\in\mathbb{N},\omega\in\mathbb{R}^d} \mathcal{ACB}_\nu(d,\omega)\ . \tag{4.15}$$

*We also define*

$$\mathcal{ACB}(d,\omega) := \bigcup_{\nu\in[0,1)} \mathcal{ACB}_\nu(d,\omega) = \bigcup_{\nu\in[0,1)} \overline{(\omega, A_{d,\nu})}$$
$$= \bigcup_{(\omega,A)\in\mathcal{SOT}^{const}_{SO(3)}(d,\omega)} \overline{(\omega,A)} = \bigcup_{(\omega,A)\in\mathcal{SOT}^{const}_{SO(2)}(d,\omega)} \overline{(\omega,A)}\ , \tag{4.16}$$

*where in the third and fourth equalities we used (4.7). We abbreviate their union by $\mathcal{ACB}$, i.e., $\mathcal{ACB} := \bigcup_{d\in\mathbb{N},\omega\in\mathbb{R}^d} \mathcal{ACB}(d,\omega)$.* □

Recalling our earlier remark, Definition 4.1 tells us that $\mathcal{ACB}$ contains the most important spin-orbit tori in applications. However as explained in Section 7.6 in [He2] and Chapter 8 of the current work, it is easy to show that spin-orbit tori exist which are not in $\mathcal{ACB}$. (???? give details about why. E.g what about prop? ??????) In fact the current work is by no means limited to the spin-orbit tori in $\mathcal{ACB}$ (see especially Chapter 9). ?

???????IMPROVE????????

Figures 1 and 2 provide symbolic illustrations of these ideas. Thus $\mathcal{SOT}^{const}_{SO(2)}(d,\omega)$ is denoted by a circle whereby each point on the circle is of the form $(\omega, A_{d,\nu})$ and belongs to the equivalence class $\overline{(\omega, A_{d,\nu})} = \mathcal{ACB}_\nu(d,\omega)$ represented shown by a curve with multiple intersections on the circle. If $(\omega, A)$ is on orbital resonance the "curve" $\mathcal{ACB}_\nu(d,\omega)$ intersects the circle at on a finite number of places times corresponding to the orbital resonance multiplicity (see [1]) whereas if $(\omega, A)$ is off orbital resonance the "curve" $\mathcal{ACB}_\nu(d,\omega)$ intersects the circle at a dense and countably infinite set. (THIS PARA NEEDS FIXING WHEN THE FIGS ARE READY?????????) Check English. Is this para too early? or ... briefly; later

Note that every $n$-turn spin transfer matrix of an $(\omega, A)$ in $\mathcal{SOT}^{const}_{SO(3)}(d,\omega)$ is constant valued since, by (2.41), it satisfies

$$\Psi_{\omega,A}(n;\phi) = A^n\ . \tag{4.17}$$

Thus for every spin-orbit torus in $\mathcal{SOT}^{const}_{SO(3)}$ every $n$-turn spin transfer matrix is constant valued so that, by Definition 4.1, every spin-orbit torus in $\mathcal{ACB}$ is similar to a spin-orbit torus for which every $n$-turn spin transfer matrix is constant valued. This motivates our acronym $\mathcal{ACB}$ in Definition 4.1 since the spin transfer matrices of the spin-orbit tori in $\mathcal{ACB}$ are so-called "almost coboundaries" (see, e.g., [KR]).

**Remark:**

(1) Definition 4.1 gives us another property shared by similar spin-orbit tori since it implies that if $(\omega, A)$ belongs to $\mathcal{ACB}$ then every spin-orbit torus in $\overline{(\omega,A)}$ belongs to $\mathcal{ACB}$. □ Transitivity?

We now formalize the transfer fields associated with $\mathcal{SOT}^{const}_H(d,\omega)$. Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$. Then we define

$$\mathcal{TF}^{const}_H(\omega, A) := \{T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3)) : R_{d,\omega}(T;\omega,A) \in \mathcal{SOT}^{const}_H(d,\omega)\}\ , \tag{4.18}$$

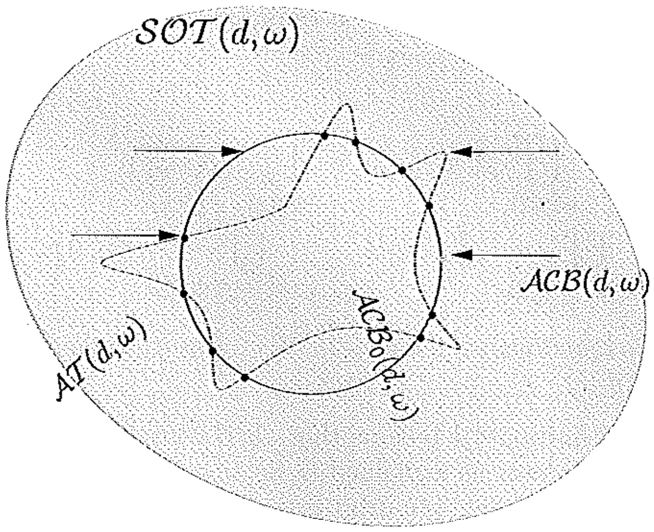



Figure 1: A symbolic representation of the relations between the sets $\mathcal{SOT}(d,\omega)$, $\mathcal{AT}(d,\omega)$, $\mathcal{ACB}(d,\omega)$ and $\mathcal{ACB}_0(d,\omega)$ defined in the text. The pink area represents a part of $\mathcal{SOT}(d,\omega)$ and the red, blue and green locii represent $\mathcal{AT}(d,\omega)$, $\mathcal{ACB}_0(d,\omega)$ and $\mathcal{ACB}(d,\omega)$ respectively. The $\mathcal{ACB}_0(d,\omega)$ crosses the $\mathcal{AT}(d,\omega)$ at.......

$\mathcal{SOT}(d,\omega)$

$\mathcal{ACB}(d,\omega)$

$\mathcal{AT}(d,\omega)$ $\mathcal{ACB}_0(d,\omega)$

Figure 2: A symbolic representation of the relations between the sets $\mathcal{SOT}(d,\omega)$, $\mathcal{AT}(d,\omega)$, $\mathcal{ACB}(d,\omega)$ and $\mathcal{ACB}_0(d,\omega)$ defined in the text. The pink area represents a part of $\mathcal{SOT}(d,\omega)$ and the red, blue and green locii represent $\mathcal{AT}(d,\omega)$, $\mathcal{ACB}_0(d,\omega)$ and $\mathcal{ACB}(d,\omega)$ respectively. The $\mathcal{ACB}_0(d,\omega)$ crosses the $\mathcal{AT}(d,\omega)$ at.......

i.e., $\mathcal{TF}_H^{const}(\omega, A)$ is the set of all transfer fields from $(\omega, A)$ to spin-orbit tori in $\mathcal{SOT}_H^{const}(d, \omega)$. In the special case $H = SO(2)$, (4.18) gives us

$$\mathcal{TF}_{SO(2)}^{const}(\omega, A) = \bigcup_{\nu\in[0,1)} \{T \in \mathcal{TF}_{d,\omega}(A, A_{d,\nu})\} . \tag{4.19}$$

By Definition 4.1 it is clear that a spin-orbit torus $(\omega, A) \in \mathcal{SOT}(d, \omega)$ belongs to $\mathcal{ACB}(d, \omega)$ iff $\mathcal{TF}_{SO(2)}^{const}(\omega, A)$ is nonempty, i.e., iff $\mathcal{TF}_{SO(3)}^{const}(\omega, A)$ is nonempty. Note also that the arguments that led to (4.4) imply that if $(\omega, A) \in \mathcal{SOT}_{SO(3)}^{const}(d, \omega)$ then $\mathcal{TF}_{SO(2)}^{const}(\omega, A)$ contains a transfer field which is constant valued.

**Remark:**

(2) Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$. The elements of $\mathcal{TF}_{SO(2)}^{const}(\omega, A)$ are the discrete-time analogues of the so-called uniform invariant frame fields introduced in [BEH]. Thus we abbreviate

$$\mathcal{UIFF}(\omega, A) := \mathcal{TF}_{SO(2)}^{const}(\omega, A) , \tag{4.20}$$

and we call every element of $\mathcal{UIFF}(\omega, A)$ a "uniform IFF of $(\omega, A)$". By the remarks after (4.19), $\mathcal{UIFF}(\omega, A)$ is nonempty iff $(\omega, A) \in \mathcal{ACB}(d, \omega)$. The basis for the analogy is explained in Section 4.3. □

One equivalence class is especially important, namely $\mathcal{ACB}_0(d, \omega)$. As we shall see this is central to the discussion of spin-orbit resonance, whence the following definition is needed.

**Definition 4.2** *($\mathcal{CB}(d, \omega), \mathcal{CB}$)*
*We denote the set of those spin-orbit tori in $\mathcal{SOT}(d, \omega)$ which are similar to the trivial spin-orbit torus $(\omega, I_{3\times3})$ by $\mathcal{CB}(d, \omega)$, i.e., by recalling Definition 4.1:*

$$\begin{aligned}\mathcal{CB}(d, \omega) &:= \mathcal{ACB}_0(d, \omega) = \overline{(\omega, A_{d,0})} \\ &= \{R_{d,\omega}(T; \omega, A_{d,0}) : T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))\} ,\end{aligned} \tag{4.21}$$

*and we denote their union by $\mathcal{CB}$, i.e.,*

$$\mathcal{CB} := \bigcup_{d\in\mathbb{N},\omega\in\mathbb{R}^d} \mathcal{CB}(d, \omega) . \tag{4.22}$$

*Note that, by Definition 4.1, $\mathcal{CB} = \mathcal{ACB}_0$. The acronym $\mathcal{CB}$ comes from the language of cocycles and will be explained further below.* □

Let $(\omega, A) \in \mathcal{CB}(d, \omega)$, thus by Definition 4.2 and (4.18) $\mathcal{TF}_{G_0}^{const}(\omega, A)$ is nonempty and every transfer field, $T$, in $\mathcal{TF}_{G_0}^{const}(\omega, A)$ satisfies

$$T^t(\phi + 2\pi\omega)AT = I_{3\times3} , \tag{4.23}$$

so that, by (2.41),

$$\Psi_{\omega,A}(n; \phi) = T(\phi + 2\pi n\omega)T^t(\phi) . \tag{4.24}$$

Eq. (4.24) motivates the acronym $\mathcal{CB}$ in Definition 4.2 since the spin transfer matrix $\Psi_{\omega,A}$ in (4.24) belongs to that class of cocycles which are called "coboundaries" (see [HK1, HK2]).

**Remark:**

Transitivity?

(3) Definition 4.2 gives us another property shared by similar spin-orbit tori. In particular, it implies that if $(\omega, A)$ belongs to $\mathcal{CB}$ then every spin-orbit torus in $\overline{(\omega, A)}$ belongs to $\mathcal{CB}$. □

?

See also (4.16)

Recall that spin-orbit tori exist which are not in $\mathcal{ACB}$ (and therefore not in $\mathcal{CB}$). In fact the problem of deciding whether a given spin-orbit torus is in $\mathcal{ACB}$ or in $\mathcal{CB}$, is fruitful both theoretically and practically. In particular the techniques of Chapter 9 are meant to head us into the study of this general problem.(IMPROVE????????)

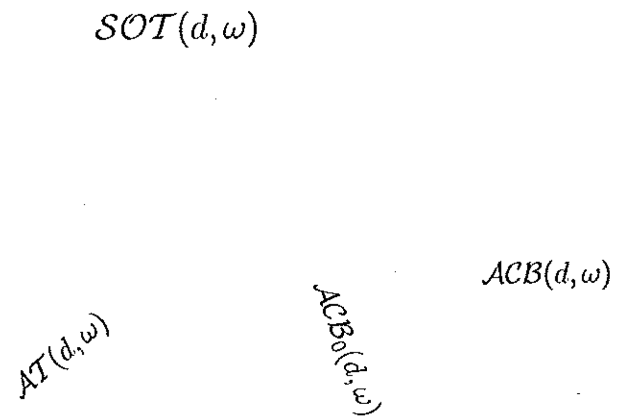



Figure 3: A symbolic representation of the relations between the sets $\mathcal{ACB}$ etc........

## 4.2 Spin tunes and spin-orbit resonances of the first kind

We now come to the definition of spin tune. A $\nu \in [0, 1)$ is said to be a spin tune for $(\omega, A) \in \mathcal{SOT}(d, \omega)$ if $(\omega, A) \sim_{d,\omega} (\omega, \exp(2\pi\nu\mathcal{J}))$. More formally we have the definition

**Definition 4.3** *(Spin tune of the first kind)*
*If $(\omega, A) \in \mathcal{SOT}(d, \omega)$ we define the set*

$$\Xi_1(\omega, A) := \{\nu \in [0, 1) : (\omega, A) \in \mathcal{ACB}_\nu\} = \{\nu \in [0, 1) : (\omega, A) \in \overline{(\omega, A_{d,\nu})}\} . \qquad (4.25)$$

*We call the elements of $\Xi_1(\omega, A)$ the spin tunes of the first kind of $(\omega, A)$.* □

It follows from Definitions 4.1 and 4.3 that, for every $(\omega, A) \in \mathcal{SOT}(d, \omega)$, the set $\Xi_1(\omega, A)$ is nonempty iff $(\omega, A) \in \mathcal{ACB}(d, \omega)$. Note also, by Definitions 4.2 and 4.3, that 0 is a spin tune of the first kind of a spin-orbit torus iff that spin-orbit torus is in $\mathcal{CB}$. Most importantly, for every $(\omega, A) \in \mathcal{ACB}(d, \omega)$ and every $\nu \in \Xi_1(\omega, A)$ we have the relation

$$\Xi_1(\omega, A) = [0, 1) \cap \left\{\varepsilon\nu + m \cdot \omega + n : \varepsilon \in \{1, -1\}, m \in \mathbb{Z}^d, n \in \mathbb{Z}\right\} . \qquad (4.26)$$

(4.26) uses $\cap$

In fact the inclusion $\Xi_1(\omega, A) \supset [0,1) \cap \left\{ \nu + m \cdot \omega + n : m \in \mathbb{Z}^d, n \in \mathbb{Z} \right\}$ is very easily demonstrated as follows.

