Quasiperiodic spin–orbit motion in storage rings: spin tunes and maps

D. P. Barber[a], J. A. Ellison[b] and K. Heinemann[b]

[a] Deutsches Elektronen–Synchrotron, DESY, 22603 Hamburg, Germany

[b] Department of Mathematics and Statistics, The University of New Mexico, Albuquerque, New Mexico 87131, U.S.A.

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This version has the examples distributed through the text

Abstract

We return to our in–depth analysis of the concept of spin precession rates and spin tune for integrable orbital motion in storage rings.

While the analysis in [1] was based on flows, here the analysis is based on multi–turn spin–orbit maps. Our map formalism can deal with the hard edge and thin lens approximations so that it delivers insights/results which are not available from an analysis in terms of smooth flows. Thus the results of the present paper are considerably broader in scope than those in [1].

As in [1] our definition of spin precession rates on nonperiodic synchro–betatron orbits is based on a generalization of the Floquet theorem which we derive from the important concept of quasiperiodicity. These circumstances lead naturally to the definition of the uniform precession rate and, as a special case, to a definition of spin tune which in turn leads to the definition of spin–orbit resonance on synchro–betatron orbits. We also carry out a spectral analysis of the spin motion and link the spectra to spin tunes.

Spin–orbit systems are classified in sets defined by the attributes (e.g., spin tune) that they possess.

We illustrate the various aspects of our formalism with four examples of model spin–orbit systems chosen to show the standard or special behavior that can arise and we allocate them to our sets with the help of a Venn diagram.

(i) IT IS LIKELY THAT WE WILL GET THE SAME REFEREES AS BEFORE. IN FACT WE SHOULD PERHAPS REQUEST THAT. BUT THEN WE MUST MAKE IT CLEAR IN THE ABSTRACT AND INTRO WHY THIS PAPER SHOULD BE PUBLISHED IN ADDITION TO BEH-I.

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Contents

1 Introduction

In [1] we undertook an extensive study of the concept of spin tune arising from solving the Thomas–Bargmann–Michel–Telegdi (T–BMT) equation [20] of spin precession. For this we assumed that the orbital motion was integrable and we introduced the invariant spin field (ISF) and exploited the concept of quasiperiodicity. In [1] it was stipulated that the electric and magnetic fields were smooth (of class C^1, i.e. continuously differentiable) both in the orbital phases and in the generalized azimuth θ and the spin–orbit motion was analyzed using differential equations and represented in terms of flows. Of course, since physical fields are smooth, this is a perfectly reasonable approach. On the other hand, practical numerical spin–orbit tracking simulations are usually carried out with fields which cut off sharply at the ends of magnets and/or with thin lens approximations, i.e., with the fields of magnets represented by step functions and/or delta functions. Thus in [1] our formalism involved smoothness in θ but the numerical calculations cited there in Sec. X had been obtained using hard edged and thin lens fields.

In this paper we provide an analysis that can include such hard edged and thin lens fields. In particular we do not require that fields and functions are smooth in θ and ϕ but merely require that the fields are continuous (i.e., of class C^0) in the orbital phases and we forego an insistence on continuity in θ. It is then convenient (and common usus) to work with 1–turn spin–orbit maps instead of with flows. In any case the use of the map formalism is perfectly suited to the fact that the fields are 2π–periodic in θ. Thus in this paper we exploit maps and do so in a way which allows us to take over quasiperiodicity and many more of the concepts of [1] by translating them into "map language". Of course, the use of

maps allows models with fields smooth in θ to be studied but it also allows rigorous study of some simple, convenient and popular models of spin motion with discontinuous fields to be studied. For details on how the map formalism derives from the flow formalism, i.e. how each specific T–BMT–equation manifests itself in a specific 1–turn spin transfer matrix, see Remark 1 in Sec. 2.1.

This paper is designed so that it can be read independently of [1]. However, we wish to avoid repeating the copious contextual material contained in [1]. We therefore invite the reader to consult the Introduction and the Summary and Conclusion in [1] in order to acquire a better appreciation of the context. In [1] we were primarily interested in assigning a Floquet-like behavior to spin motion on synchro–betatron orbits and in defining the concepts of spin tune and spin–orbit resonance via the notion of "well–tuned" tori. Well–tuning is essential for systematizing the motion of sets of spins moving on tori in phase space. We also suggested spectral analysis of spin motion for "measuring" the spin tune during simulations.

In this paper, while recasting many of these topics in the language of maps, we take the opportunity to illustrate the efficacy and relevance of the formalism by using it to make a detailed analysis of four traditional models. Two of the models deliver expected results. But the other two behave in unexpected ways (and thereby illustrate the advantages of our approach??) The use of maps and requiring only continuity in orbital phase (instead of smoothness) also allows us to make stronger statements (Theorems 4.1, 8.1a-b are striking examples!) about spin motions than we did for the flow formalism in [1]. Another difference is that we introduce some definitions which have no counterpart in [1], such as spin frequency and the SPF.

In this paper, as in [1], the orbital motion is integrable and we allow the number of angle variables, d, to be arbitrary (but ≥ 1) although for spin motion in storage rings, the case $d = 3$ is the most important. We use the symbols $\phi = (\phi_1, ..., \phi_d)$, $J = (J_1, ..., J_d)$ and $\omega(J) = (\omega_1(J), ..., \omega_d(J)) := d\phi/d\theta$ to mean respectively the lists of orbital angles, orbital actions and orbital tunes. We then write the T–BMT equation as $d\mathbf{S}/d\theta = \mathbf{\Omega}(\theta, J, \phi(\theta)) \times \mathbf{S}$ where the vector $\mathbf{S}$ is the spin expectation value ("the spin") in the rest frame of a particle and $\mathbf{\Omega}$ is the precession vector obtained as indicated in [1] from the electric and magnetic fields on the particle trajectory.

Generally, if ϕ appears as an independent variable in a function, the function will be 2π–periodic in each $\phi_1, ..., \phi_d$. In that case we say for brevity that the function is 2π–periodic in ϕ. Since we are viewing the system turn by turn using maps, it will be convenient to employ the language of the literature on so–called discrete–time dynamical systems. We therefore call the turn number the "time" and often denote it by the integer n. Of course, this "time" should not be confused with the time, t, appearing in equation (1.1) in [1]. For the purposes of this paper it will suffice to confine ourselves to motion on a single torus. Thus the actions J are just a parameters.

The paper is structured as follows. In Sec. II we introduce the notation and basic definitions for our formulation in terms of maps. We also specify four examples which will be repeatedly used in the following sections to illustrate the theory and to expose some subtleties. In Sec. III we take a first look into the main topic of this paper, namely *quasiperiodic solutions* of spin motion in preparation for their use in the rest of the paper. Sec. IV introduces the concepts of *polarization fields*, *spin fields* and *stroboscopic averaging* and contains a theorem about *invariant spin fields*. Sec. V is dedicated to the definition

Most serious people use maps

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of *spin frequency* which naturally flows from the concept of quasiperiodicity. In Sec. VI we introduce the *spin tune* and the concepts of *spin–orbit resonance* and *well–tuning.* We also see their relation to a generalized Floquet theorem. Then in Sec. VII we carry out Fourier analysis on the spin motion by exploiting quasiperiodicity. Sec. VIII deals with *frame fields* which are a natural extension of the concept of spin fields. Frame fields are an important tool for studying a spin–orbit system. Thus, for example, the Theorem 8.1, gives, among other things, a useful criterion for the existence of spin tune. In Theorem 8.2 which deals with the maximum time averaged polarization, we see that frame fields are useful also for computations. The paper is summarized in Sec. IX where we......

tools

The labels for the theorems, lemmas, corollaries, propositions and definitions are chosen in a way which indicates their relative positions in the text and for clarity, we mark the ends of these entities and remarks with the symbol □. To aid the reader we mark the key equations with a ⋆⋆⋆ on the left and we include a glossary of notation at the end. In order to maintain the flow of the arguments, some purely mathematical matters are relegated to appendices. When an asterisk appears in an argument of a function in a mathematical expression it is to be understood as representing an unspecified value of that argument.

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Well-tuning leads to physical frequencies?

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2 Spin–Orbit Motion and Quasiperiodicity in Discrete Time

2.1 Spin–Orbit Maps in Discrete Time

2

Since the orbital motion is integrable and all solutions to the T—BMT equation simply represent rotations of spins, spin–orbit motion will be represented in the present paper in terms of the following discrete-time dynamical system

$$S_{n+1} = \mathfrak{a}(\phi_n) S_n \ , \tag{2.1}$$

$$\phi_{n+1} = \phi_n + 2\pi\omega \ , \tag{2.2}$$

where $\omega \in \mathbb{R}^d$, $\mathfrak{a} : \mathbb{R}^d \to SO(3)$ is a continuous and 2π–periodic function of ϕ representing the 1–turn spin transfer matrix and where S_n is the column matrix of the three spin components at the end of the n–th turn. The set $SO(3)$ consists of those real 3×3–matrices R for which $R^T R = I$ where R^T denotes the transpose of R and for which $\det(R) = 1$. For details on

orbital problem is linear flow on the

It is well known that if $m \cdot \omega$

teger for $m \in \mathbb{Z}^d \setminus \{0\}$ then the orbit

$\omega : n \in \mathbb{Z}\}$ fills the d-torus densely,

[Ref Katok]. Since a is continuous, the orbit samples all values of a and this motivates our primary focus on this situation. It seems natural in this map context to call such an ω non resonant but we have not found such usage in the map literature. In the flow context ω is stb non resonant if $\omega \cdot m$ is not zero for $m \in \mathbb{Z}^d \setminus \{0\}$. We will stick to this terminology and thus the above situation occurs when $(\omega, 1)$ is non resonant. ~~Our spin-orbit system will be stb off orbital resonance if $(\omega, 1)$ is not resonant. Otherwise the system will be stb on orbital resonance~~ If $(\omega, 1)$ is nonresonant, our spin-orbit system is often said to be off orbital resonance. We will not use this terminology here.

$A(n-1, \varphi_0) = \mathfrak{a}(\varphi_0, 2\pi\omega(n-1)) = \mathfrak{a}(\varphi_0 - 2\pi\omega + 2\pi n\omega) = A(n, \varphi - 2\pi\omega)$

how (2.1) and (2.2) derive from the flow formalism, see Remark 1 below. We only consider real–valued S_n and for convenience we allow it to have arbitrary length so that S_n is $\mathbb{R}^3$–valued. Since $\mathfrak{a} \in SO(3)$, the length of a spin and the angle between two spins propagating with the same $\mathfrak{a}$ are invariant.

With the aid of (2.2), (2.1) can be written as a lower dimensional discrete-time dynamical system parametrized by the variable ϕ_0, i.e.

$$S_{n+1} = A(n;\phi_0)S_n \; , \tag{2.3}$$

where $A(n;\phi_0) := \mathfrak{a}(\phi_0+2\pi n\omega)$. Of course, our "spin–orbit system" is uniquely characterized by $\mathfrak{a}$ and ω. We also call S_* a solution of (2.3) "starting at ϕ_0". Note that the dynamical system (2.1),(2.2) is *autonomous* because the r.h.s. of (2.1) and (2.2) do not explicitly depend on the time n. In contrast, the dynamical system (2.3) is *nonautonomous* because the r.h.s. of (2.3) explicitly depends on the time n.

A $\chi \in \mathbb{R}^k$ is said to be *nonresonant* if the equation $m \cdot \chi = 0$, together with the condition $m \in \mathbb{Z}^k$, can only be fulfilled for $m = 0$, where $\mathbb{Z}$ denotes the set of integers and where $\cdot$ denotes the Euclidean scalar product. The spin–orbit system is then said to be *off orbital resonance* if $(1, \omega)$ is nonresonant. But we often just say that ω is off orbital resonance. Otherwise the system is *on orbital resonance.*

Remark:

(1) Although the spin–orbit system of this paper is defined solely in terms of maps using the discrete-time formalism, its physical origin lies, as pointed out in the Introduction, in the flow formalism, i.e. the continuous time formalism, in which the spin and orbital motion are governed by the equations

$$\dot{S} = \mathcal{A}(\theta,\phi)S \; , \tag{2.4}$$
$$\dot{\phi} = \omega \; , \tag{2.5}$$

where the real skew–symmetric 3×3 matrix $\mathcal{A}(\theta,\phi)$ is derived from the precession vector of the T–BMT–equation [1] and is 2π–periodic in θ and ϕ and of class C^1 in ϕ. Since we wish to work with hard–edged and/or thin–lens fields, we do not insist that $\mathcal{A}(\theta,\phi)$ is of class C^1 in θ but allow it to be piecewise C^1 on $[0, 2\pi)$. Thus, and since $\mathcal{A}(\theta,\phi)$ is 2π–periodic in θ, there at most finitely many values in every interval $[2\pi n, 2\pi(n+1))$ for which $\mathcal{A}(\theta,\phi)$ is not C^1 in θ (n an arbitrary integer). For any initial and final azimuths θ_0 and θ outside the thin lenses and outside the edges, i.e. for azimuths at which $\mathcal{A}$ is C^1 in θ, the general solution of (2.4) can be written in terms of the *Principal Solution Matrix* (PSM) Φ as

$$S(\theta) = \Phi(\theta,\theta_0;\phi(\theta_0),\omega)S(\theta_0) \; , \tag{2.6}$$

where $\Phi \in SO(3)$ satisfies (2.4) and $\Phi(\theta_0,\theta_0;\phi(\theta_0),\omega) = I$ and where $\Phi(\theta,\theta_0;\phi(\theta_0),\omega)$ is 2π–periodic and C^1 in $\phi(\theta_0)$. The 1–turn spin transfer matrix $\mathfrak{a}$ is then given by $\mathfrak{a}(\phi_0) := \Phi(\theta_0 + 2\pi, \theta_0; \phi_0, \omega)$, where θ_0 is called the "reference azimuth". This is sufficient for explaining how the map formalism derives from the flow formalism, i.e. how each specific T–BMT equation (2.4) manifests itself in a specific 1–turn spin

transfer matrix $\mathfrak{a}$. Note that we will return to the issue of the reference azimuth in Section 6.1 where we will also make use of the relations

$$\Phi(\theta,\theta_0;*,\omega) = \Phi(\theta,\theta_f;* + (\theta-\theta_0)\omega,\omega)\Phi(\theta_f,\theta_0;*,\omega)\ , \tag{2.7}$$

$$\Phi(\theta_f+2\pi,\theta_0+2\pi;*,\omega) = \Phi(\theta,\theta_0;*,\omega) \tag{2.8}$$

Because of (2.8) we can choose θ_0 to be in $[0,2\pi)$. If $\mathcal{A}$ is C^1 then the above mentioned properties of Φ follow as shown in [1], where this case was studied. If $\mathcal{A}(\theta,\phi)$ is not everywhere C^1 in θ then the above mentioned properties of Φ can be derived by using the techniques of distribution theory whereby one treats (2.4) as a generalized differential equation (see for example CHANGE REFERENCE [23]).

