Appendices

A Fourier Analysis of Quasiperiodic Functions

The purpose of Appendix A is to obtain a detailed overview of the Fourier analysis of arbitrary quasiperiodic functions and to then give some applications needed in the main text.

A.1

Before proceeding we make some comments on the difficulty of doing Fourier analysis on quasiperiodic functions in our map formalism. Note that Fourier analysis is simpler in the flow formalism.

The Fourier analysis we do in Lemma A.1 uses Weyl's equidistribution theorem, the generalized Fejér theorem and some linear algebra of $\mathbb{Z}$–modules. Of course, for a χ in $\mathbb{R}^k$ three cases arise: (i) $\chi \in \mathbb{Q}^k$, (ii) $(1,\chi)$ nonresonant, (iii) $\chi \notin \mathbb{Q}^k$ with $(1,\chi)$ resonant. We will treat these three cases separately and as a first step we make three observations aimed at aiding the understanding of Lemma A.1 and its proof.

Firstly, if χ is in $\mathbb{Q}^k$, then F is just a trigonometric polynomial. For details, see the proof of Lemma A.1a. Secondly, if $(1,\chi)$ is nonresonant then Weyl's equidistribution theorem together with the generalized Fejér theorem straightforwardly ensure that the sequence F^N, defined by

$$F^N(n) := \sum_{\substack{m\in Z^k \\ \|m\|\leq N}} A_{N,m,k}a(F, m\cdot\chi)\exp(2\pi i n m\cdot\chi) , \qquad \text{(A.1)}$$

with $A_{N,m,k}$ given in (7.13), converges uniformly to F on $\mathbb{Z}$ as $N \to \infty$. For further details of the case of nonresonant $(1,\chi)$, see the proof of Lemma A.1c. We have thus outlined how to do Fourier analysis if either $\chi \in \mathbb{Q}^k$ or if $(1,\chi)$ is nonresonant.

We now come to our third observation which deals with case (iii), i.e. the difficult case where $\chi \notin \mathbb{Q}^k$ and $(1,\chi)$ is resonant. We begin with the important subcase where $\chi = (\chi^1, \chi^2)$ with $(1,\chi^2)$ nonresonant and with χ^1 having only rational components. This subcase of (iii) is covered by Lemma A.1d whose proof rests on the observation that $F_r(n) := F(qn+r)$ is a $q\chi^2$–quasiperiodic function of n if q is chosen as a positive integer such that $q\chi^1$ has only integer components. In fact, because $(1, q\chi^2)$ is nonresonant one can then apply Lemma A.1c to the function F_r! For more details, see the proof of Lemma A.1d. To treat case (iii) in general we use some linear algebra of $\mathbb{Z}$–modules in order to reduce the problem to the above mentioned subcase. Then, it turns out that in case (iii) a unimodular $k\times k$ matrix Z exists such that $Z\chi =: (\check{\chi}^1, \check{\chi}^2)$ whereby $(1, \check{\chi}^2)$ is nonresonant and $\check{\chi}^1$ has only rational components. For details, see the proof of Lemma A.1e. Recall that a $k\times k$ matrix X is said to be unimodular if its components are integers and if $|\det(X)| = 1$. In this case X^{-1} exists and its components are integers. Then if f denotes a generator of F via the prescription $F(n) = f(2\pi n\chi)$, $\check{f}$, defined by $\check{f}(\phi) := f(Z^{-1}\phi)$, is a generator of F via the prescription $F(n) = \check{f}(2\pi n Z\chi)$. Thus F is $Z\chi$–quasiperiodic. Finally, Lemma A.1d can be applied. Although our method of handling case (iii) may seem intricate, its naturalness becomes apparent if it is applied when $k = 2$. Note also that for $k \geq 2$ all three cases occur while only the simple cases (i) and (ii) occur for $k = 1$.

A.2

DEFINE q-periodic. OR REPLACE by "WITH PERIOD q".

We now state and prove our Lemma A.1 whereby k denotes a positive integer.

Lemma A.1 *a) Let q be a positive integer. If the function $F : \mathbb{Z} \to \mathbb{C}$ is q–periodic then it is also $1/q$–quasiperiodic, $\mathfrak{M}(F) = \mathbb{R}$ and*

$$F(n) = \sum_{r=0}^{q-1} \exp(2\pi irn/q)a(F, r/q)\ , \qquad a(F, r/q) = \frac{1}{q}\sum_{p=0}^{q-1} \exp(-2\pi irp/q)F(p)\ . \quad \text{(A.2)}$$

If F is χ–quasiperiodic and $\chi \in \mathbb{Q}^k$ with $q\chi$ in $\mathbb{Z}^k$ then F is q–periodic so that $\mathfrak{M}(F) = \mathbb{R}$ and (A.2) holds. If, in addition, q is the smallest positive integer such that $q\chi$ is in $\mathbb{Z}^k$ then, for all integers r, $r/q \in Y_\chi$. The same statements hold if F is $\mathbb{C}^j$–valued or $\mathbb{C}^{j\times j}$–valued.

b) Let $G^0(*), G^1(*), G^2(*), \ldots$ be a sequence of complex valued and bounded functions such that $a(G^N, 0) = 0$. If the sequence G^N converges uniformly on $\mathbb{Z}$ as $N \to \infty$ then the limit G is a complex valued and bounded function with $a(G, 0) = 0$.

c) Let $f : \mathbb{R}^k \to \mathbb{C}$ be a continuous and 2π–periodic function. Then the sequence $f^0, f^1, f^2, \ldots$ of continuous and 2π–periodic functions $f^N : \mathbb{R}^k \to \mathbb{C}$, defined by

$$f^N(\phi) := \sum_{\substack{m\in\mathbb{Z}^k \\ ||m||\leq N}} A_{N,m,k} f_m \exp(im\cdot\phi)\ , \qquad \text{(A.3)}$$

converges uniformly to f on $\mathbb{R}^k$ as $N \to \infty$ with the m-th Fourier coefficient defined by

$$f_m := \frac{1}{(2\pi)^k}\int_0^{2\pi}\cdots\int_0^{2\pi} f(\phi)\exp(-im\cdot\phi)d\phi_1\cdots d\phi_k\ . \qquad \text{(A.4)}$$

The associated sequence of χ–quasiperiodic functions F^N, generated by f^N via the prescription $F^N(n) := f^N(2\pi n\chi)$, converges uniformly to F on $\mathbb{Z}$ as $N \to \infty$ where the χ–quasiperiodic function F is defined by the prescription $F(n) := f(2\pi n\chi)$. If $(1, \chi)$ is nonresonant then $\mathfrak{M}(F) = \mathbb{R}$ and $f_m = a(F, m\cdot\chi)$. Analogous statements hold if f is $\mathbb{C}^j$–valued or $\mathbb{C}^{j\times j}$–valued.

d) THIS THEOREM HAS THE LONGEST PROOF OF APPENDIX A (IN FACT THE LONGEST PROOF IN THE PAPER)

Let F be a χ–quasiperiodic function and let $\chi =: (\chi^1, \chi^2)$ with $(1, \chi^2)$ nonresonant and $\chi^1 \in \mathbb{Q}^{k-s}, \chi^2 \in \mathbb{R}^s$, where $0 < s < k$. Then $\mathfrak{M}(F) = \mathbb{R}$ and the sequence F^N, defined by

$$F^N(n) := \sum_{r=0}^{q-1}\sum_{\substack{m\in\mathbb{Z}^s \\ ||m||\leq N}} A_{N,m,s}\, a(F, m\cdot\chi^2 + r/q)\exp\Big(2\pi in(m\cdot\chi^2 + r/q)\Big)\ , \qquad \text{(A.5)}$$

M(F) includes the zeros between the ... . So M(F) = R *converges uniformly to F on $\mathbb{Z}$ as $N \to \infty$, if the positive integer q is chosen such that $q\chi^1$ is in $\mathbb{Z}^{k-s}$. If q is the smallest positive integer such that $q\chi^1$ in $\mathbb{Z}^{k-s}$ then, for all $m \in \mathbb{Z}^s$ and $r \in \mathbb{Z}$, $m \cdot \chi^2 + r/q \in Y_\chi$.*

e) THE PROOF OF THIS THEOREM USES THE METHOD OF LOCHAK'S APPENDIX 3.8. WHILE OUR METHOD IS THE SAME, OUR Z-MODULE IS VERY DIFFERENT FROM THE Z-MODULE IN LOCHAK: WHILE OUR MODULE IS $M_\chi := \{m \in \mathbb{Z}^k : m \cdot \chi \in \mathbb{Z}\}$, THE MODULE IN LOCHAK'S APPENDIX 3.8 IS $\{m \in \mathbb{Z}^k : m \cdot \chi = 0\}$. OF COURSE, THE REASON FOR THIS DIFFERENCE IS THAT WE ARE AIMING AT THE MAP FORMALISM WHILE LOCHAK IS AIMING AT THE FLOW FORMALISM. IT IS THEREFORE A POSITIVE SURPRISE THAT LOCHAK'S METHOD IS APPLICABLE TO OUR CONTEXT

Let χ be in $\mathbb{R}^k \setminus \mathbb{Q}^k$ and let $(1, \chi)$ be resonant. Then a unimodular $k \times k$ matrix Z exists such that $Z\chi =: (\chi^1, \chi^2)$ with $(1, \chi^2)$ nonresonant and $\chi^1 \in \mathbb{Q}^{k-s}$, where $0 < s < k$. Moreover, the positive integer s is uniquely determined by χ, i.e. independent of the choice of Z.

f) Let F be a χ–quasiperiodic function and let $(1, \chi)$ be resonant and $\chi \in \mathbb{R}^k \setminus \mathbb{Q}^k$. Thus a unimodular $k \times k$ matrix Z exists such that $Z\chi =: (\chi^1, \chi^2)$ with $\chi^1 \in \mathbb{Q}^{k-s}$ and $(1, \chi^2)$ nonresonant where s is independent of the choice of Z and $0 < s < k$. Also, $\mathfrak{M}(F) = \mathbb{R}$ and the sequence F^N, defined by (A.5), converges uniformly to F on $\mathbb{Z}$ as $N \to \infty$, if the positive integer q is chosen such that $q\chi^1$ is in $\mathbb{Z}^{k-s}$. If q is the smallest positive integer such that $q\chi^1$ in $\mathbb{Z}^{k-s}$ then, for all $m \in \mathbb{Z}^s$ and $r \in \mathbb{Z}$, $m \cdot \chi^2 + r/q \in Y_\chi$.

Proof of Lemma A.1a: We first consider the case where the complex valued function F is q–periodic, where q is a positive integer. Of course, $F(0), F(1), ..., F(q-1)$ are the only distinct values that F can assume. We now choose a complex polynomial $P(z) = \sum_{j=0}^{q-1} p_j z^j$ of degree $q-1$, where $p_j \in \mathbb{C}$ such that, for $n = 0, 1, ..., q-1$, $F(n) = P\Big(\exp(2\pi i n/q)\Big)$. Note that such an interpolating polynomial always exists. Because F is q–periodic, $F(n) = P\Big(\exp(2\pi i n/q)\Big)$ for *all* integers n so that the function $f : \mathbb{Z} \to \mathbb{C}$, defined by $f(\phi) := P\Big(\exp(i\phi)\Big)$, generates F via the prescription $F(n) = f(2\pi i n/q)$. Thus F is $1/q$–quasiperiodic. Then from the definition in Sec. VII.1, $qa(F, j/q) = \sum_{n=0}^{q-1} \exp(-2\pi i n j/q) F(n)$. We now observe that for $j, m = 0, ..., q-1$, $\sum_{n=0}^{q-1} \exp\Big(2\pi i n(j-m)/q\Big)$ vanishes if $j \neq m$.

I FEEL THAT THE LOGIC FOR THE NEXT STUFF ON THE FIRST EQUALITY OF (A.2) NEEDS IMPROVING HERE. PERHAPS THE LOGIC FOR THE 2ND EQUALITY NEEDS IMPROVING TOO.

Thus, for $j = 0, ..., q-1$, and recalling that with the relation $qp_j = qa(F, j/q)$ so that (A.2) holds. That $\mathfrak{M}(F) = \mathbb{R}$, follows then from (A.2). We now drop the assumption that F is q–periodic and instead assume that F is a complex valued and χ–quasiperiodic function where $q\chi$ is in $\mathbb{Z}^k$. It then follows that F is again q–periodic so that, by the above, $\mathfrak{M}(F) = \mathbb{R}$ and (A.2) hold. In addition we now take q to be the smallest positive integer such that $q\chi$ is in $\mathbb{Z}^k$. Then there exists a j such that the j–th component χ_j of χ equals p/q, where the integers p, q are relatively prime. Then integers $\tilde{p}, \tilde{q}$ exist such that $\tilde{p}p + \tilde{q}q = 1$ For these, Syntax?

$\tilde{p}\chi_j + \tilde{q} = 1/q$ so that $1/q \in Y_\chi$. It follows that, for every integer r, $r/q \in Y_\chi$.

For the cases where F is $\mathbb{C}^j$–valued or $\mathbb{C}^{j\times j}$–valued, the claims follow easily. □

Proof of Lemma A.1b: Let $G^0(*), G^1(*), G^2(*), \ldots$ be a sequence of complex valued and bounded functions which converges uniformly on $\mathbb{Z}$ to a complex valued and bounded function $G(*)$. Also, let $a(G^N, 0) = 0$, i.e.

$$(\forall\varepsilon > 0)(\forall N \in \mathbb{N})(\exists T(\varepsilon, N) \in \mathbb{N})(\forall T \geq T(\varepsilon, N)) \;\; |a_T(G^N, 0)| < \varepsilon/2\,. \tag{A.6}$$

The idea for the following proof is from [28, p.259]. Because G and G^N are bounded functions we can define the nonnegative number δ^N by $\delta^N := \sup_{n\in\mathbb{Z}} |G(n) - G^N(n)|$. Because $G - G^N$ converges uniformly to zero, it follows that $\lim_{N\to\infty} \delta^N = 0$, hence

$$(\forall\varepsilon > 0)(\exists N(\varepsilon))\left(\delta^{N(\varepsilon)} < \varepsilon/2\right)\,. \tag{A.7}$$

Because G and G^N are bounded functions, we also have, for all T and N,

$$\begin{aligned}|a_T(G - G^N, 0)| &\leq a_T(|G - G^N|, 0) \leq a_T(\sup_{n\in\mathbb{Z}} |G(n) - G^N(n)|, 0)\\ &= \sup_{n\in\mathbb{Z}} |G(n) - G^N(n)| = \delta^N\,,\end{aligned}$$

i.e.

$$|a_T(G, 0) - a_T(G^N, 0)| \leq \delta^N\,. \tag{A.8}$$

Combining (A.7),(A.8) yields, for all T,

$$(\forall\varepsilon > 0)\; |a_T(G, 0) - a_T(G^{N(\varepsilon)}, 0)| < \varepsilon/2\,. \tag{A.9}$$

From (A.6) and (A.9) it follows that

$$\begin{aligned}(\forall\varepsilon > 0)(\forall T \geq T(\varepsilon, N(\varepsilon)))(&\left(|a_T(G, 0) - a_T(G^{N(\varepsilon)}, 0)| < \varepsilon/2\right)\\ &\&\left(|a_T(G^{N(\varepsilon)}, 0)| < \varepsilon/2\right)\,.\end{aligned} \tag{A.10}$$

Because, for all T and N,

$$|a_T(G, 0)| = |a_T(G, 0) - a_T(G^N, 0) + a_T(G^N, 0)| \leq |a_T(G, 0) - a_T(G^N, 0)| + |a_T(G^N, 0)|\,,$$

it follows from (A.10) that $(\forall\varepsilon > 0)(\forall T \geq T(\varepsilon, N(\varepsilon)))|a_T(G, 0)| < \varepsilon$. Thus $a_T(G, 0)$ converges to zero as $T \to \infty$, i.e. $a(G, 0) = 0$. □

Proof of Lemma A.1c: Let $f : \mathbb{R}^k \to \mathbb{C}$ be a continuous and 2π–periodic function. That the sequence f^N converges uniformly to f is a multidimensional generalization of Fejér's theorem (see for example [25, Sec. III.22],[22, Sec. 79]). It trivially follows that the associated sequence F^N converges uniformly to F on $\mathbb{Z}$ as $N \to \infty$. By using a 'map' version of Weyl's equidistribution theorem ([14, Chapter 3]), we obtain that $f_m = a(F, m \cdot \chi)$.

Thus it remains to prove that $\mathfrak{M}(F) = \mathbb{R}$. Obviously, $a(F, \lambda)$ is well defined if $\lambda \in Y_\chi$. We thus consider a real λ not in Y_χ and we define

$$G(n) := \exp(-2\pi i n\lambda)F(n) , \qquad G^N(n) := \exp(-2\pi i n\lambda)F^N(n) . \qquad (A.11)$$

Because G and G^N are complex valued and bounded functions, we have the situation of Lemma A.1b so that we conclude that

$$0 = a(G, 0) = a(F, \lambda) , \qquad (A.12)$$

where in the second equation we used (A.11). From (A.12) it follows that λ is in $\mathfrak{M}(F)$ so that we have completed the proof that $\mathfrak{M}(F) = \mathbb{R}$. Thus we have proven all claims for the case where f is complex valued. For the cases where f is $\mathbb{C}^j$–valued or $\mathbb{C}^{j\times j}$–valued, the claims follow easily. □

Proof of Lemma A.1d: Let $F : \mathbb{Z} \to \mathbb{C}$ be a χ–quasiperiodic function and let $\chi =: (\chi^1, \chi^2)$ with $(1, \chi^2)$ nonresonant and $\chi^1 \in \mathbb{Q}^{k-s}, \chi^2 \in \mathbb{R}^s$, where $0 < s < k$. Let the positive integer q be chosen such that $q\chi^1$ is in $\mathbb{Z}^{k-s}$. We define the function $F_r : \mathbb{Z} \to \mathbb{C}$ by $F_r(n) := F(qn+r)$, where $r \in \mathbb{Z}$. The only trick of our proof will be to use the function F_r and to express F in terms of this function. Our proof goes in six steps. Let $f : \mathbb{R}^k \to \mathbb{C}$ be a function which generates F via $F(n) = f(2\pi n\chi)$. Then

$$F_r(n) = F(qn + r) = f(2\pi\chi[qn + r]) = f(2\pi r\chi^1, 2\pi\chi^2[qn + r]) =: g(2\pi nq\chi^2) , \qquad (A.13)$$

where the continuous and 2π–periodic function $g : \mathbb{R}^s \to \mathbb{C}$ is defined by $g(\phi) := f(2\pi r\chi^1, \phi + 2\pi r\chi^2)$. Thus F_r is a $q\chi^2$–quasiperiodic function generated by g via (A.13). Because $(1, \chi^2)$ is nonresonant, so is $(1, q\chi^2)$, hence we can apply Lemma A.1c to g. Thus the sequence F_r^N, defined by

$$F_r^N(n) := \sum_{\substack{m\in\mathbb{Z}^s\\ \|m\|\le N}} A_{N,m,s}a(F_r, qm\cdot\chi^2)\exp(2\pi i nqm\cdot\chi^2) , \qquad (A.14)$$

converges uniformly to F_r on $\mathbb{Z}$ as $N \to \infty$, i.e. for all r

$$(\forall\varepsilon > 0)(\exists M(\varepsilon, r) \in \mathbb{N})(\forall N \ge M(\varepsilon, r)) \sup_n |F_r^N(n) - F_r(n)| < \varepsilon . \qquad (A.15)$$