So let $\nu \in \Xi_1(\omega, A)$. Then, by Definition 4.3, $(\omega, A) \in \overline{(\omega, A_{d,\nu})}$ whence recalling Definition 3.1, the function $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$, defined by $T(\phi) := \exp(-\mathcal{J}m \cdot \phi))$ with $m \in \mathbb{Z}^d$, belongs to $\mathcal{TF}_{d,\omega}(A_{d,\nu}, A_{d,\nu'})$ where $\nu' \in [0,1)$ is defined by $\nu' := \nu + m \cdot \omega \bmod 1$. Thus, by Definition 3.2, $\overline{(\omega, A_{d,\nu})} = \overline{(\omega, A_{d,\nu'})}$ so that $(\omega, A) \in \overline{(\omega, A_{d,\nu'})}\}$ which implies, by Definition ~~4.3, that $\nu' \in$~~ $\Xi_1(\omega, A)$. A complete proof of (4.26), uses the tool of quasiperiodic functions, see Chapter 7.

An important implication of (4.26) is the following. If $\nu, \mu \in [0,1), \omega \in \mathbb{R}^d$ then, since by Definition 4.1, $\mathcal{ACB}_\mu(d,\omega)$ and $\mathcal{ACB}_\nu(d,\omega)$ are equivalence classes of $\sim_{d,\omega}$ they are either equal or disjoint. Moreover, it follows from (4.5), (4.26) and Definition 4.3 that if $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and $\nu \in \Xi_1(\omega, A)$ then only those $(\omega, A_{d,\mu})$ are similar to $(\omega, A)$ for which $\varepsilon \in \{1,-1\}, m \in \mathbb{Z}^d, n \in \mathbb{Z}$ exist such that $\mu = \varepsilon\nu + m \cdot \omega + n$. Thus, by Definition 4.1, for arbitrary $\nu, \mu \in [0,1)$ we have $\mathcal{ACB}_\mu(d,\omega) = \mathcal{ACB}_\nu(d,\omega)$ iff $\varepsilon \in \{1,-1\}, m \in \mathbb{Z}^d, n \in \mathbb{Z}$ exist such that $\mu = \varepsilon\nu + m \cdot \omega + n$.

IMPROVE

This is illustrated in Fig. 2 by points intersecting the curve with the circle. (IMPROVE??????)

WE MUST DECIDE ON THE NUMERING CONVENTION FOR REMARKS.

**Remark:**

(1) If one considers a family $(\omega(J), A_J)_{J\in\Lambda}$ of spin-orbit tori (see Chapters 5 and 6) and if every $(\omega(J), A_J)$ has a spin tune of the first kind, say $\nu_J$, then $\nu_J$ is called an amplitude dependent spin tune (ADST). □

We need another definition.

**Definition 4.4** *(Spin-orbit resonance of the first kind)*
*Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$. We say that $(\omega, A)$ is on spin-orbit resonance of the first kind if $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and if for every $\nu \in \Xi_1(\omega, A)$ we can find $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ such that*

$$\nu = m \cdot \omega + n\,. \tag{4.27}$$

*We say that $(\omega, A)$ is "off spin-orbit resonance of first kind" iff $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and if $(\omega, A)$ is not on spin-orbit resonance of the first kind. Note that a spin-orbit torus which has no spin tunes of first kind is neither on nor off spin-orbit resonance of the first kind.* □

**Remarks:** Define multiplicity of orbital resonance

(2) By (4.26) and Definition 4.4 an $(\omega, A) \in \mathcal{ACB}(d,\omega)$ is on spin-orbit resonance of the first kind iff (4.27) holds for some $m \in \mathbb{Z}^d, n \in \mathbb{Z}, \nu \in \Xi_1(\omega, A)$. Thus a single spin tune $\nu$ of $(\omega, A)$ of the first kind is sufficient to determine if $(\omega, A)$ is on spin-orbit resonance of the first kind. Note that, by (4.26) and Definition 4.4, a spin-orbit torus $(\omega, A) \in \mathcal{ACB}(d,\omega)$ is on spin-orbit resonance of the first kind iff $0 \in \Xi_1(\omega, A)$. Thus, by Definitions 4.2, a spin-orbit torus $(\omega, A) \in \mathcal{ACB}(d,\omega)$ is on spin-orbit resonance of the first kind iff $(\omega, A) \in \mathcal{CB}(d,\omega)$. Thus a spin-orbit torus is on a spin-orbit resonance of the first kind iff it is in $\mathcal{CB}$.

(3) As stated at the beginning of this Chapter, the phenomenon of spin-orbit resonance can lead to instability of spin motion. For example it can lead to a large spread of the ISF, to be introduced in Chapter 5, and that can lead to unacceptably low equilibrium polarization. Furthermore, in Chapter 6 spin-orbit resonances of the first kind lead us to the Uniqueness Theorem for the ISF [Yo1, DK73]. A rigorous definition, as in Definition 4.4, is therefore very relevant for understanding real spin motion. For example, in Chapter 8 we will find a case for which there is no spin tune of the first kind, for which nevertheless, certain phenomena are often misnamed as resonances.

Cancel

See [?, ?, ?, ?, ?] for formalisms and calculations which have demonstrated the potential for a large spread of the ISF near spin-orbit resonances. For detailed further comments see Section X in [BEH].

c) SPECIAL STRUCTURE IN CHAPTER 9,

(4) Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and let us perturb $\mathcal{P}_{\omega,A}(\phi, S)$ into $\mathcal{P}_{\omega,A}(\phi, S) + \varepsilon \begin{pmatrix} 2\pi a \\ B(\phi)S \end{pmatrix}$. Then on the basis of the above notion of spin-orbit resonance of the first kind, we will have motions far from leading order resonances (FLOR) and near to leading order resonances (NLOR), where $a \in \mathbb{R}^d, B(\phi) \in \mathbb{R}^{3\times3}$. For example, $\nu - m \cdot \omega - n$ will appear as a small divisor in the analysis. (IMPROVE?????) □

## 4.3 $H$ normal forms and the subsets $\mathcal{CB}_H$ of $\mathcal{SOT}$

Recall again that each spin-orbit torus shares many properties with all similar ones so that in order to study these properties of $(\omega, A)$ one should look for the simple elements of $\overline{(\omega, A)}$. In Sections 4.1 and 4.2 we studied this issue for when these simple elements belong to $\mathcal{SOT}^{const}_{SO(3)}(d,\omega)$. Of course, then $\overline{(\omega, A)}$ even contains spin-orbit tori from $\mathcal{SOT}^{const}_{SO(2)}(d,\omega) \subset \mathcal{SOT}_{SO(2)}(d,\omega)$.

Thus it is a natural to look into the more general situation when $\overline{(\omega, A)} \cap \mathcal{SOT}_{SO(2)}(d,\omega)$ is nonempty or the even more general case when $\overline{(\omega, A)} \cap \mathcal{SOT}_H(d,\omega)$ is nonempty. Note that $\mathcal{SOT}^{const}_{SO(2)}(d,\omega)$ is a proper subset of $\mathcal{SOT}_{SO(2)}(d,\omega)$. So this point of view is indeed a generalization of the one in Sections 4.1,4.2. (IMPROVE TEXT?????)

Thus in this section we discuss those $(\omega, A)$ for which $\overline{(\omega, A)}$ contains elements in $\mathcal{SOT}_H(d,\omega)$ where $H$ is a subgroup of $SO(3)$ with special emphasis on the case $H = SO(2)$. This leads to the concept of "H normal form" given by the following definition.

**Definition 4.5** *($H$ normal form, $\mathcal{CB}_H(d,\omega), \mathcal{CB}_H$)*
*Let $H$ be a subgroup of $SO(3)$ and let $(\omega, A)$ be in $\mathcal{SOT}(d,\omega)$. Then we call a $(\omega, A')$ in $\mathcal{SOT}_H(d,\omega)$ an "H normal form of $(\omega, A)$" if $(\omega, A) \sim (\omega, A')$, i.e., $(\omega, A') \in \overline{(\omega, A)}$. We denote by $\mathcal{CB}_H(d,\omega)$ the set of all spin-orbit tori in $\mathcal{SOT}(d,\omega)$ which have an $H$ normal form, i.e.,*

$$\mathcal{CB}_H(d,\omega) := \bigcup_{(\omega,A)\in\mathcal{SOT}_H(d,\omega)} \overline{(\omega, A)} \,. \tag{4.28}$$

*Thus $(\omega, A) \in \mathcal{CB}_H(d,\omega)$ iff $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exists such that*

$$T^t(\phi + 2\pi\omega)A(\phi)T(\phi) \in H \,, \tag{4.29}$$

*holds for every $\phi \in \mathbb{R}^d$. We denote, for fixed $H$, the union of all $\mathcal{CB}_H(d,\omega)$ by $\mathcal{CB}_H$. The acronym $\mathcal{CB}_H$ will be explained further below. We also define*

$$\mathcal{TF}_H(\omega, A) := \left\{ T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3)) : R_{d,\omega}(T; \omega, A) \in \mathcal{SOT}_H(d,\omega) \right\} .$$
$$= \left\{ T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3)) : T^t(\phi + 2\pi\omega)A(\phi)T(\phi) \in H \right\} . \tag{4.30}$$

*Thus $(\omega, A) \in \mathcal{CB}_H(d,\omega)$ iff $\mathcal{TF}_H(\omega, A)$ is nonempty.* □

We now make some remarks on Definition 4.5.

**Remarks:**

(1) Let $(\omega, A)$ be in $\mathcal{SOT}(d,\omega)$. It follows from Definitions 4.1, 4.2, 4.5 and (4.8) that $(\omega, A)$ has a $G_0$ normal form iff $(\omega, A) \in \mathcal{CB}(d,\omega)$. Therefore, by Definition 4.5,

$$\mathcal{CB}_{G_0}(d,\omega) = \mathcal{CB}(d,\omega) , \qquad \mathcal{CB}_{G_0} = \mathcal{CB} . \tag{4.31}$$

Thus the sets $\mathcal{CB}_H(d,\omega)$ are generalizations of the set $\mathcal{CB}(d,\omega)$ and the sets $\mathcal{CB}_H$ are generalizations of the set $\mathcal{CB}$. This circumstance motivates the acronym $\mathcal{CB}_H$.

(2) Definition 4.5 gives us another property shared by similar spin-orbit tori since it implies that if $(\omega, A)$ belongs to $\mathcal{CB}_H$ then every spin-orbit torus in $\overline{(\omega, A)}$ belongs to $\mathcal{CB}_H$.

(3) In Chapter 9 we link $H$ normal forms with the exploitation of structural equations (see Proposition 9.11) and therefore connect them with the principal bundle $\lambda_d$. One practical aspect of this link is that the structural equations are a tool for finding a transfer field in $\mathcal{TF}_H(\omega, A)$.

(4) Let $(\omega, A)$ be in $\mathcal{SOT}(d,\omega)$ and $H, H'$ be subgroups of $SO(3)$ such that $H' \subset H$. Then $\mathcal{CB}_{H'}(d,\omega) \subset \mathcal{CB}_H(d,\omega)$ and $\mathcal{CB}_{H'} \subset \mathcal{CB}_H$.

(5) Let $(\omega, A)$ be in $\mathcal{SOT}_H(d,\omega)$ where $H$ is a subgroup of $SO(3)$. Recall that we equip $H$ with the relative topology from $SO(3)$. Then, by Definition 2.5 and (2.41) the spin transfer matrix $\Psi_{\omega,A}$ of $(\omega, A)$ belongs to $COC_{per}(\mathbb{R}^d, L_\omega, H)$.

(6) We now discuss briefly the method of "rational approximation" to find an $H$ normal form of a $(\omega, A) \in \mathcal{CB}_H(d,\omega)$. If $\omega \in \mathbb{Q}^d$ and if we choose a nonzero integer $j$ such that $j\omega \in \mathbb{Z}^d$ then the criterion (4.29) simplifies to the following property for $T(\phi)$:

$$T^t(\phi)\Psi(j;\phi)T(\phi) \in H . \tag{4.32}$$

If $\omega$ is not in $\mathbb{Q}^d$ then one try (??????can) do a "rational approximation". This entails replacing $\omega$ by an element $\omega$ in $\mathbb{Q}^d$, choosing a nonzero integer $j$ such that $j\omega \in \mathbb{Z}^d$ and then solving (4.32). (IS THIS STABLE NEAR A S-O RESONANCE, IN PARTICULAR NEAR SNAKE RESONANCES??????) □

We now discuss the important case $H = SO(2)$. For this one can show, e.g., as in Appendix C in [He2], that if $g \in \mathcal{C}(\mathbb{R}^d, SO(2))$ then an $f \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$ exists such that

$$g(\phi) = \exp(\mathcal{J} 2\pi f(\phi)) \,. \tag{4.33}$$

In the special case where $g \in \mathcal{C}_{per}(\mathbb{R}^d, SO(2))$ it easily follows from (4.33) that a constant $N \in \mathbb{Z}^d$ and a $h \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ exist such that

N = index

$$g(\phi) = \exp(\mathcal{J}[N \cdot \phi + 2\pi h(\phi)]) \,. \tag{4.34}$$

Smooth a f the g ⇒ A argument)

Now let $(\omega, A) \in \mathcal{SOT}_{SO(2)}(d, \omega)$ so that, by (4.1b), $A \in \mathcal{C}_{per}(\mathbb{R}^d, SO(2))$. Then, by (4.34), $N \in \mathbb{Z}^d$ and $a \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ exist such that

$$A(\phi) = \exp(\mathcal{J}[N \cdot \phi + 2\pi a(\phi)]) \,. \tag{4.35}$$

The elements of $\mathcal{TF}_{SO(2)}(\omega, A)$ are the discrete-time analogues of the invariant frame field (IFF) described in the continuous-time formalism, e.g., in [BEH]. This can be seen as follows.