Example 1, defined in Sec. 2.3 provides an example where $\mathcal{A}$ is C^1. In contrast, Examples 2 and 3, also defined in Sec. 2.3, have their origin in $\mathcal{A}$'s which, which are not everywhere of class C^1 in θ. □

2.2 The Properties of Solutions and the n–turn Spin Transfer Matrix

Iteration of (2.3) yields the following representation of solutions:

$$S_n = M(n;\phi_0)S_0\ , \tag{2.9}$$

where $M(0;*) = I$ and, for $n>0$,

$$\begin{aligned} M(n;*) &= A(n-1;*)A(n-2;*)\cdots A(0;*)\ ,\\ M(-n;*) &= A^T(-n;*)A^T(-n+1;*)\cdots A^T(-1;*)\ . \end{aligned} \tag{2.10}$$

Moreover, for arbitrary integers N and n,

$$M(N+n;*) = M(n;*+2\pi N\omega)M(N;*)\ . \tag{2.11}$$

We call $M(n;\phi_0)$ the *n–turn spin transfer matrix.* Since it contains all the information about the solutions of (2.3) it is the central object of the formalism. If one solution is known for a ϕ_0, M takes a special form. Consider the transformation $S_n \mapsto \hat{S}_n$ with

$$S_n = T(n)\hat{S}_n\ , \tag{2.12}$$

where $T:\mathbb{Z}\to SO(3)$ and the third column, T^3, is the known solution of (2.3) and the other two columns T^1 and T^2 are chosen arbitrarily. Then

$$\hat{S}_{n+1} = T^T(n+1)A(n)T(n)\hat{S}_n \equiv \hat{A}(n)\hat{S}_n\ . \tag{2.13}$$

Because T^3 is a solution of (2.3) a real $\lambda(n)$ exists such that

$$\begin{aligned} \hat{A}(n) &= [T^T(n+1)A(n)T^1(n), T^T(n+1)A(n)T^2(n), \begin{pmatrix}0\\0\\1\end{pmatrix}]\\ &= \begin{pmatrix}\cos\lambda(n) & -\sin\lambda(n) & 0\\ \sin\lambda(n) & \cos\lambda(n) & 0\\ 0 & 0 & 1\end{pmatrix} = \exp(\mathcal{J}\lambda(n))\ . \end{aligned} \tag{2.14}$$

where $\mathcal{J} := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and where we used the fact that the third column of $\hat{A}(n)$ is $(0,0,1)^T$ and where, in the second equality we used the fact that $\hat{A}(n) \in SO(3)$. It follows that, for $n \geq 0$,

$$M(n) = T(n)\exp(\mathcal{J}[\lambda(0) + ... + \lambda(n-1)])T^T(0) ,$$
$$M(-n-1) = T(-n-1)\exp(-\mathcal{J}[\lambda(-1) + ... + \lambda(-n-1)])T^T(0) . \quad (2.15)$$

We see that T transforms A via (2.14) into a matrix $\hat{A}$ which has a simple block diagonal form and therefore we call every function $T : \mathbb{Z} \to SO(3)$ a "simple precession frame (SPF) starting at ϕ_0" if its third column is a solution of (2.3).

Remark:

(2) Of course, if T is a SPF starting at a ϕ_0 then (2.15) can be written as $M(n) = T(n)\exp(\mathcal{J}\mu(n))T^T(0)$ where $\mu(n)$ is real. Thus the corresponding "phase function", defined by $\exp(i\mu(n))$, is uniquely determined by T. In particular, for $n \geq 0$,

$$\exp(i\mu(n)) = \exp(i[\lambda(0) + ... + \lambda(n-1)]) ,$$
$$\exp(i\mu(-n-1)) = \exp(-i[\lambda(-1) + ... + \lambda(-n-1)]) .$$

The concepts of SPF and its phase function are very convenient but the reader should not read too much into the meaning of their names. Of course, owing to the arbitrariness of T^1 and T^2, both $\lambda(n)$ and $\mu(n)$ depend on the choice of T.

(2b) If μ is a real valued function on $\mathbb{Z}$ and if R is a constant $SO(3)$–matrix then T, defined by $T(n) := M(n;\phi_0)R\exp(-\mathcal{J}\mu(n))$, is a SPF starting at ϕ_0 and μ is the corresponding phase function. We see by this construction that, for every ϕ_0, a large abundance of SPF's starting at ϕ_0 exists. Thus the existence of an SPF T is a trivial matter if no further conditions (e.g. quasiperiodicity, to be discussed below) are imposed on T. □

2.3 Definition of the Examples

Throughout this paper we will illustrate our formalism by considering four examples of simple models of spin motion in storage rings, chosen to illustrate some standard or special features of spin motion. All four examples can be derived from the flow formalism as pointed out in Remark 1 but two of the examples are then best analysed directly with maps. The first three models involve the so-called "single resonance model" in some way. The fourth involves synchrotron oscillations. All four models have $d = 1$.

Example 1 This is the single resonance model used to model spin motion on vertical betatron trajectories when the spin motion is dominated by the effect of a single Fourier harmonic of the horizontal quadrupole fields on the trajectories [1, Sec. VII]. In particular $\mathcal{A} = \mathcal{A}(\phi)$ is of class C^1 and is given by

$$\mathcal{A}(\phi) := \begin{pmatrix} 0 & -\sigma_1 & \sigma_2\sqrt{2J_2}\sin(\phi) \\ \sigma_1 & 0 & -\sigma_2\sqrt{2J_2}\cos(\phi) \\ -\sigma_2\sqrt{2J_2}\sin(\phi) & \sigma_2\sqrt{2J_2}\cos(\phi) & 0 \end{pmatrix} , \quad (2.16)$$

where σ_2 is a real constant such that the quantity $\sigma_2\sqrt{2J_2}$ is just the "resonance strength" ϵ of the conventional literature on the single resonance model [17]. The parameter σ_1 is the nonzero real θ–independent angular frequency of precession of spins on the design orbit (i.e., $\sigma_2 = 0$), around the third reference axis which is perpendicular to the plane of the ring. We assume that $\sigma_2 \neq 0$. This model can be solved exactly and the PSM reads as

$$\Phi(\theta, \theta_0; \phi_0) = \exp(\mathcal{J}[\phi_0 + \omega(\theta - \theta_0)]) \exp((\theta - \theta_0)E) \exp(-\phi_0 \mathcal{J}) \ , \tag{2.17}$$

where

$$E := \begin{pmatrix} 0 & -\delta & 0 \\ \delta & 0 & -\epsilon \\ 0 & \epsilon & 0 \end{pmatrix} , \tag{2.18}$$

for which $\delta := \sigma_1 - \omega$ and where ω is the angular frequency (tune) for the dominant harmonic. According to Remark 1 we have $\mathfrak{a}(\phi_0) = \Phi(\theta_0 + 2\pi, \theta_0; \phi_0)$ which defines Example 1. With (2.10) and for all n and ϕ_0, we obtain the map

$$M(n; \phi_0) = \Phi(2\pi n + \theta_0, \theta_0; \phi_0) = \exp(\mathcal{J}[\phi_0 + 2\pi n\omega]) \exp(2\pi n E) \exp(-\phi_0 \mathcal{J}) \tag{2.19}$$

hence $\mathfrak{a}(\phi_0) = M(1; \phi_0) = \exp(\mathcal{J}[\phi_0 + 2\pi\omega]) \exp(2\pi E) \exp(-\phi_0 \mathcal{J})$. That the r.h.s. of (2.19) is independent of θ_0 comes as no surprise because $\mathcal{A}$ is independent of the azimuth, i.e., the single resonance model is autonomous. We call the subcase "off orbital resonance" Example 1a, and the subcase "on orbital resonance" Example 1b.

Example 2 This example is a modification of Example 1 in which the spin motion is described by (2.16) in most of the ring and a thin lens Siberian Snake is placed at $\theta = 0$ [1, Sec. X]. We restrict our study to the special case where $\delta = \sigma_1 - \omega = 0$ and $\epsilon = 3/2$, considered in [26, Eq. 21] and [27]. Then with the third reference axis perpendicular to the plane of the ring, Example 2 is defined by

$$\mathfrak{a}(\phi) := \exp(\mathcal{J}[N_1 + N_2\phi]) \ , \tag{2.20}$$

where N_1 is a real constant and where N_2 is a constant, nonzero integer. In contrast to [26, 27] we only consider the case where ω is off orbital resonance, i.e. irrational. More details about the values and origins of N_1 and N_2 are not needed here.

Example 3 This is a modification of Example 1 in which the spin motion is described by (2.16) in most of the ring and a thin lens Siberian Snake is placed at $\theta = 0$ with a second snake at $\theta = \pi$. The orientations of the so-called snake axes are chosen so that the spin tune on the closed orbit is 1/2 [1, Sec. III] and we set $\delta = \sigma_1 - \omega = 0$. Our primary interest is to explore the case of snake "resonance" [1, Sec. X] and [4, 5], namely at rational orbital tunes ω for which, $1/2 = m_0 + m_1\,\omega$ where m_0 and m_1 are integers and m_1 is odd. Clearly, this corresponds to orbital resonance. In order to illustrate the key points, it will be sufficient to choose $\omega = 1/2$, so that either $m_0 = 0$ and $m_1 = 1$ or $m_0 = 1$ and $m_1 = -1$. Then, with our choice of snake axes and starting just before the first snake, Example 3 is defined by

$$\mathfrak{a} := \begin{pmatrix} 1-2c^2 & 2bc & 2ac \\ 2bc & 1-2b^2 & -2ab \\ -2ac & 2ab & 2a^2-1 \end{pmatrix} =: \mathfrak{a}^{\epsilon} , \tag{2.21}$$

where

$$a(\phi) := -2\sin^2(\pi\epsilon/2)\sin(\phi)\cos(\phi) \qquad b(\phi) := -2\sin(\pi\epsilon/2)\cos(\pi\epsilon/2)\cos(\phi) ,$$
$$c(\phi) := 2\sin^2(\pi\epsilon/2)\cos^2(\phi) - 1 , \tag{2.22}$$

and where in this case, for consistency with [30] and [5] the second reference axis is perpendicular to the plane of the ring. Note that

$$|a|^2 + |b|^2 + |c|^2 = 1 . \tag{2.23}$$

In this example we are interested in elaborating on some aspects of the work described in [4, 5]. We call the subcase where ϵ is an even integer, Example 3a, the subcase where ϵ is an odd integer, Example 3b and the the subcase where ϵ is not an integer, Example 3c. To allow comparison with other work, for Example 3c, we use $\epsilon = 0.4$. Of course, we are aware that beams in storage rings are unstable for $\omega = 1/2$, that particles have three modes of oscillation and that particle motion in real rings can be nonintegrable. Nevertheless Example 3 is a useful vehicle for demonstrating the use of our formalism at snake "resonances" and some technical points not mentioned in other work [?]. Note that, as foreseen in [5] for ω's very close to, but not at snake "resonance" tunes, the spin motion is very different from that exactly at such tunes.