Having got a decent approximating sequence F_r^N for F_r we have taken the first step in our proof. The final five steps are straightforward but tedious. In the next step we will obtain a decent approximating sequence for F. Of course, if $r = 0, ..., q-1$, then $F(n) = F(q\lceil n/q\rceil + n - q\lceil n/q\rceil) = F(qN(n) + t(n)) = F_{t(n)}(N(n))$, where $t(n) := n - q\lceil n/q\rceil$ and $N(n) := \lceil n/q\rceil$ (recall that $\lceil n/q\rceil$ is the integer part of n/q). Because F_r^N converges uniformly to F_r it seems plausible that the sequence $F_{t(n)}^N(N(n))$ converges uniformly in n to $F(n)$ as $N \to \infty$ and we now will prove that in the second step of the proof. First of all, it trivially follows from (A.15) that, for all r,

$$(\forall\varepsilon > 0)(\forall N \ge M(\varepsilon, r)) \sup_n |F_r^N(N(n)) - F_r(N(n))| < \varepsilon , \qquad (A.16)$$

hence, for $0 \leq r \leq q-1$,

$$(\forall \varepsilon > 0)(\forall N \geq M(\varepsilon)) \max_{r=0,\ldots,q-1} \sup_n |F_r^N(N(n)) - F_r(N(n))| < \varepsilon \,, \tag{A.17}$$

where $M(\varepsilon) := \max_{r=0,\ldots,q-1} M(\varepsilon, r)$. Because $0 \leq t(n) \leq q-1$ we have that

$$\sup_n |F_{t(n)}^N(N(n)) - F_{t(n)}(N(n))| \leq \max_{r=0,\ldots,q-1} \sup_n |F_r^N(N(n)) - F_r(N(n))| \,,$$

so that (A.17) yields

$$(\forall \varepsilon > 0)(\forall N \geq M(\varepsilon)) \sup_n |F_{t(n)}^N(N(n)) - F_{t(n)}(N(n))| < \varepsilon \,, \tag{A.18}$$

i.e.

$$(\forall \varepsilon > 0)(\forall N \geq M(\varepsilon)) \sup_n |F_{t(n)}^N(N(n)) - F(n)| < \varepsilon \,. \tag{A.19}$$

With (A.19) we have shown that the sequence F^N, defined by

$$F^N(n) := F_{t(n)}^N(N(n)) = \sum_{\substack{m \in Z^s \\ \|m\| \leq N}} A_{N,m,s} a(F_{t(n)}, qm \cdot \chi^2) \exp(2\pi i N(n) qm \cdot \chi^2) \,, \tag{A.20}$$

converges uniformly to F on $\mathbb{Z}$ as $N \to \infty$. Note that in the second equation of (A.20) we used (A.14). With (A.20) and with the uniform convergence of F^N we have taken the second step in our proof. In the next three steps we will show that the approximating sequence F^N in (A.20) has the form of (A.5) and in passing we will also prove that $\mathfrak{M}(F) = \mathbb{R}$. Defining the complex valued function G_m by $G_m(n) := a(F_{t(n)}, qm \cdot \chi^2) \exp(-2\pi i t(n) m \cdot \chi^2)$, we obtain from (A.20) that

$$F^N(n) = \sum_{\substack{m \in Z^s \\ \|m\| \leq N}} A_{N,m,s} G_m(n) \exp(2\pi i n m \cdot \chi^2) \,. \tag{A.21}$$

Note that G_m is q–periodic because t is q–periodic, i.e. $t(n+q) = t(n)$. Thus by Lemma A.1a

$$\begin{aligned} G_m(n) &= \textstyle\sum_{r=0}^{q-1} \exp(2\pi i r n/q) a(G_m, r/q) \,, \\ a(G_m, r/q) &= \textstyle\frac{1}{q} \sum_{p=0}^{q-1} \exp(-2\pi i r p/q) G_m(p) \\ &= \textstyle\frac{1}{q} \sum_{p=0}^{q-1} \exp(-2\pi i r p/q) a(F_{t(p)}, qm \cdot \chi^2) \exp(-2\pi i t(p) m \cdot \chi^2) \\ &= \textstyle\frac{1}{q} \sum_{p=0}^{q-1} \exp(-2\pi i r p/q) a(F_p, qm \cdot \chi^2) \exp(-2\pi i p m \cdot \chi^2) \,, \end{aligned} \tag{A.22}$$

where we also used that, for $0 \leq p \leq q-1$, $t(p) = p$. Inserting (A.22) into (A.21) yields

$$F^N(n) = \frac{1}{q} \sum_{r=0}^{q-1} \sum_{p=0}^{q-1}$$
$$\exp\left(2\pi i r(n-p)/q\right) \sum_{\substack{m \in Z^s \\ \|m\| \leq N}} A_{N,m,s} a(F_p, qm \cdot \chi^2) \exp(2\pi i (n-p) m \cdot \chi^2) \,. \tag{A.23}$$

With (A.23) we have taken the third step of the proof and have come closer to our goal of showing that F^N has the form of (A.5). Before we bring the r.h.s. of (A.23) into the desired form, we prove in the next step that $\mathfrak{M}(F) = \mathbb{R}$. In fact, by (A.23) it is clear that F^N is a trigonometric polynomial so that $\mathfrak{M}(F^N) = \mathbb{R}$. In particular, for real λ we get by (A.21)

$$\begin{aligned} a(F^N, \lambda) &= \lim_{T\to\infty} \frac{1}{T+1} \sum_{n=0}^{T} F^N(n) \exp(-2\pi i n\lambda) \\ &= \lim_{T\to\infty} \frac{1}{T+1} \sum_{n=0}^{T} \sum_{\substack{m\in Z^s \\ \|m\|\le N}} A_{N,m,s} G_m(n) \exp\Big(2\pi i n(m\cdot\chi^2 - \lambda)\Big) \\ &= \lim_{T\to\infty} \frac{1}{T+1} \sum_{n=0}^{T} \sum_{\substack{m\in Z^s \\ \|m\|\le N}} A_{N,m,s} \sum_{r=0}^{q-1} a(G_m, r/q) \exp\Big(2\pi i n(m\cdot\chi^2 + r/q - \lambda)\Big) \\ &= \sum_{\substack{m\in Z^s \\ \|m\|\le N}} A_{N,m,s} \sum_{r=0}^{q-1} a(G_m, r/q) \lim_{T\to\infty} \frac{1}{T+1} \sum_{n=0}^{T} \exp\Big(2\pi i n(m\cdot\chi^2 + r/q - \lambda)\Big) \,, \end{aligned} \tag{A.24}$$

where in the third equation we used (A.22). If $\lambda \notin Y_{(1/q,\chi^2)}$ then, by (A.24), $a(F^N, \lambda) = 0$ and with the abbreviations $G(n) := \exp(-2\pi i n\lambda) F(n)$, $G^N(n) := \exp(-2\pi i n\lambda) F^N(n)$, we observe that G and G^N are complex valued and bounded functions and that $a(G^N, 0) = a(F^N, \lambda) = 0$. Because G^N converges uniformly on $\mathbb{Z}$ to G as $N \to \infty$, we have the situation of Lemma A.1b so that we conclude, if $\lambda \notin Y_{(1/q,\chi^2)}$, that

$$0 = a(G, 0) = a(F, \lambda) \,, \tag{A.25}$$

hence $\lambda \in \mathfrak{M}(F)$. Having thus shown that $\mathbb{R} \setminus Y_{(1/q,\chi^2)} \subset \mathfrak{M}(F)$ we now consider a λ in $Y_{(1/q,\chi^2)}$. Then $\lambda = k_2 + k_1 \cdot \chi^2 + k_0/q$, where $k_1 \in \mathbb{Z}^s, k_0, k_2 \in \mathbb{Z}$ and so, by (A.24),

$$\begin{aligned} a(F^N, k_2 + k_1\cdot\chi^2 + k_0/q) &= \sum_{\substack{m\in Z^s \\ \|m\|\le N}} A_{N,m,s} \sum_{r=0}^{q-1} a(G_m, r/q) \lim_{T\to\infty} \frac{1}{T+1} \sum_{n=0}^{T} \\ &\exp\Big(2\pi i n[(m-k_1)\cdot\chi^2 + (r-k_0)/q]\Big) = A_{N,k_1,s}\, a(G_{k_1}, t(k_0)/q) \,, \end{aligned} \tag{A.26}$$

where in the second equation we used that $(1, \chi^2)$ is nonresonant and that the relation: $(r - k_0)/q$ =integer, implies that $r = t(k_0)$. With the abbreviations $G(n) := \exp\Big(-2\pi i n(k_1 \cdot \chi^2 + k_0/q)\Big) F(n) - a(G_{k_1}, t(k_0)/q)$ and $G^N(n) := \exp\Big(-2\pi i n(k_2 + k_1\cdot\chi^2 + k_0/q)\Big) F^N(n) - A_{N,k_1,s}\, a(G_{k_1}, t(k_0)/q)$, we observe that G and G^N are complex valued and bounded functions and that, by (A.26), $a(G^N, 0) = 0$. Because $F^N - F$ converges uniformly on $\mathbb{Z}$ to 0 as $N \to \infty$, so does $G^N - G$ (note that $A_{N,k_1,s}$ converges to 1 as $N \to \infty$). Thus we again have the situation of Lemma A.1b and we conclude that

$$0 = a(G, 0) = a(F, k_2 + k_1\cdot\chi^2 + k_0/q) - a(G_{k_1}, t(k_0)/q) \,, \tag{A.27}$$

hence $k_2 + k_1 \cdot \chi^2 + k_0/q \in \mathfrak{M}(F)$. With (A.27) we have shown that $Y_{(1/q,\chi^2)} \subset \mathfrak{M}(F)$ so that we have completed the proof that $\mathfrak{M}(F) = \mathbb{R}$ and we thus have taken the fourth step of the proof. In the next step we now bring, with (A.27) at hand, the r.h.s. of (A.23) into the desired form. Using (A.22) and (A.27) we get

$$a(F, k_1 \cdot \chi^2 + k_0/q) = \frac{1}{q}\sum_{p=0}^{q-1} a(F_p, qk_1 \cdot \chi^2)\exp(-2\pi i p[k_1 \cdot \chi^2 + t(k_0)/q]) \ . \quad (A.28)$$

Setting $k_1 = m, k_0 = r$ in (A.28), where $0 \le r \le q-1$, results in

$$a(F, m \cdot \chi^2 + r/q) = \frac{1}{q}\sum_{p=0}^{q-1} a(F_p, qm \cdot \chi^2)\exp(-2\pi i p[m \cdot \chi^2 + r/q]) \ . \quad (A.29)$$

Inserting (A.29) into (A.23) yields (A.5) and we thus have taken the fifth step of the proof. We now assume that q is the smallest positive integer such that $q\chi^1$ is in $\mathbb{Z}^{k-s}$. Then there exists a j such that the j–th component χ^1_j of χ^1 satisfies $\chi^1_j = p/q$, where the integers p, q are relatively prime, i.e. integers $\tilde{p}, \tilde{q}$ exists such that $\tilde{p}p + \tilde{q}q = 1$. Thus $\tilde{p}\chi^1_j + \tilde{q} = 1/q$ so that $1/q \in Y_{\chi^1}$. It follows, for every $m \in \mathbb{Z}^s$ and $r \in \mathbb{Z}$, that $m \cdot \chi^2 + r/q \in Y_\chi$ and we thus have taken the sixth and final step of our proof. We therefore have proven all claims in the case where F is complex valued. For the cases where F is $\mathbb{C}^j$–valued or $\mathbb{C}^{j\times j}$–valued, the claims follow easily. □

Proof of Lemma A.1e: Let χ be in $\mathbb{R}^k \setminus \mathbb{Q}^k$ and let $(1,\chi)$ be resonant. First of all we observe that $\mathbb{Z}^k$ is a $\mathbb{Z}$–module, i.e. a module over the ring of integers [21],[24, Appendix 3], where the addition is the same as in $\mathbb{R}^k$ (recall that a $\mathbb{Z}$–module is an abelian group where the multiplication of an element x with a positive integer n is defined as the n–fold sum: $x + ... + x$). Moreover $e^1, ..., e^k$ generate $\mathbb{Z}^k$, i.e. every m in $\mathbb{Z}^k$ is a linear combination of $e^1, ..., e^k$ with integer coefficients (note that $e^j := (0, ..., 0, 1, 0, ...0)$ with the j–th component equal to 1). The $\mathbb{Z}$–module $\mathbb{Z}^k$ is not only finitely generated (by $e^1, ..., e^k$) but $e^1, ..., e^k$ form a basis of $\mathbb{Z}^k$ (hence $\mathbb{Z}^k$ is k–dimensional) because they are linearly independent over $\mathbb{R}$. We next observe that the set M_χ, defined by

$$M_\chi := \{m \in \mathbb{Z}^k : m \cdot \chi \in \mathbb{Z}\} \ , \quad (A.30)$$

is a $\mathbb{Z}$–module and that $M_\chi \subset \mathbb{Z}^k$. It follows (see [21] or [24, Appendix 3]) that the $\mathbb{Z}$–module M_χ has a basis and that its dimension r satisfies $r \le k$. Also we have $0 < r$, because $(1,\chi)$ is resonant. Because M_χ is r–dimensional it has a basis $k_1, ..., k_r$. Abbreviating $Z_1 := [k^1, ..., k^r]^T$, the $r \times k$ matrix Z_1 satisfies

$$M_\chi = Z_1^T \mathbb{Z}^r \ , \qquad Z_1\chi \in \mathbb{Z}^r \ . \quad (A.31)$$

By some linear algebra of $\mathbb{Z}$–modules it follows (see [21, Sec. III.8] or [24, Appendix 3.8]) that the $r \times k$ matrix Z_1 can be factorized via $Z_1 =: Z_2 Z_3 Z$, where the $r \times r$ matrix Z_2 and the $k \times k$ matrix Z are unimodular and where the $r \times k$ matrix Z_3 has the form $[l_1 e^1, ..., l_r e^r]^T$ with nonzero integers $l_1, ..., l_r$. Note that [21, Sec. III.8] contains a constructive method of computing Z_2, Z_3, Z when Z_1 is given. Abbreviating $\check{\chi} := Z\chi$, it follows from (A.31) that

$$Z_2 Z_3 \check{\chi} = Z_2 Z_3 Z\chi = Z_1\chi \in \mathbb{Z}^r \ . \quad (A.32)$$

Because Z_2 is unimodular, (A.32) yields $Z_3\check{\chi} \in \mathbb{Z}^r$, hence $(\check{\chi}_1, ..., \check{\chi}_r) \in \mathbb{Q}^r$. It follows that $r < k$ because the equation: $r = k$ would lead to the relation: $\check{\chi} \in \mathbb{Q}^k$ which is wrong because $\chi \notin \mathbb{Q}^k$. We abbreviate

$$\check{\chi} =: (\check{\chi}_1, ..., \check{\chi}_r, \hat{\chi}) \ , \tag{A.33}$$

where $\hat{\chi} \in \mathbb{R}^{k-r}$. We now will show that $(1, \hat{\chi})$ is nonresonant. Note for every $m \in \mathbb{Z}^k$ that the relation $m \in M_\chi$ is equivalent to $(Z^T)^{-1}m \in M_{\check{\chi}}$. Thus we conclude from (A.31)

$$M_{\check{\chi}} = (Z^T)^{-1}M_\chi = (Z^T)^{-1}Z_1^T\mathbb{Z}^r = (Z^T)^{-1}(Z_2Z_3Z)^T\mathbb{Z}^r = Z_3^TZ_2^T\mathbb{Z}^r = Z_3^T\mathbb{Z}^r \ , \tag{A.34}$$

where we also used the factorization formula for Z_1 and the unimodularity of Z_2. We are now in a position to show that $(1, \hat{\chi})$ is nonresonant. In fact, if $\hat{m} \in M_{\hat{\chi}}$, then $\hat{m} \cdot \hat{\chi} \in \mathbb{Z}$, hence by (A.33): $(0, \hat{m}) \cdot \check{\chi} \in \mathbb{Z}$. Thus $(0, \hat{m}) \in M_{\check{\chi}}$, i.e. by (A.34): $(0, \hat{m}) \in Z_3^T\mathbb{Z}^r$. On the other hand, by the definition of Z_3, the set $Z_3^T\mathbb{Z}^r$ is a r–dimensional $\mathbb{Z}$–module with a basis consisting of the vectors $l_1e^1, ..., l_re^r$. Therefore $(0, \hat{m}) \in Z_3^T\mathbb{Z}^r$ implies $\hat{m} = 0$, hence $(1, \hat{\chi})$ is nonresonant. We can summarize: the unimodular $k \times k$ matrix Z satisfies $Z\chi = (\chi^1, \chi^2)$ where $\chi^1 = (\check{\chi}_1, ..., \check{\chi}_r)$ and $\chi^2 = \hat{\chi}$. Thus $\chi^1 \in \mathbb{Q}^r$ and $(1, \chi^2)$ is nonresonant; moreover $s = k - r$ (hence $0 < s < k$). Note also that $M_{\check{\chi}} = \{(m_1, ..., m_r, 0, ..., 0) \in \mathbb{Z}^k : m_1\check{\chi}_1 + ... + m_r\check{\chi}_r \in \mathbb{Z}\}$. It remains to be shown that: $s = k - r$ holds for every choice of Z. Thus let $\tilde{Z}$ be a unimodular $k \times k$ matrix such that $\tilde{Z}\chi =: (\chi^1, \chi^2)$ with $(1, \chi^2)$ nonresonant and $\chi^1 \in \mathbb{Q}^{k-s}$ where $0 < s < k$. Then $M_{\tilde{\chi}} = (\tilde{Z}^T)^{-1}M_\chi$, where $\tilde{\chi} := \tilde{Z}\chi$. Thus $M_{\tilde{\chi}}$ is a r–dimensional $\mathbb{Z}$–module. Because $(1, \chi^2)$ is nonresonant, $M_{\tilde{\chi}} = \{(m_1, ..., m_{k-s}, 0, ..., 0) \in \mathbb{Z}^k : m_1\tilde{\chi}_1 + ... + m_{k-s}\tilde{\chi}_{k-s} \in \mathbb{Z}\}$. Thus the dimension of $M_{\tilde{\chi}}$ fulfills $r \leq k - s$. Writing $\tilde{\chi}_1 =: p_1/q_1, ..., \tilde{\chi}_{k-s} =: p_{k-s}/q_{k-s}$ with integers $q_i \neq 0, p_i$, we observe that $q_1e^1, ..., q_{k-s}e^{k-s}$ belong to $M_{\tilde{\chi}}$. Because $q_1e^1, ..., q_{k-s}e^{k-s}$ are linearly independent over $\mathbb{R}$, the dimension of $M_{\tilde{\chi}}$ fulfills $r \geq k - s$. We conclude that $r = k - s$. □

Proof of Lemma A.1f: Let F be a complex valued χ–quasiperiodic function and let $(1, \chi)$ be resonant and $\chi \in \mathbb{R}^k \setminus \mathbb{Q}^k$. Then, by Lemma A.1e, a unimodular $k \times k$ matrix Z exists such that $Z\chi =: (\chi^1, \chi^2)$ with $\chi^1 \in \mathbb{Q}^{k-s}$ and $(1, \chi^2)$ nonresonant where $0 < s < k$ and where the positive integer s is uniquely determined by χ. Let F be generated by f via $F(n) = f(2\pi n\chi)$ where $f : \mathbb{R}^k \to \mathbb{C}$ is a continuous and 2π–periodic function. Then

$$F(n) = f(2\pi n\chi) = f(2\pi nZ^{-1}Z\chi) = f(2\pi nZ^{-1}(\chi^1, \chi^2)) =: \tilde{f}(2\pi n(\chi^1, \chi^2)) \ , \tag{A.35}$$

where the continuous and 2π–periodic function $\tilde{f} : \mathbb{R}^k \to \mathbb{C}$ is defined by $\tilde{f}(\phi) := f(Z^{-1}\phi)$. By (A.35), F is generated by $\tilde{f}$ via $F(n) = \tilde{f}(2\pi n(\chi^1, \chi^2))$ so that F is (χ^1, χ^2)–quasiperiodic. Thus, by Lemma A.1d, $\mathfrak{M}(F) = \mathbb{R}$ and the sequence F^N, defined by (A.5), converges uniformly to F on $\mathbb{Z}$ as $N \to \infty$, if the positive integer q is chosen such that $q\chi^1$ is in $\mathbb{Z}^{k-s}$. Again using Lemma A.1d it follows, for all $m \in \mathbb{Z}^s$ and $r \in \mathbb{Z}$, that $m \cdot \chi^2 + r/q \in Y_\chi$, if q is the smallest positive integer such that $q\chi^1$ in $\mathbb{Z}^{k-s}$. □

Remark:

(1) If $\chi \in \mathbb{R}^k$, then M_χ is a finite–dimensional $\mathbb{Z}$–module of dimension $\leq k$. This property follows, as pointed out in the proof of Lemma A.1e, from the facts that $M_\chi \subset \mathbb{Z}^k$ and that $\mathbb{Z}^k$ is a finite–dimensional $\mathbb{Z}$–module. In fact, the condition that M_χ is contained

in a finite–dimensional $\mathbb{Z}$–module is not superfluous because a $\mathbb{Z}$–module, even if it is finitely generated, might have no basis. For example, the $\mathbb{Z}$–module $\mathbb{Z}_2$ has no basis (hence it is not finite–dimensional) because it contains only finitely many elements (namely two: the set of even integers and the set of odd integers).