Let $(\omega, A) \in \mathcal{CB}_{SO(2)}(d, \omega)$ and let us pick a $T \in \mathcal{TF}_{SO(2)}(\omega, A)$. Then, by (3.10), (4.1b), (4.35) and (4.38), $N \in \mathbb{Z}^d$ and $h \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ exist such that (4.38???????)

$$T^t(\phi + 2\pi\omega) A(\phi) T(\phi) = \exp(\mathcal{J}[N \cdot \phi + 2\pi h(\phi)]) \,. \tag{4.36}$$

Of course the rhs of (4.36) is the 1-turn spin transfer matrix of $R_{d,\omega}(T; \omega, A)$. If $S$ is a spin trajectory of $(\omega, A)$ over $\phi_0$, then by our transformation rule (3.4) we can transform $S$ into a spin trajectory $S'(n) = T^t(\phi_0 + 2\pi\omega n) S(n)$ of $R_{d,\omega}(T; \omega, A)$ and we see by (2.44) and (4.36) that $S'$ obeys the simple EOM:

$$S'(n+1) = \exp\Big(\mathcal{J}[N \cdot (\phi_0 + 2\pi\omega n) + 2\pi h(\phi_0 + 2\pi\omega n)]\Big) S'(n) \,. \tag{4.37}$$

We now define

$$\begin{aligned} \mathcal{IFF}(\omega, A) &:= \mathcal{TF}_{SO(2)}(\omega, A) \\ &= \{T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3)) : R_{d,\omega}(T; \omega, A) \in \mathcal{SOT}_{SO(2)}(d, \omega)\} \,. \end{aligned} \tag{4.38}$$

We call every element of $\mathcal{IFF}(\omega, A)$ an "IFF of $(\omega, A)$". Clearly, by Definition 4.5, $\mathcal{IFF}(\omega, A)$ is nonempty iff $(\omega, A) \in \mathcal{CB}_{SO(2)}(d, \omega)$.

(MENTION QUAIPERIODICITY??????????)

For the case when $T$ is chosen so that the argument of the exponential is independent of $\phi$, $T(\phi)$ is analogous to the uniform IFF of the continuous-time formalism [BEH]. In that case we can write the argument as $2\pi\nu$ where $\nu$ is the ADST. Of course since, by (4.18) and (4.30), $\mathcal{TF}^{const}_{SO(2)}(\omega, A) \subset \mathcal{TF}_{SO(2)}(\omega, A)$ we have, by (4.20) and (4.38),

$$\mathcal{UIFF}(\omega, A) \subset \mathcal{IFF}(\omega, A) \,. \tag{4.39}$$

It is noteworthy that the constant $N$ in (4.36) carries interesting information about $A$. For example for $(\omega, A)$ to be proper it is necessary that all $d$ components of $N$ are even integers.

$S^3$

This is shown in Section 7.2 of [He2] by using simple arguments from Homotopy Theory. We will return to (4.36) later on. CLS

O

Let $(\omega, A) \in \mathcal{ACB}(d, \omega)$. By Remark 2 in Section 4.1 the set $\mathcal{UIFF}(\omega, A)$ is nonempty so that, by (4.39), $\mathcal{IFF}(\omega, A)$ is nonempty. Then we have

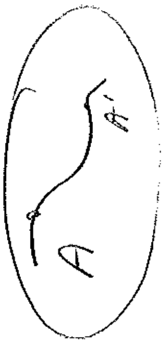
$$\mathcal{ACB}(d, \omega) \subset \mathcal{CB}_{SO(2)}(d, \omega) . \tag{4.40}$$

Let $(\omega, A) \in \mathcal{ACB}(d, \omega)$. We will now briefly discuss how IFFs are important from a practical point of view. In fact in the computer code SPRINT [EPAC98, BHV98, Ho, Vo, BHV00, BEH00] one computes a spin tune $\nu$ of the first kind in two steps. By (4.40) $\mathcal{IFF}(\omega, A)$ is nonempty. Then in the first step of SPRINT one computes an IFF of $(\omega, A)$, say $T$. By (4.36), an $N \in \mathbb{Z}^d$ and a $h \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ exist such that (4.36) holds. In the second step of SPRINT one computes $\nu$ by doing some averaging (NEEDS FIXING ?????) Fourier Analysis of $h$. For more remarks on the first step see Remark 6 in Chapter 5. $\nabla_0$

When N is even the quaternions are periodic

# 5 Polarization fields and invariant spin fields

## 5.1 Generalities



High precision measurements of the spin-dependent properties of the interactions of colliding protons, deuterons, electrons and positrons in storage rings depend on the equilibrium spin polarization being maximized. This, in turn, is facilitated by an understanding of the meaning of the term equilibrium, both as it applies to the value of the polarization and to its direction at each point in phase space. We will return to these matters in Section 5.4 but continue now with a definition and an exploration of the effects of maps.

Suppose therefore that $(\omega, A) \in \mathcal{SOT}(d, \omega)$ and that, at time $n = 0$, a spin vector $S_{\phi_0}$ has been assigned to every point $\phi_0 \in \mathbb{R}^d$ of the "orbital torus" and consider its evolution according to (2.44). Let $S_{\phi_0}(n)$ denote the spin trajectory over $\phi_0$ and for the initial value $S_0 = S_{\phi_0}$. Of course $S_{\phi_0}$ is supposed to be $2\pi$-periodic in $\phi_0$ and we define the field $\mathcal{S} = \mathcal{S}(n, \phi)$ by $\mathcal{S}(n, \phi + 2\pi n\omega) = S_\phi(n)$ where $n, \phi$ vary over $\mathbb{Z}, \mathbb{R}^d$ respectively. Clearly $\mathcal{S}(n, \cdot)$ is the distribution of spins which started at $n = 0$ with the assignment $S_{\phi_0}$ and evolved under the dynamics of (2.44). Since (2.44) gives us $S_\phi(n+1) = A(\phi + 2\pi n\omega) S_\phi(n)$, we have

SOT

Same homotopy

$$\mathcal{S}(n+1, \phi) = A(\phi - 2\pi\omega) \mathcal{S}(n, \phi - 2\pi\omega) . \tag{5.1}$$

**Definition 5.1** *(Polarization field, invariant spin field) Let $(\omega, A) \in \mathcal{SOT}(d, \omega)$. We call a function $\mathcal{S} : \mathbb{Z} \times \mathbb{R}^d \longrightarrow \mathbb{R}^3$ a "polarization field of $(\omega, A)$", if it satisfies the evolution equation (5.1) and if $\mathcal{S}(0, \cdot) \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$. Clearly $\mathcal{S}(n, \cdot) \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ and we call $\mathcal{S}(0, \cdot)$ the "generator of $\mathcal{S}$".*

*A polarization field $\mathcal{S}$ is called "invariant" if it is time-independent, i.e., if $\mathcal{S}(n, \cdot) = \mathcal{S}(0, \cdot)$. We call a polarization field $\mathcal{S}$ a "spin field" if $|\mathcal{S}(0, \cdot)| = 1$ and then $|\mathcal{S}(n, \cdot)| = 1$. If an invariant polarization field is a spin field we call it an "invariant spin field (ISF)".* □

**Remark:**

(1) According to Definition 5.1, every polarization field $\mathcal{S}$ fulfills three different conditions: the "dynamical" condition (5.1), the "kinematical" condition that $\mathcal{S}(0,\cdot)$ is $2\pi$–periodic, and the "regularity" condition that $\mathcal{S}(0,\cdot)$ is continuous. However, in contrast to the dynamical and kinematical conditions, the regularity condition is a matter of choice. While in this work, and in [He2], we choose continuity as the regularity property of the functions $A$ and $\mathcal{S}(0,\cdot)$ etc., this property is a matter of choice and can basically vary between the extremes "Borel measurable" and "of class $C^\infty$". □

We define the function $\tilde{\mathcal{P}}_{\omega,A} : \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3) \to \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$ by

$$\tilde{\mathcal{P}}_{\omega,A}(g) := A(\cdot - 2\pi\omega)g(\cdot - 2\pi\omega) , \tag{5.2}$$

where $g \in \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$. Clearly $\tilde{\mathcal{P}}_{\omega,A}$ is a bijection since $\tilde{\mathcal{P}}_{-\omega,A'}$ is its inverse where $A' := A^t(\cdot - 2\pi\omega)$. Thus the function $\tilde{L}_{\omega,A} : \mathbb{Z} \times \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3) \to \mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$, defined by

$$\tilde{L}_{\omega,A}(n;\cdot) := \tilde{\mathcal{P}}^n_{\omega,A} , \tag{5.3}$$

where $\tilde{\mathcal{P}}^n_{\omega,A}$ denotes the $n$-th iteration of $\tilde{\mathcal{P}}_{\omega,A}$, is a $\mathbb{Z}$-action on $\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3)$, i.e., $(\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3),\tilde{L}_{\omega,A})$ is a $\mathbb{Z}$-set. Note that, by (2.41), (5.2) and (5.3),

$$\tilde{L}_{\omega,A}(n;g) = \Psi_{\omega,A}(n;\cdot - 2\pi n\omega)g(\cdot - 2\pi n\omega) . \tag{5.4}$$

Of course, with (5.2) the evolution equation (5.1) can be written as $\mathcal{S}(n+1,\cdot) = \tilde{\mathcal{P}}_{\omega,A}(\mathcal{S}(n,\cdot))$ whence, by (5.3), for every polarization field $\mathcal{S}$

$$\mathcal{S}(n,\cdot) = \tilde{L}_{\omega,A}(n;\mathcal{S}(0,\cdot)) . \tag{5.5}$$

**Remarks:**

(2) Let $(\omega,A) \in \mathcal{SOT}(d,\omega)$. Comparing (5.1) and (2.44) we see they are both linear systems. However (5.1) is more complex in that it depends on two independent variables but it is simpler in that it is autonomous. Accordingly, the transformation rule (5.20) of the polarization field is autonomous while the transformation rule (3.4) of the spin trajectories is nonautonomous.

?

Before we take a closer look at the linearity of (5.1) we make some general comments on linearity. Let $G$ be a group and $(E,L)$ be a left $G$ set. Also let $E$ be a vector space and let every $L(h,\cdot)$ be linear where, of course, $h \in G$. Recalling Definition 2.3, the $L(h,\cdot)$ are bijections whence, since they are linear, they are automorphisms of the vector space $E$, i.e., $L(h,\cdot) \in GL(E)$ where $GL(E)$ denotes the set of automorphisms of the vector space $E$. Recalling Remark 1 in Section 2.3, one can cast the data of $L$ into the function $L^{hom} : G \to GL(E)$ defined by $L^{hom}(h) := L(h,\cdot)$. The key point here is that, since $L(h,\cdot) \in GL(E)$, the function $L^{hom}$ is a homomorphism from the group $\mathbb{Z}$ into the group $GL(E)$ where the group multiplication in $GL(E)$ is understood to be the composition of functions. In other words $L^{hom}$ is a so-called "representation" of the group $\mathbb{Z}$ on the linear space $E$. Thus the notion of group representation emerges as a specialization of the notion of left group action. In fact the first contact of a physicist with group actions typically occurs via group representations.

Therefore the linearity of (5.1) has the following implication. Since $\mathbb{R}^3$ has a natural structure of a real vector space, the set $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ is a real linear space in a natural way. Moreover, by (5.4), we have for for $n \in \mathbb{Z}, x, x' \in \mathbb{R}, g, g' \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$,

$$\tilde{L}_{\omega,A}(n; xg + x'g') = x\tilde{L}_{\omega,A}(n; g) + x'\tilde{L}_{\omega,A}(n; g') \,, \tag{5.6}$$

whence every $\tilde{L}_{\omega,A}(n; \cdot)$ is a linear function. It thus (then ?????)follows from the above that $\tilde{L}_{\omega,A}(n; \cdot) \in GL(\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3))$ and that the function $\tilde{L}^{hom}_{\omega,A}$ defined above is a representation of the group $\mathbb{Z}$ on the linear space $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$. Of course we also have $\tilde{P}_{\omega,A} \in GL(\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3))$.

(3) The linearity of (5.1) has even more interesting consequences which we mention here although they are beyond the scope of this work. Thus we shall only give the flavor and we will use terminology that we do not define.

Since $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ is a linear space it is, in particular, an Abelian group and so, by (5.6), the linear function $\tilde{L}_{\omega,A}(n; \cdot)$ is a homomorphism from the Abelian group $\mathbb{Z}$ into the Abelian group $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$. This allows one, via standard techniques from Dynamical Systems Theory, to implement a so-called module structure on $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ with respect to the ring $\mathbb{Z}\mathbb{Z}$ (the so-called integer group ring of $\mathbb{Z}$). Feldman and Moore [FM] seem to have been the first to apply, in the 1970's, this encoding technique to the field of Dynamical Systems. See also [HK1, HK2, Zi1]. Having thus encoded $(\omega, A)$ into a module structure one arrives, via standard techniques from Homological Algebra, at the cohomology groups $Ext^k_{\mathbb{Z}\mathbb{Z}}(\mathbb{Z}, \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3))$. Interestingly, one can then rather easily show [He1] that if $(\omega, A)$ is off orbital resonance,it has an ISF iff the zeroth cohomology group is nontrivial, i.e., iff $Ext^0_{\mathbb{Z}\mathbb{Z}}(\mathbb{Z}, \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3))$ has more than one element. Note that these standard constructions can be performed within our context because, recalling Section 3.2, the $\mathbb{Z}$ space $(\mathbb{R}^{d+3}, L_{\omega,A})$ is a skew product over the $\mathbb{Z}$ space $(\mathbb{R}^d, L_\omega)$. Note also that this skew product property will play a role in Chapter 9. □

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Denoting the set of polarization fields of $(\omega, A)$ by $\mathcal{PF}(\omega, A)$ and referring to (5.5) and Definition 5.1 we have

$$\mathcal{PF}(\omega, A) = \left\{ \mathbb{Z} \times \mathbb{R}^d \xrightarrow{\mathcal{S}} \mathbb{R}^3 : \mathcal{S}(0, \cdot) \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3) \,,\ (\forall n \in \mathbb{Z})\ \mathcal{S}(n, \cdot) = \tilde{L}_{\omega,A}(n; \mathcal{S}(0, \cdot)) \right\} . \tag{5.7}$$

Moreover, denoting the set of invariant spin fields of $(\omega, A)$ by $\mathcal{ISF}(\omega, A)$ and referring to (5.7) and Definition 5.1 we have

$$\mathcal{ISF}(\omega, A) = \{\mathcal{S} \in \mathcal{PF}(\omega, A) : \mathcal{S}(0, \cdot) \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2) \,,\ (\forall n \in \mathbb{Z})\ \mathcal{S}(0, \cdot) = \tilde{L}_{\omega,A}(n; \mathcal{S}(0, \cdot))\} \,, \tag{5.8}$$

where the 2-sphere is defined by $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ and is equipped with the relative topology from $\mathbb{R}^3$ (i.e., a subset of $\mathbb{S}^2$ is open iff it is the intersection of $\mathbb{S}^2$ with an open subset of $\mathbb{R}^3$). It follows from (5.2) and Definition 5.1 that any polarization field $\mathcal{S}$ of $(\omega, A)$ is invariant iff

$$\mathcal{S}(0,\cdot) = \tilde{\mathcal{P}}_{\omega,A}(\mathcal{S}(0,\cdot)) \,. \tag{5.9}$$

So from (5.8),(5.9) we have

$$\mathcal{ISF}(\omega, A) = \{\mathcal{S} \in \mathcal{PF}(\omega, A) : \mathcal{S}(0,\cdot) \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2) \ , \ \mathcal{S}(0,\cdot) = \tilde{\mathcal{P}}_{\omega,A}(\mathcal{S}(0,\cdot))\} \,. \tag{5.10}$$

Since $\mathcal{S}(0,\cdot)$ is the generator of $\mathcal{S}$, (5.10) delivers the following criterion, which we call the "ISF criterion":

- (ISF criterion) A spin-orbit torus $(\omega, A)$ in $\mathcal{SOT}(d,\omega)$ has an ISF iff a function $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ exists which satisfies the functional equation

$$g(\cdot + 2\pi\omega) = Ag \,. \tag{5.11}$$

Note that if a function $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ satisfies the ISF criterion of $(\omega, A)$ then $g$ is the generator of an ISF $\mathcal{S}$, i.e.,

$$\mathcal{S}(n,\cdot) = \tilde{L}_{\omega,A}(n; g) \,. \tag{5.12}$$

Clearly the generator of every ISF satisfies the ISF criterion. The ISF criterion (5.9) will play a major role in Chapter 9.