Example 4 This example illustrates the meaning of our spin precession rates when synchrotron oscillations cause $\mathcal{A}(\theta,\phi)$ in (2.4) to be modulated harmonically in the way often used for estimating the strength of the so–called synchrotron sideband resonances [33]. It corresponds to the "oscillator approximation" of synchrotron motion. See [34, 35, 36]. In particular $\mathcal{A} = \mathcal{A}(\phi)$ is of class C^1 and is given by

$$\mathcal{A}(\phi) := \mathcal{J}\sigma_1(1+\epsilon'\cos\phi) = \begin{pmatrix} 0 & -\sigma_1(1+\epsilon'\cos\phi) & 0 \\ \sigma_1(1+\epsilon'\cos\phi) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{2.24}$$

where $\epsilon' = \sigma_3\sqrt{2J_3}$ is the amplitude of the fractional energy deviation and whereby $\sigma_3 \neq 0$ is a scale factor. The third reference axis is perpendicular to the plane of the ring, $\omega \notin \mathbb{Q}$ is the synchotron tune and the nonzero real constant σ_1 has the same meaning as in Example 1. This model can be solved exactly and the PSM reads as

$$\Phi(\theta,\theta_0;\phi_0) = \exp(\mathcal{J}[\sigma_1(\theta-\theta_0) + \frac{\sigma_1\epsilon'}{\omega}\sin(\phi_0 + \omega(\theta-\theta_0))]) . \tag{2.25}$$

According to Remark 1 we have $\mathfrak{a}(\phi_0) = \Phi(\theta_0 + 2\pi, \theta_0; \phi_0)$ which defines Example 4. With (2.10) and for all n and ϕ_0, we obtain the map

$$M(n;\phi_0) = \Phi(2\pi n + \theta_0, \theta_0; \phi_0) = \exp(\mathcal{J}[2\pi n\sigma_1 + \frac{\sigma_1\epsilon'}{\omega}\sin(\phi_0 + 2\pi n\omega)]) , \tag{2.26}$$

Hence $\mathfrak{a}(\phi_0) = M(1;\phi_0) = \exp(\mathcal{J}[2\pi\sigma_1 + \frac{\sigma_1\epsilon'}{\omega}\sin(\phi_0 + 2\pi\omega)])$. That the r.h.s. of (2.26) is independent of θ_0 comes as no surprise because $\mathcal{A}$ is independent of the azimuth.

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2.4 Quasiperiodic Functions in Discrete Time

We now introduce the concept of quasiperiodicity for maps. This will be a key tool for our analysis.

Definition 2.1: a) A function $F : \mathbb{Z} \to \mathbb{C}$ is said to be χ*–quasiperiodic*, $\chi \in \mathbb{R}^k$, if a continuous function $f : \mathbb{R}^k \to \mathbb{C}$ exists such that

$$F(n) = f(2\pi n\chi + c), \tag{2.27}$$

where f is continuous and 2π–periodic (i.e., 2π–periodic in each argument) and where c is some constant in $\mathbb{R}^k$. We say that "F is generated by f" and that f is the "generator" of F. More generally, a function $F : \mathbb{Z} \to X$ is said to be χ*–quasiperiodic* if either $X = \mathbb{C}^j$ or $X = \mathbb{C}^{j\times j}$ and if all of its components are χ–quasiperiodic (j positive integer).
b) A function is said to be *quasiperiodic*, if it is χ*–quasiperiodic* for some χ. □

For $k = d$ and $\chi = \omega$, the sets of χ–quasiperiodic functions for maps parallel the sets $\mathcal{Q}(1, \omega; d + 1)$ of smooth functions encountered for flows in [1].

Remark:

(3) Obviously, the generator f of a quasiperiodic function F can be chosen such that $c = 0$ but nonzero c is sometimes convenient. Note also that we choose the sets $\mathbb{R}$ and $\mathbb{C}$ such that $\mathbb{R} = \{x \in \mathbb{C} : \Im m\{x\} = 0\}$, i.e. $\mathbb{R} \subset \mathbb{C}$. Thus if F is a χ–quasiperiodic function whose components are real then a generator f exists whose components are real (just take the real part of a given generator!). □

From the definition of quasiperiodicity we see that $A(*; \phi_0)$ in (2.3) is ω–quasiperiodic and is generated by $\mathfrak{a}$ in (2.2). Analytical experience with simple models suggests that, for every ϕ_0, an ω–quasiperiodic SPF exists hence that for every ϕ_0 a normalized solution of (2.3) exists. In particular this is the case for our four examples as we show later. Analytical experience with simple models also suggests that generalized Floquet parameters, as defined in Section 6.1, exist at every ϕ_0 but we will see with Examples 2,3 and 4 that exceptions exist.

3 Interlude on quasiperiodic solutions

In this section we take a first look at quasiperiodic solutions of (2.3). For more detailed studies of these matters, see Secs. 4 to 8.

Suppose that $t : \mathbb{R}^d \to SO(3)$ is a continuous and 2π–periodic function and suppose that a ξ exists such that the third column, t^3, generates a ω–quasiperiodic solution of (2.3), starting at this ξ, i.e., $S_n = t^3(\xi + 2\pi n\omega)$. Note that (recall Sec. 2.2) T_0, defined by $T_0(n) := t(\xi + 2\pi n\omega)$ is a SPF starting at ξ. Let $G : \mathbb{R}^d \to \mathbb{R}$ be a continuous and 2π–periodic function. Then u, defined by $u(\phi) := t(\phi)\exp(\mathcal{J}G(\phi))$, is continuous and 2π–periodic. Let $U(n) = u(\xi + 2\pi n\omega)$ and consider the transformation $S_n \mapsto \hat{S}_n$ such that $S_n = U(n)\hat{S}_n$. Then $\hat{S}_{n+1} = \hat{A}(n)\hat{S}_n$ where $\hat{A}(n) := U^T(n+1)A(n)U(n) = \exp(-\mathcal{J}G(\phi + 2\pi\omega))t^T(\phi + 2\pi\omega)\mathfrak{a}(\phi)t(\phi)\exp(\mathcal{J}G(\phi))$ and where $\phi = \xi + 2\pi n\omega$. If we can find G and ν such that $\hat{A}(n) =$

$\exp(\mathcal{J}2\pi\nu)$ then $S_n = U(n)\exp(\mathcal{J}2\pi\nu n)U^T(0)S_0 = u(\xi + 2\pi n\omega)\exp(\mathcal{J}2\pi\nu n)U^T(0)S_0$, so that all solutions for our ξ are χ–quasiperiodic where $\chi = (\omega, \nu)$.

We now need

Lemma 3.1 *Let ω be off orbital resonance. Then, with the above assumptions on t and ξ, a unique $N \in \mathbb{Z}^d$ and a continuous and 2π–periodic function $\lambda_p : \mathbb{R}^d \to \mathbb{R}$ exist such that for all ϕ*

$$t^T(\phi + 2\pi\omega)\mathfrak{a}(\phi)t(\phi) = \exp(\mathcal{J}[\lambda_p(\phi) + N \cdot \phi]) \ . \tag{3.1}$$

Proof: We define, for every integer j, the function $T_j : \mathbb{Z} \to SO(3)$ by $T_j(n) := t(\xi + 2\pi(n + j)\omega)$. Because of (2.11), the third column, T_j^3, of T_j is a ω–quasiperiodic solution of (2.3) starting at $\xi + 2\pi j\omega$. Clearly, for $\phi = \xi + 2\pi j\omega + 2\pi m$ $(m \in \mathbb{Z}^d, j \in \mathbb{Z})$,

$$t^T(\phi + 2\pi\omega)\mathfrak{a}(\phi)t(\phi)e^3 = e^3 \ , \tag{3.2}$$

where $e^3 := (0, 0, 1)$. Note that T_j is a SPF starting at $\xi + 2\pi j\omega + 2\pi m$. Because (3.2) holds for a dense set of ϕ's, (3.2) holds for every ϕ by continuity of t and $\mathfrak{a}$. Applying Appendix C to (3.2) then yields (3.1). □

Thus we have, for $\phi = \xi + 2\pi n\omega$,

$$\begin{aligned}\hat{A}(n) &= \exp(-\mathcal{J}G(\phi + 2\pi\omega))\exp(\mathcal{J}(\lambda_p(\phi) + N \cdot \phi))\exp(\mathcal{J}G(\phi)) \\ &= \exp(\mathcal{J}N \cdot \phi)\exp(-\mathcal{J}[G(\phi + 2\pi\omega) - G(\phi) - \lambda_p(\phi)]) \ .\end{aligned}$$

We now assume that G satisfies the *homological equation*

$$G(\phi + 2\pi\omega) - G(\phi) = \lambda_p(\phi) - 2\pi\nu \ , \tag{3.3}$$

for every ϕ. We then obtain, for $\phi = \xi + 2\pi n\omega$,

$$\hat{A}(n) = \exp(\mathcal{J}N \cdot [\xi + 2\pi n\omega])\exp(\mathcal{J}2\pi\nu) \ .$$

Finally, by simple Fourier analysis, the average of the l.h.s. of (3.3) is zero so that the average of λ_p is $2\pi\nu$, i.e.,

$$2\pi\nu = \frac{1}{(2\pi)^d}\int_0^{2\pi} \cdots \int_0^{2\pi} \lambda_p(\phi)d\phi_1 \cdots d\phi_k \ . \tag{3.4}$$

Thus we have proven

just A

Theorem 3.2 *Let ω be off orbital resonance and suppose that $t : \mathbb{R}^d \to SO(3)$ is a continuous and 2π–periodic function. Assume also that there exists a ξ such that $S_n = t^3(\xi + 2\pi n\omega)$ is a ω–quasiperiodic solution of (2.3) starting at ξ. Then there exists a continuous and 2π–periodic function $\lambda_p : \mathbb{R}^d \to \mathbb{R}$ and a $N \in \mathbb{Z}^d$ such that, for all ϕ, (3.1) holds. Furthermore, if $N = 0$ and if the homological equation (3.3) has a continuous and 2π–periodic solution G then all solutions of (2.3) starting at ξ are (ω, ν)–quasiperiodic with ν given by (3.4).* □

Corollary 3.3 *Under the conditions of Theorem 3.2 every solution of (2.3) starting at an arbitrary ϕ_0 is (ω, ν)–quasiperiodic with ν given by (3.4).*

Proof: Consider a solution of (2.3) starting at an arbitrary ϕ_0 and make the transformation $S_n \mapsto \hat{S}_n$ such that $S_n = u(\phi_0 + 2\pi\omega n)\hat{S}_n$. As above we write $\hat{S}_{n+1} = \hat{A}(n)\hat{S}_n$ where, as before, $\hat{A}(n) := \exp(-\mathcal{J}G(\phi + 2\pi\omega))t^T(\phi + 2\pi\omega)\mathfrak{a}(\phi)t(\phi)\exp(\mathcal{J}G(\phi))$ except that $\phi = \phi_0 + 2\pi\omega n$. Using the fact that (3.1) and (3.3) hold for all ϕ we have $\hat{A}(n) = \exp(\mathcal{J}2\pi\nu)$ for all n. . □

Remark:

(1) The assumption underlying Corollary 3.3 that a continuous and 2π–periodic function $t : \mathbb{R}^d \to SO(3)$ and a ξ exist such that $S_n = t^3(\xi + 2\pi n\omega)$ is a ω–quasiperiodic solution of (2.3) starting at ξ, is always fulfilled if a ω–quasiperiodic SPF (recall Sec. 2.2) exists which starts at ξ. Proof: Suppose that T is a ω–quasiperiodic SPF starting at ξ. Then (recall Remark 2 in Sec. 2) a continuous and 2π–periodic function $t : \mathbb{R}^d \to \mathbb{R}^{3\times 3}$ exists which generates T via the prescription $T(n) = t(\xi + 2\pi n\omega)$. Because T is $SO(3)$–valued, t is $SO(3)$–valued on a dense set, i.e. at $\phi = \xi + 2\pi n\omega + 2\pi m$ $(m \in \mathbb{Z}^d, n \in \mathbb{Z})$. Because t is continuous, it then follows that t is $SO(3)$–valued. Finally, because T is a SPF, it follows that $S_n := t^3(\xi + 2\pi n\omega)$ is a ω–quasiperiodic solution of (2.3) starting at ξ, which completes the proof. Recall also the remarks at the end of Sec. II. □

The following theorem specifies a sufficient condition for the fulfillment of the homological equation (3.3).

Theorem 3.4 *If ω is in an appropriate Diophantine set (to be defined in Appendix B) and if $\lambda_p : \mathbb{R}^d \to \mathbb{R}$ is a 2π–periodic function of class C^k with sufficiently large k then the homological equation (3.3) has a continuous and 2π–periodic solution $G : \mathbb{R}^d \to \mathbb{R}$ such that the Fourier coefficients G_m and $(\lambda_p)_m$ of G and λ_p satisfy the relation*

$$G_m = \frac{(\lambda_p)_m}{\exp(2\pi i m \cdot \omega) - 1} .$$

A proof is given in Appendix B. □

4 ω–Quasiperiodic Solutions and the Invariant Spin Field

As mentioned at the end of Section 2, we expect that ω–quasiperiodic solutions of (2.3) are abundant. In this section we show, among other things, that, off orbital resonance, the existence of an ω–quasiperiodic solution to (2.3) at one ϕ_0 implies the existence of an ω–quasiperiodic solution at every ϕ_0. It even implies the existence of an invariant spin field which is defined via the concepts of the polarization field and the spin field.