A.3

While Lemma A.1 gives the foundations of Fourier analysis of quasiperiodic functions the following corollary of Lemma A.1 draws some important conclusions.

Corollary A.2 *a) Let F be a χ–quasiperiodic function. Then $\mathfrak{M}(F) = \mathbb{R}$. If $\Lambda(F) = \emptyset$ then $F = 0$. If F is complex valued and $\Lambda(F)$ is nonempty then a sequence of complex valued and χ–quasiperiodic functions $F^0, F^1, \ldots$ exists such that F^N converges uniformly to F on $\mathbb{Z}$ as $N \to \infty$ and such that*

$$F^N(n) = \sum_{\lambda \in \Lambda^N(F)} b_{N,\lambda} a(F,\lambda) \exp(2\pi i n \lambda) , \tag{A.36}$$

where $b_{N,\lambda} \in \mathbb{C}$ is independent of F and where $\Lambda^N(F)$ is a finite subset of $\Lambda(F) \cap Y_\chi$.
b) If F is a $\mathbb{C}^j$–valued quasiperiodic function then $\Lambda(F) = \Lambda(F_1) \cup \Lambda(F_2) \cup \ldots$, where $F_1, F_2, \ldots$ denote the components of F. The same claim holds if F is $\mathbb{C}^{j\times j}$–valued. If G and H are complex valued quasiperiodic functions then $\Lambda(G + H) \subset (\Lambda(G) \cup \Lambda(H))$. The same claim holds if G and H are both $\mathbb{C}^j$–valued or both $\mathbb{C}^{j\times j}$–valued.
c) If F is a χ–quasiperiodic function, then $\Lambda(F) \subset Y_\chi$.
d) Let $F : \mathbb{Z} \to \mathbb{C}$ be defined by $F(n) := \exp(2\pi i(\alpha n + \beta n^2))$ where α, β are real constants with irrational β. Then $\Lambda(F) = \emptyset$. THE LATTER IMPLICATION IS QUITE A FAMOUS RESULT (SO IN PRINCIPLE WE CAN DROP OUR PROOF OF IT) WHICH WAS FIRST PROVEN IN THE 1910's BY WEYL AND, INDEPENDENTLY, BY HARDY AND LITTLEWOOD. OUR PROOF SOLVES THE UNDERLYING SMALL DIVISOR PROBLEM ALONG THE LINES OF WEYL. UNLIKE OUR PROOF, THE PROOF OF HARDY AND LITTLEWOOD USED CAUCHY'S THEOREM FROM COMPLEX ANALYSIS
Thus F is not quasiperiodic.
e) Let $f : \mathbb{R}^k \to \mathbb{C}$ be a continuous and 2π–periodic function and let χ be in $\mathbb{R}^k$ with $(1, \chi)$ nonresonant (positive integer k). Then f is a constant function, if $f = f(* + 2\pi\chi)$.

Proof of Corollary A.2a: If G is a quasiperiodic function then it follows from Lemmas A.1a,c,f that $\mathfrak{M}(G) = \mathbb{R}$ and that $\Lambda(G) = \emptyset$ implies $G = 0$. To prove the remaining claims we now consider a complex valued and χ–quasiperiodic function F, where $\chi \in \mathbb{R}^k$ and where $\Lambda(F)$ is nonempty. We first consider the case where $\chi \in \mathbb{Q}^k$. Then we define

$$F^N(n) := F(n) = \sum_{r=0}^{q-1} \exp(2\pi i r n/q) a(F, r/q) , \tag{A.37}$$

where in the second equation we used Lemma A.1a and where q is the smallest positive integer such that $q\chi$ in $\mathbb{Z}^k$. Trivially, F^N converges uniformly to F on $\mathbb{Z}$ as $N \to \infty$. We also define

$$\Lambda^N(F) := \{r/q : a(F, r/q) \neq 0, r = 0, \ldots, q-1\} , \qquad b_{N,\lambda} := 1 . \tag{A.38}$$

We conclude from (A.37) and (A.38) that (A.36) holds. It also follows from Lemma A.1a that $\Lambda^N(F) \subset \Lambda(F) \cap Y_\chi$. We secondly consider the case where $(1, \chi)$ is nonresonant. Then we define F^N as in (A.1). By Lemma A.1c, F^N converges uniformly to F on $\mathbb{Z}$ as $N \to \infty$. We also define

$$\Lambda^N(F) := \{\lfloor m \cdot \chi \rfloor : a(F, m \cdot \chi) \neq 0, ||m|| \leq N, m \in \mathbb{Z}^k\}\,, \qquad b_{N,\lambda} := A_{N,m(\lambda),k}\,, \tag{A.39}$$

where $m(\lambda)$ is that unique element of $\mathbb{Z}^k$ for which $\lambda = \lfloor m(\lambda) \cdot \chi \rfloor$. We conclude from (A.1) and (A.39) that (A.36) holds. It is obvious that $\Lambda^N(F) \subset \Lambda(F) \cap Y_\chi$. We thirdly consider the case where $(1, \chi)$ is resonant and $\chi \in \mathbb{Q}^k$. Then, by Lemma A.1e, a unimodular $k \times k$ matrix Z exists such that $Z\chi =: (\mu^1, \mu^2)$ with $\mu^1 \in \mathbb{Q}^{k-s}$ and $(1, \mu^2)$ nonresonant where $0 < s < k$. Then we define F^N by (A.5), where q is the smallest positive integer such that $q\mu^1$ in $\mathbb{Z}^{k-s}$. By Lemma A.1f, F^N converges uniformly to F on $\mathbb{Z}$ as $N \to \infty$. We also define

$$\begin{aligned}\Lambda^N(F) := \{\lfloor m \cdot \mu^2 + r/q \rfloor : a(F, m \cdot \mu^2 + r/q) \neq 0, ||m|| \leq N,\\ m \in \mathbb{Z}^s, r = 0, ..., q-1\}\,, \qquad b_{N,\lambda} := A_{N,m(\lambda),k}\,,\end{aligned} \tag{A.40}$$

where $m(\lambda)$ is that unique element of $\mathbb{Z}^s$ for which $\lambda = \lfloor m(\lambda) \cdot \mu_2 + r(\lambda)/q \rfloor$, where the integer $r(\lambda)$ is unique by the condition $0 \leq r(\lambda) \leq q-1$. We conclude from (A.5) and (A.40) that (A.36) holds. It is also follows from Lemma A.1f that $\Lambda^N(F) \subset \Lambda(F) \cap Y_\chi$. □

Proof of Corollary A.2b: Let F be a $\mathbb{C}^j$–valued quasiperiodic function. Then, by Corollary A.2a, $\mathfrak{M}(F) = \mathbb{R}$ so that $\Lambda(F) = \Lambda(F_1) \cup \Lambda(F_2) \cup ...$, where $F_1, F_2, ...$ denote the components of F. If F is $\mathbb{C}^{j \times j}$–valued then the proof goes the same. Let G and H be complex valued quasiperiodic functions and let λ be in $\Lambda(G+H)$, i.e. let $a(G+H, \lambda) \neq 0$. Because of Corollary A.2a we have that $a(G, \lambda)$ and $a(H, \lambda)$ are well defined so that $a(G, \lambda) + a(H, \lambda) = a(G+H, \lambda) \neq 0$. Thus $\lambda \in (\Lambda(G) \cup \Lambda(H))$, i.e. we have shown that $\Lambda(G+H) \subset (\Lambda(G) \cup \Lambda(H))$. It is now easy to prove the claim if G and H are multicomponent with $G+H$ well defined. By the above, $\Lambda(G+H) = \Lambda(G_1+H_1) \cup \Lambda(G_2+H_2) \cup ...$, where $G_1, G_2, ..., H_1, H_2, ...$ denote the components of G and H. We also know that $\Lambda(G_1 + H_1) \subset \Lambda(G_1) \cup \Lambda(H_1)$ etc. so that $\Lambda(G+H) \subset \Lambda(G_1) \cup \Lambda(H_1) \cup \Lambda(G_2) \cup \Lambda(H_2) \cup ... = \Lambda(G) \cup \Lambda(H)$. Thus we have shown the claim also in the case where G, H are multicomponent. □

Proof of Corollary A.2c: Let λ be in $\mathbb{R}$ and let F be a complex valued and χ–quasiperiodic function. Because the sequence F^N in Corollary A.2a converges uniformly to F on $\mathbb{Z}$ as $N \to \infty$ and since $a(F^N, \lambda)$ and $a(F, \lambda)$ exist, we have

$$\begin{aligned}&|a(F^N, \lambda) - a(F, \lambda)| = |a(F^N - F, \lambda)|\\ &= |\lim_{T \to \infty} \frac{1}{T+1} \sum_{n=0}^{T} (F^N(n) - F(n)) \exp(-2\pi i \lambda n)| \leq \sup_n |F^N(n) - F(n)|\,,\end{aligned}$$

where we also used that F^N and F are bounded functions. It follows that

$$\lim_{N \to \infty} a(F^N, \lambda) = a(F, \lambda)\,, \tag{A.41}$$

since F^N converges uniformly to F on $\mathbb{Z}$ as $N \to \infty$. If $\lambda \notin Y_\chi$, then, by Corollary A.2a, $a(F^N, \lambda) = 0$ and (A.41) gives $a(F, \lambda) = 0$. Thus if $\lambda \notin Y_\chi$, then $\lambda \notin \Lambda(F)$, so that $\Lambda(F) \subset Y_\chi$.

It is now easy to prove the claims for every χ–quasiperiodic function F. Let F be $\mathbb{C}^j$–valued or $\mathbb{C}^{j\times j}$–valued. From the above we know that

$$\Lambda(F_n) \subset Y_\chi \,, \tag{A.42}$$

where the $F_1, F_2, \ldots$ denote the components of F. We conclude from (A.42) and Corollary A.2b that $\Lambda(F) \subset Y_\chi$. □

Proof of Corollary A.2d: For every function $G : \mathbb{Z} \to \mathbb{R}$ and every T in $\mathbb{N}$ one obtains

$$|\sum_{n=0}^{T} \exp(2\pi i G(n))|^2 = \sum_{r=-T}^{T} \sum_{k=\max\{0,-r\}}^{\min\{T,T-r\}} \exp(2\pi i[G(k+r) - G(k)]) \,. \tag{A.43}$$

Note that (A.43), which can be easily proved by induction in T, is often called 'Weyl differencing' since it appeared in [31]. Note also that the expression $\sum_{n=0}^{T} \exp(2\pi i G(n))$ is sometimes called an 'exponential sum' or a 'Weyl sum'. For a recent textbook on exponential sums, see [19]. Let λ be in $\mathbb{R}$. To come to our case of interest we set $G(n) = (\alpha - \lambda)n + \beta n^2$ in (A.43) and thus obtain

$$\begin{aligned}(T+1)^2|a_T(F,\lambda)|^2 &= |\sum_{n=0}^{T} F(n)\exp(-2\pi i\lambda n)|^2 \\ &= \sum_{r=-T}^{T} \exp(2\pi i[(\alpha-\lambda)r + \beta r^2]) \sum_{k=\max\{0,-r\}}^{\min\{T,T-r\}} \exp(4\pi i\beta r k) \,,\end{aligned} \tag{A.44}$$

which implies that, for $T \in \mathbb{N}$,

$$(T+1)^2|a_T(F,\lambda)|^2 \leq \sum_{r=-T}^{T} |\sum_{k=\max\{0,-r\}}^{\min\{T,T-r\}} \exp(4\pi i\beta r k)| \,. \tag{A.45}$$

Of course our claims are proven if $|a_T(F,\lambda)|^2 \to 0$ as $T \to \infty$ and (A.45) is the starting point to achieve this. Note that the sum over k in (A.45) is just a geometrical sum whose summation results in

$$\begin{aligned}|\sum_{k=\max\{0,-r\}}^{\min\{T,T-r\}} \exp(4\pi i\beta r k)| &= |\frac{1-\exp(4\pi i\beta r[T+1-|r|])}{1-\exp(4\pi i\beta r)}| \\ &= |\frac{1-\exp(4\pi i\beta r[T+1-|r|])}{2\sin(2\pi\beta r)}| \,.\end{aligned} \tag{A.46}$$

Because β is irrational the denominator $\sin(2\pi\beta r)$ on the r.h.s. of (A.46) acquires arbitrary small values when r varies over the integers. Therefore the basic inequality (A.45) confronts us with a small divisor problem and to cope with this problem we use a clever idea from Weyl's proof of 'Satz 9' in [31].

NOTE THAT 'Satz 9' IN [31] CLAIMS MORE THAN WE DO. THUS OUR PROOF IS A DESTILATE FROM THE MORE INTRICATE PROOF OF 'Satz 9' IN [31]

Following his idea we split the sum over r in (A.45) into two parts. In fact defining, for $0 < \varepsilon < 1/2$ and $T \in \mathbb{N}$, the set $Z_1(\varepsilon, T) := \{r \in \mathbb{Z} : |r| \leq T, \varepsilon \leq \lfloor 2\beta r \rfloor \leq 1 - \varepsilon\}$, (A.45) can be written as

$$
\begin{aligned}
(T+1)^2 |a_T(F,\lambda)|^2 \leq & \sum_{r \in Z_1(\varepsilon,T)} \Big| \sum_{k=\max\{0,-r\}}^{\min\{T,T-r\}} \exp(4\pi i \beta r k) \Big| \\
& + \sum_{r \in Z_2(\varepsilon,T)} \Big| \sum_{k=\max\{0,-r\}}^{\min\{T,T-r\}} \exp(4\pi i \beta r k) \Big| , \qquad \text{(A.47)}
\end{aligned}
$$

where $Z_2(\varepsilon, T) := \{r \in \mathbb{Z} : |r| \leq T, r \notin Z_1(\varepsilon, T)\}$. To exploit (A.47) we now will estimate the two terms on its r.h.s. and beginning with the first term we obtain from (A.46) that

$$
\sum_{r \in Z_1(\varepsilon,T)} \Big| \sum_{k=\max\{0,-r\}}^{\min\{T,T-r\}} \exp(4\pi i \beta r k) \Big| \leq \sum_{r \in Z_1(\varepsilon,T)} \Big| \frac{1}{\sin(2\pi\beta r)} \Big| . \qquad \text{(A.48)}
$$

Note that, for $r \in Z_1(\varepsilon, T)$,

$$
\begin{aligned}
|\sin(2\pi\beta r)| &= \Big|\sin\Big(\pi \lfloor 2\beta r \rfloor + \pi \lceil 2\beta r \rceil\Big)\Big| = \Big|(-1)^{\lceil 2\beta r \rceil} \sin\Big(\pi \lfloor 2\beta r \rfloor\Big)\Big| \\
&= \Big|\sin\Big(\pi \lfloor 2\beta r \rfloor\Big)\Big| = \sin\Big(\pi \lfloor 2\beta r \rfloor\Big) \geq \sin(\pi\varepsilon) , \qquad \text{(A.49)}
\end{aligned}
$$

where in the inequality we used the definition of $Z_1(\varepsilon, T)$. Inserting (A.49) into (A.48) yields

$$
\sum_{r \in Z_1(\varepsilon,T)} \Big| \sum_{k=\max\{0,-r\}}^{\min\{T,T-r\}} \exp(4\pi i \beta r k) \Big| \leq \sum_{r \in Z_1(\varepsilon,T)} \frac{1}{\sin(\pi\varepsilon)} \leq \sum_{r=-T}^{T} \frac{1}{\sin(\pi\varepsilon)} = \frac{2T+1}{\sin(\pi\varepsilon)} \qquad \text{(A.50)}
$$