If a spin-orbit torus $(\omega, A)$ has an ISF $\mathcal{S}$ then $-\mathcal{S}$ is also an ISF of $(\omega, A)$. So since $\mathcal{S} \neq -\mathcal{S}$, if $(\omega, A)$ has a finite number of ISF's, then this number is even. The important subcase where $(\omega, A)$ has exactly two ISF's is dealt with in Chapter 6. Chapter 8 provides an example for the subcase where $(\omega, A)$ has no ISF. Note that the spin-orbit torus of this example is on orbital resonance. There are some indications, mainly from numerical computations on ISF's, that practically relevant spin-orbit tori which have no ISF are "rare". Thus we state the following conjecture, which we call the "ISF-conjecture": If a spin-orbit torus $(\omega, A)$ is off orbital resonance, then it has an ISF. The ISF-conjecture is, at least to our knowledge, unsettled. The existence problem of the ISF is important both theoretically and practically and Chapter 9 presents a new framework for discussing it. and [illegible]
/ for [illegible]

We now make some remarks on the relationship between ISFs and IFFs.

**Remarks:**

(3) Let $(\omega, A) \in \mathcal{CB}_{SO(2)}(d,\omega)$. Then $(\omega, A)$ has an ISF. To show that, we recall from Section 4.3 that $(\omega, A)$ has an IFF, say $T$, whence $N \in \mathbb{Z}^d$ and $h \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R})$ exist such that (4.36) holds. This implies that

$$A(\phi)T(\phi) = T(\phi + 2\pi\omega)\exp(\mathcal{J}[N \cdot \phi + 2\pi h(\phi)]) \,. \tag{5.13}$$

Then by multiplying (5.13) from the right by $(0,0,1)^t$ and by using (4.3) we have,

$$\begin{aligned} A(\phi)T(\phi)(0,0,1)^t &= T(\phi + 2\pi\omega)\exp(\mathcal{J}[N \cdot \phi + 2\pi h(\phi)])(0,0,1)^t \\ &= T(\phi + 2\pi\omega)(0,0,1)^t \,. \end{aligned} \tag{5.14}$$

Next, with $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ defined by $g(\phi) := T(\phi)(0,0,1)^t$, we see that $|g(\phi)| = |T(\phi)(0,0,1)^t| = |(0,0,1)^t| = 1$ so that $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$. Then from (5.14) $g(\cdot + 2\pi\omega) = Ag$. So $g$ satisfies the ISF criterion for $(\omega, A)$ whence $(\omega, A)$ indeed has an ISF. We have thereby shown that the third column of every IFF is the generator of an ISF.

(4) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$. We now prove the converse of Remark 3 above. Thus let $(\omega, A)$ have an ISF so that, by the ISF criterion, a $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ exists such that $g(\cdot + 2\pi\omega) = Ag$. Let $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ and let the third column of $T$ be $g$. Then by the ISF criterion

$$T(\phi + 2\pi\omega)(0,0,1)^t = g(\phi + 2\pi\omega) = A(\phi)g(\phi) ,$$

whence $(0,0,1)^t = T^t(\phi + 2\pi\omega)A(\phi)T(\phi)(0,0,1)^t$. This implies, by (4.3), that $T^t(\phi + 2\pi\omega)A(\phi)T(\phi)$ so that by Definition 4.5, $T \in \mathcal{TF}_{SO(2)}(\omega, A)$ whence $T$ is an IFF of $(\omega, A)$. This raises the natural question about the conditions under which a spin-orbit torus with ISF has an IFF. Of course by Remark 3 above we know that there are many situations where such a $T$ exists. However, as shown in [He2], it can happen that $(\omega, A)$ has an ISF but no IFF. We thus formulate the following problem which we call the "IFF problem": Under which conditions on a $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ does a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ exist such that $g = T(0,0,1)^t$? In Chapter 7 of [He2] this question was studied by using simple arguments from Homotopy Theory. In Chapter 9 we will consider this question by using principal bundles.

(5) Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$ so that, by (4.40), $(\omega, A) \in \mathcal{CB}_{SO(2)}(d,\omega)$. Then $(\omega, A)$ has an IFF whence, by Remark 3 above, $(\omega, A)$ has an ISF.

(6) Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$. Recall that we briefly discussed at the end of Section 4.3 the computer code SPRINT that computes a spin tune of first kind in two steps and that in the first step an IFF of $(\omega, A)$ is computed, say $T$. Note that by Remark 5 above an IFF exists. The first step of SPRINT consists of two substeps. In the first substep a generator $g$ of an ISF is computed by using the fact that $g$ satisfies the ISF criterion. This makes it possible to apply the technique of stroboscopic averaging to compute $g$. The second substep is a simple orthonormalization procedure to obtain a $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ whose third column is $g$. Thus, by Remark 4 above, $T$ is an IFF (needs work???????). $\square$

## 5.2 Transforming polarization fields

We now derive a natural rule for transforming polarization fields according to our transformation rule for a spin-orbit torus from Chapter 3. Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and let $(\omega, A') := R_{d,\omega}(T; \omega, A)$ where, of course, $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$. Then from (3.10) and (5.2) and for $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ we obtain

$$(\tilde{\mathcal{P}}_{0,T} \circ \tilde{\mathcal{P}}_{0,T^t})(g) = \tilde{\mathcal{P}}_{0,T}(T^t g) = TT^t g = g ,$$

$$(\tilde{\mathcal{P}}_{0,T^t} \circ \tilde{\mathcal{P}}_{\omega,A} \circ \tilde{\mathcal{P}}_{0,T})(g) = (\tilde{\mathcal{P}}_{0,T^t} \circ \tilde{\mathcal{P}}_{\omega,A})(Tg) = \tilde{\mathcal{P}}_{0,T^t}\Big(A(\cdot - 2\pi\omega)T(\cdot - 2\pi\omega)g(\cdot - 2\pi\omega)\Big)$$

$$= T^t A(\cdot - 2\pi\omega)T(\cdot - 2\pi\omega)g(\cdot - 2\pi\omega) = A'(\cdot - 2\pi\omega)g(\cdot - 2\pi\omega) = \tilde{\mathcal{P}}_{\omega,A'}(g) \,,$$

so that

$$\tilde{\mathcal{P}}_{0,T^t} = \tilde{\mathcal{P}}_{0,T}^{-1} \,, \qquad \tilde{\mathcal{P}}_{0,T}^{-1} \circ \tilde{\mathcal{P}}_{\omega,A} \circ \tilde{\mathcal{P}}_{0,T} = \tilde{\mathcal{P}}_{\omega,A'} \,. \tag{5.15}$$

It follows from (5.3),(5.15) that

$$\tilde{L}_{\omega,A'}(n;\cdot) = \tilde{\mathcal{P}}^n_{\omega,A'} = \tilde{\mathcal{P}}_{0,T}^{-1} \circ \tilde{\mathcal{P}}^n_{\omega,A} \circ \tilde{\mathcal{P}}_{0,T} = \tilde{\mathcal{P}}_{0,T}^{-1} \circ \tilde{L}_{\omega,A}(n,\cdot) \circ \tilde{\mathcal{P}}_{0,T} \,. \tag{5.16}$$

Thus, recalling Definition 3.3, $\tilde{\mathcal{P}}_{0,T}^{-1}$ is a $\mathbb{Z}$ map from the $\mathbb{Z}$ set $(\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3), \tilde{L}_{\omega,A})$ to the $\mathbb{Z}$ set $(\mathcal{C}_{per}(\mathbb{R}^d,\mathbb{R}^3), \tilde{L}_{\omega,A'})$. Since $\tilde{\mathcal{P}}_{0,T}^{-1}$ is a bijection both $\mathbb{Z}$ sets are isomorphic as $\mathbb{Z}$ sets.

With (5.16) it is simple to transform polarization fields. In fact if $g$ is the generator of a polarization field $\mathcal{S}$ of $(\omega, A)$ then we can relate it to the polarization field $\mathcal{S}'$ of $(\omega, A')$ whose generator is $\tilde{\mathcal{P}}_{0,T}^{-1}(g) = T^t g$. Thus, by (5.5),

$$\mathcal{S}(n,\cdot) = \tilde{L}_{\omega,A}(n;g) \,, \quad \mathcal{S}'(n,\cdot) = \tilde{L}_{\omega,A}(n;T^t g) \,, \tag{5.17}$$

whence, by (5.3),(5.15),

$$\begin{aligned}
\mathcal{S}'(n,\cdot) &= \tilde{L}_{\omega,A'}(n;\mathcal{S}'(0,\cdot)) = \tilde{\mathcal{P}}^n_{\omega,A'}(\mathcal{S}'(0,\cdot)) = (\tilde{\mathcal{P}}_{0,T}^{-1} \circ \tilde{\mathcal{P}}^n_{\omega,A} \circ \tilde{\mathcal{P}}_{0,T})(\mathcal{S}'(0,\cdot)) \\
&= (\tilde{\mathcal{P}}_{0,T}^{-1} \circ \tilde{\mathcal{P}}^n_{\omega,A} \circ \tilde{\mathcal{P}}_{0,T} \circ \tilde{\mathcal{P}}_{0,T}^{-1})(g) = \tilde{\mathcal{P}}_{0,T}^{-1}(\tilde{\mathcal{P}}^n_{\omega,A}(g) \\
&= \tilde{\mathcal{P}}_{0,T}^{-1}(\tilde{L}_{\omega,A}(n,g) = \tilde{\mathcal{P}}_{0,T}^{-1}(\mathcal{S}(n,\cdot)) \,,
\end{aligned} \tag{5.18}$$

i.e.,

$$\mathcal{S}'(n,\cdot) = \tilde{\mathcal{P}}_{0,T}^{-1}(\mathcal{S}(n,\cdot)) \,. \tag{5.19}$$

We conclude from (5.19) that if $\mathcal{S}$ is a polarization field of $(\omega, A)$ then $\mathcal{S}'$, defined by the lhs of (5.19), is a polarization field of $(\omega, A')$. Thus with (5.19) we have a natural transformation rule of polarization fields. Note that, by (5.2) and (5.15), one can write (5.19) as

$$\mathcal{S}'(n,\phi) = T^t(\phi)\mathcal{S}(n,\phi) \,. \tag{5.20}$$

**Remark:**

(7) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$. It is clear, by (5.7), that (5.19) maps the set $\mathcal{PF}(\omega, A)$ of polarization fields of $(\omega, A)$ bijectively onto the set $\mathcal{PF}(\omega, A')$. Moreover it is clear, by (5.10), that (5.19) maps $\mathcal{ISF}(\omega, A)$ bijectively onto $\mathcal{ISF}(\omega, A')$. In particular two similar spin-orbit tori have the same number of ISFs. Thus we arrived at another property shared by similar spin-orbit tori.