4.1 Polarization Fields and Spin Fields

Suppose that at $n = 0$ a spin vector $S_0(\phi)$ has been assigned to every point ϕ of the orbital torus and consider its evolution according to (2.3). Let $S_n(\phi_0)$ denote the solution of (2.3), starting at ϕ_0 and for the initial value $S_0(\phi_0)$. Define the *field* $\mathcal{S} = \mathcal{S}(n, \phi)$ by $\mathcal{S}(n, \phi_0 + 2\pi n\omega) = S_n(\phi_0)$. Since $S_{n+1}(\phi_0) = \mathfrak{a}(\phi_0 + 2\pi n\omega)S_n(\phi_0)$ we have $\mathcal{S}(n+1, \phi_0 + 2\pi n\omega + 2\pi\omega) = \mathfrak{a}(\phi_0 + 2\pi n\omega)\mathcal{S}(n, \phi_0 + 2\pi n\omega)$ or $\mathcal{S}(n+1, \phi) = \mathfrak{a}(\phi - 2\pi\omega)\mathcal{S}(n, \phi - 2\pi\omega)$. Although the spins $S_0(\phi_0)$ can be chosen arbitrarily we elect for continuity in ϕ_0. — i.e. no filling-up with

Thus we consider functions $\mathcal{S} : \mathbb{Z}\times\mathbb{R}^d \to \mathbb{R}^3$ where $\mathcal{S}(n, *)$ is continuous and 2π–periodic and where

$$\mathcal{S}(n+1, \phi) = \mathfrak{a}(\phi - 2\pi\omega)\mathcal{S}(n, \phi - 2\pi\omega) . \tag{4.1}$$

We call such a $\mathcal{S}$, a *polarization field* and say that $\mathcal{S}(n, *)$ is the polarization field "at time n". It evolves according to (4.1), which corresponds to (2.3). From (2.11) and (4.1) it is clear that a polarization field $\mathcal{S}$ satisfies for every integer n

$$\mathcal{S}(n, *) = M(n, * - 2\pi n\omega)\mathcal{S}(0, * - 2\pi n\omega) . \tag{4.2}$$

Of course, given a polarization field $\mathcal{S}$ then S_*, defined by $S_n := \mathcal{S}(n, \phi_0 + 2\pi n\omega)$, is a solution of (2.3) starting at ϕ_0. A polarization field $\mathcal{S}$ is called a *spin field* if $|\mathcal{S}| = 1$, where $|\cdot|$ denotes the Euclidean norm.

Before we continue we point out that the above definition of the polarization field has two very different parts: the dynamical condition (4.1) and a regularity condition. In contrast to the dynamical condition, the regularity condition is to a certain extent a matter of convenience. For example in this paper, we choose the condition that the polarization field is a continuous function of the phase.

The $S_0(\phi)$ can have very different lengths in ϕ range

4.2 Invariant Spin Fields

It is now natural to check if there are spin fields which are invariant from turn to turn. We call a polarization field $\mathcal{S}$ *invariant* if $\mathcal{S}(n, *) = \mathcal{S}(0, *)$ for all n. Then with (4.1) $\mathcal{S}(0, *)$ fulfills

$$\mathcal{S}(0, *) = \mathfrak{a}(* - 2\pi\omega)\mathcal{S}(0, * - 2\pi\omega) . \tag{4.3}$$

Equation (4.3) has already been extensively used (studied?), see [3, 18, 30, 32]. An invariant polarization field which is a spin field, i.e., normalized to unity, is called an *invariant spin field* (ISF). More generally we say that a spin field $\mathcal{S}$ is a "k–turn invariant spin field" if k is a nonzero integer and if $\mathcal{S}(n+k, *) = \mathcal{S}(n, *)$ for all n. Then

$$\mathcal{S}(0, *) = M(k, * - 2\pi k\omega)\mathcal{S}(0, * - 2\pi k\omega) . \tag{4.4}$$

Of course, the concepts ISF and 1–turn invariant spin field are equivalent and every k–turn invariant spin field is a $(-k)$–turn invariant spin field.

Remark:

(1) A spin field $\mathcal{S}$ is a k–turn invariant spin field if (4.4) holds. In particular it is an ISF if (4.3) holds. Note that a k–turn invariant spin field with $k \neq 1$ need not be an ISF. See Example 3. Note also that every ISF is a k–turn invariant spin field for all k. □

The fact that, by definition, polarization fields are 2π–periodic and continuous in phase has the following important implication. Let $\mathcal{S}$ be an invariant polarization field. By Section 4.1 it is clear that S_*, defined by $S_n := \mathcal{S}(n, \phi_0 + 2\pi n\omega)$, is a solution of (2.3) starting at ϕ_0. Since $\mathcal{S}$ is invariant, we have $S_n = \mathcal{S}(0, \phi_0 + 2\pi n\omega)$ so that S_* is ω–quasiperiodic. We thus see that an invariant spin field supplies, for every ϕ_0, a generator of a normalized ω–quasiperiodic solution of (2.3). Most interestingly, off orbital resonance, the converse also holds as the following theorem shows.

Theorem 4.1 *Let ω be off orbital resonance and suppose that there exists a ϕ_0 for which (2.3) has a ω–quasiperiodic and $\mathbb{S}^2$–valued solution starting at ϕ_0, where $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$. Then an ISF exists.*

Proof: Let a ω–quasiperiodic solution S_* of (2.3) for our ϕ_0 be generated by the generator u through the assignment $S_n = u(2\pi n\omega)$. Thus at time n where at $\phi = \phi_0 + 2\pi n\omega$, the spin is $u(2\pi n\omega)$. We can assume (recall Remark 3 in Sec. 2) that u is $\mathbb{R}^3$–valued. Since ω is off orbital resonance it defines spin vectors uniquely on a dense set, i.e. at $\phi = \phi_0 + 2\pi n\omega + 2\pi m$ $(m \in \mathbb{Z}^d, n \in \mathbb{Z})$. We now use u to define a spin field $\mathcal{S}$. Let

$$\mathcal{S}(0, \phi) = u(\phi - \phi_0) . \tag{4.5}$$

Then for each $\phi = \phi_0 + 2\pi n\omega + 2\pi m$, $\mathcal{S}(0, \phi)$ is just S_n. However (4.5) can be employed for all ϕ and $\mathcal{S}(0, *)$ is continuous and 2π–periodic since u is.

We now show that $\mathcal{S}$, defined by (4.5), is an ISF. Because, by assumption, S obeys (2.3), $S_{n+1} = \mathfrak{a}(\phi_0 + 2\pi n\omega)S_n$ so that $u(2\pi(n+1)\omega) = \mathfrak{a}(\phi_0 + 2\pi n\omega)u(2\pi n\omega)$. Let $\phi - 2\pi\omega = \phi_0 + 2\pi n\omega + 2\pi m$ then

$$u(\phi - \phi_0) = \mathfrak{a}(\phi - 2\pi\omega)u(\phi - \phi_0 - 2\pi\omega) , \tag{4.6}$$

i.e. $\mathcal{S}(0, \phi) \equiv u(\phi - \phi_0)$ satisfies (4.3) on a dense set of ϕ's. But $\mathfrak{a}$ and u are continuous so that (4.6) is satisfied for all ϕ. It remains to be proven that u is $\mathbb{S}^2$–valued. Since ω is off orbital resonance we have a dense set of points ϕ for which $u(\phi)$ is $\mathbb{S}^2$–valued. Thus, by the continuity of u, u is $\mathbb{S}^2$–valued for all ϕ. Then $\mathcal{S}(0, \phi)$ is a normalized solution of (4.3) for all ϕ. It is therefore an ISF. □

It is widely believed that it is a nontrivial problem, by analytical means, to discover whether a spin-orbit system has an ISF. Therefore, in the light of Theorem 4.1 and off orbital resonance, the matter of the existence of a nonzero ω–quasiperiodic solution of (2.3) is nontrivial. Clearly the matter of the existence of ω–quasiperiodic SPF's is also nontrivial. Recall also the remarks at the end of Sec. II. Note that in the proof of Theorem 4.1 we profit from the fact that the ISF need only be continuous in the phase. In contrast, in [1] the ISF is C^1 in the phase so that the statement of Theorem 4.1, which cannot be found in [1], would be more difficult to prove in that context.

Remarks:

(2) By the comments just before Theorem 4.1 and by Theorem 4.1 it is clear that, off orbital resonance, a nonzero ω–quasiperiodic solution of (2.3) exists for *every* ϕ_0, if a nonzero ω–quasiperiodic solution exists for *some* ϕ_0.

(3) If there are two linearly independent quasiperiodic solutions of (2.3) for a ϕ_0 then all solutions for ϕ_0 are quasiperiodic: if S_n^1 and S_n^2 are linearly independent solutions then the vector product $S_n^3 := S_n^1 \times S_n^2$ is a solution and the general solution has the form $\alpha_1 S_n^1 + \alpha_2 S_n^2 + \alpha_3 S_n^3$ for real $\alpha_1, \alpha_2, \alpha_3$. Thus if S_n^1 is χ^1–quasiperiodic and S_n^2 is χ^2–quasiperiodic then S_n^3 is χ–quasiperiodic and then all solutions are χ–quasiperiodic, where $\chi := (\chi^1, \chi^2)$ and where (χ^1, χ^2) denotes the Cartesian product of χ_1 and χ_2. In particular if we have two linearly independent ω–quasiperiodic solutions then all solutions are ω–quasiperiodic.

(4) From the proof of Theorem 4.1 it follows that the function $\mathcal{S}$, defined by $\mathcal{S}(n,\phi) := t^3(\phi)$, is an ISF if t^3 is the function in Theorem 3.2. Note also that $\mathcal{S}(n,\phi) = u^3(\phi)$ if u^3 is the third column of the function u which occurs in the proof of Theorem 3.2. □

?

4.3 Stroboscopic sequences

For every polarization field $\mathcal{S}$ we define its "stroboscopic sequence" $\{\mathcal{S}^N\}_{N=1}^{\infty}$ by $\mathcal{S}^N(n,\phi) := (1/N)\sum_{k=0}^{N-1} \mathcal{S}(n+k,\phi)$. Of course, $\mathcal{S}^1 = \mathcal{S}$ and, in the special case when $\mathcal{S}$ is n–independent (e.g. an ISF), we have $\mathcal{S}^N = \mathcal{S}$. We now show that every $\mathcal{S}^N$ is a polarization field. Because the dynamical system (2.1),(2.2) is autonomous one expects that the functions $\tilde{\mathcal{S}}$, defined by $\tilde{\mathcal{S}}(n,\phi) := \mathcal{S}(n+k,\phi)$ for integer k, are polarization fields. In fact this is true since it follows from (4.1) that $\tilde{\mathcal{S}}(n+1,\phi) = \mathfrak{a}(\phi - 2\pi\omega)\tilde{\mathcal{S}}(n, \phi - 2\pi\omega)$. On the other hand, because $\mathcal{S}^N$ is a linear superposition of $\tilde{\mathcal{S}}$'s and because (4.1) is linear in $\mathcal{S}$, we obtain

$$\mathcal{S}^N(n+1,\phi) = \mathfrak{a}(\phi - 2\pi\omega)\mathcal{S}^N(n,\phi - 2\pi\omega) \,, \qquad (4.7)$$

which proves that every $\mathcal{S}^N$ is a polarization field. If, for a given ϕ, the stroboscopic sequence $\mathcal{S}^N(n,\phi)$ converges for every n as $N \to \infty$, then the limit is called the *stroboscopic average* of $\mathcal{S}$ at ϕ. Since, in any case, one has

So as $N \to \infty$ this LHS $\to 0$

$$\mathcal{S}^N(n+1,\phi) - \mathcal{S}^N(n,\phi) = \frac{1}{N}\Big(\mathcal{S}(n+N,\phi) - \mathcal{S}(n,\phi)\Big) \,, \qquad (4.8)$$

$O(S(n,\phi))$ if finite

it follows from (4.7) that, if the stroboscopic average exists at every ϕ, then it is a time invariant polarization field. It is this that makes the stroboscopic sequences important [17]. In particular if the stroboscopic average, say $\hat{\mathcal{S}}$, exists for every ϕ and is nowhere zero then, as follows from Section 4.2, $\hat{\mathcal{S}}/|\hat{\mathcal{S}}|$ is an ISF. Therefore stroboscopic sequences are used, especially in simulations, to compute ISF's (see also Section 8.3). It is also clear by (4.7) that the stroboscopic average exists for every ϕ iff (=if and only if) $\mathcal{S}^N(0,\phi)$ converges for every ϕ as $N \to \infty$.

The following algorithm for computing $\mathcal{S}^N(0,\phi)$ is often useful. We first conclude from (4.2) and from the definition of $\mathcal{S}^N(0,\phi)$ that

$$\mathcal{S}^N(0,\phi) = \frac{1}{N}\sum_{k=0}^{N-1} \mathcal{S}(k,\phi) = \frac{1}{N}\sum_{k=0}^{N-1} M(k, \phi - 2\pi k\omega)\mathcal{S}(0, \phi - 2\pi k\omega) \,. \qquad (4.9)$$

Using (2.11) we also have

$$I = M(0;\phi) = M(k;\phi - 2\pi k\omega)M(-k;\phi) ,$$

so that (4.9) yields

$$\mathcal{S}^N(0,\phi) = \frac{1}{N}\sum_{k=0}^{N-1} (M(-k,\phi))^T \mathcal{S}(0,\phi - 2\pi k\omega) , \qquad (4.10)$$

which is the promised algorithm.

For a detailed study of polarization fields and their stroboscopic averages, see also [?]. We return to stroboscopic averaging in Section 8 where we discuss the time averaged polarization and in Section F where we apply it to Example 3.