To estimate the second term on the r.h.s. of (A.47) we compute

$$
\begin{aligned}
\sum_{r \in Z_2(\varepsilon,T)} \Big| \sum_{k=\max\{0,-r\}}^{\min\{T,T-r\}} \exp(4\pi i \beta r k) \Big| &\leq \sum_{r \in Z_2(\varepsilon,T)} \sum_{k=\max\{0,-r\}}^{\min\{T,T-r\}} |\exp(4\pi i \beta r k)| \\
&= \sum_{r \in Z_2(\varepsilon,T)} (T+1-|r|) \leq \sum_{r \in Z_2(\varepsilon,T)} (T+1) = (T+1)\#Z_2(\varepsilon,T) , \qquad \text{(A.51)}
\end{aligned}
$$

where in the first equation we used that the sum over k has $T+1-|r|$ terms and where $\#Z_2$ denotes the number of elements of Z_2. To estimate $\#Z_2$ we conclude from the definition of Z_2 that

$$
\begin{aligned}
Z_2(\varepsilon, T) = &\{r \in \mathbb{N} : r \leq T, \lfloor 2\beta r \rfloor < \varepsilon\} \cup \{r \in \mathbb{Z} : -T \leq r \leq 0, \lfloor 2\beta r \rfloor < \varepsilon\} \\
&\cup \{r \in \mathbb{N} : r \leq T, 1-\varepsilon < \lfloor 2\beta r \rfloor\} \cup \{r \in \mathbb{Z} : -T \leq r \leq 0, 1-\varepsilon < \lfloor 2\beta r \rfloor\} ,
\end{aligned}
$$

so that

$$\begin{aligned}\#Z_2(\varepsilon,T) &\leq \#\{r\in\mathbb{N}: r\leq T, \lfloor 2\beta r\rfloor<\varepsilon\} + \#\{r\in\mathbb{Z}: -T\leq r\leq 0, \lfloor 2\beta r\rfloor<\varepsilon\}\\ &+\#\{r\in\mathbb{N}: r\leq T, 1-\varepsilon<\lfloor 2\beta r\rfloor\} + \#\{r\in\mathbb{Z}: -T\leq r\leq 0, 1-\varepsilon<\lfloor 2\beta r\rfloor\}\\ &= \#\{r\in\mathbb{N}: r\leq T, \lfloor 2\beta r\rfloor<\varepsilon\} + \#\{r\in\mathbb{N}: r\leq T, \lfloor -2\beta r\rfloor<\varepsilon\}\\ &+\#\{r\in\mathbb{N}: r\leq T, 1-\varepsilon<\lfloor 2\beta r\rfloor\} + \#\{r\in\mathbb{N}: r\leq T, 1-\varepsilon<\lfloor -2\beta r\rfloor\}\,.\\ &=: \#Z_3(\varepsilon,T)+\#Z_4(\varepsilon,T)+\#Z_5(\varepsilon,T)+\#Z_6(\varepsilon,T)\,,\end{aligned} \tag{A.52}$$

where in the first equation we used the fact that the number of elements of a finite set is invariant under a bijection of that set. Because 2β and -2β are irrational the equidistribution theorem in one dimension (see for example 'Satz 2' in [31] or [22, Sec. 3]) yields

$$\varepsilon = \lim_{T\to\infty}\frac{\#Z_3(\varepsilon,T)}{T+1} = \lim_{T\to\infty}\frac{\#Z_4(\varepsilon,T)}{T+1} = \lim_{T\to\infty}\frac{\#Z_5(\varepsilon,T)}{T+1} = \lim_{T\to\infty}\frac{\#Z_6(\varepsilon,T)}{T+1}\,,$$

so that there exists a $N(\varepsilon)\in\mathbb{N}$ such that, for $i=3,4,5$ and 6, $(\forall T\geq N(\varepsilon))\ |\frac{\#Z_i(\varepsilon,T)}{T+1}-\varepsilon|<\varepsilon$. Thus, for $i=3,4,5$ and 6,

$$(\forall T\geq N(\varepsilon))\ \frac{\#Z_i(\varepsilon,T)}{T+1}<2\varepsilon\,. \tag{A.53}$$

From (A.52) and (A.53) it follows that

$$(\forall T\geq N(\varepsilon))\ \frac{\#Z_2(\varepsilon,T)}{T+1}<8\varepsilon\,. \tag{A.54}$$

From (A.51) and (A.54) it follows that

$$(\forall T\geq N(\varepsilon))\ \sum_{r\in Z_2(\varepsilon,T)}\ |\sum_{k=\max\{0,-r\}}^{\min\{T,T-r\}}\exp(4\pi i\beta rk)|<8\varepsilon(T+1)^2\,. \tag{A.55}$$

Having estimated both terms on the r.h.s. of (A.47) we obtain from (A.50) and (A.55) that

$$(\forall T\geq N(\varepsilon))\ (T+1)^2|a_T(F,\lambda)|^2<\frac{2T+1}{\sin(\pi\varepsilon)}+8\varepsilon(T+1)^2\,,$$

hence

$$(\forall T\geq N(\varepsilon))\ |a_T(F,\lambda)|^2<\frac{2}{(T+1)\sin(\pi\varepsilon)}+8\varepsilon\,. \tag{A.56}$$

To get rid of the T–dependence on the r.h.s. of (A.56) we estimate

$$\frac{2}{(T+1)\sin(\pi\varepsilon)}=\frac{2\varepsilon}{(T+1)\varepsilon\sin(\pi\varepsilon)}<\frac{1}{(T+1)\varepsilon}\,, \tag{A.57}$$

where in the inequality we used that $0<\varepsilon<1/2$. From (A.56) and (A.57) we obtain that

$$(\forall T\geq\max\{N(\varepsilon),1/\varepsilon^2\})\ |a_T(F,\lambda)|^2<9\varepsilon\,. \tag{A.58}$$

Because (A.58) holds for all ε in $(0,1/2)$ we conclude that $|a_T(F,\lambda)|^2 \to 0$ as $T \to \infty$, i.e. $a(F,\lambda) = 0$. Because the real λ is arbitrary we thus have shown that the spectrum $\Lambda(F)$ is empty. This implies, by Corollary A.2a and the fact that F is not the zero function, that F is not quasiperiodic. □

Proof of Corollary A.2e: Let $f : \mathbb{R}^k \to \mathbb{C}$ be a continuous and 2π–periodic function and let χ be in $\mathbb{R}^k$ with $(1,\chi)$ nonresonant (positive integer k). Then the χ–quasiperiodic function F, generated by f via $F(n) = f(2\pi n\chi)$, is constant. Thus, and because $(1,\chi)$ is nonresonant, one has, for all $m \in \mathbb{Z}^k \setminus \{0\}$, $a(F, m\cdot\chi) = 0$. It then follows, by Lemma A.1c, that f is constant. □

B Diophantine Conditions

In this appendix we prove a lemma which is used in the proof of Theorem 4.4.

Definition B.1: The tune vector $\omega \in \mathbb{R}^d$ is said to satisfy a Diophantine condition if, for $\tau \in \mathbb{R}, \tau > 0$, we have $\omega \in \mathcal{O}(\tau)$, where $\mathcal{O}(\tau) := \bigcup_{\gamma>0} \mathcal{O}(\tau,\gamma)$ and where for $\gamma \in \mathbb{R},\ \ \gamma > 0$, we have $\mathcal{O}(\tau,\gamma) := \{\omega \in \mathbb{R}^d : |\exp(2\pi i m\cdot\omega) - 1| \geq \gamma ||m||^{-\tau} \text{ for all } m \in \mathbb{Z}^d,\ m \neq 0\}$. □

Remark:

(1) The condition 'on orbital resonance' is incompatible with the fulfillment of the Diophantine condition $\omega \in \mathcal{O}(\tau)$. In particular, for $\omega \in \mathcal{O}(\tau), m \in \mathbb{Z}^d \setminus \{0\}$, $\exp(2\pi i m\cdot\omega) \neq 1$.

We now can state the lemma we aim at.

Lemma B.2 *Let $F : \mathbb{R}^d \to \mathbb{R}$ be of class C^n and 2π–periodic and let $F_0 = 0$, where F_0 is the zeroth Fourier coefficient of F (recall Lemma A.1c). Let $\omega \in \mathcal{O}(\tau)$ where $0 < \tau < n-d$. Then the sequence $\sigma^0, \sigma^1, \sigma^2, \ldots$ of functions $\sigma^N : \mathbb{R}^d \to \mathbb{R}$, given by*

$$\sigma^N(\phi) := \sum_{\substack{m\in Z^d \\ 0<||m||\leq N}} (\exp(2\pi i m\cdot\omega) - 1)^{-1} F_m \exp(i m\cdot\phi) , \tag{B.1}$$

converges uniformly on $\mathbb{R}^d$ as $N \to \infty$ to a continuous function G which is 2π–periodic. Moreover

$$G(* + 2\pi\omega) - G = F , \tag{B.2}$$

and, for $m \neq 0$, the m–th Fourier coefficient of G is

$$G_m = (\exp(2\pi i m\cdot\omega) - 1)^{-1} F_m . \tag{B.3}$$

Proof: Note that, by Remark 1, σ^N is well defined by (B.1). Because a constant $M \geq 0$ exists such that $||m||^n |F_m| \leq M$ (see [22, p.409]), we have, for every $\tau > 0$,

$$\sum_{\substack{m\in Z^d \\ 0<||m||\leq N}} ||m||^\tau |F_m| \leq M \sum_{\substack{m\in Z^d \\ 0<||m||\leq N}} ||m||^{\tau-n} = M \sum_{j=1}^{N} j^{\tau-n} \sum_{\substack{m\in Z^d \\ ||m||=j}} 1 . \tag{B.4}$$

A combinatorial argument gives

$$\sum_{\substack{m\in Z^d\\ ||m||=j}} 1 \le 3^d d j^{d-1}\,. \tag{B.5}$$

Moreover, if $\omega \in \mathcal{O}(\tau,\gamma)$, then $|(\exp(2\pi i m\cdot\omega)-1)^{-1}| \le \gamma^{-1}||m||^\tau$ and (B.4) and (B.5) give

$$\sum_{\substack{m\in Z^d\\ 0<||m||\le N}} |(\exp(2\pi i m\cdot\omega)-1)^{-1}F_m| \le 3^d d M\gamma^{-1}\sum_{j=1}^{N} j^{\tau-n+d-1}\,. \tag{B.6}$$

Since $\tau < n-d$ we conclude that the l.h.s. of (B.6) converges as $N\to\infty$ for every $\omega\in\mathcal{O}(\tau)$. It follows (see for example [15, Sec. VII.1]) that σ^N converges uniformly on $\mathbb{R}^d$ as $N\to\infty$ to a continuous function G which is 2π–periodic. Note that, by (B.1),

$$\sigma^N(*+2\pi\omega)-\sigma^N = F^N\,, \tag{B.7}$$

where

$$F^N(\phi) := \sum_{\substack{m\in Z^d\\ 0<||m||\le N}} F_m \exp(i m\cdot\phi)\,. \tag{B.8}$$

Because F is of class C^d the sequence F^N converges uniformly on $\mathbb{R}^d$ to F as $N\to\infty$. Taking the limit of (B.7) as $N\to\infty$, we thus obtain (B.2). Because of (B.1) we have, for $0<||m||\le N$, that the m–th Fourier coefficient of σ^N is $(\exp(2\pi i m\cdot\omega)-1)^{-1}F_m$. Because σ^N converges uniformly to G it then follows that the m–th Fourier coefficient of G satisfies (B.3). □

Remarks:

(1) To prove (B.5) we define the sets $s := \{m\in\mathbb{Z}^d : ||m|| = j\}$ and $s_i := \{m\in s : |m_i| = j\}$. It follows that $s = \bigcup_{i=1}^d s_i$ and that s_i contains $2(2j+1)^{d-1}$ elements. Then s contains no more than $2d(2j+1)^{d-1}$ elements and because

$$2(2j+1)^{d-1} \le 3(2j+1)^{d-1} \le 3(3j)^{d-1} = 3^d j^{d-1}\,,$$

we conclude that s contains no more than $3^d d j^{d-1}$ elements, thus proving (B.5).

(2) Lemma B.2 provides the basic framework that we need for discussing the uniform convergence of the sequence σ^N. In particular it shows that if n increases beyond a certain value, then the small divisor problem loses much of its potency. This comes as no surprise because the inequality $||m||^n|F_m| \le M$ implies that the Fourier coefficients decrease with increasing $||m||$ more rapidly as n increases. Then with growing n the small divisor in (B.1) can come closer to zero without destroying the convergence.

However, although Lemma B.2 takes much of the mystery out of the working of the Diophantine condition, it puts the burden on determining which ω are in the set $\mathcal{O}(\tau)$ and it is not so easy to decide, off orbital resonance, if ω is in $\mathcal{O}(\tau)$. But some relief comes from a theorem (not proven here), which shows that if $\tau > d$ then the complement, $\mathcal{O}^c(\tau)$, of $\mathcal{O}(\tau)$ is a small set in terms of the Lebesgue measure of Borel measurable sets in $\mathbb{R}^d$ (see [12, 16]). Then, if in addition $\tau < n-d$, the sequence σ^N converges uniformly for almost every ω. For these two conditions to be consistent we thus need $n > 2d$.

C The Exponential Lemma

In this appendix we prove a lemma concerning the exponential function $\exp(ix)$. If k denotes a positive integer, then the following holds.

Lemma C.1 *a) Let $f : \mathbb{R}^k \to U(1)$ be a continuous function, where $U(1) := \{x \in \mathbb{C} : |x| = 1\}$ is the unit circle in the complex plane. Then a continuous function $\lambda : \mathbb{R}^k \to \mathbb{R}$ exists such that $f = \exp(i\lambda)$. When $\tilde{\lambda} : \mathbb{R}^k \to \mathbb{R}$ is an arbitrary continuous function such that $f = \exp(i\tilde{\lambda})$, then an integer N_1 exists such that, for all ϕ, $\tilde{\lambda}(\phi) = \lambda(\phi) + 2\pi N_1$. When, for all ϕ, $f(\phi) = 1$ then $\lambda = 2\pi N_2$, where N_2 is a constant integer.*
b) Let $f : \mathbb{R}^k \to U(1)$ be a continuous and 2π–periodic function. Then a continuous and 2π–periodic function $\hat{\lambda} : \mathbb{R}^k \to \mathbb{R}$ and a $\hat{N} \in \mathbb{Z}^k$ exist such that, for all ϕ, $\lambda(\phi) = \hat{\lambda}(\phi) + \hat{N} \cdot \phi$, where λ is the function in Lemma C.1a. Also, $\hat{N}$ is uniquely determined by f, i.e. independent of the choice of λ.
c) Let R be in $SO(3)$. When $Re^3 = e^3$, then a y exists in $\mathbb{R}$ such that $R = \exp(y\mathcal{J})$.
d) Let $u : \mathbb{R}^k \to SO(3)$ be a continuous and 2π–periodic function such that $ue^3 = e^3$. Then a $\hat{N}$ in $\mathbb{Z}^k$ and a continuous and 2π–periodic function $\hat{\lambda} : \mathbb{R}^k \to \mathbb{R}$ exist such that, for all ϕ, $u(\phi) = \exp(\mathcal{J}[\hat{\lambda}(\phi) + \hat{N} \cdot \phi])$.

Proof of Lemma B.1a: Let $f : \mathbb{R}^k \to U(1)$ be a continuous function with positive integer k. Then (see for example [13, Sec. III.4],[29, Sec. 6]) a continuous function $\lambda : \mathbb{R}^k \to \mathbb{R}$ exists such that $f = \exp(i\lambda)$. When, for all ϕ, $f(\phi) = 1$ then, for all ϕ, a function $N_2 : \mathbb{R}^k \to \mathbb{Z}$ exists such that $\lambda(\phi) = 2\pi N_2(\phi)$. Because λ is continuous and because $\mathbb{Z}$ is a discrete subspace of $\mathbb{R}$, N_2 is continuous w.r.t. the discrete topology of $\mathbb{Z}$, hence it is constant (recall that if $g : X \to Y$ is a continuous function, where X is connected and Y is discrete, then g is a constant function). If $\tilde{\lambda} : \mathbb{R}^k \to \mathbb{R}$ is an arbitrary continuous function such that $f = \exp(i\tilde{\lambda})$, then $\exp(i(\tilde{\lambda} - \lambda)) = 1$ hence, due to the above, an integer N_1 exists such that, for all ϕ, $\tilde{\lambda}(\phi) = \lambda(\phi) + 2\pi N_1$. □
Proof of Lemma B.1b: Let $f : \mathbb{R}^k \to U(1)$ be a continuous and 2π–periodic function with positive integer k. It follows, due to Lemma B.1a, that an integer $\hat{N}_i$ exists such that, for all ϕ, $\lambda(\phi + 2\pi e^i) - \lambda(\phi) = 2\pi\hat{N}_i$. Therefore the function $\hat{\lambda} : \mathbb{R}^k \to \mathbb{R}$, defined by $\hat{\lambda}(\phi) := \lambda(\phi) - \hat{N} \cdot \phi$, is continuous and 2π–periodic, where $\hat{N} := (\hat{N}_1, ..., \hat{N}_k)$. That $\hat{N}$ is uniquely determined by f follows by, applying again, Lemma B.1a. □
Proof of Lemma B.1c: Let R be in $SO(3)$. When $Re^3 = e^3$, then also $R^T e^3 = e^3$ so that R is of the simple form

$$R = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{C.1}$$

with real numbers a, b, c and d. Because R is in $SO(3)$, it follows from (C.1) that $y \in \mathbb{R}$ exists such that $R = \exp(y\mathcal{J})$, if $Re^3 = e^3$. □
Proof of Lemma B.1d: Let $u : \mathbb{R}^k \to SO(3)$ be a continuous and 2π–periodic function such that $ue^3 = e^3$. Then by Lemma B.1c, for all ϕ, $u(\phi) = \exp(y(\phi)\mathcal{J})$, where $y(\phi)$ is real. Then the function $f : \mathbb{R}^k \to U(1)$, defined by $f(\phi) := \exp(iy(\phi))$, is continuous and 2π–periodic

and satisfies

$$\begin{pmatrix} \Re e\{f\} & -\Im m\{f\} & 0 \\ \Im m\{f\} & \Re e\{f\} & 0 \\ 0 & 0 & 1 \end{pmatrix} = u \, . \tag{C.2}$$

Then, due to Lemma B.1b, a continuous and 2π–periodic function $\hat{\lambda} : \mathbb{R}^k \to \mathbb{R}$ exists such that, for all ϕ, $f(\phi) = \exp(i[\hat{\lambda}(\phi) + \hat{N} \cdot \phi])$, where $\hat{N} \in \mathbb{Z}^k$. Thus (C.2) yields, for all ϕ, $u(\phi) = \exp(\mathcal{J}[\hat{\lambda}(\phi) + \hat{N} \cdot \phi])$. □

D Proof of Theorem 7.1

We first need the following

Lemma D.1 *Let $\Xi(\phi_0)$ be nonempty for $\phi_0 = \xi$ and let ν be in $\Xi(\xi)$. Then $[\nu] \subset \Xi(\xi)$.*

Proof of Lemma D.1: Let $\nu \in \Xi(\xi)$. By the assumptions we have a ω–quasiperiodic UPF V starting at ξ with UPR ν so that V has a generator $v =: [v^1, \ v^2, \ v^3]$ which is a continuous and 2π–periodic function $v : \mathbb{R}^d \to \mathbb{R}^{3\times 3}$ which satisfies $V(n) = v(2\pi n\omega)$. Let $\hat{\nu}$ be in $[\nu]$ so that $\varepsilon \in \{-1, 1\}, j_0 \in \mathbb{Z}, j \in \mathbb{Z}^d$ exist such that $\hat{\nu} = \varepsilon\nu + j_0 + j \cdot \omega$. We now define the continuous and 2π–periodic function $\hat{v} : \mathbb{R}^d \to \mathbb{R}^{3\times 3}$ by $\hat{v}(\phi) := [v^1(\phi), \varepsilon v^2(\phi), \varepsilon v^3(\phi)] \exp(-\mathcal{J} j \cdot \phi)$ and we obtain, for every integer n,

$$\begin{aligned} &\hat{v}^T(2\pi n\omega) M(n;\xi) \hat{v}(0) = \hat{v}^T(2\pi n\omega) v(2\pi n\omega) \exp(2\pi n\nu\mathcal{J}) v^T(0) \hat{v}(0) \\ &= \exp(2\pi n(\hat{\nu} - j_0)\mathcal{J}) = \exp(2\pi n\hat{\nu}\mathcal{J}) \, , \end{aligned} \tag{D.1}$$

where in the last equation we used that j_0 is an integer. Because, for every integer n, $v(2\pi n\omega)$ is in $SO(3)$ the same holds for $\hat{v}(2\pi n\omega)$. Thus we conclude from (D.1) that

$$M(n;\xi) = \hat{v}(2\pi n\omega) \exp(2\pi n\hat{\nu}\mathcal{J}) \hat{v}^T(0) \, . \tag{D.2}$$

Equation (D.2) implies that $\hat{v}(2\pi n\omega)$ is a ω–quasiperiodic UPF starting at ξ and has UPR $\hat{\nu}$. Therefore we have shown that $\hat{\nu} \in \Xi(\xi)$ so that $[\nu] \subset \Xi(\xi)$. □

We are now equipped to prove Theorem 7.1.