(8) Clearly the transformation rule (5.20) of the polarization field and the transformation rule (3.4) of spin trajectories are very similar. Unsurprisingly, one can even show the following. Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and let $\mathcal{S}$ be a polarization field of $(\omega; A)$. Also, $(\omega, A') := R_{d,\omega}(T;\omega, A)$ where of course $T \in \mathcal{C}_{per}(\mathbb{R}^d, SO(3))$ and let $\mathcal{S}'$ be the polarization field of $(\omega, A')$ which is the transform of $\mathcal{S}$ as in (5.19). Clearly by Definition 5.1, if we pick $\phi \in \mathbb{R}^d$ then the function $S_\phi$ defined by $S_\phi(n) := \mathcal{S}(n, \phi + 2\pi n\omega)$ is a spin trajectory of $(\omega, A)$ over $\phi$ and the function $S'_\phi$ defined by $S'_\phi(n) := \mathcal{S}'(n, \phi + 2\pi n\omega)$ is a spin trajectory of $(\omega, A')$ over $\phi$. The point here is that $S'_\phi$ is the transform of $S_\phi$ via (3.4). □

## 5.3 $k$-turn ISFs

Consider again $(\omega, A) \in \mathcal{SOT}(d, \omega)$. For nonnegative integer $k$ we call a spin field, $\mathcal{S}$, of $(\omega, A)$ a "$k$–turn invariant spin field" of $(\omega, A)$ if, for all $\phi \in \mathbb{R}^d$ and every integer $n$,

$$\mathcal{S}(n + k, \phi) = \mathcal{S}(n, \phi) . \tag{5.21}$$

As explained in Section X in [BEH], a $k$–turn invariant spin field arises naturally if particles return to their original positions in orbital phase space after $k$ turns. It can then be found by solving an eigenproblem. We do this in Chapter 8). However, a $k$–turn invariant spin field is more likely to occur without the aformentioned constraint on the orbital motion. We denote the set of $k$–turn invariant spin fields of $(\omega, A)$ by $\mathcal{ISF}_k(\omega, A)$. Of course, the concepts ISF and 1–turn invariant spin field are identical, i.e., $\mathcal{ISF}(\omega, A) = \mathcal{ISF}_1(\omega, A)$. and every ISF is a $k$–turn invariant spin field for all $k$, i.e.,

$$\mathcal{ISF}(\omega, A) \subset \mathcal{ISF}_k(\omega, A) . \tag{5.22}$$

This is important for Section 8.2 since if, for example, one can find all 2–turn invariant spin fields then it is sometimes simple to obtain all ISF's by just checking which of the 2–turn invariant spin fields are ISF's. Note that a $k$–turn invariant spin field with $k \neq 1$ need not be an ISF. In fact this happens in Section 8.2. It is easy to show, by (5.1)and (5.21), that a $k$–turn invariant spin field of $(\omega, A)$ exists iff a function $g \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{S}^2)$ exists such that, for all $\phi \in \mathbb{R}^d$,

$$g(\phi + 2\pi k\omega) = \Psi_{\omega,A}(k; \phi) g(\phi) . \tag{5.23}$$

If that is the case then the function $\mathcal{S}$, defined for $\phi \in \mathbb{R}^d, n \in \mathbb{Z}$, by $\mathcal{S}(n, \phi) := g(\phi)$, is a $k$–turn invariant spin field of $(\omega, A)$. Note that, when $k = 1$, (5.23) is the ISF criterion (5.11) in Chapter 5. Denoting the set of $k$–turn invariant spin fields of $(\omega, A)$ by $\mathcal{ISF}_k(\omega, A)$ we obtain the definition

$$\mathcal{ISF}_k(\omega, A) \; := \; \{\mathcal{S} \in \mathcal{SF}(\omega, A) : \mathcal{S}(0, \cdot) = \tilde{L}_{\omega,A}(1; \mathcal{S}(0, \cdot))\} . \tag{5.24}$$

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## 5.4 Polarization

We now tie together the concepts of polarization field and polarization. Thus consider a family $(\omega(J), A_J)_{J \in \Lambda}$ of spin-orbit tori where $(\omega(J), A_J) \in \mathcal{SOT}(d, \omega(J))$ and $\Lambda$ is the set of action values. We note (see also [BH, ?]) that for every $J \in \Lambda$, we have a so-called "local polarization", say $\mathcal{S}_{loc,J}$ which by definition is a polarization field of $(\omega(J), A_J)$ satisfying

$$|\mathcal{S}_{loc,J}| \leq 1 . \tag{5.25}$$

The associated bunch polarization is then given by

$$P(n) = \left| \int_\Lambda dJ \rho_{eq}(J) \int_{[0,2\pi]^d} d\phi \mathcal{S}_{loc,J}(n, \phi) \right| , \tag{5.26}$$

where $\rho_{eq} = \rho_{eq}(J)$ is the equilibrium orbital phase space density. In the so-called "spin equilibrium" the polarization fields $\mathcal{S}_{loc,J}$ are, by the definition of the spin equilibrium, invariant.

Thus the bunch polarization for the combined beam equilibrium and spin equilibrium reads as

$$P(n) = P(0) = \left| \int_\Lambda dJ \rho_{eq}(J) \int_{[0,2\pi]^d} d\phi S_{loc,J}(0,\phi) \right| , \tag{5.27}$$

whence

$$P(0) \le \int_\Lambda dJ \rho_{eq}(J) \left| \int_{[0,2\pi]^d} d\phi S_{loc,J}(0,\phi) \right| . \tag{5.28}$$

Note that we assume that the function $\rho_{eq}$ is regular enough to ensure that the integrals in (5.26), (5.27) and (5.28) are meaningful. Then under the assumption that every $(\omega(J), A_J)$ has an ISF and since $|\mathcal{S}_{loc,J}| \le 1$, with (5.28) we have

$$\mathcal{P}(0) \le \mathcal{P}_{max}(0) , \tag{5.29}$$

where

$$\mathcal{P}_{max}(0) := \sup\left( \left\{ \int_\Lambda dJ \rho_{eq}(J) \left| \int_{[0,2\pi]^d} d\phi \mathcal{S}_J(0,\phi) \right| : \mathcal{S}_J \in \mathcal{ISF}(\omega(J), A_J) \right\} \right) . \tag{5.30}$$

Thus the ISFs provide an upper bound for the polarization and this is why they are so important in practice. In Chapter 6 we will see how the Uniqueness Theorem leads to simplification of the rhs of (5.30).

# 6 Uniqueness of invariant spin fields

We saw with formula (5.29) in Chapter 5 that the invariant spin fields play a role in the estimation of the polarization of the bunch. We now reconsider (5.29) using the fact that the invariant spin fields also play a role for spin tunes of the first kind.

Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$. Then, by Remark 3 in Chapter 5, $(\omega, A)$ has an ISF. Since, as shown in Chapter 4, every spin-orbit torus in $\mathcal{ACB}$ has spin tunes of first kind and, as just shown, it has an ISF it is, for every given $(\omega, A) \in \mathcal{ACB}(d,\omega)$, natural to ask about the impact of the set $\Xi_1(\omega, A)$ on $\mathcal{ISF}(\omega, A)$. In fact, if $(\omega, A)$ is off orbital resonance, this question is partially answered by the following theorem whose proof can be found in Section F.10 of [He2].

**Theorem 6.1** (*The Uniqueness Theorem*) *Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ be off orbital resonance, i.e., let $(1,\omega)$ be nonresonant. Also, let $(\omega, A)$ be off spin-orbit resonance of first kind. Then $(\omega, A)$ has an ISF, say $\mathcal{S}$, and $\mathcal{S}$ and $-\mathcal{S}$ are the only ISF's of $(\omega, A)$.* □

Note that the claim of Theorem 6.1 that $(\omega, A)$ has an ISF is trivial since, from Remark 3 in Chapter 5, we know that every $(\omega, A) \in \mathcal{ACB}(d,\omega)$ has an ISF. Thus the meat of the claim of Theorem 6.1 is that $(\omega, A)$ has two ISF's. Recall also from Chapter 5 that the set of ISF's of a spin-orbit torus is either infinite or contains an even number of elements.

Don't need ISF to get spin tune of 2nd kind.

The Uniqueness Theorem allows us to simplify the formula for the maximum polarization (5.30). For that purpose we assume that the proper spin-orbit tori $(\omega(J), A_J)$ in (5.30) satisfy the assumptions of Theorem 6.1 for almost every $J$, i.e., we assume that a set $M \subset \Lambda$ exists which has Lebesgue measure zero and such that, for every $J \in (\Lambda \setminus M)$, the spin-orbit torus $(\omega(J), A_J)$ is off orbital resonance and off spin-orbit resonance of first kind. Thus, by Theorem 6.1 and for every $J \in \Lambda \setminus M)$, the spin-orbit torus $(\omega(J), A_J)$ has an ISF, say $\mathcal{S}_J$, and $\mathcal{ISF}(\omega(J), A_J)$ just has the two elements $\mathcal{S}_J, -\mathcal{S}_J$. Thus (5.30) simplifies to

max permissible (attainable)
equil beam pol.

$$\mathcal{P}_{max}(0) = \int_\Lambda dJ \rho_{eq}(J) \left| \int_{[0,2\pi]^d} d\phi \mathcal{S}_J(0,\phi) \right| , \tag{6.1}$$

where we assume that the functional dependences of $\rho_{eq}(J)$ and $\mathcal{S}_J(0,\phi)$ on $J$ are regular enough to ensure that the integrals in (6.1) are meaningful.

# 7 Quasiperiodicity and spectra

In Remark X of Section Y we explained how spin tunes of the first kind can be extracted from tracking simulations within the SPRINT algorithm. In this section we explain another method whereby spin tunes are extracted from the Fourier spectrum of the spin motion. This leads us to consider the notions of quasiperiodic function and spectral analysis and relate them in the context of spin-orbit tori. Note that similar studies are performed, in [BEH], for the continuous-time formalism.

Our basic IVP is of the form

$$S(n+1) = A(\phi_0 + 2\pi n\omega) S(n) , \qquad S(0) = S_0 \in \mathbb{R}^3 , \tag{7.1}$$

and in a simulation of it it is natural to obtain the frequency content of the data by using a discrete Fourier transform (DFT). The DFT of $S(0), ..., S(N)$ is defined by $\hat{S}$ where

$$\hat{S}(k) := \frac{1}{N+1} \sum_{n=0}^{N} S(n) \exp(-2\pi i n k/(N+1)) , \tag{7.2}$$

and where $k = 0, ..., N$. However in order to study spectra in the context of spin-orbit tori we define spectra in more general terms as follows.

Dependence of spectra on $\phi_0$, $\phi$; context a.e. ...

If $F : \mathbb{Z} \to X$ is a function with $X = \mathbb{R}^j$ or $X = \mathbb{R}^{j\times j}$ for some positive integer $j$ and if $\lambda \in [0,1)$ and $N$ is a nonnegative integer, then we define

$$a_N(F,\lambda) := (N+1)^{-1} \sum_{n=0}^{N} F(n) \exp(-2\pi i n\lambda) . \tag{7.3}$$

We denote by $\Lambda_{tot}(F)$ the set of those $\lambda \in [0,1)$ for which $a_N(F,\lambda)$ converges as $N \to \infty$. If $\lambda \in \Lambda_{tot}(F)$ we denote the limit of $a_N(F,\lambda)$ by $a(F,\lambda)$ and we define the "spectrum $\Lambda(F)$ of $F$" by $\Lambda(F) := \{\lambda \in \Lambda_{tot}(F) : a(F,\lambda) \neq 0\}$. It is clear by (7.2) and (7.3) how $a(S,\lambda)$ can be approximated by using a DFT.

We will show below that if $S$ is a spin trajectory of a spin-orbit torus $(\omega, A) \in \mathcal{ACB}(d,\omega)$ then $\Lambda_{tot}(S) = [0,1)$ and

$$\Lambda(S) \subset \Xi_1(\omega, A) \cup \{l \cdot \omega + n : l \in \mathbb{Z}^d, n \in \mathbb{Z}\} . \tag{7.4}$$

so that spin tunes of the first kind belong to the spectrum of spin trajectories of spin-orbit tori in $\mathcal{ACB}$.

Note that (7.4) is very important when one wants to numerically compute a spin tune of first kind by using the spectrum of $S$ (e.g., via the DFT technique). Figures...show DFT's of spin trajectories where $N = 10000, 100000$.(REDO THIS ?????)

To come further we need to return to quasiperiodicity. So let $f \in C_{per}(\mathbb{R}^d, X)$ with $X = \mathbb{R}^j$ or $X = \mathbb{R}^{j\times j}$ for some positive integer $j$. If $\chi \in \mathbb{R}^d$ then $f$ is called the "$\chi$-generator" of the function $F : \mathbb{Z} \to X$ defined by $F(n) = f(2\pi n\chi)$. A function $F : \mathbb{Z} \to X$ is called "$\chi$–quasiperiodic" if it has a $\chi$-generator and it is called "quasiperiodic" if it has a $\chi$-generator for some $\chi$. Note that $a(F, \lambda)$ is meaningful for every quasiperiodic function $F$ and every $\lambda \in [0,1)$. In fact [He2] if $F$ is $\chi$–quasiperiodic then $\Lambda_{tot}(F) = [0,1)$ and

$$\Lambda(F) \subset \{l \cdot \chi + n : l \in \mathbb{Z}^d, n \in \mathbb{Z}\} . \tag{7.5}$$

We now introduce the so-called uniform precession frames (UPF's) among which the $\omega$–quasiperiodic ones are useful in many ways. In particular $\omega$–quasiperiodic UPF's will allow us to derive (4.26) and (7.4). Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and let $\phi_0 \in \mathbb{R}^d$. Then a function $t : \mathbb{Z} \to SO(3)$ is called a "uniform precession frame (UPF) of $(\omega, A)$ over $\phi_0$" iff a $\nu \in [0,1)$ exists such that

$$t^t(n+1)A(\phi_0 + 2\pi n\omega)t(n) = \exp(2\pi\nu\mathcal{J}) . \tag{7.6}$$

Of course $\nu$ is uniquely determined by $t$ and so we call $\nu$ the "uniform precession rate (UPR) of $t$". We denote by $\Xi_2(\omega, A, \phi_0)$ the set of those UPR's which correspond to $\omega$–quasiperiodic UPF's over $\phi_0$ and we define $\Xi_2(\omega, A) := \bigcup_{\phi_0 \in \mathbb{R}^d} \Xi_2(\omega, A, \phi_0)$. One can prove the following theorem (see Sections 8.1 and 8.3 in [He2]).

**Theorem 7.1** *a) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$. If $\nu \in \Xi_2(\omega, A, \phi_0)$ for some $\phi_0 \in \mathbb{R}^d$ then every spin trajectory of $(\omega, A)$ over $\phi_0$ is $(\omega, \nu)$–quasiperiodic.*

*b) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and $\phi_0 \in \mathbb{R}^d$. Then*

$$\Xi_1(\omega, A) \subset \Xi_2(\omega, A, \phi_0) . \tag{7.7}$$

*In addition let $(\omega, A) \in \mathcal{ACB}_\nu(d,\omega)$ and $T \in \mathcal{TF}_{d,\omega}(A, A_{d,\nu})$ with $\nu \in [0,1)$. Then the function $t : \mathbb{Z} \to SO(3)$, defined by $t(n) := T(\phi_0 + 2\pi n\omega)$, is an $\omega$–quasiperiodic UPF over $\phi_0$ with UPR $\nu$.*

Couldn't find it in this paper

*c) Let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ be off orbital resonance, i.e., let $(1,\omega)$ be nonresonant, and for some $\phi_0 \in \mathbb{R}^d$, let $(\omega, A)$ have an $\omega$–quasiperiodic spin trajectory $S$ over $\phi_0$. Then $(\omega, A)$ has an invariant polarization field $\mathcal{S}$ such that, for all integers $n$,*

$$S(n) = G(\phi_0 + 2\pi n\omega) . \tag{7.8}$$

*If in addition $S(0)$ is normalized to 1 then $|S(n)| = 1$ and $S$ is an ISF of $(\omega, A)$.* □

Part a) of the above theorem implies that if $S$ is a spin trajectory of a spin-orbit torus $(\omega, A) \in \mathcal{ACB}(d,\omega)$ then $\Lambda_{tot}(S) = [0,1)$.