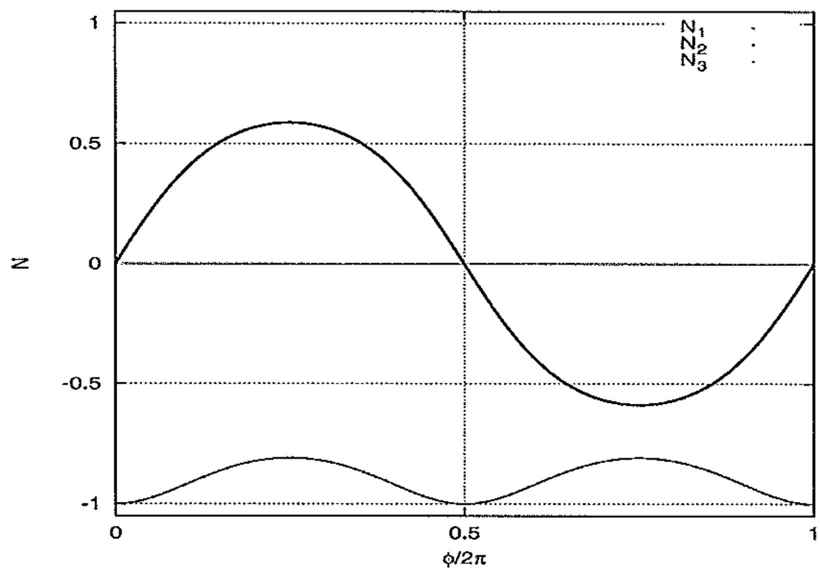

Figure 1: The three components of the function $\mathcal{N}$ from (F.8) for the case $\epsilon = 0.4$ vs. the fractional normalised phase $\lfloor\phi/2\pi\rfloor$, generated as described in Remark 4 of Section 4.4. Note that $\mathcal{N} = \mathcal{S}(0,*)$ where $\mathcal{S}$ is a 2-turn ISF for Example 3 with $\epsilon = 0.4$

4.4 The Examples

We now seek n–turn invariant spin fields for the examples.

Th. 4.1

Example 1 Since E is real and skew–symmetric, there is a constant y in $\mathbb{S}^2$ such that $Ey = 0$. If we choose a spin field $\mathcal{S}$ such that $\mathcal{S}(0,\phi) = \exp(\mathcal{J}\phi)y$ we obtain, by (2.17),

$$\mathfrak{a}(\phi - 2\pi\omega)\mathcal{S}(0,\phi - 2\pi\omega) = \Phi(2\pi;\phi - 2\pi\omega)\exp(\mathcal{J}[\phi - 2\pi\omega])y$$
$$= \exp(\mathcal{J}\phi)\exp(2\pi E)y = \exp(\mathcal{J}\phi)y = \mathcal{S}(0,\phi) ,$$

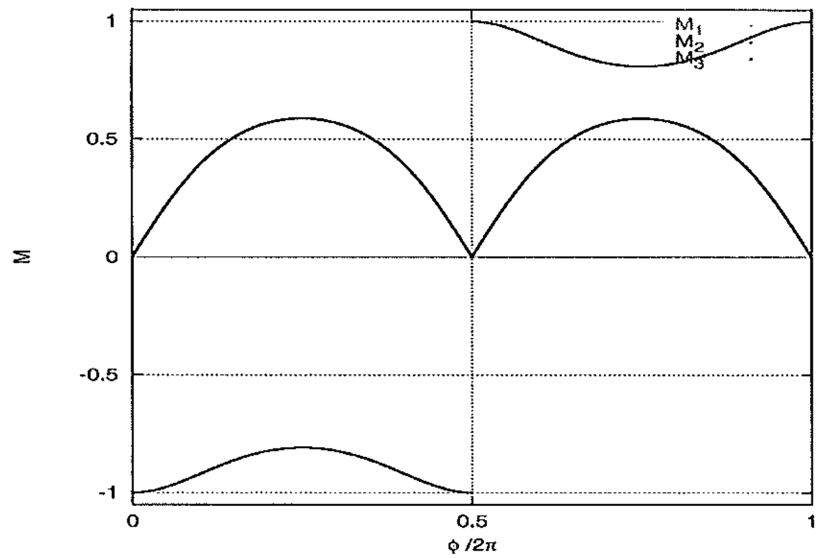

Figure 2: The three components of the function $\mathcal{M}$ vs. the fractional normalised phase $\lfloor \phi/2\pi \rfloor$, generated as described in Remark 4 of Section 4.4. Note that $\mathcal{M}$ fulfills, for Example 3 with $\epsilon = 0.4$, all conditions of an ISF except for continuity

so that the chosen $\mathcal{S}$ is an ISF. Moreover, by Remark 1 a k–turn invariant spin field exists for every k. Thus by Section 4.2, a normalized ω–quasiperiodic solution of (2.3) exists for every ϕ_0.

Example 2 We choose a spin field $\mathcal{S}$ such that $\mathcal{S}(0,\phi) = e^3$ where $e^3 := (0,0,1)$ gives the components of the unit vector perpendicular to the plane of the ring.

Then by (2.10), (2.20) and the relation $\mathcal{J}e^3 = 0$ we obtain $M(n;\phi)e^3 = e^3$ so that by (4.2)

$$\mathcal{S}(n,\phi) = e^3 = \mathcal{S}(0,\phi)\ .$$

Thus the chosen $\mathcal{S}$ is an ISF. Recalling Section 4.2, it follows that a normalized ω–quasiperiodic solution of (2.3) exists for every ϕ_0. Moreover, by Remark 1 a k–turn invariant spin field exists for every k.

Example 3 For every ϵ, a 2–turn invariant spin field exists. An ISF exists iff ϵ is an integer. Moreover, for every ϕ_0, a normalized ω–quasiperiodic solution of (2.3) exists. These facts will be proved in Appendix F. Since no ISF exists if ϵ is not an integer we see that the assumption that ω is off orbital resonance is indispensable in Theorem 4.1. We will see in Section 6.2 that a normalized ω–quasiperiodic solution of (2.3) exists for every ϕ_0.

Example 4 As in Example 2, $\mathcal{S} = e^3$, the perp to the plane of the ring, is an ISF. Recalling Section 4.2, it follows that a normalized ω–quasiperiodic solution of (2.3) exists for every ϕ_0. Moreover, by Remark 1 a k–turn invariant spin field exists for every k.

BLABLA: ISF is parallel to e^3, the perpendicular to the plane of the ring.

Remark:

(4) The significance of the brief but precise statements on Example 3 will now be outlined as follows. Recalling Remark 1 in Section 4.2, every ISF $\mathcal{S}$ is a 2-turn ISF so that by (4.4)

$$\mathcal{S}(0,\phi) = M(2,\phi)\mathcal{S}(0,\phi) \; .$$

Then the ISF for each ϕ can be obtained as a normalized eigenvector of $M(2,\phi)$. This is a specific case of the use of multi–turn maps suggested in [1, Sec. X] and experience shows that such a procedure is often valid.

However, since for Example 3c no ISF exists, care is needed as we now show for the case $\epsilon = 0.4$. In Section F it is shown that in this case exactly two 2–turn invariant spin fields $\pm\mathcal{S}$ exist where $\mathcal{N} = \mathcal{S}(0,*)$ is given by (F.8). Figure 1 shows the components of $\mathcal{N}(\phi)$ plotted versus $\lfloor\phi/2\pi\rfloor$ on the interval $[0,1)$. Since $\mathcal{N}$ is 2π–periodic and continuous we obtain curves which are single valued and continuous.

The non-existence of a 1–turn ISF is illustrated in figure 2 which shows the components of a function $\mathcal{M}$ plotted versus $\lfloor\phi/2\pi\rfloor$ on the interval $[0,1)$ which fulfills all conditions of an ISF except for continuity. The curves are obtained in two steps. First, the $\mathcal{M}$ is constructed for $0 \le \phi/2\pi < 1/2$ from the $\mathcal{N}$ of (F.8). Then $\mathcal{M}$ is propagated for two or more turns and we obtain curves representing a function which are single valued and which are continuous except at $\lfloor\phi/2\pi\rfloor = 0, 1/2$ where the third component has a discontinuity in sign. If the $\mathcal{N}$ of (F.8) had been evaluated for the range $0 \le \phi/2\pi < 1$ and if it would have been propagated for two or more turns, a double valued function in $\lfloor\phi/2\pi\rfloor$ would have been generated.

The function of figure 2 provides a demonstration of the assertion that Example 3c has no ISF of the kind defined in this paper. However, this function would be an ISF if one would relax the regularity condition (i.e., continuity in ϕ) of the present paper to a weaker one as was done in [11, 5] where sign discontinuities are allowed. Moreover, if one allows for discontinuous ISF's then one observes that besides the function of figure 2 there are many other ISF's. In fact, in [4, 5] it is demonstrated that for rational tunes, an arbitrary number of different ISF's of the kind defined there can be generated by introducing sign discontinuities by hand. In contrast, as explained in Remark 4 in Sec. VIII, for the regularity condition of the present paper, off orbital resonance, an ISF is unique up to a global sign.

Figure 2 illustrate the claim first made in [4] that for snake "resonances" the discontinuities can be shifted but not eliminated. Analogous phenomena are seen for the ω of other snake "resonances". More details can be found in [5] where emphasis is put on the case of $\omega = 1/6$ and where the cases of $\omega = 1/3$ and $1/4$ are treated too. In these latter cases, ISF's of the kind defined here do exist.

PUT IN SOME STUFF ON INTEGER ϵ

The unavoidability of sign discontinuities at snake "resonance" tunes is contrary to the claim in [6]. In particular, the fields of spins represented by the smooth curves given there in figures 7 and 8 are not 1–turn invariant. Therefore they represent neither the ISF's of the kind defined in this paper nor those defined in [4, 5].

5 Spin frequencies

5.1

Now that we have encountered an example of (ω, ν)–quasiperiodicity, where ν is a real number, i.e., $(\omega, \nu) \in \mathbb{R}^{d+1}$, we are led to suggest the following definition.

Definition 5.1 (Spin frequency): We denote by $\hat{\Xi}(\phi_0)$ the set of those ν in $[0, 1)$ with the following property: all solutions of (2.3), starting at ϕ_0, are (ω, ν)–quasiperiodic. If a real constant ν exists which is in each $\hat{\Xi}(\phi_0)$ then we call ν a "spin frequency". Thus a real constant ν in $[0, 1)$ is a spin frequency iff, for every ϕ_0, the solutions of (2.3) are (ω, ν)–quasiperiodic. □

Remarks:

(1) The restriction of ν to the interval $[0, 1)$ suffices since, as is clear from (2.27), the integer parts of the components of the tune vector χ have no effect. Since $M(n; \phi_0)$ is 2π–periodic in ϕ_0 it is also clear that $\hat{\Xi}(\phi_0)$ is 2π–periodic in ϕ_0.

(2) It follows easily from Definition 5.1 that if ν is, for some ϕ_0, in $\hat{\Xi}(\phi_0)$ then an element in $[0, 1)$ of the form $\varepsilon\nu + j_0 + j \cdot \omega$ is in $\hat{\Xi}(\phi_0)$, if $(\varepsilon, j_0, j) \in \{-1, 1\} \times \mathbb{Z} \times \mathbb{Z}^d$. In fact if S_* is a solution of (2.3) starting at ϕ_0 then a generator f of S_* exists such that $S_n = f(2\pi\omega n, 2\pi\nu n)$ so that g, defined by $g(\phi, \psi) := f(\phi, \epsilon\psi - \epsilon j \cdot \phi)$, is a generator of S_* such that $S_n = g(2\pi\omega n, 2\pi(\epsilon\nu + j_0 + j \cdot \omega)n)$. Thus it is convenient to introduce an equivalence relation on $[0, 1)$ by defining constants ν_1 and ν_2 in $[0, 1)$ to be equivalent - and writing $\nu_1 \sim \nu_2$ - iff there exist $(\varepsilon, j_0, j) \in \{-1, 1\} \times \mathbb{Z} \times \mathbb{Z}^d$ such that $\nu_2 = \varepsilon\nu_1 + j_0 + j \cdot \omega$. The equivalence relation partitions the interval into equivalence classes $[\nu] := \{\mu \in [0, 1) : \mu \sim \nu\}$ such that $[\nu_1] = [\nu_2]$ iff $\nu_1 \sim \nu_2$ and $[\nu_1] \cap [\nu_2] = \emptyset$ iff $\nu_1 \not\sim \nu_2$. Thus if, for a ϕ_0, ν is in $\hat{\Xi}(\phi_0)$ then $[\nu] \subset \hat{\Xi}(\phi_0)$. It follows that, if ν is a spin frequency, then every element in $[\nu]$ is a spin frequency, too. □

With Definition 5.1 of spin frequency we can now reexamine the results of Sec. 4. Then from Corollary 3.3 it is clear that the sets $\hat{\Xi}(\phi_0)$ have the element $\lfloor\nu\rfloor$ in common where ν is given by (3.4) and where $\lfloor\nu\rfloor$ denotes the *fractional part* of ν. This is therefore a spin frequency. Note also that if, for a ϕ_0, a solution S_* of (2.3) starting at ϕ_0 is ω–quasiperiodic, then for every $\nu \in [0, 1)$, a generator f exists for this ϕ_0 such that $S_n = f(2\pi\omega n, 2\pi\nu n)$. Thus if for a ϕ_0 every solution S_* of (2.3) starting at ϕ_0 is ω–quasiperiodic then $\hat{\Xi}(\phi_0) = [0, 1)$. In particular if for every ϕ_0 every solution S_* of (2.3) starting at ϕ_0 is ω–quasiperiodic then every element in $[0, 1)$ is a spin frequency.

5.2 The Examples

We now check if the examples have spin frequencies, and related questions.