Proof of Theorem 7.1a: Let U be a ω–quasiperiodic UPF starting at ξ with UPR ν. Then, for all integers n,

$$M(n;\xi) = U(n) \exp(\mathcal{J}\nu 2\pi n) U^T(0) \, . \tag{D.3}$$

With

$$\Delta_\pm := \frac{1}{2} \begin{pmatrix} 1 & \pm i & 0 \\ \mp i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad \Delta_0 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{D.4}$$

we have

$$\exp(\mathcal{J}\nu 2\pi n) = \Delta_+ \exp(2\pi i n\nu) + \Delta_- \exp(-2\pi i n\nu) + \Delta_0 \, . \tag{D.5}$$

By (D.3) and (D.5) we have, for all n,

$$M(n;\xi) = U(n)\Big(\Delta_+ \exp(2\pi i n\nu) + \Delta_- \exp(-2\pi i n\nu) + \Delta_0\Big) U^T(0)$$
$$=: \mathcal{M}_+(n) + \mathcal{M}_-(n) + \mathcal{M}_0(n)\ . \qquad \text{(D.6)}$$

By the definition of $\mathcal{M}_+$ it is clear that if $\lambda \in \Lambda(\mathcal{M}_+)$, then

$$\lfloor \lambda - \nu \rfloor \in \Lambda\Big(U\Delta_+ U^T(0)\Big)\ . \qquad \text{(D.7)}$$

Because U is ω–quasiperiodic, $U\Delta_+ U^T(0)$ is ω–quasiperiodic so that, by Corollary A.2c, $\Lambda(U\Delta_+U^T(0)) \subset Y_\omega$. Thus with (D.7), $\lambda \in \{\nu + \mu : \mu \in Y_\omega\}$ so that we have shown that

$$\Lambda(\mathcal{M}_+) \subset \{\nu + \mu : \mu \in Y_\omega\}\ . \qquad \text{(D.8)}$$

In the same way one can show that

$$\Lambda(\mathcal{M}_-) \subset \{-\nu + \mu : \mu \in Y_\omega\}\ , \qquad \Lambda(\mathcal{M}_0) \subset Y_\omega\ . \qquad \text{(D.9)}$$

Because $\mathcal{M}_+, \mathcal{M}_-$ and $\mathcal{M}_0$ are quasiperiodic functions it follows, by (D.6), that

$$\Lambda\Big(M(*;\xi)\Big) = \Lambda(\mathcal{M}_+ + \mathcal{M}_- + \mathcal{M}_0) \subset \Lambda(\mathcal{M}_+) \cup \Lambda(\mathcal{M}_-) \cup \Lambda(\mathcal{M}_0)\ , \qquad \text{(D.10)}$$

where in the inequality we used Corollary A.2b. From (D.8), (D.9) and (D.10) it follows that $\Lambda\Big(M(*;\xi)\Big) \subset \{(\varepsilon\nu + \lambda) \in [0,1) : \varepsilon \in \{0,1,-1\}, \lambda \in Y_\omega\}$
$= (Y_\omega \cap [0,1)) \cup \{(\varepsilon\nu + \lambda) \in [0,1) : \varepsilon \in \{1,-1\}, \lambda \in Y_\omega\}$
$= (Y_\omega \cap [0,1)) \cup [\nu] \subset Y_\omega \cup [\nu]$. We conclude that $\Lambda\Big(M(*;\xi)\Big) \setminus Y_\omega \subset [\nu] \setminus Y_\omega \subset [\nu] \subset \Xi(\xi)$, where in the last inequality we used Lemma D.1. □

Proof of Theorem 7.1b: Let U_1 and U_2 be ω–quasiperiodic UPF's starting at ξ with UPR's ν_1 and ν_2. Then, for all integers n,

$$M(n;\xi) = U_1(n)\exp(\mathcal{J}\nu_1 2\pi n)U_1^T(0) = U_2(n)\exp(\mathcal{J}\nu_2 2\pi n)U_2^T(0)\ ,$$

i.e.

$$\tilde{M}(n;\xi) := \exp(\mathcal{J}\nu_2 2\pi n) = U_2^T(n)U_1(n)\exp(\mathcal{J}\nu_1 2\pi n)U_1^T(0)U_2(0)\ . \qquad \text{(D.11)}$$

Of course, $\tilde{M}$ is the spin transfer matrix of the spin–orbit system $(\exp(\mathcal{J}\nu_2 2\pi), \omega)$. The trick of the proof is to interprete (D.11) in a convenient way. In fact, due to (D.11), the spin–orbit system $(\exp(\mathcal{J}\nu_2 2\pi), \omega)$ has a ω–quasiperiodic UPF equal to $U_2^T U_1$ (with UPR ν_1). It then follows by Theorem 7.1a that the spectrum of $\tilde{M}$, i.e. the set $\{\nu_2, -\nu_2\} \cup \mathbb{Z}$, is a subset of $Y_\omega \cup [\nu_1]$. Therefore $\nu_1 \sim \nu_2$. We thus have shown that $\Xi(\xi) \subset [\nu]$. Applying Theorem 7.1a yields $[\nu] = \Xi(\xi)$. □

Proof of Theorem 7.1c1: Let the spin–orbit system be well–tuned. Then, by Theorem 7.1b, $\Xi = [\nu]$ where ν is a spin tune. Thus all elements of Ξ are equivalent w.r.t. $\sim$. Let all $\Xi(\phi_0)$ be nonempty and let all elements of Ξ be equivalent w.r.t. $\sim$. Thus, if $\nu \in \Xi(\phi_0)$ for a ϕ_0 then $\Xi \subset [\nu] = \Xi(\phi_0)$, where in the equality we used Theorem 7.1b. Since, trivially, $\Xi(\phi_0) \subset \Xi$ we conclude that $\Xi = \Xi(\phi_0)$, i.e. all $\Xi(\phi_0)$ are equal. This proves the first claim. The second claim trivially follows from Theorem 7.1b. □

Proof of Theorem 7.1c2: The third claim follows from Theorem 7.1b and the fact that each equivalence class w.r.t. $\sim$ has at most countably many elements. To prove the fourth claim we conclude from the third claim that Ξ has at most countably many elements if the spin–orbit system is well–tuned. The fourth claim now follows by contraposition. The fifth and final claim follows trivially from the definition of the equivalence relation $\sim$. □

Proof of Theorem 7.1d: Let U be a ω–quasiperiodic UPF starting at ξ with UPR ν and let ν be not in Y_ω. We define

$$F(n) := U(n)\exp(\mathcal{J}\nu 2\pi n)(e^1 - ie^2) = \exp(i\nu 2\pi n)U(n)(e^1 - ie^2) \; . \qquad \text{(D.12)}$$

Due to Corollary A.2a, $\Lambda(U(e^1 - ie^2))$ is nonempty so let y be in $\Lambda(U(e^1 - ie^2))$. Thus, by (D.12), $\lambda := \lfloor \nu + y \rfloor$ is in $\Lambda(F)$. Because of Corollary A.2c, $y \in Y_\omega$ and because $\nu \notin Y_\omega$ we obtain that $\lambda \notin Y_\omega$. Note that $U\exp(\mathcal{J}\nu 2\pi *)e^1$ and $-iU\exp(\mathcal{J}\nu 2\pi *)e^2$ are quasiperiodic functions so that, by (D.12) and Corollary A.2b,

$$\begin{aligned}
\Lambda(F) &\subset \Lambda\Big(U\exp(\mathcal{J}\nu 2\pi *)e^1\Big) \cup \Lambda\Big(-iU\exp(\mathcal{J}\nu 2\pi *)e^2\Big) \\
&= \Lambda\Big(U\exp(\mathcal{J}\nu 2\pi *)e^1\Big) \cup \Lambda\Big(U\exp(\mathcal{J}\nu 2\pi *)e^2\Big) \\
&= \Lambda\Big(M(*;\xi)U(0)e^1\Big) \cup \Lambda\Big(M(*;\xi)U(0)e^2\Big) \; ,
\end{aligned}$$

i.e.

$$\lambda \in \Lambda\Big(M(*;\xi)U(0)e^1\Big) \cup \Lambda\Big(M(*;\xi)U(0)e^2\Big) \; . \qquad \text{(D.13)}$$

Because, by Corollary A.2a, $\mathfrak{M}(M(*;\xi)) = \mathbb{R}$, we have that $a(M(*;\xi),\lambda)$ exists so that $a(M(*;\xi)U(0)e^1,\lambda) = a(M(*;\xi),\lambda)U(0)e^1$ and $a(M(*;\xi)U(0)e^2,\lambda) = a(M(*;\xi),\lambda)U(0)e^2$. Thus (D.13) implies that $a(M(*;\xi),\lambda) \neq 0$, i.e. $\lambda \in \Lambda\Big(M(*;\xi)\Big)$. Because $\lambda \notin Y_\omega$ we conclude from Theorem 7.1a that $\lambda \in \Xi(\xi)$. Therefore $\Lambda(M(*;\xi)) \cap \Xi(\xi) \neq \emptyset$. □

Proof of Theorem 7.1e: Let Ξ have uncountably many elements. We first of all define $F := \{\phi_0 \in \mathbb{R}^d : \Xi(\phi_0) \not\subset Y_\omega\}$. Of course F is nonempty because otherwise each $\Xi(\phi_0)$ would be a subset of Y_ω which would lead to the result $\Xi \subset Y_\omega$ which is wrong because Y_ω has at most countably many elements. Because of Theorem 7.1d a function $f : F \to [0,1)$ exists such that, for every $\phi_0 \in F$, $f(\phi_0) \in \Lambda(M(*;\phi_0)) \cap \Xi(\phi_0)$. To prove that the range $f(F)$ of f has uncountably many elements we now assume that it doesn't. Thus F has a nonempty subset F' which has at most countably many elements and such that $f(F) = f(F')$. We then compute $\bigcup_{\phi_0 \in F} \Xi(\phi_0) = \bigcup_{\phi_0 \in F}[f(\phi_0)] = \bigcup_{\phi_0 \in F'}[f(\phi_0)]$, where in the first equation we used Theorem 7.1b. Thus $\bigcup_{\phi_0 \in F} \Xi(\phi_0)$ has at most countably many elements. Because Ξ has uncountably many elements it then follows that $\bigcup_{\phi_0 \in \mathbb{R}^d \setminus F} \Xi(\phi_0)$ has uncountably many elements. However the latter set is a subset of Y_ω which is a contradiction. Therefore our assumption that $f(F)$ has at most countably many elements was wrong. Because $f(F)$ has uncountably many elements and because $f(F) \subset \bigcup_{\phi_0 \in \mathbb{R}^d} \Lambda(M(*;\phi_0))$ we obtain that $\bigcup_{\phi_0 \in \mathbb{R}^d} \Lambda(M(*;\phi_0))$ has uncountably many elements. To prove that no spin frequency exists we now assume that a spin frequency μ exists. Then every solution of (2.3) is (ω,μ)–quasiperiodic so that, by Corollary A.2c, $\bigcup_{\phi_0 \in \mathbb{R}^d} \Lambda(M(*;\phi_0)) \subset Y_{(\omega,\mu)}$. This implies that $\bigcup_{\phi_0 \in \mathbb{R}^d} \Lambda(M(*;\phi_0))$ has at most countably elements which is a contradiction. Thus our assumption that a spin frequency exists, was wrong. □

E Proof of Theorem 7.2

Proof of Theorem 7.2a: Because $S.$ is ω–quasiperiodic we have (see the proof of Theorem 4.1) a $\mathbb{S}^2$–valued generator u of S_* via $S_n = u(2\pi n\omega)$. Applying then Lemma A.1c, the m-th Fourier coefficient u_m of u reads as

$$u_m = a(S_*, m\cdot\omega) \ . \tag{E.1}$$

Because of Theorem 4.1 an ISF $\mathcal{S}$ exists and (see the proof of Theorem 4.1) $\mathcal{S}(0,*) = u(*-\xi)$. It follows that $S_n = \mathcal{S}(n, \xi + 2\pi n\omega)$. Using (E.1) and the equation $\mathcal{S}(0,*) = u(*-\xi)$, yields that $a(S, m\cdot\omega)\exp(-im\cdot\xi)$ is the m-th Fourier coefficient of $\mathcal{S}(0,*)$. □
Proof of Theorem 7.2b: Let $\phi_0 = \xi$ and let U be a ω–quasiperiodic UPF starting at ξ with UPR ν. Let S_* be a solution of (2.3). Then

$$S_n = \tilde{S}_+(n) + \tilde{S}_-(n) + \tilde{S}_0(n) \ , \tag{E.2}$$

where

$$\tilde{S}_+ := \mathcal{M}_+ S_0 \ , \qquad \tilde{S}_- := \mathcal{M}_- S_0 \ , \qquad \tilde{S}_0 := \mathcal{M}_0 S_0 \ ,$$

with $\mathcal{M}_+, \mathcal{M}_-, \mathcal{M}_0$ given by (D.6).

First of all we assume that $0 \in [\nu]$. Then S_* is ω–quasiperiodic so that, by Lemma A.1c, the sequence S^N, defined by (7.14), converges uniformly on $\mathbb{Z}$ to S_* as $N \to \infty$. Thus in this case $\hat{S} = S$. In particular $\hat{S}$ is a ω–quasiperiodic solution of (2.3).

We now assume that $0 \notin [\nu]$. Because $\tilde{S}_+$ is quasiperiodic, $a(\tilde{S}_+, m\cdot\omega)$ is well–defined for every $m \in \mathbb{Z}^d$. We then obtain by (E.2) that

$$a(\tilde{S}_+, m\cdot\omega) = a(U\Delta_+ U^T(0)S_0, m\cdot\omega - \nu) \ . \tag{E.3}$$

Because $U\Delta_+ U^T(0)S_0$ is ω–quasiperiodic and because, in the present case, $(m\cdot\omega - \nu) \notin Y_\omega$ it follows, by Corollary A.2c, that $a(U\Delta_+ U^T(0)S_0, m\cdot\omega - \nu) = 0$ so that, by (E.3), $a(\tilde{S}_+, m\cdot\omega) = 0$. Analogously one shows that $a(\tilde{S}_-, m\cdot\omega) = 0$, i.e. we obtain that

$$a(\tilde{S}_+, m\cdot\omega) = a(\tilde{S}_-, m\cdot\omega) = 0 \ . \tag{E.4}$$

Because $\tilde{S}_0$ is ω–quasiperiodic we have, by Lemma A.1c, that τ^N, defined by

$$\tau^N(n) := \sum_{\substack{m\in Z^d \\ \|m\|\le N}} A_{N,m,d} \exp(2\pi i n m\cdot\omega) a(\tilde{S}_0, m\cdot\omega)$$
$$= \sum_{\substack{m\in Z^d \\ \|m\|\le N}} A_{N,m,d} \exp(2\pi i n m\cdot\omega) a(S_., m\cdot\omega) = S^N(n) , \tag{E.5}$$

converges uniformly on $\mathbb{Z}$ to $\tilde{S}_0$ as $N \to \infty$, where in the second equation we used (E.4) and where in the third equation we used (7.14). We thus have shown, for the present case, that S^N converges uniformly on $\mathbb{Z}$ to the function $\tilde{S}_0$ as $N \to \infty$. Thus, in the present case, $\hat{S} = \tilde{S}_0$. We now have to show that $\tilde{S}_0$ is a solution of (2.3). Because $\tilde{S}_0(0) = U(0)\Delta_0 U^T(0) S_0$ we obtain

$$M(n,\xi)\tilde{S}_0(0) = U(n)\Big(\Delta_+ \exp(2\pi i n\nu) + \Delta_- \exp(-2\pi i n\nu) + \Delta_0\Big)\Delta_0 U^T(0) S_0$$
$$= U(n)\Delta_0^2 U^T(0) S_0 = U(n)\Delta_0 U^T(0) S_0 = \tilde{S}_0(n) , \tag{E.6}$$

where in the first equation we used (D.5) and the fact that U is a UPF starting at ξ with UPR ν. With (E.6) we have shown that $\tilde{S}_0$ is a (ω–quasiperiodic) solution of (2.3) (note that this even holds if $0 \in [\nu]$). We therefore have shown, for arbitrary values of ν, that the sequence S^N converges uniformly on $\mathbb{Z}$ as $N \to \infty$ to a ω–quasiperiodic solution $\hat{S}$ of (2.3).

To complete the proof we have to show that (7.15) holds. We first assume that $|\hat{S}| \neq 0$. Then $s := \hat{S}/|\hat{S}|$ is a normalized solution of (2.3). It follows, by the proof of Theorem 4.1, that an ISF $\mathcal{S}$ exists such that, for all integers n, $s(n) = \mathcal{S}(n, \xi + 2\pi n\omega)$. Multiplying this equation by $|\hat{S}|$ yields (7.15). If $|\hat{S}| = 0$, then (7.15) holds trivially. □

Proof of Theorem 7.2c: Let ν be in $\Xi(\phi_0)$ for $\phi_0 = \xi$. It then follows by the proof of Theorem 7.1b that a uniform IFF v exists such that (8.5) holds for all ϕ so that, by Remark 3 and for all ϕ_0, U_{ϕ_0} is a ω–quasiperiodic UPF starting at ϕ_0 with UPR ν, where $U_{\phi_0}(n) := v(0, \phi_0 + 2\pi n\omega)$. By Corollary A.2a we have, for all ϕ_0, $\mathfrak{M}(U_{\phi_0}) = \mathbb{R}$ so that $a(U_{\phi_0}, \lambda)$ exists. Of course, for all ϕ_0 and n,

$$M(n;\phi_0) = U_{\phi_0}(n) \exp(\mathcal{J}\nu 2\pi n) U^T_{\phi_0}(0)$$
$$= U_{\phi_0}(n)\Big(\Delta_+ \exp(2\pi i n\nu) + \Delta_- \exp(-2\pi i n\nu) + \Delta_0\Big) U^T_{\phi_0}(0) , \tag{E.7}$$

where in the second equation we used (D.5). Because $U_{\phi_0}\Delta_+ \exp(2\pi i\nu *) U^T_{\phi_0}(0)$ is quasiperiodic we obtain, for real λ and every ϕ_0,

$$a\Big(U_{\phi_0}\Delta_+ \exp(2\pi i\nu *) U^T_{\phi_0}(0), \lambda\Big) = a\Big(U_{\phi_0} \exp(2\pi i\nu *), \lambda\Big)\Delta_+ U^T_{\phi_0}(0)$$
$$= a(U_{\phi_0}, \lambda - \nu)\Delta_+ U^T_{\phi_0}(0) , \tag{E.8}$$

and analogously

$$\begin{aligned} &a\Big(U_{\phi_0}\Delta_- \exp(-2\pi i\nu *) U^T_{\phi_0}(0), \lambda\Big) = a(U_{\phi_0}, \lambda + \nu)\Delta_- U^T_{\phi_0}(0) , \\ &a\Big(U_{\phi_0}\Delta_0 U^T_{\phi_0}(0), \lambda\Big) = a(U_{\phi_0}, \lambda)\Delta_0 U^T_{\phi_0}(0) . \end{aligned} \tag{E.9}$$