Next, let $(\omega, A) \in \mathcal{SOT}(d,\omega)$ and let $\phi_0 \in \mathbb{R}^d$. Then as in Section 8.3 in [He2], one can prove that if $\mu \in \Xi_2(\omega, A, \phi_0)$

$$\Xi_2(\omega, A, \phi_0) = [0,1) \cap \{\varepsilon\mu + m \cdot \omega + n : \varepsilon \in \{1,-1\}, m \in \mathbb{Z}^d, n \in \mathbb{Z}\} . \qquad (7.9)$$

Thus $\Xi_2(\omega, A, \phi_0)$ has only countably many elements. It could even be empty. It easily follows from Definition 4.3 that if $\nu \in \Xi_1(\omega, A)$ then

$$[0,1) \cap \{\varepsilon\nu + m \cdot \omega + n : \varepsilon \in \{1,-1\}, m \in \mathbb{Z}^d, n \in \mathbb{Z}\} \subset \Xi_1(\omega, A) . \qquad (7.10)$$

Let $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and let $\phi_0 \in \mathbb{R}^d$. Then we can pick a $\nu \in \Xi_1(\omega, A)$ which, by (7.7), belongs to $\Xi_2(\omega, A, \phi_0)$ so that, by (7.9),

$$\Xi_2(\omega, A, \phi_0) = [0,1) \cap \{\varepsilon\nu + m \cdot \omega + n : \varepsilon \in \{1,-1\}, m \in \mathbb{Z}^d, n \in \mathbb{Z}\} , \qquad (7.11)$$

whence, by (7.10), if $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and $\phi_0 \in \mathbb{R}^d$

$$\Xi_2(\omega, A, \phi_0) \subset \Xi_1(\omega, A) . \qquad (7.12)$$

It follows from (7.7) and (7.12) that if $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and $\phi_0 \in \mathbb{R}^d$ then

$$\Xi_2(\omega, A, \phi_0) \subset \Xi_1(\omega, A) \subset \Xi_2(\omega, A, \phi_0) , \qquad (7.13)$$

so that, if $(\omega, A) \in \mathcal{ACB}(d,\omega)$ and $\phi_0 \in \mathbb{R}^d$,

$$\Xi_2(\omega, A, \phi_0) = \Xi_1(\omega, A) . \qquad (7.14)$$

We conclude from (7.14) that if $\nu \in \Xi_1(\omega, A)$ and $\phi_0 \in \mathbb{R}^d$ then $\nu \in \Xi_2(\omega, A, \phi_0)$ whence, by (7.9)and (7.14),

$$\Xi_1(\omega, A) = \Xi_2(\omega, A, \phi_0) = [0,1) \cap \{\varepsilon\nu + m \cdot \omega + n : \varepsilon \in \{1,-1\}, m \in \mathbb{Z}^d, n \in \mathbb{Z}\} , (7.15)$$

which implies (4.26).

One can show [He1] that if $(\omega, A) \in \mathcal{SOT}(d,\omega), \phi_0 \in \mathbb{R}^d, \nu \in \Xi_2(\omega, A, \phi_0)$ and $S$ is a spin trajectory of $(\omega, A)$ over $\phi_0$ then

$$\Lambda(S) \subset \Xi_2(\omega, A, \phi_0) \cup \{l \cdot \omega + n : l \in \mathbb{Z}^d, n \in \mathbb{Z}\} . \qquad (7.16)$$

Thus if $(\omega, A) \in \mathcal{ACB}(d,\omega)$ then (7.14) and (7.16) imply (7.4).

We thus have attained our two main aims for this section. In the remaining parts of this section we discuss some more facts. Proved ?

(1) Although for a spin-orbit torus $(\omega, A)$ in $\mathcal{ACB}(d,\omega)$ every $\Xi_2(\omega, A, \phi_0)$ is nonempty due to (7.14), the converse need not hold, i.e., there can be spin-orbit tori $(\omega, A)$ in $\mathcal{SOT}(d,\omega) \setminus \mathcal{ACB}(d,\omega)$ for which all $\Xi_2(\omega, A, \phi_0)$ are nonempty. See Chapter 8. Thus $\Xi_2(\omega, A, \phi_0)$ is a tool which refines the tool $\Xi_1(\omega, A)$.

(2) If one merely assumes that a spin trajectory $S$ of $(\omega, A) \in \mathcal{SOT}(d,\omega)$ over $\phi_0$ is $(\omega,\nu)$–quasiperiodic with $\nu \in [0,1)$ then, by (7.5),

$$\Lambda(S) \subset \{l\nu + m \cdot \omega + n : m \in \mathbb{Z}^d, l, n \in \mathbb{Z}\} . \qquad (7.17)$$

Note that this assumption is weaker than the assumption that $\nu \in \Xi_2(\omega, A, \phi_0)$ whence it is no surprise that (7.17) is weaker than (7.16).

(3) If one only assumes that a spin trajectory $S$ of $(\omega, A) \in \mathcal{SOT}(d,\omega)$ over $\phi_0$ is $\chi$–quasiperiodic for some $\chi \in \mathbb{R}^j$ then, by (7.5),

$$\Lambda(S) \subset \{l \cdot \chi + n : l \in \mathbb{Z}^d, n \in \mathbb{Z}\} . \qquad (7.18)$$

(4) There are spin-orbit tori which have nonzero spin trajectories $S$ for which $\Lambda(S)$ is empty [He1]. Note that such spin trajectories are not quasiperiodic at all since a quasiperiodic function can only have an empty spectrum if it vanishes [He1].

(5) With the notation $\Xi_2(\omega, A, \phi_0)$ at hand we come to a second notion of spin tune, the "spin tune of the second kind". In fact a $(\omega, A) \in \mathcal{SOT}(d,\omega)$ is said to be "well-tuned" if all $\Xi_2(\omega, A, \phi_0)$ are nonempty and equal. Otherwise $(\omega, A)$ is said to be "ill-tuned". Of course, if $(\omega, A)$ is well–tuned, then all $\Xi_2(\omega, A, \phi_0)$ are equal to $\Xi_2(\omega, A)$. For a well–tuned spin-orbit torus we call the elements of $\Xi_2(\omega, A)$ "spin tunes of the second kind". The definition of spin tune of the second kind transfers the definition of spin tune in [BEH] from the continuous-time formalism of spin-orbit systems to the discrete-time formalism of spin-orbit tori. Of course, it follows from (7.14) that if a spin-orbit torus is in $\mathcal{ACB}$ then it is well-tuned and then the spin tunes of first and second kind are the same. Whole section in thesis

Mention all Thm 6.2.

(6) Since every $\Xi_2(\omega, A, \phi_0)$ has only countably many elements, every spin-orbit torus has only countably many spin tunes of the second kind, and may even have none. Thus if $\Xi_2(\omega, A)$ has uncountably many elements then $(\omega, A)$ is ill–tuned. Examples of this situation are studied in Chapter 8.

(7) Let $(\omega, A)$ and $(\omega, A') \in \mathcal{SOT}(d,\omega)$ be similar and $\phi_0 \in \mathbb{R}^d$. Then $\Xi_2(\omega, A, \phi_0) = \Xi_2(\omega, A', \phi_0)$. This implies by (7.14) that $\Xi_1(\omega, A) = \Xi_1(\omega, A')$. These are again properties which are shared by similar spin-orbit tori.

(8) It is easy to obtain a connection between Floquet theory and UPF's as follows. We say that $(\omega, A) \in \mathcal{SOT}(d,\omega)$ satisfies the generalized Floquet Theorem over $\phi_0 \in \mathbb{R}^d$ if a quasiperiodic $SO(3)$–valued function $p$ and a real $3 \times 3$ matrix $B$ exist such that $p(0) = I_{3\times3}$ and such that, for all integers $n$, $\Psi_{\omega,A}(n; \phi_0) = p(n)\exp(nB)$. In fact if $t$ is an $\omega$–quasiperiodic UPF over $\phi_0$ with UPR $\nu$, then the generalized Floquet Theorem holds over $\phi_0$ since one can define $p$ and $B$ by $p(n) := t(n)t^t(0), B := 2\pi\nu t(0)\mathcal{J}t^t(0)$. In particular one concludes from Theorem 7.1b that if $(\omega, A) \in \mathcal{ACB}(d,\omega)$ then the generalized Floquet Theorem is satisfied over every $\phi_0 \in \mathbb{R}^d$. □

Theorem 6.2?

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# 8 A simple class of spin-orbit tori

UNDER CONSTRUCTION

## 8.1 Defining the spin-orbit torus $(1/2, A^\epsilon)$

We now illustrate the theory of spin-orbit tori with an example of the kind that has arisen in connection with maintaining polarization in high energy proton rings and we will use it to check the existence of the ISF. This example gives rise to the spin-orbit torus $(1/2, A^\epsilon)$ belonging to $\mathcal{SOT}(1, 1/2)$ where $A^\epsilon$ is defined by

$$A^\epsilon := \begin{pmatrix} 1-2c^2 & 2ac & -2bc \\ -2ac & 2a^2-1 & -2ab \\ -2bc & 2ab & 1-2b^2 \end{pmatrix} , \tag{8.1}$$

for which

$$a(\phi) := -2\sin^2(\pi\epsilon/2)\sin(\phi)\cos(\phi) , \qquad b(\phi) := -2\sin(\pi\epsilon/2)\cos(\pi\epsilon/2)\cos(\phi) ,$$
$$c(\phi) := 2\sin^2(\pi\epsilon/2)\cos^2(\phi) - 1 . \tag{8.2}$$

The meaning of the parameter $\epsilon \in \mathbb{R}$ is explained below and

$$a^2 + b^2 + c^2 = 1 . \tag{8.3}$$

Numerical studies with the spin-orbit torus $(1/2, A^\epsilon)$ show that it has no ISF of the kind defined in the paper, i.e., $\mathcal{ISF}(1/2, A^\epsilon)$ is empty, for most values of $\epsilon$. Therefore the aim of this section is to study this matter carefully. For this we compute, for every value of $\epsilon$, the set of all ISF's of $(1/2, A^\epsilon)$. Our strategy, to obtain all ISF's, is simple and involves computing the the set $\mathcal{ISF}_2(1/2, A^\epsilon)$ of 2-turn ISF's defined in Chapter 5. We then determine the subset $\mathcal{ISF}(1/2, A^\epsilon)$ of $\mathcal{ISF}_2(1/2, A^\epsilon)$. Our results are listed in Proposition 8.1, below, which, in particular, tells us for which values of $\epsilon$ no ISF exists.

THESE REMAINING PARTS OF SECTION 8.1 HAVE TO BE ADAPTED TO THE FACT THAT WE HAVE A NEW COORDINATE SYSTEM UNDERLYING THE DEFINITION OF $A^\epsilon$

Before proceeding we give a brief outline of the relevance of the spin-orbit torus $(1/2, A^\epsilon)$ for the phenomenology of polarized beams in rings. For more details, see [BV] and the references therein. Note that [BV] considers the spin-orbit torus $(1/2, \tilde{A}^\epsilon)$ defined by

$$\tilde{A}^\epsilon(\phi) = T^t A^\epsilon(\phi) T , \tag{8.4}$$

where $T$ is the constant valued function with value $T := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. This amounts to an exchange of the second and third coordinate axes while keeping the coordinate system right handed. Note that $T \in \mathcal{TF}_{d,\omega}(A^\epsilon, \tilde{A}^\epsilon)$. Of course, by Chapter 5, the cardinality of $\mathcal{ISF}(1/2, A^\epsilon)$ is the same as that of $\mathcal{ISF}(1/2, \tilde{A}^\epsilon)$). Thus Proposition 8.1 would also hold if $(1/2, A^\epsilon)$ were replaced by $(1/2, \tilde{A}^\epsilon)$.

To arrive at (8.1) we begin with the so-called "single resonance model" used to model spin motion on a vertical betatron trajectory when the spin motion is dominated by the effect of a single Fourier harmonic of the horizontal quadrupole fields on the trajectory (see Section 7 in [BEH, ?]). The size of this harmonic is customarily called the resonance strength and is denoted by $\epsilon$ [HH96, ?]. Here, for convenience, we have taken to be purely real and, of course, $d = 1$.

The family of spin-orbit tori, $(\omega, A_{\omega,\nu_0,\epsilon})_{\epsilon\in[0,\infty)}$, of the single resonance model is defined such that each $(\omega, A_{\omega,\nu_0,\epsilon})$ belongs to $\mathcal{SOT}_{prop}(1,\omega) \subset \mathcal{SOT}(1,\omega)$ where $\omega \in \mathbb{R}$ is fixed but arbitrary and as shown in [BEH] we have

$$A_{\omega,J,\sigma_1,\sigma_2}(\phi) := \exp(\mathcal{J}[\phi + 2\pi\omega])\exp(2\pi E)\exp(-\phi\mathcal{J}) , \tag{8.5}$$

where

$$E := \begin{pmatrix} 0 & -\nu_0 + \omega & 0 \\ \nu_0 - \omega & 0 & -\epsilon \\ 0 & \epsilon & 0 \end{pmatrix} . \tag{8.6}$$

and where $\nu_0$ is the spin tune on the design orbit. [1]

Of course, by (3.2) and (8.5), the spin-orbit torus $(\omega, \exp(2\pi E))$ in $\mathcal{SOT}(1,\omega)$ is similar to the spin-orbit torus $(\omega, A_{\omega,\nu_0,\epsilon})$ of the single resonance model. On the other hand, since the 1-turn spin transfer matrix, $\exp(2\pi E)$, of $(\omega, \exp(2\pi E))$ is a constant function it follows from a remark after Definition 4.1 that every spin-orbit torus which is similar to $(\omega, \exp(2\pi E))$ is in $\mathcal{ACB}$. Thus $(\omega, A_{\omega,\nu_0,\epsilon})$ is in $\mathcal{ACB}$ for every choice of its parameters $\omega, \nu_0, \epsilon$. Moreover it always has an ISF.