Example 1 By (2.18),

$$W^{-1} E W = \sigma \mathcal{J} , \tag{5.1}$$

where the $SO(3)$–matrix W is defined by $W := [w_1, w_2, w_3]$ with

$$w_1 := (\delta, 0, -\epsilon)/\sigma \ , \qquad w_2 := e^2 := (0,1,0) \ , \qquad w_3 := w_1 \times w_2 \ ,$$

and where $\sigma := \sqrt{\delta^2 + \epsilon^2}$. Then, by (2.17) and (2.19) we can write,

$$M(n;\phi_0) = \Phi(2\pi n;\phi_0) = \exp(\mathcal{J}[\phi_0 + 2\pi n\omega])W\exp(\mathcal{J}\sigma 2\pi n)W^T\exp(-\mathcal{J}\phi_0) \ . \quad (5.2)$$

Thus every solution of (2.3) is $(\omega, \lfloor\sigma\rfloor)$–quasiperiodic. In particular, $\lfloor\sigma\rfloor$ is a spin frequency. Note that, because an ISF exists (recall Sec. 4.4), we have a normalized ω–quasiperiodic solution of (2.3) for every ϕ_0.

Example 2 By (2.20) and (2.10), and for arbitrary n and ϕ_0,

$$M(n;\phi_0) = \exp(\mathcal{J}[nN_1 + N_2 n\phi_0 + \pi N_2 n(n-1)\omega]) \ . \quad (5.3)$$

We now assume that there is a ϕ_0 where $M(*;\phi_0)$ is quasiperiodic. Then with $e^1 := (1,0,0)$ giving the components of the radial unit vector in the plane of the ring, $M(*;\phi_0)e^1$ is quasiperiodic so that, by (5.3), $\exp(i[nN_1 + N_2 n\phi_0 + \pi N_2 n(n-1)\omega])$ is a quasiperiodic function of n. However, by Corollary A.2d, this function cannot be quasiperiodic since $N_2\omega/2$ is irrational. Thus our assumption that $M(*;\phi_0)$ is quasiperiodic, is false. Since $M(*;\phi_0)$ is not quasiperiodic, every $\hat{\Xi}(\phi_0)$ is empty so that no spin frequency exists. However, recalling Sec. 4.4, we know that for every ϕ_0 at least one normalized ω–quasiperiodic solution of (2.3) exists.

Example 3 CHECK

As demonstrated in Appendix F, each $\hat{\Xi}(\phi_0)$ is nonempty, so that all solutions of (2.3) are quasiperiodic. Appendix F also shows that no spin frequency exists unless ϵ is an even integer.

Example 4 As shown in Section 8.4, $\lfloor\sigma_1 + N\omega\rfloor$ is a spin frequency for every integer N.

6 Uniform precession rates and spin tunes

Since our dynamical system (2.3) is ω–quasiperiodic in time it is natural to seek a generalized Floquet theorem, i.e., to factorize $M(*,\phi_0)$ into a ω–quasiperiodic part and a part which oscillates with a generalized Floquet frequency. Moreover, as we shall see, a generalized Floquet structure provides a clean and effective way to characterize the frequency content of the solutions and to recognize exceptions. It will become clear that this Floquet ansatz is closely related to the (ω,ν)–quasiperiodicity of Sec. V. We are then led to the concepts of "spin tune" and "spin–orbit resonance".

6.1

From Sec. 2.2 and Sec. 3 it is clear $t(\phi_0 + 2\pi\omega n)$ in Theorem 3.2 is an SPF starting at ϕ_0 and that $\exp(i[\lambda_p(\phi_0 + 2\pi\omega\, n) + N\cdot\phi_0 + 2\pi N\cdot\omega n])$ is the phase function of the SPF.

Moreover, the SPF $u(\phi_0 + 2\pi\omega n)$ which occurs in the proof of Corollary 3.3 has the simple phase function $\exp(i2\pi n\nu)$ in which the phase advances uniformly with the constant rate per turn, ν given by (3.4). Therefore we call a SPF starting at ϕ_0, U, a "uniform precession frame (UPF) starting at ϕ_0" if the phase function is of the form $\exp(i2\pi n\nu)$ where $\nu \in [0,1)$, i.e., for all n

$$M(n;\phi_0) = U(n)\exp(\mathcal{J}\nu 2\pi n)U^T(0) . \tag{6.1}$$

Clearly the restriction of the real constant ν to the interval $\nu \in [0,1)$ brings no loss of generality. We call ν the "uniform precession rate (UPR)" corresponding to U. Of course any UPR is uniquely determined by the corresponding UPF, as can be shown by solving (6.1) for $\exp(\mathcal{J}\nu 2\pi n)$. The converse is not true, i.e. different UPF's can have the same UPR.

Of basic interest are those UPF's which are ω–quasiperiodic since, as will become clear in this section, they are closely linked with the concept of generalized Floquet parameters. We have already encountered ω–quasiperiodic UPF's in Section 3. In particular, the proof of Corollary 3.3 showed for arbitrary ϕ_0 that $u(\phi_0 + 2\pi\omega n)$, as a function of n, is a ω–quasiperiodic UPF starting at ϕ_0 with UPR $\lfloor\nu\rfloor$ where ν is given by (3.4). However ω–quasiperiodic UPF's arise even in more general situations, e.g. on orbital resonance. Note that ω–quasiperiodic UPF's parallel the "proper UPF's " in [1].

The following remark is useful for checking that an SPF is a UPF.

Remark:

(1) Let U be a SPF starting at a ϕ_0. Then if U is a UPF starting at that ϕ_0 and with UPR ν then, for all n,

$$U^T(n+1)\mathfrak{a}(\phi_0 + 2\pi n\omega)U(n) = \exp(2\pi\nu\mathcal{J}) . \tag{6.2}$$

Conversely, if a ν exists in $[0,1)$ such that (6.2) holds for all n, then U is a UPF starting at ϕ_0 and with UPR ν. These statements follow from (2.10) and (2.11). □

We denote by $\Xi(\phi_0)$ the set of those UPR's which correspond to an ω–quasiperiodic UPF starting at ϕ_0 and the set, $\bigcup_{\phi_0\in\mathbb{R}^d}\Xi(\phi_0)$, of all those UPR's is denoted by Ξ. Since $M(n;\phi_0)$ is 2π–periodic in ϕ_0 it is clear that $\Xi(\phi_0)$ is 2π–periodic in ϕ_0.

The connection of the $\Xi(\phi_0)$ with the concept of generalized Floquet parameters comes about as follows. We say that the spin-orbit system has a "generalized Floquet parametrization at ϕ_0" if an ω–quasiperiodic $SO(3)$–matrix p and a constant real matrix B exist such that $p(0) = I$ and such that, for all n, $M(n;\phi_0) = p(n)\exp(nB)$. It is then easy to see with (6.1) that $\Xi(\phi_0)$ is nonempty iff a generalized Floquet parametrization exists at ϕ_0. In fact, if U is an ω–quasiperiodic UPF starting at ϕ_0 with UPR ν then p and B defined by $p(n) := U(n)U^T(0), B := \nu U(0)\mathcal{J}U^T(0)$, provide a generalized Floquet parametrization at ϕ_0. Conversely, if generalized Floquet parameters p and B are given at ϕ_0 then (see e.g. [1, Lemma 2.1]) then a constant $SO(3)$–matrix W exist and a $\nu \in [0,1)$ exist such that $W^TBW = 2\pi\nu\mathcal{J}$, whence U, defined by $U(n) := p(n)W$, is a ω–quasiperiodic UPF starting at ϕ_0 with UPR ν. An analogous result was obtained in [1, Sec. V].

In this paper we adopt the point of view that usually a generalized Floquet parametrization exists at every ϕ_0. Among our examples, only Example 2 turns out to be a counterexample.

The concept of the ω–quasiperiodic UPF leads us to the following

Definition 6.1 (spin tune): The spin–orbit system is said to be "well–tuned" if all $\Xi(\phi_0)$ are nonempty and equal. Otherwise it is said to be "ill–tuned". Of course, for a well tuned spin–orbit system all $\Xi(\phi_0)$ are equal to Ξ. For a well–tuned spin–orbit system we call the elements of Ξ "spin tunes". If the spin–orbit system is well–tuned then it is said to be *on spin–orbit resonance* if 0 is a spin tune. If the spin–orbit system is well–tuned and not on spin–orbit resonance, then it is said to be *off spin–orbit resonance.* □

Defined before?

The above definition of well-tuning is related to the issue of the "reference azimuth" θ_0 underlying our map formalism (recall Remark 1 in Section 2.1). We denote by Θ the set of those azimuth values θ where $\mathcal{A}$ is of class C^1. Of course, Θ is an open set, $\theta_0 \in \Theta$ and the set $[0, 2\pi) \setminus \Theta$ has at most finitely many elements. For a reference azimuth $\theta \in \Theta$ we denote by $\Xi_\theta(\phi_0)$ the set of those UPR's which correspond to ω–quasiperiodic UPF's starting at ϕ_0. Clearly, $\Xi(\phi_0) = \Xi_{\theta_0}(\phi_0)$. The following proposition gives strength to the concept of well–tuning.

Proposition 6.1: For every $\theta \in \Theta$ and every ϕ_0,

$$\Xi_\theta(\phi_0 + (\theta - \theta_0)\omega) = \Xi(\phi_0) \, . \tag{6.3}$$

Moreover, if the spin-orbit system is well-tuned then, for every $\theta \in \Theta$ and every ϕ_0,

$$\Xi_\theta(\phi_0) = \Xi(\phi_0) \, . \tag{6.4}$$

Monster theorem?

Also, if $d = 1, \omega \neq 0$ and $\Xi \neq \emptyset$ then the spin-orbit system is well-tuned iff (6.4) holds for every $\theta \in \Theta$ and every ϕ_0.

Proof: We first express the $\Xi_\theta(\phi_0)$ in terms of the $\Xi(\phi_0)$. We define the "n–turn spin transfer matrix" for the reference azimuth $\theta \in \Theta$ by $M_\theta(n;\phi_0) := \Phi(\theta + 2\pi n, \theta; \phi_0, \omega)$. It is clear by Remark 1 in Sec. II.1 and by the definition of M that $M = M_{\theta_0}$. It also follows that, for every $\theta \in \Theta$ and every ϕ_0,

$$M_\theta(n;\phi_0) =$$
$$\Phi(\theta, \theta_0; \phi_0 + (2\pi n + \theta_0 - \theta)\omega, \omega) M(n; \phi_0 + (\theta_0 - \theta)\omega) \Phi^T(\theta, \theta_0; \phi_0 + (\theta_0 - \theta)\omega, \omega) \, . \tag{6.5}$$

Let ν be in $\Xi(\phi_0)$. Then a corresponding ω–quasiperiodic UPF U exists which starts at ϕ_0 and which satisfies (6.1). It follows from (6.1) and (6.5) that

$$\begin{aligned} M_\theta(n; \phi_0 + (\theta - \theta_0)\omega) &= \Phi(\theta, \theta_0; \phi_0 + 2\pi n\omega, \omega) M(n;\phi_0) \Phi^T(\theta, \theta_0; \phi_0, \omega) \\ &= \Phi(\theta, \theta_0; \phi_0 + 2\pi n\omega, \omega) U(n) \exp(\mathcal{J}\nu 2\pi n) U^T(0) \Phi^T(\theta, \theta_0; \phi_0, \omega) \\ &=: U_\theta(n) \exp(\mathcal{J}\nu 2\pi n) U_\theta^T(0) \, , \end{aligned} \tag{6.6}$$

where

$$U_\theta(n) := \Phi(\theta, \theta_0; \phi_0 + 2\pi n\omega, \omega)U(n) \ . \tag{6.7}$$

Since $\Phi(\theta, \theta_0; \phi, \omega)$ is 2π–periodic and continuous in ϕ, (6.7) means that U_θ is ω–quasiperiodic. Thus, and by (6.6), U_θ is, for the reference azimuth θ, an ω–quasiperiodic UPF starting at $\phi_0 + (\theta - \theta_0)\omega$ whose UPR is ν so that $\nu \in \Xi_\theta(\phi_0 + (\theta - \theta_0)\omega)$. Because ν is an arbitrary element in $\Xi(\phi_0)$ we obtain $\Xi(\phi_0) \subset \Xi_\theta(\phi_0 + (\theta - \theta_0)\omega)$. Since θ and θ_0 play symmetric roles we also have the reverse inclusion $\Xi_\theta(\phi_0 + (\theta - \theta_0)\omega) \subset \Xi(\phi_0)$ so that (6.3) is valid for every ϕ_0 and for every $\theta \in \Theta$.

It is now easy to prove the last two claims of the proposition. If the spin-orbit system is well-tuned then (6.4) follows from (6.3). To prove the third and last claim, let $d = 1, \omega \neq 0$ and $\Xi \neq \emptyset$. If the spin-orbit system is well-tuned then we already know from the second claim that (6.4) holds. To complete the proof of the third claim, we assume that (6.4) holds for every $\theta \in \Theta$ and every ϕ_0. Thus, and by (6.3), we have, for every $\theta \in \Theta$ and every ϕ_0,

$$\Xi(\phi_0 + (\theta - \theta_0)\omega) = \Xi(\phi_0) \ . \tag{6.8}$$

Since $[0, 2\pi)$ is an open set there exists an open interval $\mathcal{I} := (\theta_0 - \delta\theta, \theta_0 + \delta\theta)$ around θ_0 and such that $\mathcal{I} \subset \Theta$. Because (6.8) holds for every $\theta \in \mathcal{I}$ and every ϕ_0 we have, for every ϕ_0 and every $\phi \in (\phi_0 - \delta\theta|\omega|, \phi_0 + \delta\theta|\omega|)$,

$$\Xi(\phi) = \Xi(\phi_0) \ . \tag{6.9}$$

Since ϕ_0 is arbitrary in (6.9) it follows that (6.9) holds for every ϕ and every ϕ_0. Thus the spin-orbit system is well-tuned and the third claim is proved. □

Note that the main content of Proposition 6.1 is the statement about the case $d = 1$ and that this therefore covers all four of our examples. We leave open here the question if the statements of Proposition 6.1 would hold in the flow formalism of [?].