It follows from (E.7),(E.8) and (E.9), for real λ and every ϕ_0, that

$$a\Big(M(*;\phi_0),\lambda\Big) = a(U_{\phi_0},\lambda-\nu)\Delta_+U^T_{\phi_0}(0) + a(U_{\phi_0},\lambda+\nu)\Delta_-U^T_{\phi_0}(0)$$
$$+a(U_{\phi_0},\lambda)\Delta_0U^T_{\phi_0}(0)\ . \tag{E.10}$$

Because, as will be shown below, $a(U_{\phi_0},\lambda)$ is continuous in ϕ_0 we conclude from (E.10) that, for every real λ, $a\Big(M(*;\phi_0),\lambda\Big)$ is continuous in ϕ_0. We also have, for all integers k and real λ,

$$\exp(2\pi ik\lambda)a\Big(M(*;\xi),\lambda\Big) = a\Big(M(*+k;\xi),\lambda\Big) = a\Big(M(*;\xi+2\pi k\omega)M(k;\xi),\lambda\Big)$$
$$= a\Big(M(*;\xi+2\pi k\omega),\lambda\Big)M(k;\xi)\ , \tag{E.11}$$

where in the second equation we used (2.11). Taking the Euclidean norm of (E.11) we obtain that

$$|a\Big(M(*;\xi),\lambda\Big)| = |\exp(2\pi ik\lambda)a\Big(M(*;\xi),\lambda\Big)| = |a\Big(M(*;\xi+2\pi k\omega),\lambda\Big)M(k;\xi)|$$
$$\le \sqrt{3}|a\Big(M(*;\xi+2\pi k\omega),\lambda\Big)|\ , \tag{E.12}$$

where in the inequality we used that $M(k;\xi)$ is in $SO(3)$. Thus for all ϕ in $\{\xi+2\pi k\omega+2\pi m : m\in\mathbb{Z}^d, k\in\mathbb{Z}\}$ we have that

$$|a\Big(M(*;\phi),\lambda\Big)| \ge \frac{1}{\sqrt{3}}|a\Big(M(*;\xi),\lambda\Big)|\ . \tag{E.13}$$

Because (E.13) holds for a dense set of ϕ's and because $a\Big(M(*;\phi),\lambda\Big)$ is continuous in ϕ we conclude that (E.13) holds for all ϕ. If a λ is in $\Lambda\Big(M(*;\xi)\Big)$ then $|a\Big(M(*;\xi),\lambda\Big)| > 0$ so that, by (E.13) and for every ϕ, $|a\Big(M(*;\phi),\lambda\Big)| > 0$, i.e. $\lambda\in\Lambda\Big(M(*;\phi)\Big)$. Thus, for every ϕ,

$$\Lambda\Big(M(*;\xi)\Big) \subset \Lambda\Big(M(*;\phi)\Big)\ . \tag{E.14}$$

Because, by Theorem 7.1b, every $\Xi(\phi_0)$ is nonempty, our argumentation leading to (E.14) can be repeated for arbitrary ξ, i.e. (E.14) holds for every $\xi\in\mathbb{R}^d$. Thus under the assumption that $a(U_{\phi_0},\lambda)$ is continuous in ϕ_0, we have shown that all $\Lambda\Big(M(*;\phi)\Big)$ are equal.

Thus it remains to be proven that this assumption is true. If λ is not in Y_ω then, by Corollary A.2c, $a(U_{\phi_0},\lambda)=0$ so that in this case the continuity in ϕ_0 is obvious. We thus have to consider only the case when $\lambda \in Y_\omega$, i.e. $\lambda = m \cdot \omega + m_0$ with $m \in \mathbb{Z}^d, m_0 \in \mathbb{Z}$. We first define, for every ϕ_0, the continuous and 2π–periodic function $g(*;\phi_0) : \mathbb{R}^d \to SO(3)$ by $g(\phi;\phi_0) := v(0, \phi_0 + \phi)$. We obtain, for all ϕ_0, that

$$\begin{aligned} a(U_{\phi_0}, m\cdot\omega + m_0) &= a(U_{\phi_0}, m\cdot\omega) = a\Big(g(2\pi\omega *;\phi_0), m\cdot\omega\Big) = g_m(*;\phi_0) \\ &= \exp(im\cdot\phi_0) g_m(*;0)\ , \end{aligned} \tag{E.15}$$

where in the third and fifth equations we used Lemma A.1c and where $g_m(*;\phi_0)$ denotes the m–th Fourier coefficient of $g(*;\phi_0)$. By (E.15), $a(U_{\phi_0}, m\cdot\omega + m_0)$ is continuous in ϕ_0 which completes the proof. Note that (E.15) implies that $a(U_{\phi_0}, m\cdot\omega+m_0) = \exp(im\cdot\phi_0)a(U_0, m\cdot\omega + m_0)$. □

F Example 3

In this appendix we investigate various aspects of the spin–orbit system of Example 3. We denote this system by $(\mathfrak{a}^\epsilon, 1/2)$. We begin by checking whether it has n–turn invariant spin fields and n–turn invariant frame fields. Then the quasiperiodicity of the solutions of (2.3) is investigated. We finish by looking at the stroboscopic sequences of polarization fields.

F.1 n–turn fields

If $\mathcal{S}$ is an ISF then by Remark 1 in Sec. 4, $\mathcal{S}$ is a 2-turn invariant spin field hence with (4.4) $\mathcal{S}(0,*)$ satisfies the eigenproblem $M^\epsilon(2;*)\mathcal{S}(0,*) = \mathcal{S}(0,*)$, where M^ϵ denotes the spin transfer matrix of $(\mathfrak{a}^\epsilon, 1/2)$. Thus we start by searching for all 2-turn invariant spin fields. For this we need the 2–turn spin transfer matrix and we note that from (2.21) and (2.22)

$$\mathfrak{a}^\epsilon(*+\pi) = \begin{pmatrix} 1-2c^2 & -2bc & 2ac \\ -2bc & 1-2b^2 & 2ab \\ -2ac & -2ab & 2a^2-1 \end{pmatrix} . \tag{F.1}$$

From (2.21), (2.10) and (F.1) it follows that the 2–turn spin transfer matrix is

$$M^\epsilon(2;*) = \mathfrak{a}^\epsilon(*+\pi)\mathfrak{a}^\epsilon = \begin{pmatrix} 1-8c^2+8c^4 & 4bc(1-2c^2) & 4ac(1-2c^2) \\ -4bc(1-2c^2) & 1-8b^2c^2 & -8abc^2 \\ -4ac(1-2c^2) & -8abc^2 & 1-8a^2c^2 \end{pmatrix} . \tag{F.2}$$

The 2–turn eigenproblem can be written in the form

$$M^\epsilon(2;\phi) f(\phi) = f(\phi)\ , \tag{F.3}$$

where we aim to find a nonzero $f(\phi)$. Since $M^\epsilon(2;\phi) \in SO(3)$, it follows that $f(\phi)$ is an eigenvector with eigenvalue of zero of the skew–symmetric part of $M^\epsilon(2;*)$ which is

$$\Big(M^\epsilon(2;*) - (M^\epsilon)^T(2;*)\Big)/2 = 4c(1-2c^2)\begin{pmatrix} 0 & b & a \\ -b & 0 & 0 \\ -a & 0 & 0 \end{pmatrix} , \tag{F.4}$$

why $M^\epsilon = I$?
$(M^\epsilon)^2 f = f$

So the eigenproblem of (F.3) is solved with

$$f \equiv 4c(1-2c^2)(0,a,-b)\ . \tag{F.5}$$

Multiplying this solution by a scalar gives another solution of (F.3) - in particular $f(\phi) \equiv h(\phi)$ satisfies (F.3), if h denotes the continuous and 2π–periodic function $h : \mathbb{R} \to \mathbb{R}^3$, defined by

$$h := (0, ab, -b^2)\ . \tag{F.6}$$

Note that, by (2.23) and (F.6),

$$|h| = |b|\sqrt{1-c^2}\ . \tag{F.7}$$

Since h is not $\mathbb{S}^2$–valued it is still not admissable (the same holds for the r.h.s. of (F.5)). We therefore normalize it to obtain $f(\phi) \equiv h(\phi)/|h(\phi)|$ and thereby get the required solution

$$\begin{aligned}\mathcal{N}(\phi) &:= \frac{h(\phi)}{|h(\phi)|}\\ &= \frac{\cos(\pi\epsilon/2)}{|\cos(\pi\epsilon/2)|\sqrt{1-\sin^2(\pi\epsilon/2)\cos^2(\phi)}}\Big(0, \sin(\pi\epsilon/2)\sin(\phi), -\cos(\pi\epsilon/2)\Big)\ ,\end{aligned} \tag{F.8}$$

to the eigenproblem (F.3) providing that $|h(\phi)| \neq 0$.

If $|\sin(\pi\epsilon/2)|$ neither equals 0 nor 1, i.e. if ϵ is not an integer, then $n(\phi)$ is well–defined and satisfies the relation

$$M^\epsilon(2;\phi)\mathcal{N}(\phi) = \mathcal{N}(\phi)\ , \tag{F.9}$$

for all ϕ. We must now consider two cases: Case 1 (ϵ is not an integer) and Case 2 (ϵ is an integer).

We begin with Case 1. Then $\mathcal{N}$, defined by (F.8), is a continuous and 2π–periodic function $\mathcal{N} : \mathbb{R} \to \mathbb{S}^2$ which satisfies the eigenproblem (F.9). It is now clear how to construct a 2–turn invariant spin field. In fact the spin field $\mathcal{S}$, defined by $\mathcal{S}(0,*) := \mathcal{N}$, is a 2–turn invariant spin field since (4.4) holds. Because the first component of $\mathcal{S}$ vanishes for all ϕ, $\mathcal{S}$ is never parallel to the radial direction. Then v, defined in terms of the radial unit vector e^1 by

$$v := [e^1 \times \mathcal{S}, e^1, \mathcal{S}]\ , \tag{F.10}$$

is a 2–turn invariant frame field. This will be useful later.

We will now show that exactly two 2-turn invariant spin fields exist namely $\pm\mathcal{N}$. However to do so we have to face the fact that $M^\epsilon(2;\phi)$ can be the unit matrix for some values of ϕ. Let $f : \mathbb{R} \to \mathbb{S}^2$ be a continuous and 2π–periodic function which satisfies (F.3) for all ϕ. Defining $M := \{\phi \in \mathbb{R} : M^\epsilon(2;\phi) = I\}$ and recalling (F.9), we see that for $\phi \in \mathbb{R} \setminus M$, $|f(\phi)\cdot\mathcal{N}(\phi)| = 1$. Note that, by (F.2), $M = \{\phi \in \mathbb{R} : c(\phi)(c^2(\phi)-1) = 0\}$ so that M consists only of isolated points. i.e., each point of M is contained in an open interval which contains no other point of M. Because $|f(\phi)\cdot\mathcal{N}(\phi)|$ is continuous in ϕ, it follows that $|f(\phi)\cdot\mathcal{N}(\phi)| = 1$

for every ϕ. Thus either $f = \mathcal{N}$ or $f = -\mathcal{N}$ so that $\mathcal{S}$ and $-\mathcal{S}$ (TRIVIAL?) are the only 2–turn invariant spin fields. We conclude from (2.21), (4.1) and (F.8) that

$$\mathcal{S}(1, * + \pi) = \mathfrak{a}^{\epsilon}\mathcal{S}(0, *) = -\mathcal{S}(0, * + \pi) . \tag{F.11}$$

With (4.3), (F.11) implies that neither $\mathcal{S}$ nor $-\mathcal{S}$ is an ISF. Thus we have shown that none of the 2–turn invariant spin fields is an ISF and we conclude that $(\mathfrak{a}^{\epsilon}, 1/2)$ has no ISF.

We now come to Case 2. Then ϵ is an integer so that, by (2.21), the 1–turn spin transfer matrix has the simple form $\mathfrak{a}^{\epsilon} = \begin{pmatrix} 1-2c^2 & 0 & 2ac \\ 0 & 1 & 0 \\ -2ac & 0 & 2a^2-1 \end{pmatrix}$, which implies that e^2 is an ISF. Thus $[e^3, e^1, e^2]$ is an IFF. If ϵ is even, $\mathfrak{a}^{\epsilon}$ represents a rotation of π around the vertical.

This completes the study of the matter of the existence the ISF and IFF and we can summarize the above by the following

Proposition F.1 a) The spin–orbit system $(\mathfrak{a}^{\epsilon}, 1/2)$ has an ISF, iff ϵ is an integer. If ϵ is an integer, then an IFF exists.
b) For every value of ϵ, $(\mathfrak{a}^{\epsilon}, 1/2)$ has a 2–turn invariant frame field. □

Odd / even integer?

Remark:

(1) We can generalize Case 1 by considering a spin–orbit system $(\mathfrak{a}, \omega)$ for which $d = 1$, and $\omega = p/q$, where $q > 0, p$ are integers so that the system is on orbital resonance. Of course, $(\mathfrak{a}^{\epsilon}, 1/2)$ is a special case of this. Recalling from (2.11) that

? Where is "a"? q = integer So it's OK

$$\mathfrak{a}M(q; *)\mathfrak{a}^T = M(q; * + 2\pi\omega) , \tag{F.12}$$

we see that if $\mathcal{S}$ is a q–turn invariant spin field,

Think.

$$M(q; *)\mathfrak{a}^T\mathcal{S}(0, * + 2\pi\omega) = \mathfrak{a}^T M(q; * + 2\pi\omega)\mathcal{S}(0, * + 2\pi\omega) = \mathfrak{a}^T\mathcal{S}(q, * + 2\pi\omega(q+1)) = \mathfrak{a}^T\mathcal{S}(0, * + 2\pi\omega) . \tag{F.13}$$

This implies that the spin field $\tilde{\mathcal{S}}$, given by $\tilde{\mathcal{S}}(0, *) := \mathfrak{a}^T\mathcal{S}(0, * + 2\pi\omega)$, is a q–turn invariant spin field so that, if $\mathcal{S}$ and $-\mathcal{S}$ are the only q–turn invariant spin fields, then $\tilde{\mathcal{S}}$ is either equal to $\mathcal{S}$ or $-\mathcal{S}$, i.e. a ξ exists in $\{-1, 1\}$ such that

$$\mathfrak{a}\mathcal{S}(0, *) = \xi\mathcal{S}(0, * + 2\pi\omega) . \tag{F.14}$$

This is the situation of Case 1 and it is therefore no coincidence that the r.h.s. of (F.11) has the form $\xi\mathcal{S}(0, * + 2\pi\omega)$.

□

F.2

We now investigate the quasiperiodicity of the solutions of (2.3) and related questions. As in the previous section, we consider Case 1 and Case 2 separately.

We begin with Case 1, where ϵ is not an integer and search for an ω–quasiperiodic SPF. For this we consider the 2–turn invariant frame field v given by (F.10) where the 2–turn invariant spin field $\mathcal{S}$ is taken to be $\mathcal{S}(0,*) = \mathcal{N}$ with $\mathcal{N}$ given by (F.8). Because v is a frame field, it follows from Remark 1 in Sec. VI that U_{ϕ_0}, defined by $U_{\phi_0}(n) := v(n, \phi_0+\pi n)$, is an SPF starting at ϕ_0, as we now show.

Because v is a 2–turn invariant frame field we have

$$\begin{aligned} U_{\phi_0}(2n) &= v(2n, \phi_0 + 2\pi n) = v(0, \phi_0) = U_{\phi_0}(0) \, , \\ U_{\phi_0}(2n+1) &= v(2n+1, \phi_0 + \pi(2n+1)) = v(1, \phi_0 + \pi) = U_{\phi_0}(1) \, . \end{aligned} \tag{F.15}$$

arbitrary integer n. Also, by (F.10) and (F.11) we have

$$\begin{aligned} v(1, \phi_0+\pi) &= [e^1 \times \mathcal{S}(1, \phi_0+\pi), e^1, \mathcal{S}(1, \phi_0+\pi)] \\ &= [-e^1 \times \mathcal{S}(0, \phi_0+\pi), e^1, -\mathcal{S}(0, \phi_0+\pi)] \\ &= [e^1 \times \mathcal{S}(0, \phi_0+\pi), e^1, \mathcal{S}(0, \phi_0+\pi)] \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ &= v(0, \phi_0+\pi) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} =: v(0, \phi_0+\pi)\hat{\mathcal{J}} \, . \end{aligned} \tag{F.16}$$

It follows from (F.15) and (F.16) that for all n

$$U_{\phi_0}(n) = v(0, \phi_0+\pi n) \begin{pmatrix} \cos(\pi n) & 0 & -\sin(\pi n) \\ 0 & 1 & 0 \\ \sin(\pi n) & 0 & \cos(\pi n) \end{pmatrix} = v(0, \phi_0+\pi n)\hat{\mathcal{J}}^n \, , \tag{F.17}$$

so that U_{ϕ_0} is generated by u_{ϕ_0}, defined by $u_{\phi_0}(\phi) := v(0, \phi_0+\phi) \begin{pmatrix} \cos(\phi) & 0 & -\sin(\phi) \\ 0 & 1 & 0 \\ \sin(\phi) & 0 & \cos(\phi) \end{pmatrix}$,
i.e., $U_{\phi_0}(n) = u_{\phi_0}(\pi n)$. Thus U_{ϕ_0} is an ω–quasiperiodic SPF starting at ϕ_0.