The spin-orbit torus $(1/2, A^\epsilon)$ is a modification of the spin-orbit torus $(\omega, A_{\nu_0,\epsilon})$ in which the spin motion is described by the single resonance model in most of the ring and a special magnet, namely a thin-lens Siberian Snake, is placed at $\theta = 0$ with a second snake at $\theta = \pi$. The orientations of the so-called snake axes are chosen so that the spin tune on the closed orbit is $1/2$. For more details, see, for example [BV]). The primary interest of this work is to explore a scenario for which the system is traditionally said to be on "snake resonance" (see, e.g., Section X in [BEH]), namely at rational orbital tunes $\omega$ for which, $1/2 = m_0 + m_1\,\omega$ where $m_0$ and $m_1$ are integers and $m_1$ is odd. Clearly, this corresponds to orbital resonance. For the spin-orbit torus $(1/2, A^\epsilon)$ we obviously chose the special case: $\omega = 1/2$. With our choice of snake axes and starting just before the first snake, we obtain the spin-orbit torus $(1/2, A^\epsilon)$. It is well known that orbital motion is not stable for this case. However, here we are just concerned with the spin motion and we note that such models have proved instructive for presaging loss of polarization in rings close to snake resonance. Note however, that as we shall see, the term "resonance" in this context is usually a misnomer.

END ADAPT

## 8.2 Computing the ISF's of $(1/2, A^\epsilon)$

As mentioned above, our strategy for obtaining all ISF's for $(1/2, A^\epsilon)$ is to first obtain all 2–turn ISF's. In fact for $k = 2$ with $\omega = 1/2$ for the spin-orbit torus $(1/2, A^\epsilon)$, (5.22) reads

[1] Note that in [BEH] $\epsilon$ was written as $\sigma_2\sqrt{2J}$ and $\nu_0$ was written as $\sigma_1$

as

$$\mathcal{ISF}(1/2, A^\epsilon) \subset \mathcal{ISF}_2(1/2, A^\epsilon) , \tag{8.7}$$

To obtain the 2–turn ISF's we will use (5.24) which reads for $k = 2$ and for the spin-orbit torus $(1/2, A^\epsilon)$ as

$$\begin{aligned} \mathcal{ISF}_2(1/2, A^\epsilon) &= \{\mathcal{S} \in \mathcal{PF}(1/2, A^\epsilon) : |\mathcal{S}| = 1 \text{ and } \mathcal{S}(0,\cdot) = A^\epsilon(\cdot)\mathcal{S}(0,\cdot)\} \\ &= \Big\{\mathbb{Z} \times \mathbb{R} \xrightarrow{\mathcal{S}} \mathbb{R}^3 : |\mathcal{S}(0;\cdot)| = 1 \text{ and } \mathcal{S}(0,\cdot) = A^\epsilon(\cdot)\mathcal{S}(0,\cdot) \text{ and} \\ &\quad \mathcal{S}(0,\cdot) \in \mathcal{C}_{per}(\mathbb{R},\mathbb{R}^3) \text{ and } (\forall n \in \mathbb{Z})\ \mathcal{S}(n,\cdot) = A^\epsilon(\cdot - \pi n)\mathcal{S}(0,\cdot - \pi n)\Big\} . \end{aligned} \tag{8.8}$$

Thus we must solve the eigenproblem (5.23), i.e.,

$$g(\phi) = \Psi_{1/2,A^\epsilon}(2;\phi)\ g(\phi) , \qquad |g(\phi)| = 1 , \tag{8.9}$$

where $g \in \mathcal{C}_{per}(\mathbb{R},\mathbb{R}^3)$. Since (8.9) is an eigenproblem for every $g(\phi)$ our task is simple.

To obtain the 2–turn spin transfer matrix in (8.9) we first conclude from (8.1) and (8.2) that

$$A^\epsilon(\cdot + \pi) = \begin{pmatrix} 1 - 2c^2 & 2ac & 2bc \\ -2ac & 2a^2 - 1 & 2ab \\ 2bc & -2ab & 1 - 2b^2 \end{pmatrix} . \tag{8.10}$$

We also deduce from (2.41), (8.1) and(8.10) that the 2–turn spin transfer matrix reads as

$$\Psi_{1/2,A^\epsilon}(2;\cdot) = A^\epsilon(\cdot + \pi)A^\epsilon = \begin{pmatrix} 1 - 8c^2 + 8c^4 & 4ac(1-2c^2) & -4bc(1-2c^2) \\ -4ac(1-2c^2) & 1 - 8a^2c^2 & 8abc^2 \\ 4bc(1-2c^2) & 8abc^2 & 1 - 8b^2c^2 \end{pmatrix} . \tag{8.11}$$

Note that for the spin-orbit torus $(1/2, A^\epsilon)$, (5.10) reads as

$$\begin{aligned} \mathcal{ISF}(1/2, A^\epsilon) &= \{\mathcal{S} \in \mathcal{PF}(1/2, A^\epsilon) : |\mathcal{S}| = 1 \text{ and } \mathcal{S}(0,\cdot) = A^\epsilon(\cdot)\mathcal{S}(0,\cdot)\} \\ &= \Big\{\mathbb{Z} \times \mathbb{R} \xrightarrow{\mathcal{S}} \mathbb{R}^3 : |\mathcal{S}(0;\cdot)| = 1 \text{ and } \mathcal{S}(0,\cdot) = A^\epsilon(\cdot - \pi)\mathcal{S}(0,\cdot - \pi) \text{ and} \\ &\quad \mathcal{S}(0,\cdot) \in \mathcal{C}_{per}(\mathbb{R},\mathbb{R}^3) \text{ and } (\forall n \in \mathbb{Z})\ \mathcal{S}(n,\cdot) = \mathcal{S}(0,\cdot)\Big\} . \end{aligned} \tag{8.12}$$

To compute $\mathcal{ISF}_2(1/2, A^\epsilon)$ and $\mathcal{ISF}(1/2, A^\epsilon)$ we consider two separate cases: Case 1 for which $\epsilon$ is not an integer, and Case 2 for when $\epsilon$ is an integer.

We begin with Case 1. Then $\epsilon$ is not an integer so that $|\sin(\pi\epsilon/2)|$ equals neither 0 or 1, and we define the function $g^1 \in \mathcal{C}_{per}(\mathbb{R},\mathbb{R}^3)$ by

$$g^1(\phi) := \frac{\cos(\pi\epsilon/2)}{|\cos(\pi\epsilon/2)|\sqrt{1 - \sin^2(\pi\epsilon/2)\cos^2(\phi)}} \Big(0, \cos(\pi\epsilon/2), \sin(\pi\epsilon/2)\sin(\phi)\Big) . \tag{8.13}$$

It is easy to show that $g^1$ satisfies the eigenproblem (8.9) whence, by (8.8), the function $\mathcal{S}^1$, defined by $\mathcal{S}^1(n,\phi) := g^1(\phi)$, is in $\mathcal{ISF}_2(1/2, A^\epsilon)$. Of course, by (8.8), $-\mathcal{S}^1$ is in $\mathcal{ISF}_2(1/2, A^\epsilon)$, too. Let $g \in C_{per}(\mathbb{R}, \mathbb{R}^3)$ be an arbitrary solution of (8.9), so that $g$ is not necessarily equal to $g^1$. In fact there are only a very few such $g$ and to show this we use the simple fact that if $R \neq I_{3\times 3}$ is a matrix in $SO(3)$ and if $\mathrm{v}, \mathrm{v}' \in \mathbb{R}^3$ are eigenvectors of $R$ with the eigenvalue 1, then $\mathrm{v} = \pm\mathrm{v}'$ so that $|\mathrm{v}\cdot\mathrm{v}'| = 1$. Then defining the set $M := \{\phi \in \mathbb{R} : \Psi_{1/2,A^\epsilon}(2;\phi) = I_{3\times 3}\}$ we see from (8.9) that, for $\phi \in \mathbb{R}\setminus M$, $|g(\phi)\cdot g^1(\phi)| = 1$. Note that, by (8.2) and (8.11),

$$M = \{\phi \in \mathbb{R} : c(\phi)(c^2(\phi) - 1) = 0\} = \{\phi \in \mathbb{R} : \cos^2(\phi) = \frac{1}{2\sin^2(\pi\epsilon/2)}\} , \quad (8.14)$$

whence $M$ consists only of isolated points, i.e., each point of $M$ is contained in an open interval which contains no other point of $M$. Thus $\mathbb{R}\setminus M$ is dense in $\mathbb{R}$. Since $\mathbb{R}\setminus M$ is dense in $\mathbb{R}$ and $|g(\phi)\cdot g^1(\phi)|$ is continuous in $\phi$, it follows that $|g(\phi)\cdot g^1(\phi)| = 1$ holds for every $\phi$. Thus either $g = g^1$ or $g = -g^1$ so that, by (8.8), $\mathcal{S}^1$ and $-\mathcal{S}^1$ are, for Case 1, the only 2–turn invariant spin fields of $(1/2, A^\epsilon)$, i.e., $\mathcal{ISF}_2(1/2, A^\epsilon) = \{\mathcal{S}^1, -\mathcal{S}^1\}$. We conclude from (5.1) with $\omega = 1/2$, and from (8.1) and (8.13) that

$$\mathcal{S}^1(1, \phi+\pi) = A^\epsilon(\phi)\mathcal{S}^1(0,\phi) = -\mathcal{S}^1(0, \phi+\pi) . \quad (8.15)$$

Since $\mathcal{S}^1$ is nonzero it follows from (8.15) that neither $\mathcal{S}^1$ nor $-\mathcal{S}^1$ are invariant for Case 1, i.e., $\mathcal{ISF}(1/2, A^\epsilon)\cap\mathcal{ISF}_2(1/2, A^\epsilon) = \emptyset$ whence, by (5.22), $\mathcal{ISF}(1/2, A^\epsilon) = \emptyset$. We have thus shown that $(1/2, A^\epsilon)$ has no ISF in Case 1.

We now come to Case 2 and and consider two separate cases: Case 2a, for which $\epsilon$ is an even integer and Case 2b, for which $\epsilon$ is an odd integer.

We begin with Case 2a. By (8.1) and (8.2) the 1–turn spin transfer matrix has the simple constant form

$$A^\epsilon = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \quad (8.16)$$

We also deduce from (2.41), (8.1) and (8.10) that the 2–turn spin transfer matrix reads as

$$\Psi_{1/2,A^\epsilon}(2;\cdot) = A^\epsilon(\cdot+\pi)A^\epsilon = I_{3\times 3} . \quad (8.17)$$

Thus for Case 2a, (8.9) reads as $g(\phi) = g(\phi)$ with $|g(\phi)| = 1$ so that the set of 2–turn ISF's is equal to the set of spin fields, i.e., for Case 2a

$$\mathcal{ISF}_2(1/2, A^\epsilon) \;=\; \{\mathcal{S} \in \mathcal{PF}(1/2, A^\epsilon) : |\mathcal{S}| = 1\} . \quad (8.18)$$

So for Case 2a, the set of 2–turn ISF's carries no useful information about the set of ISF's. However, we can apply the ISF criterion (5.11) which, for the spin-orbit torus $(1/2, A^\epsilon)$, reads as

$$g(\phi) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g(\phi-\pi) , \qquad |g(\phi)| = 1 , \quad (8.19)$$

where $g \in \mathcal{C}_{per}(\mathbb{R}, \mathbb{R}^3)$ and where we used (8.16). Then, by (8.12), the set of all ISF's of $(1/2, A^\epsilon)$ for Case 2a, reads as

$$\begin{aligned}\mathcal{ISF}(1/2, A^\epsilon) &= \{\mathcal{S} \in \mathcal{PF}(1/2, A^\epsilon) : |\mathcal{S}| = 1 \text{ and } \mathcal{S}(0, \cdot) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathcal{S}(0, \cdot - \pi)\} \\ &= \Big\{\mathbb{Z} \times \mathbb{R} \xrightarrow{\mathcal{S}} \mathbb{R}^3 : |\mathcal{S}(0; \cdot)| = 1 \text{ and } \mathcal{S}_1(0, \cdot) = -\mathcal{S}_1(0, \cdot - \pi)\,, \mathcal{S}_2(0, \cdot) = -\mathcal{S}_2(0, \cdot - \pi)\,, \\ &\quad \mathcal{S}_3(0, \cdot) = \mathcal{S}_3(0, \cdot - \pi) \text{ and } \mathcal{S}(0, \cdot) \in \mathcal{C}_{per}(\mathbb{R}, \mathbb{R}^3) \text{ and } (\forall n \in \mathbb{Z})\ \mathcal{S}(n, \cdot) = \mathcal{S}(0, \cdot)\Big\}\,,\end{aligned} \tag{8.20}$$

where $\mathcal{S} =: (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)^t$. This means that, for Case 2a, the spin-orbit torus $(1/2, A^\epsilon)$ has infinitely many ISF's. One of them is the constant function with value $(0, 0, 1)^t$.