Theorem 6.2 *For every ϕ_0, $\Xi(\phi_0) \subset \hat{\Xi}(\phi_0)$. Thus if $\nu \in \Xi(\phi_0)$ then every solution of (2.3) is (ω, ν)–quasiperiodic. Moreover, every spin tune is a spin frequency.*

Proof of Theorem 6.2: Let $\nu \in \Xi(\phi_0)$ for a ϕ_0. Then there exists a ω–quasiperiodic UPF starting at ϕ_0 and, for every integer n, (6.1) holds and $U(n) = u(2\pi n\omega)$ where u is a generator of U. Thus m, defined by $m(x, y) := u(x)\exp(\mathcal{J}y)u^T(0)$, is a generator of $M(*; \phi_0)$ since $M(n; \phi_0) = m(2\pi n\omega, 2\pi n\nu)$. Thus every solution of (2.3) starting at ϕ_0 is (ω, ν)–quasiperiodic and therefore $\nu \in \hat{\Xi}(\phi_0)$. This proves that $\Xi(\phi_0) \subset \hat{\Xi}(\phi_0)$. Let now ν be a spin tune. Then, for all ϕ_0, ν is in $\Xi(\phi_0)$. Since $\Xi(\phi_0) \subset \hat{\Xi}(\phi_0)$ it thus follows that, for all ϕ_0, ν is in $\hat{\Xi}(\phi_0)$. This proves that ν is a spin frequency. □

Remark:

(2) It is clear by Theorem 6.2 that, if the spin–orbit system is well–tuned, then for every ϕ_0, every solution of (2.3) starting at ϕ_0 is (ω, ν)–quasiperiodic where ν is a spin tune. It also follows from Theorem 6.2 that, on spin–orbit resonance, 0 is a spin frequency so that every solution of (2.3) is ω–quasiperiodic. Conversely, if every solution of (2.3) is ω–quasiperiodic then $M(*; \phi_0)$ is a ω–quasiperiodic UPF starting at ϕ_0 and the UPR is 0. Thus the spin–orbit system is on spin–orbit resonance iff every solution of (2.3) is ω–quasiperiodic. □

6.2 The Examples

We now confront our examples with the results of Sec. VI.1 and the sets $\Xi(\phi_0)$.

Example 1 By (5.2), and for every ϕ_0, $U(n) := \exp(\mathcal{J}[\phi_0 + 2\pi n\omega])W$ is a ω–quasiperiodic UPF starting at ϕ_0 and $\lfloor\sigma\rfloor$ is the corresponding UPR. Thus $\lfloor\sigma\rfloor$ is contained in every $\Xi(\phi_0)$.

Example 2 As shown in Sec. 5.2, every $\hat{\Xi}(\phi_0)$ is empty. Hence, by Theorem 6.2, every $\Xi(\phi_0)$ is empty so that no generalized Floquet parametrization exists at any ϕ_0. In particular, the system is ill–tuned and has no spin tune.

Example 3 Every $\Xi_\theta(\phi_0)$ is nonempty. If ϵ is an even integer then this spin–orbit system is well–tuned and on spin–orbit resonance and 1/2 is a spin tune. If ϵ is not an even integer then Ξ has uncountably many elements. These claims are demonstrated in Appendix F. It follows from these claims and Proposition 6.1 that, if ϵ is not an even integer, the system exhibits dependence of $\Xi_\theta(\phi_0)$ on θ. This behavior is typical for orbital resonance with rational orbital tunes.

Example 4 As shown in Section 8.4, the spin–orbit system is well–tuned and $\Xi = [\lfloor\sigma_1\rfloor]$. In particular the spin–orbit system is on spin–orbit resonance iff $\sigma_1 \in Y_\omega$. ✗

7 Fourier Analysis

The $u(\phi_0 + 2\pi\omega n)$ which occurs in the proof of Corollary 3.3 is, as a function of n, an ω–quasiperiodic UPF starting at ϕ_0. Thus in the generic situation of Corollary 3.3 ω–quasiperiodic solutions of (2.3) and ω–quasiperiodic UPF's invite further study. In particular since quasiperiodic functions can be Fourier analyzed (see Appendix A) we can use Fourier analysis to state and prove two theorems on spectral values and tunes.

7.1

We begin by defining the mean of a quasiperiodic function. For every function $F : \mathbb{Z} \to \mathbb{C}$ we define

$$a_T(F,\lambda) := (T+1)^{-1}\sum_{n=0}^{T} F(n)\exp(-2\pi i n\lambda) \ ,$$

where $\lambda \in \mathbb{R}, T \in \mathbb{N}$ and where $\mathbb{N}$ denotes the set of nonnegative integers. We denote by $\mathfrak{M}(F)$ the set of those real numbers λ for which $a_T(F,\lambda)$ converges as $T \to \infty$. We denote this limit by $a(F,\lambda)$ and we define the "spectrum $\Lambda(F)$ of F" by $\Lambda(F) := \{\lambda \in [0,1)\cap\mathfrak{M}(F) : a(F,\lambda) \neq 0\}$. Note that if $a(F,\lambda)$ exists then, for every integer n, $a(F,\lambda+n) = a(F,\lambda)$. Thus the convention $\Lambda(F) \subset [0,1)$ underlying the definition of the spectrum is not only convenient but is also well founded. If $0 \in \mathfrak{M}(F)$, then $a(F,0)$ exists and is called the *mean of F*.

Furthermore for a multicomponent function F on $\mathbb{Z}$ we define $\mathfrak{M}(F) := \mathfrak{M}(F_1)\cap\mathfrak{M}(F_2)\cap$ ... and $\Lambda(F) := \mathfrak{M}(F)\cap(\Lambda(F_1)\cup\Lambda(F_2)\cup ...)$, where $F_1, F_2, ...$ denote the components of F.

✗ Spin tunes are of the form $\lfloor\sigma_1 + N\omega\rfloor$. So the "sidebands" of the central tune σ_1 are simply spin tunes

Note that (see Corollary A.2c) for a χ–quasiperiodic function F with $\chi \in \mathbb{R}^k$ we have $\Lambda(F) \subset Y_\chi$ where

$$Y_\chi := \{m \cdot \chi + n : m \in \mathbb{Z}^k, n \in \mathbb{Z}\} . \tag{7.10}$$

With these definitions we state the following theorem which reveals relations between the sets Y_ω, $\Xi(\phi_0)$, $\hat{\Xi}(\phi_0)$ and $\Lambda\Big(M(*;\phi_0)\Big)$.

MENTION THAT SPECTRUM OF $M(\cdot,\phi_0)$ IS NONEMPTY IF $\Xi(\phi_0)$ IS NONEMPTY

Theorem 7.1 *a) If ν is in $\Xi(\phi_0)$ then $\Lambda\Big(M(*;\phi_0)\Big) \subset Y_\omega \cup [\nu]$ and $\Lambda\Big(M(*;\phi_0)\Big) \setminus Y_\omega \subset \Xi(\phi_0)$, where the set Y_ω is defined by (7.10). If the spin–orbit system is well–tuned and if ν' denotes a spin tune then, for all ϕ, $\Lambda\Big(M(*;\phi)\Big) \subset Y_\omega \cup [\nu']$.*
b) If ν is in $\Xi(\phi_0)$ then $[\nu] = \Xi(\phi_0)$.
c1) The spin–orbit system is well–tuned iff all $\Xi(\phi_0)$ are nonempty and the elements of Ξ are equivalent w.r.t. $\sim$. The spin–orbit system is well–tuned iff the $\Xi(\phi_0)$ have a common element.
c2) In any case every $\Xi(\phi_0)$ has at most countably many elements. The spin–orbit system is ill–tuned if Ξ has uncountably many elements. Moreover, the spin–orbit system is on spin–orbit resonance iff $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ exist such that

$$\nu = m \cdot \omega + n \tag{7.11}$$

d) If $\Xi(\phi_0) \setminus Y_\omega \neq \emptyset$ then $\Lambda(M(*;\phi_0)) \cap \Xi(\phi_0) \neq \emptyset$.
e) If Ξ has uncountably many elements then no spin frequency exists.

The proof of Theorem 7.1 can be found in Appendix D. □

Remarks:

(0) It follows from Theorem 7.1b that the spin–orbit resonance condition of Definition 6.1 could be replaced by the condition (7.11). In fact, this is the usual practice.

(1) Theorem 7.1 is useful for the interpretation of the spectrum and for the Fourier analysis of solutions of (2.3) in numerical simulations. In particular, Theorem 7.1a suggests how spin tunes can be "measured" by Fourier analysis. ADD STUFF ON PHYSICALITY.

(2) We now draw another, very simple conclusion from Theorem 7.1. While, by Theorem 6.2, $\Xi(\phi_0) \subset \hat{\Xi}(\phi_0)$, Theorem 7.1 means that $\Xi(\phi_0) \neq \hat{\Xi}(\phi_0)$. This can be shown as follows. If all solutions of (2.3) for our ϕ_0 are ω–quasiperiodic (e.g., for the trivial case that $\mathfrak{a} = I$!), then $\Xi(\phi_0) \neq [0,1) = \hat{\Xi}(\phi_0)$, where in the inequality we used the fact that $\Xi(\phi_0)$ is countable by Theorem 7.1c and the fact that $[0,1)$ is uncountable, and where in the equality we used Sec. 5.1. □

It is now clear that the ν, which occurs in Corollary 3.3 and which is defined by (3.4), has the property that, for all ϕ_0, its fractional part $\lfloor\nu\rfloor$ is contained in every $\Xi(\phi_0)$. Hence, and by Theorem 7.1c, $\lfloor\nu\rfloor$ is a spin tune.

In this paper we adopt the point of view that, off orbital resonance, usually a spin tune exists. This is, off orbital resonance, a strengthening of our belief, mentioned in Section 6.1, that usually a generalized Floquet parametrization exists.

The following theorem continues the Fourier analysis contained in Theorem 7.1 by focusing on the case where ω is off orbital resonance.

Theorem 7.2 *If ω is off orbital resonance then the following hold.*
a) Let S_* be a normalized and ω–quasiperiodic solution of (2.3) for a ϕ_0. Then the sequence $\sigma^0, \sigma^1, \sigma^2, ...$ of continuous and 2π–quasiperiodic functions $\sigma^N : \mathbb{R}^d \to \mathbb{R}^3$, defined by

$$\sigma^N(\phi) := \sum_{\substack{m\in Z^d \\ ||m||\le N}} A_{N,m,d} a(S, m\cdot\omega) \exp(-im\cdot\phi_0) \exp(im\cdot\phi) \ , \tag{7.12}$$

where

$$A_{N,m,k} := \prod_{n=1}^{k} \frac{N+1-|m_n|}{N+1} \ , \qquad ||m|| := max_{j=1,...,k}|m_j| \ , \tag{7.13}$$

converges uniformly on $\mathbb{R}^d$ to a continuous and 2π–quasiperiodic function $\sigma : \mathbb{R}^d \to \mathbb{S}^2$ as $N \to \infty$. Moreover an ISF S exists such that $S(0,\phi) = \sigma(\phi)$. Also $S_n = S(n, \phi_0 + 2\pi n\omega)$ and $a(S, m\cdot\omega)\exp(-im\cdot\phi_0)$ is the m-th Fourier coefficient of σ, i.e.

$$a(S, m\cdot\omega)\exp(-im\cdot\phi_0) = \frac{1}{(2\pi)^d}\int_0^{2\pi}\cdots\int_0^{2\pi}\sigma(\phi)\exp(-im\cdot\phi)d\phi_1\cdots d\phi_d \ .$$

b) Let S_* be a solution of (2.3) for a ϕ_0 and let $\Xi(\phi_0)$ be nonempty. Then the sequence $S^0, S^1, S^2, ...$ of functions $S^N : \mathbb{Z} \to \mathbb{R}^3$, defined by

$$S^N(n) := \sum_{\substack{m\in Z^d \\ ||m||\le N}} A_{N,m,d} \exp(2\pi i n m\cdot\omega) a(S_*, m\cdot\omega) \ , \tag{7.14}$$

converges uniformly on $\mathbb{Z}$ as $N \to \infty$ to a function $\hat{S} : \mathbb{Z} \to \mathbb{R}^3$ which is a ω–quasiperiodic solution of (2.3) starting at our ϕ_0. Moreover an ISF S exists such that, for all integers n,

$$\hat{S}(n) = \mathcal{S}(0, \phi_0 + 2\pi n\omega)\hat{S}(0)\cdot\mathcal{S}(0,\phi_0) \ . \tag{7.15}$$

c) If Ξ is nonempty then $\Lambda(M(*;\phi_0))$ is independent of ϕ_0.

The proof of Theorem 7.2 can be found in Appendix E. □

Remarks:

(1) Theorem 7.2a shows how to reconstruct, off orbital resonance, an ISF from an ω–quasiperiodic solution of (2.3).