We now check whether this SPF is a UPF by applying Remark 1 in Sec. VI. Thus, for every ϕ_0 and every integer n we compute

$$\begin{aligned} & U^T_{\phi_0}(n+1)\mathfrak{a}^\epsilon(\phi_0+\pi n)U_{\phi_0}(n) \\ & \quad = \hat{\mathcal{J}}^{n+1} v^T(0, \phi_0+\pi(n+1))\mathfrak{a}^\epsilon(\phi_0+\pi n)v(0, \phi_0+\pi n)\hat{\mathcal{J}}^n \, , \end{aligned} \tag{F.18}$$

where we used (F.17). We also have

$$\begin{aligned} & v^T(0, *+\pi)\mathfrak{a}^\epsilon v(0,*) \\ & \quad = \begin{pmatrix} 0 & -n_3 & -n_2 \\ 1 & 0 & 0 \\ 0 & -n_2 & n_3 \end{pmatrix} \begin{pmatrix} 1-2c^2 & 2bc & 2ac \\ 2bc & 1-2b^2 & -2ab \\ -2ac & 2ab & 2a^2-1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -n_3 & 0 & n_2 \\ n_2 & 0 & n_3 \end{pmatrix} \\ & = \begin{pmatrix} 2c^2-1 & \frac{\sqrt{2}bc\sqrt{1-c}}{|\cos(\pi\epsilon/2)|} & 0 \\ \frac{\sqrt{2}bc\sqrt{1-c}}{|\cos(\pi\epsilon/2)|} & 1-2c^2 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \end{aligned} \tag{F.19}$$

where in the second equality we used (2.22). From (F.18) and (F.19) it follows that for every ϕ_0 and every integer n,

$$
\begin{aligned}
&U^T_{\phi_0}(n+1)\mathfrak{a}^\epsilon(\phi_0+\pi n)U_{\phi_0}(n) \\
&= \hat{\mathcal{J}}^{n+1}\begin{pmatrix} 2c^2(\phi_0+\pi n)-1 & \frac{\sqrt{2}b(\phi_0+\pi n)c(\phi_0+\pi n)\sqrt{1-c(\phi_0+\pi n)}}{|\cos(\pi\epsilon/2)|} & 0 \\ \frac{\sqrt{2}b(\phi_0+\pi n)c(\phi_0+\pi n)\sqrt{1-c(\phi_0+\pi n)}}{|\cos(\pi\epsilon/2)|} & 1-2c^2(\phi_0+\pi n) & 0 \\ 0 & 0 & -1 \end{pmatrix}\hat{\mathcal{J}}^n \\
&= \hat{\mathcal{J}}^{n+1}\begin{pmatrix} 2c^2(\phi_0)-1 & (-1)^n\frac{\sqrt{2}b(\phi_0)c(\phi_0)\sqrt{1-c(\phi_0)}}{|\cos(\pi\epsilon/2)|} & 0 \\ (-1)^n\frac{\sqrt{2}b(\phi_0)c(\phi_0)\sqrt{1-c(\phi_0)}}{|\cos(\pi\epsilon/2)|} & 1-2c^2(\phi_0) & 0 \\ 0 & 0 & -1 \end{pmatrix}\hat{\mathcal{J}}^n \\
&= \hat{\mathcal{J}}^{n+1}\begin{pmatrix} -2c^2(\phi_0)+1 & (-1)^n\frac{\sqrt{2}b(\phi_0)c(\phi_0)\sqrt{1-c(\phi_0)}}{|\cos(\pi\epsilon/2)|} & 0 \\ -(-1)^n\frac{\sqrt{2}b(\phi_0)c(\phi_0)\sqrt{1-c(\phi_0)}}{|\cos(\pi\epsilon/2)|} & 1-2c^2(\phi_0) & 0 \\ 0 & 0 & 1 \end{pmatrix}\hat{\mathcal{J}}^{n+1} \\
&= \begin{pmatrix} -2c^2(\phi_0)+1 & \frac{\sqrt{2}b(\phi_0)c(\phi_0)\sqrt{1-c(\phi_0)}}{|\cos(\pi\epsilon/2)|} & 0 \\ -\frac{\sqrt{2}b(\phi_0)c(\phi_0)\sqrt{1-c(\phi_0)}}{|\cos(\pi\epsilon/2)|} & 1-2c^2(\phi_0) & 0 \\ 0 & 0 & 1 \end{pmatrix} , \qquad \text{(F.20)}
\end{aligned}
$$

where in the second equality we used (2.22). Therefore $U^T_{\phi_0}(n+1)\mathfrak{a}^\epsilon(\phi_0+\pi n)U_{\phi_0}(n)$ is independent of n so that, by Remark 1 in Sec. VI, U_{ϕ_0} is a UPF starting at ϕ_0. Moreover, by using Remark 1 in Sec. VI again, we get

$$U^T_{\phi_0}(n+1)\mathfrak{a}^\epsilon(\phi_0+\pi n)U_{\phi_0}(n) = \exp(2\pi\nu_{\phi_0}\mathcal{J}) , \qquad \text{(F.21)}$$

for all integers n, where ν_{ϕ_0} denotes the UPR corresponding to U_{ϕ_0}. From (2.22), (F.20) and (F.21) it follows that, for every ϕ_0,

$$\cos(2\pi\nu_{\phi_0}) = 1-2c^2(\phi_0) = -1-8\sin^2(\pi\epsilon/2)\cos^2(\phi_0)\Big(\sin^2(\pi\epsilon/2)\cos^2(\phi_0)-1\Big) \qquad \text{(F.22)}$$

With (F.22) we see that Ξ has uncountably many elements so that, by Theorem 6.1d, the spin–orbit system $(\mathfrak{a}^\epsilon, 1/2)$ has no spin frequency and is therefore ill–tuned.

We now come to Case 2 so that ϵ is an integer and begin with an even ϵ. Then, by (2.21), the 1–turn spin transfer matrix has the simple form $\mathfrak{a}^\epsilon = \hat{\mathcal{J}}$, implying that $w := [e^3, e^1, e^2]$ is a uniform IFF. Then, by Remark 3 in Sec. VI, the spin–orbit system $(\mathfrak{a}^\epsilon, 1/2)$ is well–tuned.

If ϵ is an odd integer, then, by (2.21), the 1–turn spin transfer matrix has the simple form

$$\mathfrak{a}^\epsilon(\phi) = \begin{pmatrix} -\cos(4\phi) & 0 & -\sin(4\phi) \\ 0 & 1 & 0 \\ \sin(4\phi) & 0 & -\cos(4\phi) \end{pmatrix} , \qquad \text{(F.23)}$$

which implies that $w = [e^3, e^1, e^2]$ is an IFF and that

$$w^T(0, \phi + \pi)\mathfrak{a}^\epsilon(\phi)w(0, \phi) = \begin{pmatrix} -\cos(4\phi) & \sin(4\phi) & 0 \\ -\sin(4\phi) & -\cos(4\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp(-4\phi\mathcal{J}) \; . \tag{F.24}$$

Because w is an IFF and by (F.24) and the remarks after (8.4), for arbitrary n, ϕ_0 we have

$$w^T(0, \phi_0 + \pi n)M^\epsilon(n; \phi_0)w(0, \phi_0) = \exp(-4n\phi_0\mathcal{J}) \; . \tag{F.25}$$

Also, by Remark 2 in Sec. VI, T_{ϕ_0}, defined by $T_{\phi_0}(n) := w(0, \phi_0 + \pi n)$, is an ω–quasiperiodic SPF starting at ϕ_0. Because of (F.25) and Remark 3 in Sec. II the phase function μ_{ϕ_0} of T_{ϕ_0} reads as

$$\mu_{\phi_0}(n) = \exp(-4in\phi_0) =: \exp(2\pi i n\kappa_{\phi_0}) \; , \tag{F.26}$$

where $\kappa_{\phi_0} := \lfloor -\frac{2\phi_0}{\pi} \rfloor$. We conclude (recall Remark 1 in Sec. VI) that T_{ϕ_0} is an ω–quasiperiodic UPF starting at ϕ_0 with UPR κ_{ϕ_0}. From the special form of κ_{ϕ_0} we see that Ξ has uncountably many elements so that, by Theorem 6.1d, the spin–orbit system $(\mathfrak{a}^\epsilon, 1/2)$ has no spin frequency. Hence it is ill-tuned.

This completes the study of well–tuning and quasiperiodicity and we summarize the above by the following

Proposition F.2 a) The spin–orbit system $(\mathfrak{a}^\epsilon, 1/2)$ is well–tuned and has a uniform IFF if ϵ is an even integer. If ϵ is not an even integer then no spin frequency exists and in particular $(\mathfrak{a}^\epsilon, 1/2)$ is ill-tuned.
b) For every ϕ_0 and for every value of ϵ, the spin–orbit system $(\mathfrak{a}^\epsilon, 1/2)$ has, an ω–quasiperiodic UPF starting at ϕ_0. Thus, for all values of ϵ and ϕ_0, $\Xi(\phi_0)$ (and therefore $\hat{\Xi}(\phi_0)$) is nonempty, all solutions of (2.3) are quasiperiodic and a normalized ω–quasiperiodic solution of (2.3) exists. □

Proper UPF

F.3 Stroboscopic sequences

Another use of IFF

We now study the stroboscopic sequence defined in Section IV.3, for an arbitrary polarization field $\mathcal{P}$ with the aim of proving Proposition F.3, stated below. For brevity we only consider the subcase of Case 1 where $0 < \epsilon < 1/2$.

Let N be a positive integer. Then, by (2.11), $M^\epsilon(2N + 2; *) = M^\epsilon(2; *)M^\epsilon(2N; *)$ so that by induction in N,

$$M^\epsilon(2N; *) = (M^\epsilon(2; *))^N \; , \tag{F.27}$$

which, with (4.1) and (4.2) gives

Pola fields are 2π-periodic

$$\mathcal{P}(2N; *) = M^\epsilon(2N; *)\mathcal{P}(0; *) = (M^\epsilon(2; *))^N\mathcal{P}(0; *) \; , \tag{F.28}$$

$$\mathcal{P}(2N - 1; *) = \mathfrak{a}^\epsilon(* + \pi)\mathcal{P}(2N - 2; * + \pi) \; . \tag{F.29}$$

We begin by showing, among other things, that the stroboscopic average of $\mathcal{P}$ exists, i.e. that the stroboscopic sequence $\mathcal{P}^N(n, \phi)$ converges for every n and ϕ as $N \to \infty$. By Section 4.3 it is clear that this happens iff $\mathcal{P}^N(0, \phi)$ converges for every ϕ as $N \to \infty$.

As a first step we will show that $\mathcal{P}^{2N}(0,\phi)$ converges for every ϕ as $N \to \infty$. Since, by (F.28)

$$\mathcal{P}^{2N}(0,\phi) = \frac{1}{2N}\sum_{n=0}^{N-1}\Big(\mathcal{P}(2n;\phi) + \mathcal{P}(2n+1;\phi)\Big) , \tag{F.30}$$

where

$$\frac{1}{2N}\sum_{n=0}^{N-1} \mathcal{P}(2n;\phi) = \frac{1}{2N}\sum_{n=0}^{N-1} (M^{\epsilon}(2;\phi))^n \mathcal{P}(0;\phi) , \tag{F.31}$$

we must show that $(1/2N)\sum_{n=0}^{N-1} \mathcal{P}(2n;\phi)$ converges for every ϕ as $N \to \infty$. We compute the n-th powers of $M^{\epsilon}(2;\phi)$ by diagonalizing it as follows. By (F.19)

$$\mathfrak{a}^{\epsilon}(\phi) = v(0,\phi+\pi)M_1(\phi)v^T(0,\phi) , \tag{F.32}$$

where

$$M_1 := \begin{pmatrix} a_1 & a_2 & 0 \\ a_2 & -a_1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \qquad a_1 := 2c^2 - 1 , \qquad a_2 := \frac{\sqrt{2}bc\sqrt{1-c}}{|\cos(\pi\epsilon/2)|} . \tag{F.33}$$

It follows from (F.32) that

$$\mathfrak{a}^{\epsilon}(*+\pi) = v(0,*)M_1(*+\pi)v^T(0,*+\pi) , \tag{F.34}$$

so that with (F.2) and (F.32) and the $SO(3)$–property of v

$$M^{\epsilon}(2;*) = v(0,*)M_1(*+\pi)M_1 v^T(0,*) . \tag{F.35}$$

Since, by (2.22), we have

$$a_1(\phi+\pi) = a_1(\phi) , \qquad a_2(\phi+\pi) = -a_2(\phi) , \tag{F.36}$$

we obtain

$$M_1(*+\pi) = \begin{pmatrix} a_1 & -a_2 & 0 \\ -a_2 & -a_1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \tag{F.37}$$

so that

$$M_1(*+\pi)M_1(*) = \begin{pmatrix} a_3 & -a_4 & 0 \\ a_4 & a_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{F.38}$$

where

$$a_3 := a_1^2 - a_2^2 , \; a_4 := -2a_1a_2 . \tag{F.39}$$

With the unitary matrix

$$M_2 := \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} , \tag{F.40}$$

we have

$$M_2^\dagger M_1(*+\pi)M_1(*)M_2 = M_3 := \begin{pmatrix} M_5 & 0 & 0 \\ 0 & M_5^* & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{F.41}$$

where $M_5 := a_3 - ia_4$. Equation (F.35) then yields the desired diagonalization of $M^\epsilon(2;*)$, i.e.

$$M^\epsilon(2;*) = v(0,*)M_2 M_3 M_2^\dagger v^T(0,*) = M_4 M_3 M_4^\dagger , \tag{F.42}$$

where in the first equality we used the unitarity of M_2 and where for the second equality we define

$$M_4 := v(0,*)M_2 . \tag{F.43}$$

To complete the proof that $(1/2N)\sum_{n=0}^{N-1} \mathcal{P}(2n;\phi)$ converges we use (F.41) and (F.42) to write

$$(M^\epsilon(2;*))^n = M_4(M_3)^n M_4^\dagger = M_4 \begin{pmatrix} (M_5)^n & 0 & 0 \\ 0 & (M_5^*)^n & 0 \\ 0 & 0 & 1 \end{pmatrix} M_4^\dagger , \tag{F.44}$$

so that

$$\begin{aligned} \frac{1}{2N}\sum_{n=0}^{N-1} (M^\epsilon(2;*))^n &= M_4 \frac{1}{2N}\sum_{n=0}^{N-1} \begin{pmatrix} (M_5)^n & 0 & 0 \\ 0 & (M_5^*)^n & 0 \\ 0 & 0 & 1 \end{pmatrix} M_4^\dagger \\ &= M_4 \begin{pmatrix} \frac{1}{2N}\sum_{n=0}^{N-1} (M_5)^n & 0 & 0 \\ 0 & \frac{1}{2N}\sum_{n=0}^{N-1} (M_5^*)^n & 0 \\ 0 & 0 & 1/2 \end{pmatrix} M_4^\dagger . \end{aligned} \tag{F.45}$$

Since M_1 is in $SO(3)$, by (F.38) we have $a_3^2 + a_4^2 = 1$. Then by (F.40),

$$|M_5| = 1 . \tag{F.46}$$

Moreover, by (2.22), (F.33), (F.39) and (F.40) and noting that a_1, a_2, a_3 and a_4 are real, we have

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$$\begin{aligned} &\{\phi \in \mathbb{R} : M_5(\phi) = 1\} = \{\phi \in \mathbb{R} : a_3(\phi) = 1, a_4(\phi) = 0\} = \{\phi \in \mathbb{R} : a_1^2(\phi) = 1, a_2(\phi) = 0\} \\ &= \{\phi \in \mathbb{R} : b(\phi) = 0\} \cup \{\phi \in \mathbb{R} : c(\phi) = 0\} \cup \{\phi \in \mathbb{R} : c(\phi) = 1\} \\ &= \{\phi \in \mathbb{R} : b(\phi) = 0\} = \{\pi/2 + j\pi : j \in \mathbb{Z}\} , \end{aligned} \tag{F.47}$$

where in the fourth equality we used the fact that $0 < \epsilon < 1/2$. It follows from (F.47) that $M_5(\phi) = 1$ iff $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$. Thus if $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$, then

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} (M_5(\phi))^n = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} (M_5^*(\phi))^n = \frac{1}{2} , \tag{F.48}$$

and, if $\phi \notin \{\pi/2 + j\pi : j \in \mathbb{Z}\}$,

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} (M_5(\phi))^n = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} (M_5^*(\phi))^n = 0 , \tag{F.49}$$

where we used (F.46). We conclude from (F.45) and (F.48) that, if $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$,

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} (M^\epsilon(2;\phi))^n = \frac{I}{2} , \tag{F.50}$$

so that by (F.31) and for $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$,

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} \mathcal{P}(2n;\phi) = \frac{1}{2}\mathcal{P}(0;\phi) . \tag{F.51}$$

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Of course, since $M^\epsilon(2;\phi) = I$ for $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$, this result can be obtained immediately from (F.31).

STRAY POINTS IN FIGURE

However, for $\phi \notin \{\pi/2 + j\pi : j \in \mathbb{Z}\}$ and using (F.45) and (F.49)

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} (M^\epsilon(2;\phi))^n = M_4(\phi) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} M_4^\dagger(\phi) , \tag{F.52}$$

so that by (F.31),

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} \mathcal{P}(2n;\phi) = M_4(\phi) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} M_4^\dagger(\phi)\mathcal{P}(0;\phi) . \tag{F.53}$$

By (F.10), (F.40) and (F.43) and since $\mathcal{S}(0;*) = \mathcal{N}(*)$,

$$M_4 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} M_4^\dagger = v(0,*)M_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} M_2^\dagger v^T(0,*)$$

$$= v(0,*) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} v^T(0,*) = [0,0,\mathcal{S}(0,*)]v^T(0,*) = [0,0,\mathcal{N}(*)]v^T(0,*) , \tag{F.54}$$

so that with (F.53) and if $\phi \notin \{\pi/2 + j\pi : j \in \mathbb{Z}\}$,

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} \mathcal{P}(2n;\phi) = [0, 0, \frac{\mathcal{N}(\phi)}{2}] v^T(0,\phi)\mathcal{P}(0,\phi) = \frac{\mathcal{N}(\phi)}{2}\mathcal{N}(\phi) \cdot \mathcal{P}(0,\phi) , \quad \text{(F.55)}$$

where in the second equality we used (F.10). We have thus shown that $(1/2N)\sum_{n=0}^{N-1} \mathcal{P}(2n;\phi)$ converges for every ϕ as $N \to \infty$ and the limit is given by (F.51) and (F.55).

We now will show that $(1/2N)\sum_{n=0}^{N-1} \mathcal{P}(2n+1;\phi)$ converges for every ϕ as $N \to \infty$. We have, by (F.29),

$$\frac{1}{2N}\sum_{n=0}^{N-1} \mathcal{P}(2n+1;\phi) = \frac{1}{2N}\sum_{n=1}^{N} \mathcal{P}(2n-1;\phi) = \mathfrak{a}^{\epsilon}(\phi+\pi)\frac{1}{2N}\sum_{n=1}^{N} \mathcal{P}(2n-2;\phi+\pi)$$
$$= \mathfrak{a}^{\epsilon}(\phi+\pi)\frac{1}{2N}\sum_{n=0}^{N-1} \mathcal{P}(2n;\phi+\pi) . \quad \text{(F.56)}$$

It follows from (F.51) that, if $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$,

$$\lim_{N\to\infty} \frac{1}{2N}\sum_{n=0}^{N-1} \mathcal{P}(2n;\phi+\pi) = \frac{1}{2}\mathcal{P}(0;\phi+\pi) , \quad \text{(F.57)}$$

whence, by (F.56) and if $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$,

$$\lim_{N\to\infty} \frac{1}{2N}\sum_{n=0}^{N-1} \mathcal{P}(2n+1;\phi) = \frac{1}{2}\mathfrak{a}^{\epsilon}(\phi+\pi)\mathcal{P}(0;\phi+\pi) = \frac{1}{2}\hat{J}\mathcal{P}(0;\phi+\pi) , \quad \text{(F.58)}$$

where in the second equality we used (2.21)and (2.22) and the definition of $\hat{J}$ in (F.16). If $\phi \notin \{\pi/2 + j\pi : j \in \mathbb{Z}\}$ then $\phi + \pi \notin \{\pi/2 + j\pi : j \in \mathbb{Z}\}$ and, by (F.55),

$$\lim_{N\to\infty} \frac{1}{2N}\sum_{n=0}^{N-1} \mathcal{P}(2n;\phi+\pi) = \frac{\mathcal{N}(\phi+\pi)}{2}\mathcal{N}(\phi+\pi) \cdot \mathcal{P}(0,\phi+\pi) , \quad \text{(F.59)}$$

so that, by (F.56) and if $\phi \notin \{\pi/2 + j\pi : j \in \mathbb{Z}\}$,

$$\lim_{N\to\infty} \frac{1}{2N}\sum_{n=0}^{N-1} \mathcal{P}(2n+1;\phi) = \frac{1}{2}\mathfrak{a}^{\epsilon}(\phi+\pi)\mathcal{N}(\phi+\pi)\mathcal{N}(\phi+\pi) \cdot \mathcal{P}(0,\phi+\pi)$$
$$= -\frac{1}{2}\mathcal{N}(\phi)\mathcal{N}(\phi+\pi) \cdot \mathcal{P}(0,\phi+\pi) , \quad \text{(F.60)}$$

where in the second equality we used (F.11). We have thus shown that $(1/2N)\sum_{n=0}^{N-1} \mathcal{P}(2n+1;\phi)$ converges for every ϕ as $N \to \infty$ and that the limit is given by (F.58) and (F.60).