We now consider Case 2b. Then, by (8.1) and (8.2), the 1–turn spin transfer matrix has the simple form

$$\begin{aligned}A^\epsilon(\phi) &= \begin{pmatrix} -\cos(4\phi) & -\sin(4\phi) & 0 \\ \sin(4\phi) & -\cos(4\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(4\phi + \pi) & \sin(4\phi + \pi) & 0 \\ -\sin(4\phi + \pi) & \cos(4\phi + \pi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \exp(-\mathcal{J}(4\phi + \pi))\,,\end{aligned} \tag{8.21}$$

whence, by (2.41), the 2–turn spin transfer matrix has the form

$$\begin{aligned}\Psi_{1/2,A^\epsilon}(2; \phi) &= A^\epsilon(\phi + \pi) A^\epsilon(\phi) = \exp(-\mathcal{J}(4\phi + 5\pi)) \exp(-\mathcal{J}(4\phi + \pi)) = \exp(-\mathcal{J}8\phi) \\ &= \begin{pmatrix} \cos(8\phi) & \sin(8\phi) & 0 \\ -\sin(8\phi) & \cos(8\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}\,.\end{aligned} \tag{8.22}$$

To find all 2–turn ISF's we must solve (8.9) which in this case reads as

$$g(\phi) = \begin{pmatrix} \cos(8\phi) & \sin(8\phi) & 0 \\ -\sin(8\phi) & \cos(8\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} g(\phi)\,, \qquad |g(\phi)| = 1\,, \tag{8.23}$$

where $g \in \mathcal{C}_{per}(\mathbb{R}, \mathbb{R}^3)$. One solution of this is the constant function $(0, 0, 1)^t$ in $\mathcal{C}_{per}(\mathbb{R}, \mathbb{R}^3)$ which we denote by $\bar{g}$. Then by (8.8), the polarization field $\mathcal{S}^2$ with the initial value $\mathcal{S}^2(0, \phi) = (0, 0, 1)^t$, is in $\mathcal{ISF}_2(1/2, A^\epsilon)$. Of course, by (8.8), $-\mathcal{S}^2$ is in $\mathcal{ISF}_2(1/2, A^\epsilon)$, too. Let $g \in \mathcal{C}_{per}(\mathbb{R}, \mathbb{R}^3)$ be an arbitrary solution of (8.23), so that $g$ is not necessarily equal to $\bar{g}$. However there are only very few of those $g$ and to show that we recall that $M = \{\phi \in \mathbb{R} : \Psi_{1/2,A^\epsilon}(2; \phi) = I_{3\times3}\}$ whence, by (8.22),

$$M = \{\frac{k\pi}{4} : k \in \mathbb{Z}\}\,, \tag{8.24}$$

so that $M$ consists only of isolated points (whence $\mathbb{R} \setminus M$ is dense in $\mathbb{R}$). Using the same argument about $\mathbb{R} \setminus M$ that we used in Case 1, we observe, by (8.9), that, for $\phi \in \mathbb{R} \setminus M$, we have $|g(\phi)\cdot\bar{g}(\phi)| = 1$. Since $\mathbb{R}\setminus M$ is dense in $\mathbb{R}$ and $|g(\phi)\cdot\bar{g}(\phi)|$ is continuous in $\phi$, it follows that $|g(\phi)\cdot\bar{g}(\phi)| = 1$ for every $\phi$. Thus either $g = \bar{g}$ or $g = -\bar{g}$ so that, by (8.8), $\mathcal{S}^2$ and $-\mathcal{S}^2$ are, for Case 2b, the only 2–turn invariant spin fields of $(1/2, A^\epsilon)$, i.e., $\mathcal{ISF}_2(1/2, A^\epsilon) = \{\mathcal{S}^2, -\mathcal{S}^2\}$. We conclude from (8.21) that

$$\mathcal{S}^2(1,\cdot) = (0,1,0)^t = A^\epsilon(\cdot-\pi)\mathcal{S}^2(0,\cdot-\pi)\,. \tag{8.25}$$

Eq. (8.12) and (8.25) imply that for Case 2b, $\mathcal{S}^2$ and $-\mathcal{S}^2$ are ISF's of $(1/2, A^\epsilon)$ whence $\mathcal{ISF}(1/2, A^\epsilon) = \mathcal{ISF}_2(1/2, A^\epsilon) = \{\mathcal{S}^2, -\mathcal{S}^2\}$.

This completes our study of the spin-orbit torus $(1/2, A^\epsilon)$ and we can summarize the above by the following

**Proposition 8.1** *For every value of $\epsilon$, the spin-orbit torus $(1/2, A^\epsilon)$ has a 2–turn invariant spin field. Moreover the spin-orbit torus $(1/2, A^\epsilon)$ has an ISF, iff $\epsilon$ is an integer.* □

One can also demonstrate the following facts about $(1/2, A^\epsilon)$ [He1]:

(1) $(1/2, A^\epsilon) \in \mathcal{CB}_{SO(2)}$ iff $\epsilon$ is an integer.

(2) $(1/2, A^\epsilon) \in \mathcal{ACB}$ iff $\epsilon$ is an even integer. If $\epsilon$ is an even integer then $1/2$ is a spin tune of the first kind whence, by (4.26), the set of all spin tunes of the first kind reads as:

$$\Xi_1(1/2, A^\epsilon) = [0,1)\cap\{\frac{m}{2} : m\in\mathbb{Z}\} = \{\frac{1}{2}\}\,,$$

i.e., $1/2$ is the only spin tune of the first kind.

(3) For every value of $\epsilon$ and for every $\phi_0$,$(1/2, A^\epsilon)$ has an $\omega$–quasiperiodic UPF over $\phi_0$ with UPR $\nu(\phi_0)$ which is uniquely determined by requiring

$$\exp(\mathcal{J}2\pi\nu(\phi_0)) = \begin{pmatrix} -2c^2(\phi_0)+1 & \frac{\sqrt{2}b(\phi_0)c(\phi_0)\sqrt{1-c(\phi_0)}}{|\cos(\pi\epsilon/2)|} & 0 \\ -\frac{\sqrt{2}b(\phi_0)c(\phi_0)\sqrt{1-c(\phi_0)}}{|\cos(\pi\epsilon/2)|} & 1-2c^2(\phi_0) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $0 \le \nu(\phi_0) < 1$.

In particular, for all values of $\epsilon$, every spin trajectory is quasiperiodic.

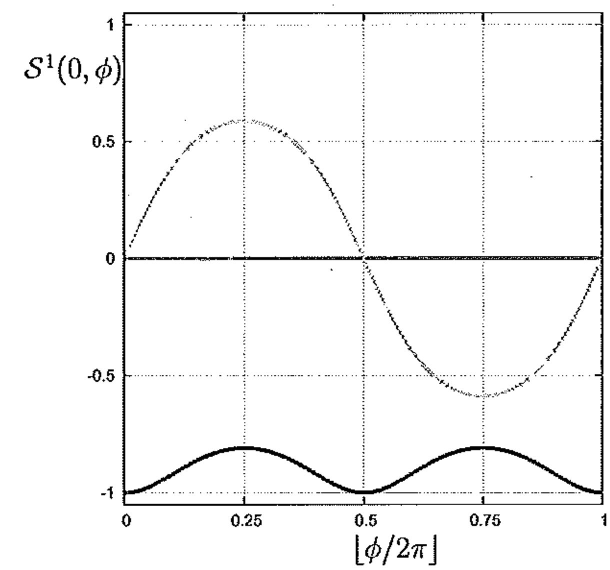
(4) $(1/2, A^\epsilon)$ is ill-tuned iff $\epsilon$ is not an even integer, i.e., $(1/2, A^\epsilon)$ is well-tuned iff $\epsilon$ is an even integer. (UNCOUNTABLY MANY ELEMENTS??????)

Note that following the arguments in Section X of [BEH] for orbital resonance, it should come as no surprise that the system is usually ill-tuned. □

By item 2 there is no spin tune of first kind when $\epsilon$ is not an even integer whence in this case "resonance" in the term "snake resonance" is a misnomer.

SHOULD WE SAY SOMETHING ABOUT the NON-PROPER STUFF????????

Although this model has provided a useful example, the underlying single resonance model is usually only valid for values $\epsilon \ll 1$, i.e., in a regime where resonances do not overlap.



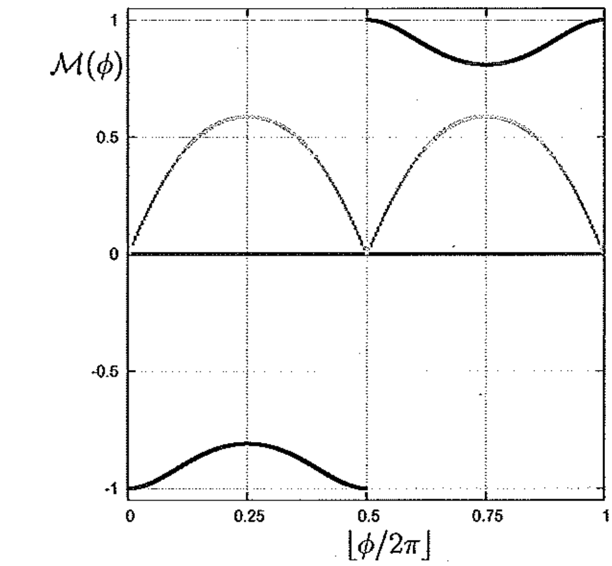




Figure 4: (a) Components 1 ( red), 2 (green) and 3 (blue) of the function $\mathcal{S}^1(0,\phi)$ derived from (8.13) for the case $\epsilon = 0.4$. (b) Components 1 ( red), 2 (green) and 3 (blue) of the function $\mathcal{M}$ generated as described in the text with $\epsilon = 0.4$.

## 8.3 Numerical investigations for $(1/2, A^\epsilon)$

Following our treatment of the spin-orbit torus $(1/2, A^\epsilon)$ using the concepts of the previous sections we now return to some earlier, numerical investigations for systems on snake resonance and look at the spin-orbit torus $(1/2, A^\epsilon)$ in those terms. It suffices to begin with [?, ?] where it was found that at a snake resonance with $\omega = 1/6$ the ISF defined there was irreducibly discontinuous as a function of $\phi$ for a non-integer value of $\epsilon$, namely 0.4. A similar but unpublished observation was made for $\omega = 1/2$. However, in this work we only allow ISF continuous in $\phi$ so that there is no contradiction with our finding above that there is no ISF for the spin-orbit torus $(1/2, A^\epsilon)$.

Figure 2a (left) shows the components the function $\mathcal{S}^1(0,\phi)$ given by the $g^1(\phi)$ of (8.13) for the $\epsilon = 0.4$ used in [?, ?] and plotted versus the normalized fractional phase $\lfloor\phi/2\pi\rfloor$ on the interval $[0,1)$. Since our $\mathcal{S}^1$ is $2\pi$-periodic and continuous we obtain curves which are single valued and continuous.

The non-existence of a 1-turn ISF is illustrated in figure 2b (right) which shows the components of a function $\mathcal{M}$ plotted versus $\lfloor\phi/2\pi\rfloor$ on the interval $[0,1)$. The curves are obtained in two steps. First, $\mathcal{M}$ is constructed for $0 \leq \lfloor\phi/2\pi\rfloor < 1/2$ by the definition $\mathcal{M}(\phi) := \mathcal{S}^1(0,\phi)$. Then $\mathcal{M}$ is propagated for one or more turns using the first equality of (8.15). This gives curves which are single valued and which are continuous except at $\lfloor\phi/2\pi\rfloor = 0$ and $\lfloor\phi/2\pi\rfloor = 1/2$ where the second and third components have a discontinuity in sign. If the $\mathcal{M}$ from the $\mathcal{S}^1(\phi)$ had been evaluated for the range $0 \leq \lfloor\phi/2\pi\rfloor < 1$ and if it had then been propagated for one or more turns, a double valued function of $\lfloor\phi/2\pi\rfloor$ would have been generated. The resulting curves would then not have represented an ISF. The function of figure 2b provides a demonstration of the assertion that Case 1 has no ISF of the kind defined in this work. Note that an unlimited number of different $\mathcal{M}$-like functions can be constructed by choosing the starting range in the first step to be $\Delta \leq \lfloor \phi/2\pi \rfloor < 1/2 + \Delta$ for any $0 < \Delta \leq 1/2$. Then the discontinuities are at $\Delta$ and $\Delta + 1/2$.

The function $\mathcal{M}$ of figure 2b also emerges when stroboscopic averaging [HH96] is used in an attempt to construct an ISF for $\epsilon = 0.4$. It may be shown that for Cases 1 and 2, the stroboscopic sequence of [HH96, equation?] converges to a continuous $2\pi$-periodic function at all $\phi$. However, for Case 1 the stroboscopic sequence vanishes at $\lfloor \phi/2\pi \rfloor$. Then when the sequence is normalized to unity, the discontinuity appears.

An impression of some pitfalls surrounding this topic can be obtained by reviewing [?, ?] in conjunction with [BV, ?, ?].

Note that in the Borel formalism, outlined in Chapter 7(????????), Proposition 8.1 would look different since in that formalism every 2-turn ISF is an ISF whence, in the Borel formalism, $(1/2, A^\epsilon)$ has as ISF for every value of $\epsilon$.

MENTION treating this with BOREL measurable technology. See also pages 32 +33 in the Feb 2007 version.

ALSO: stuff on extra discontinuities as J increases. See the e-mail from 2005.

JUNK

¿ your predictions. In fact, I now looked at my old file which tells me that there are altogether nine cases. Using the parameter $eta := sin^2(epsilon * pi/2)$ these cases are defined as follows:

Case 1: $eta = 0$ Case 2: $0 < eta < 1/2 - sqrt(1/8)$ Case 3: $eta = 1/2 - sqrt(1/8)$ Case 4: $1/2 - sqrt(1/8) < eta < 1/2$ Case 5: $eta = 1/2$ Case 6: $1/2 < eta < 1/2 + sqrt(1/8)$ Case 7: $eta = 1/2 + sqrt(1/8)$ Case 8: $1/2 + sqrt(1/8) < eta < 1$ Case 9: $eta = 1$

With growing case number, the number of those phi's in [0,2pi] grows where the 2-turn spin transfer matrix is the unit matrix. ¿

END JUNK

# 9 Invariant reductions of the principal bundle $\lambda_d$. The existence of an ISF as a property of $\lambda_d$

## 9.1 Preliminaries

It is well known from Dynamical Systems Theory [Fe, HK1, He2] that principal-bundle techniques can be used to study skew product dynamics. Thus, by recalling from Section 3.2 that the $\mathbb{Z}$ space $(\mathbb{R}^{d+3}, L_{\omega,A})$ defined by

$$L_{\omega,A}(n;\phi,S) := \begin{pmatrix} L_\omega(n;\phi) \\ \Psi_{\omega,A}(n;\phi)S \end{pmatrix} , \qquad L_\omega(n;\phi) := \phi + 2\pi n\omega , \tag{9.1}$$

is a skew product over the $\mathbb{Z}$ space $(\mathbb{R}^d, L_\omega)$, we now revisit the theory of the previous chapters in terms of the principal bundle $\lambda_d$. In fact $\lambda_d$ allows us to specify the dynamics of $(\omega, A)$ as a well defined and infinite collection of $\mathbb{Z}$ sets containing the two examples $(\mathbb{R}^{d+3}, L_{\omega,A})$ and $(\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3), \tilde{L}_{\omega,A})$ discussed in Sections 2 and 5. Easily digestable introductions to principal bundles can be found, for example, in [Sc, NS, Na1, Na2] while the mathematically oriented reader may also consult [Hus1, Hus2, tDi2, Ma].