We can now make a first summary: $\mathcal{P}^{2N}(0,\phi)$ converges for every ϕ as $N \to \infty$ and if $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$, and with (F.30), (F.51) and (F.58), the limit is given by

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$$\bar{\mathcal{P}}(\frac{\pi}{2} + j\pi) := \lim_{N\to\infty} \mathcal{P}^{2N}(0, \frac{\pi}{2} + j\pi) = \frac{1}{2}\mathcal{P}(0; \frac{\pi}{2} + j\pi) + \frac{1}{2}\hat{J}\mathcal{P}(0; \frac{\pi}{2} + j\pi + \pi) , \quad \text{(F.61)}$$

Furthermore, if $\phi \notin \{\pi/2 + j\pi : j \in \mathbb{Z}\}$, and with (F.30), (F.55) and (F.60) the limit is given by

Depends on N

$$\begin{aligned}\bar{\mathcal{P}}(\phi) &:= \lim_{N\to\infty} \mathcal{P}^{2N}(0,\phi) \\ &= \frac{\mathcal{N}(\phi)}{2}\Big(\mathcal{N}(\phi)\cdot\mathcal{P}(0,\phi) - \mathcal{N}(\phi+\pi)\cdot\mathcal{P}(0,\phi+\pi)\Big)\,. \qquad \text{(F.62)}\end{aligned}$$

We can now show that $\mathcal{P}^N(0,\phi)$ converges for every ϕ as $N \to \infty$. We note, to begin, that $\mathcal{P}(0,*)$ is a bounded function since it is 2π–periodic and continuous. Thus with (4.2), a positive real constant a_5 exists such that, for all n and ϕ, we have $|\mathcal{P}(n,\phi)| \le a_5$. Then,

$$\begin{aligned}|\mathcal{P}^{2n+1}(0,\phi) - \mathcal{P}^{2n}(0,\phi)| &= |\frac{1}{2n+1}\sum_{k=0}^{2n}\mathcal{P}(k;\phi) - \frac{1}{2n}\sum_{k=0}^{2n-1}\mathcal{P}(k;\phi)| \\ &= |\frac{1}{2n+1}\mathcal{P}(2n;\phi) + (\frac{1}{2n+1} - \frac{1}{2n})\sum_{k=0}^{2n-1}\mathcal{P}(k;\phi)| \\ &\le |\frac{1}{2n+1}\mathcal{P}(2n;\phi)| + |(\frac{1}{2n+1} - \frac{1}{2n})\sum_{k=0}^{2n-1}\mathcal{P}(k;\phi)| \\ &\le \frac{1}{2n+1}|\mathcal{P}(2n;\phi)| + (\frac{1}{2n} - \frac{1}{2n+1})\sum_{k=0}^{2n-1}|\mathcal{P}(k;\phi)| \le \frac{a_5}{2n+1} + (1 - \frac{2n}{2n+1})a_5 \\ &= \frac{2a_5}{2n+1} \le \frac{a_5}{n}\,. \qquad \text{(F.63)}\end{aligned}$$

Since, as we have shown, $\mathcal{P}^{2N}(0,*)$ converges everywhere to $\bar{\mathcal{P}}(*)$ as $N \to \infty$, for every $\delta > 0$ and every ϕ, an integer $N_1(\phi)$ exists, such that, for all $n \ge N_1(\phi)$,

$$|\mathcal{P}^{2n}(0,\phi) - \bar{\mathcal{P}}(\phi)| \le \delta \le 2\delta\,. \qquad \text{(F.64)}$$

Clearly, since a_5/n converges to zero as $n \to \infty$, another integer N_2 exists such that, for all $n \ge N_2$, we have $a_5/n \le \delta$. Defining $N_3(\phi) := \max(N_1(\phi), N_2)$ we conclude that, if $n \ge N_3(\phi)$,

$$\begin{aligned}|\mathcal{P}^{2n+1}(0,\phi) - \bar{\mathcal{P}}(\phi)| &\le |\mathcal{P}^{2n+1}(0,\phi) - \mathcal{P}^{2n}(0,\phi)| + |\mathcal{P}^{2n}(0,\phi) - \bar{\mathcal{P}}(\phi)| \\ &\le \frac{a_5}{n} + \delta \le 2\delta\,, \qquad \text{(F.65)}\end{aligned}$$

where in the second inequality we used (F.63) and (F.64). We conclude from (F.64) and (F.65) that, if $k \ge 2N_3(\phi)$, then

$$|\mathcal{P}^k(0,\phi) - \bar{\mathcal{P}}(\phi)| \le 2\delta\,. \qquad \text{(F.66)}$$

and ϕ

Thus, since δ is arbitrary, we have shown with (F.66) that, for every ϕ, $\mathcal{P}^N(0,\phi)$ converges to $\bar{\mathcal{P}}(\phi)$ as $N \to \infty$.

We now investigate, the conditions on $\mathcal{P}$ which ensure that the limit function $\bar{\mathcal{P}}$ is continuous. Since $\mathcal{S}(0,\phi)$ and $\mathcal{P}(0,\phi)$ are continuous in ϕ we see by (F.62) that $\bar{\mathcal{P}}$ is continuous at every ϕ which is not in $\{\pi/2 + j\pi : j \in \mathbb{Z}\}$ and that $\bar{\mathcal{P}}$ converges at every $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$, i.e.

$$\begin{aligned}\lim_{\phi\to\pi/2+j\pi} \bar{\mathcal{P}}(\phi) &= \lim_{\phi\to\pi/2+j\pi} \frac{\mathcal{N}(\phi)}{2}\mathcal{N}(\phi)\cdot\mathcal{P}(0,\phi) \\ &-\frac{1}{2}\mathcal{N}(\phi)\mathcal{N}(\phi+\pi)\cdot\mathcal{P}(0,\phi+\pi) = \frac{\mathcal{N}(\pi/2+j\pi)}{2}\mathcal{N}(\pi/2+j\pi)\cdot\mathcal{P}(0,\pi/2+j\pi) \\ &-\frac{1}{2}\mathcal{N}(\pi/2+j\pi)\mathcal{N}(\pi/2+j\pi+\pi)\cdot\mathcal{P}(0,\pi/2+j\pi+\pi)\,, \end{aligned} \tag{F.67}$$

where in the first equality we used (F.62). Of course, $\bar{\mathcal{P}}$ is everywhere continuous iff, at every $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$, it is equal to its limit at those ϕ. In other words: $\bar{\mathcal{P}}$ is everywhere continuous iff, for every integer j,

$$\bar{\mathcal{P}}(\pi/2+j\pi) = \lim_{\phi\to\pi/2+j\pi} \bar{\mathcal{P}}(\phi)\,. \tag{F.68}$$

With (F.61) and (F.67) we have

$$\begin{aligned}\bar{\mathcal{P}}(\pi/2+j\pi) - \lim_{\phi\to\pi/2+j\pi:j} \bar{\mathcal{P}}(\phi) &= \frac{1}{2}\mathcal{P}(0;\pi/2+j\pi) + \frac{1}{2}\hat{J}\mathcal{P}(0;\pi/2+j\pi+\pi) \\ &-\frac{\mathcal{N}(\pi/2+j\pi)}{2}\mathcal{N}(\pi/2+j\pi)\cdot\mathcal{P}(0,\pi/2+j\pi) \\ &+\frac{1}{2}\mathcal{N}(\pi/2+j\pi)\mathcal{N}(\pi/2+j\pi+\pi)\cdot\mathcal{P}(0,\pi/2+j\pi+\pi)\,. \end{aligned} \tag{F.69}$$

Thus $\bar{\mathcal{P}}$ is everywhere continuous, iff, for every integer j, $\mathcal{P}(0,*)$ solves the following linear problem for $\mathcal{P}$

$$\begin{aligned} 0 &= \mathcal{P}(0;\pi/2+j\pi) + \hat{J}\mathcal{P}(0;\pi/2+j\pi+\pi) \\ &-\mathcal{N}(\pi/2+j\pi)\mathcal{N}(\pi/2+j\pi)\cdot\mathcal{P}(0,\pi/2+j\pi) \\ &+\mathcal{N}(\pi/2+j\pi)\mathcal{N}(\pi/2+j\pi+\pi)\cdot\mathcal{P}(0,\pi/2+j\pi+\pi)\,. \end{aligned} \tag{F.70}$$

By (F.8) and (F.16), we have

$$\begin{aligned} &\hat{J}\mathcal{P}(0;\pi/2+j\pi+\pi) + \mathcal{N}(\pi/2+j\pi)\mathcal{N}(\pi/2+j\pi+\pi)\cdot\mathcal{P}(0,\pi/2+j\pi+\pi) \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos^2(\pi\epsilon/2) & -\sin(\pi\epsilon/2)\cos(\pi\epsilon/2)(-1)^j \\ 0 & \sin(\pi\epsilon/2)\cos(\pi\epsilon/2)(-1)^j & -\sin^2(\pi\epsilon/2) \end{pmatrix} \mathcal{P}(0,\frac{\pi}{2}+j\pi+\pi) \\ &=: M_6^j\mathcal{P}(0,\frac{\pi}{2}+j\pi+\pi)\,, \end{aligned} \tag{F.71}$$

and

$$\begin{aligned} &\mathcal{N}(\pi/2+j\pi)\mathcal{N}(\pi/2+j\pi)\cdot\mathcal{P}(0,\pi/2+j\pi) - \mathcal{P}(0,\pi/2+j\pi) \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos^2(\pi\epsilon/2) & -\sin(\pi\epsilon/2)\cos(\pi\epsilon/2)(-1)^j \\ 0 & -\sin(\pi\epsilon/2)\cos(\pi\epsilon/2)(-1)^j & -\sin^2(\pi\epsilon/2) \end{pmatrix} \mathcal{P}(0,\frac{\pi}{2}+j\pi) \\ &=: M_7^j\mathcal{P}(0,\frac{\pi}{2}+j\pi)\,. \end{aligned} \tag{F.72}$$

It follows from (F.70), (F.71) and (F.72) that $\bar{\mathcal{P}}$ is continuous everywhere, iff, for every integer j,

$$M_6^j \mathcal{P}(0, \frac{\pi}{2} + j\pi + \pi) = M_7^j \mathcal{P}(0, \frac{\pi}{2} + j\pi) \,. \tag{F.73}$$

Since

$$M_7^j = -\hat{J} M_6^{j+1} \,, \tag{F.74}$$

with (F.73) we find that $\bar{\mathcal{P}}$ is continuous everywhere, iff, for every integer j,

$$M_6^j \mathcal{P}(0, \frac{\pi}{2} + j\pi + \pi) = -\hat{J} M_6^{j+1} \mathcal{P}(0, \frac{\pi}{2} + j\pi) \,. \tag{F.75}$$

Because $\mathcal{P}(0,\phi)$ is 2π–periodic in ϕ, we conclude that (F.75) holds for every integer j iff it holds just for $j=0$. Then $\bar{\mathcal{P}}$ is everywhere continuous, iff

$$M_6^0 \mathcal{P}(0, \frac{\pi}{2} + \pi) = -\hat{J} M_6^1 \mathcal{P}(0, \frac{\pi}{2}) \,. \tag{F.76}$$

By (F.71), (F.76) is equivalent to the requirement that

$$\cos(\pi\epsilon/2) e^2 \cdot \Big(\mathcal{P}(0, \frac{\pi}{2} + \pi) + \mathcal{P}(0, \frac{\pi}{2})\Big) = \sin(\pi\epsilon/2) e^3 \cdot \Big(\mathcal{P}(0, \frac{\pi}{2} + \pi) - \mathcal{P}(0, \frac{\pi}{2})\Big) \,,$$
$$0 = e^1 \cdot \Big(\mathcal{P}(0, \frac{\pi}{2} + \pi) - \mathcal{P}(0, \frac{\pi}{2})\Big) \,. \tag{F.77}$$

We thus have proved:

Proposition F.3 Let $0 < \epsilon < 1/2$ and let $\mathcal{P}$ be a polarization field. Then the following holds.

a) For every ϕ, $\mathcal{P}^N(0,\phi)$ converges to $\bar{\mathcal{P}}(\phi)$ as $N \to \infty$ where $\bar{\mathcal{P}}(\phi)$ is defined by (F.61) and (F.62).

b) $\bar{P}$ is everywhere continuous, iff (F.77) holds.

c) If the function $\mathcal{P}(0,*)$ is constant, then $\bar{P}$ is continuous everywhere iff $0 = e^2 \cdot \mathcal{P}(0,*)$. Furthermore, if $\mathcal{P}(0,*)$ is constant and $\bar{P}$ is continuous everywhere, then $\bar{P} = 0$. □

Remark:

(2) Due to Proposition F.3a and since the rhs of (F.62) is continuous in ϕ, $\bar{\mathcal{P}}(\phi)$ is, apart from the points $\phi = \pi/2 + k\pi$ (with integer k), a continuous function. Moreover, the first component of $\bar{\mathcal{P}}(\phi)$ vanishes when ϕ is different from $\phi = \pi/2 + k\pi$ with integer k. □

Figure 4 shows the components of the stroboscopic average $\bar{\mathcal{P}}(\phi)$ taken at each ϕ in the range $0 \le \lfloor \phi/2\pi \rfloor < 1$ with $\epsilon = 0.4$ with a constant starting spin field $\mathcal{P}(0,\phi)$ perpendicular to the plane of the ring, i.e., with components $(0,1,0)$. Except at $\lfloor \phi/2\pi \rfloor = 1/4$ and $3/4$, the components are continuous. That $\bar{\mathcal{P}}$ is not continuous is predicted by Proposition F.3c.

THEN SOME MORE ON THIS.

Figure 5 shows the components of the stroboscopic average $\bar{\mathcal{P}}(\phi)$ taken at each ϕ in the range $0 \le \lfloor \phi/2\pi \rfloor < 1$ with $\epsilon = 0.4$ and with the starting polarization field $\mathcal{P}(0, \phi) = (0, \cos^2(\phi), 0)$ perpendicular to the plane of the ring. That the components are continuous functions is predicted by Proposition F.3b.

The NSA is a DISCONT "ISF"? Proof?

The "ISF" (=NSA) where the 2-turn map = I is OK if the SA is OK.

MENTION EP?

THE OTHE SPECIAL EPSILONS WHERE THE MAP IS THE IDENTITY AT SOME PHI.

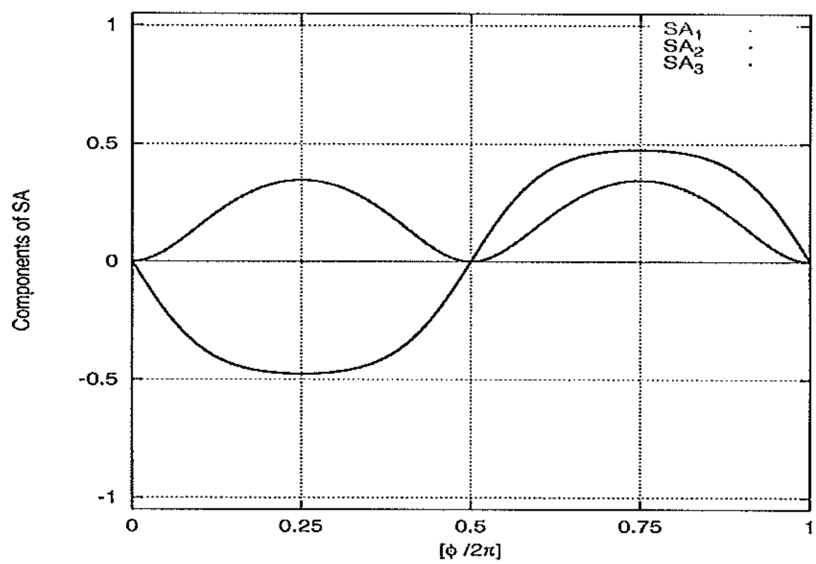

Figure 4: The three components of the stroboscopic average $\bar{\mathcal{P}}(\phi)$ vs. the fractional normalised phase $\lfloor \phi/2\pi \rfloor$, obtained from a constant starting vertical spin field $\mathcal{P}(0, \phi) = (0, 1, 0)$ for the model of Example 3 in the case $\epsilon = 0.4$.

Acknowledgments

We wish to thank...

Guide for the reader

The following list summarizes the conventions and notation used in this paper:

- Sec. 2.1: spin–orbit system, $\mathfrak{a}(\phi)$, $A(n; \phi)$, resonant, nonresonant, off orbital resonance, on orbital resonance, $SO(3)$, $\mathbb{Z}$, transpose of a matrix.

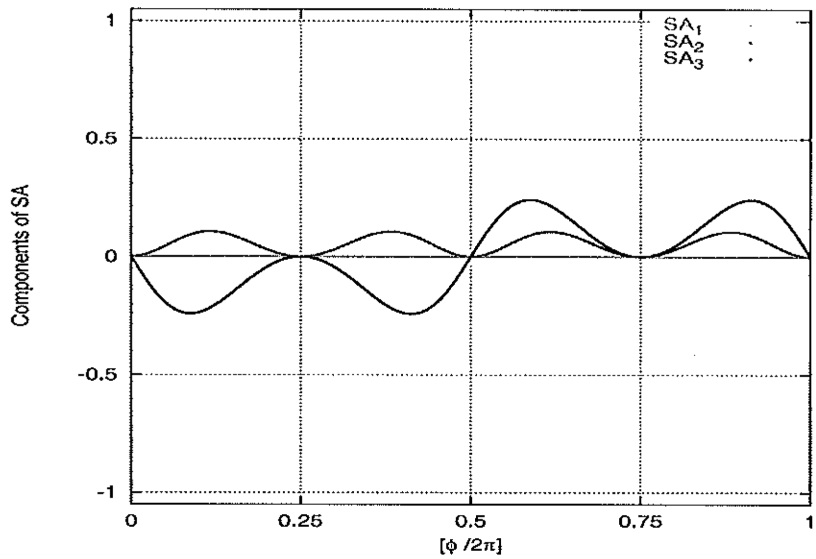

Figure 5: The three components of the stroboscopic average $\bar{\mathcal{P}}(\phi)$ vs. the fractional normalised phase $\lfloor\phi/2\pi\rfloor$, obtained from the starting vertical polarization field $\mathcal{P}(0,\phi) = (0, \cos^2(\phi), 0)$ for the model of Example 3 in the case $\epsilon = 0.4$.

- Sec. 2.2: quasiperiodic function, χ–quasiperiodic function.
- Sec. 2.3: spin transfer matrix $M(n;\phi)$, SPF, phase function of SPF.
- Sec. 2.4: $\mathcal{J}$.
- Sec. 3.1: spin field, $\mathbb{S}^2$, Euclidean norm $|\cdot|$.
- Sec. 3.2: ISF, n–turn invariant spin field.
- Sec. 4.1: spin frequency, $\hat{\Xi}(\phi_0)$, $\sim$, $[\nu]$.
- Sec. 4.2: integer part $\lceil ... \rceil$, fractional part $\lfloor ... \rfloor$.
- Sec. 5.1: UPF, UPR, $\Xi(\phi_0)$, Ξ, spin tune, well–tuned, ill–tuned, spin–orbit resonance.
- Sec. 5.2: $\mathfrak{M}(F)$, spectrum $\Lambda(F)$, $a_T(F,\lambda)$, $\mathbb{N},a(F,\lambda)$, mean of F, Y_χ.
- Sec. 6.1: frame field, IFF, n–turn invariant frame field, uniform IFF.
- Sec. A.1: $A_{N,m,k}$, $||\cdot||$.
- Sec. A.2: M_ν, unimodular, e^j.
- Sec. C: $U(1)$.

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