(2) Theorem 7.2b shows how to construct, off orbital resonance, a ω–quasiperiodic solution of (2.3) from a solution of (2.3) which is not necessarily ω–quasiperiodic.

(3) Theorem 7.2c shows, off orbital resonance, that if the spectrum of the maps $M(*;\phi_0)$ is not the same for all ϕ_0 then no ω–quasiperiodic UPF exists, i.e. $\Xi = \emptyset$. □

Theorems 7.1 and 7.2 cover and extend , in the map formalism, the material of Sec. IX in[1] for the flow formalism.

WHAT DOES THIS TELL US ABOUT OUR ABILITY To DISCOVER SPIN TUNES BY FA? MEASUREABILITY ETC AS IN PAPER I.

7.2 The Examples

We now look at the examples within the context of well-tuning, spin–orbit resonance and the sets $\Xi(\phi_0)$ and $\Lambda(M(*;\phi_0))$.

Example 1 From Section 6.2 we know that $\lfloor\sigma\rfloor$ is contained in every $\Xi(\phi_0)$. Thus, by Theorem 7.1b, this spin–orbit system is well–tuned and $\Xi = [\lfloor\sigma\rfloor]$. In particular $\lfloor\sigma\rfloor$ is a spin tune and by Theorem 7.1a $\Lambda\Big(M(*;\phi_0)\Big) \subset Y_\omega \cup [\lfloor\sigma\rfloor]$. Moreover, it follows from Theorem 7.1c that this spin–orbit system is on spin–orbit resonance iff $\sigma \in Y_\omega$.

Example 2 From (5.3) it follows that a constant $SO(3)$–matrix R exists such that, for arbitrary n and ϕ_0,

$$RM(n;\phi_0)R^{-1} = \begin{pmatrix} \exp(ix) & 0 & 0 \\ 0 & \exp(-ix) & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

where $x := nN_1 + N_2 n\phi_0 + \pi N_2 n(n-1)\omega$. Since $N_2\omega/2$ is irrational, and by Corollary A.2d this means that $\Lambda(M(*;\phi_0))$ is empty for every ϕ_0. In [6] a "spin tune" is defined as the ϕ_0–dependent eigentune of the 1–turn spin map. As we pointed out in [1], this is obviously not a spin tune in our sense. In any case, there is no spin tune in this model: $\Lambda(M(*;\phi_0))$ is empty for every ϕ_0 so that that spin motion simply cannot be characterized by a precession frequency.

Example 3 From Section 6.2 we know that Ξ has uncountably many elements if ϵ is not an even integer. It thus can be concluded from Theorem 7.1c that this spin–orbit system is ill–tuned if ϵ is not an even integer.

Example 4 Since the spin–orbit system is well–tuned and $\lfloor\sigma_1\rfloor$ is a spin tune we have by Theorem 7.1a that $\Lambda\Big(M(*;\phi_0)\Big) \subset Y_\omega \cup [\lfloor\sigma_1\rfloor]$. Mention that the a's are Bessels ?

Remark:

(2b) It is not a coincidence that Example 3 is on orbital resonance, since, as we shall see in Section 8.1, off orbital resonance it cannot happen that Ξ has uncountably many elements. However it can happen that off orbital resonance there are too few elements

in Ξ, e.g. that Ξ is empty. This is the case for Example 2 as we have shown in Sec. 6.2. □

8 Invariant Frame Fields

We now reveal further structures implicit in Sec. 3. In particular we consider the so–called "invariant frame field" (IFF) and related concepts.

8.1

As mentioned in Sec. 3, the function $\mathcal{S}$, defined by $\mathcal{S}(n,\phi) := t^3(\phi)$, is an ISF if t^3 is the function in Theorem 3.2. Analogously, $\mathcal{S}(n,\phi) = u^3(\phi)$ since $u^3(\phi) = t^3(\phi)$. This suggests some further definitions. We call a function $v : \mathbb{Z} \times \mathbb{R}^d \to SO(3)$ a "frame field" if its third column is a spin field and if $v(n,*)$ is continuous and 2π–periodic. A frame field v is called an "invariant frame field" (IFF) if $v(n,*) = v(0,*)$. Of course, the third column of an IFF is an ISF. The concept of frame field is very convenient but the reader should not read too much into the meaning of its name. Because the third column v^3 of a frame field v is a spin field we see that

$$v^T(n+1, * + 2\pi\omega)\mathfrak{a}v(n,*)e^3 = e^3 \ . \tag{8.1}$$

Using the fact that $v^T(n+1,\phi+2\pi\omega)\mathfrak{a}v(n,\phi)$ is continuous and 2π–periodic in ϕ, the application of Appendix C to (8.1) yields the result that a $m(n) \in \mathbb{Z}^d$ and a function $\lambda : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}$ exist such that, for all n and ϕ,

$$v^T(n+1,\phi+2\pi\omega)\mathfrak{a}(\phi)v(n,\phi) = \exp(\mathcal{J}[\lambda(n,\phi) + m(n)\cdot\phi]) \ , \tag{8.2}$$

and such that $\lambda(n,\phi)$ is continuous and 2π–periodic in ϕ. Note that $m(n)$ and $\exp(i\lambda)$ are uniquely determined by v. If the frame field in (8.2) is an IFF, then, for all ϕ,

$$v^T(0,\phi+2\pi\omega)\mathfrak{a}(\phi)v(0,\phi) = \exp(\mathcal{J}[\lambda(0,\phi) + m(0)\cdot\phi]) \ . \tag{8.3}$$

It follows from (8.3) that, for arbitrary n, ϕ,

$$v^T(0,\phi+2\pi n\omega)M(n;\phi)v(0,\phi) = \exp(\mathcal{J}[\mu_n(\phi) + nm(0)\cdot\phi]) \ , \tag{8.4}$$

where $\mu_n : \mathbb{R}^d \to \mathbb{R}$ is a continuous and 2π–periodic function uniquely determined by the conditions: $\mu_0 = 0, \mu_{n+1} = \mu_n + \lambda(0, * + 2\pi n\omega)$. This will be useful in Appendix F. From the remarks at the beginning of this section it is clear that the function v, defined by $v(n,\phi) := t(\phi)$, is an IFF if t is the function in Theorem 3.2. An IFF v is said to be "uniform" if in (8.3) $m(0)$ vanishes and $\lambda(0,\phi)$ is independent of ϕ, i.e., if a constant ν exists in $[0,1)$ such that, for all ϕ,

$$v^T(0,\phi+2\pi\omega)\mathfrak{a}(\phi)v(0,\phi) = \exp(2\pi\nu\mathcal{J}) \ . \tag{8.5}$$

It then follows from (8.5) that, for arbitrary integer n,

$$v^T(0, * + 2\pi n\omega)M(n;*)v(0,*) = \exp(2\pi n\nu\mathcal{J}) \ , \tag{8.6}$$

where ν is a constant. Thus the function v, defined by $v(n,\phi) := u(\phi)$, is a uniform IFF if u is the function in the proof of Theorem 3.2.

Remarks:

(1) Let v be a frame field. Then its third column, v^3, is a spin field. Thus, as mentioned at the end of Sec. 4.1 and for every ϕ_0, the function S_*, defined by $S_n := v^3(n,\phi_0+2\pi n\omega)$ is a normalized solution of (2.3) starting at ϕ_0. Therefore T, defined by $T(n) := v(n,\phi_0+2\pi\omega n)$, is a SPF starting at ϕ_0.

(2) Let v be an IFF. It follows from Remark 1 that T, defined by $T(n) := v(0,\phi_0+2\pi\omega n)$, is an ω–quasiperiodic SPF starting at ϕ_0.

(3a) Let v be a uniform IFF. Then (8.5) holds for all ϕ, where ν is in $[0,1)$. Also, by Remark 2 and for every ϕ_0, the function T, defined by $T(n) := v(0,\phi_0+2\pi\omega n)$, is an ω–quasiperiodic SPF starting at ϕ_0. It follows from (8.5) that, for all n and ϕ_0,

$$\begin{aligned}\exp(2\pi\nu\mathcal{J}) &= v^T(0,\phi_0+2\pi(n+1)\omega)\mathfrak{a}(\phi_0+2\pi n\omega)v(0,\phi_0+2\pi n\omega)\\ &= T^T(n+1)\mathfrak{a}(\phi_0+2\pi n\omega)T(n)\ .\end{aligned}$$

Thus, by Remark 1 in Sec. VI, T is a ω–quasiperiodic UPF starting at ϕ_0 with UPR ν. In particular the spin–orbit system is well–tuned and ν is a spin tune.

(3b) Clearly, an ISF exists if there is an IFF. Nevertheless, the reverse is not always true. However, the proof of this fact is beyond the scope of this paper. It is also clear that if there exists a unit vector e such that a given ISF is nowhere parallel to e then, by a simple orthonormalization, an IFF can be constructed. □

The concept of the IFF leads, off orbital resonance, to the following important theorem which complements Theorem 4.1.

Theorem 8.1 *If ω is off orbital resonance then the following hold.*
a) Let a ω–quasiperiodic SPF start at a ϕ_0. Then an IFF exists. Moreover, for every ϕ_0, an ω–quasiperiodic SPF exists which starts at ϕ_0.
b) Let a ω–quasiperiodic UPF start at a ϕ_0. Then a uniform IFF exists and the spin–orbit system is well–tuned.
c) Let (2.3) have two ω–quasiperiodic solutions S^1_*, S^2_* starting at a ϕ_0 such that $S^1_0\times S^2_0\neq 0$. Then a uniform IFF exists and the spin–orbit system is well–tuned and on spin–orbit resonance.

Proof of Theorem 8.1a: Let T be a ω–quasiperiodic SPF starting at ϕ_0. Then (recall Remark 1 in Sec. 3) a continuous and 2π–periodic function $t:\mathbb{R}^d\to SO(3)$ exists which generates T via the prescription $T(n) = t(\phi_0+2\pi n\omega)$. Defining the function v by $v(n,\phi) := t(\phi)$, it follows from the proof of Theorem 4.1 that the third column of v is an ISF, so that v is an IFF. From Remark 2 it then follows for arbitrary ϕ_0 that T, defined by $T(n) := t(\phi_0+2\pi\omega n)$ is a ω–quasiperiodic SPF starting at ϕ_0. □
Proof of Theorem 8.1b: Let T be a ω–quasiperiodic UPF starting at ϕ_0 with UPR ν. Then, by the proof of Theorem 8.1a, an IFF v exists such that $T(n) = v(0,\phi_0+2\pi n\omega)$. Thus, by

(2.11) and for every integer j, the function T_j, defined by $T_j(n) := v(0, \phi_0 + 2\pi(n+j)\omega)$, is a ω–quasiperiodic UPF starting at $\phi_0 + 2\pi j\omega$ with UPR ν. Thus (8.5) holds for $\phi = \phi_0 + 2\pi j\omega + 2\pi m$ ($m \in \mathbb{Z}^d, j \in \mathbb{Z}$). Because (8.5) holds for a dense set of ϕ's and by the continuity of $v(0,*)$ and $\mathfrak{a}$, (8.5) holds for every ϕ. Thus v is a uniform IFF. Then, by Remark 3, the spin–orbit system is well–tuned. □

Proof of Theorem 8.1c: Because $S_0^1 \times S_0^2 \neq 0$, S_0^1, S_0^2 can be normalized. Then, by the proof of Theorem 4.1, ISF's $\mathcal{S}^1, \mathcal{S}^2$ exist such that $S^1(n) = \mathcal{S}^1(0, \phi_0 + 2\pi n\omega)$ and $S^2(n) = \mathcal{S}^2(0, \phi_0 + 2\pi n\omega)$. Thus by (4.3) the continuous and 2π-periodic function $f : \mathbb{R}^d \to \mathbb{R}$, defined by $f(*) := |\mathcal{S}^1(0,*) \times \mathcal{S}^2(0,*)|$, satisfies the relation $f(*) = f(* + 2\pi\omega)$. Then, by Corollary A.2e, f is constant. Because $|\mathcal{S}^1(0,\phi_0) \times \mathcal{S}^2(0,\phi_0)| = S_0^1 \times S_0^2 \neq 0$, we see that $f \neq 0$ vanishes nowhere. Hence we can define a function $\mathcal{S}^3 : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{S}^2$ by $\mathcal{S}^3(n,*) := (\mathcal{S}^1(0,*) \times \mathcal{S}^2(0,*))/f$. Of course, $\mathcal{S}^3(n,*)$ is continuous and 2π-periodic and for all ϕ one obtains

$$
\begin{aligned}
&\mathfrak{a}(\phi)\mathcal{S}^3(0,\phi) - \mathcal{S}^3(0,\phi+2\pi\omega) = \mathfrak{a}(\phi)\Big(\mathcal{S}^1(0,\phi) \times \mathcal{S}^2(0,\phi)\Big)/f(\phi) \\
&\quad - \Big(\mathcal{S}^1(0,\phi+2\pi\omega) \times \mathcal{S}^2(0,\phi+2\pi\omega)\Big)/f(\phi+2\pi\omega) \\
&= \Big(\mathfrak{a}(\phi)\mathcal{S}^1(0,\phi) \times \mathfrak{a}(\phi)\mathcal{S}^2(0,\phi)\Big)/f(\phi) - \Big(\mathcal{S}^1(0,\phi+2\pi\omega) \times \mathcal{S}^2(0,\phi+2\pi\omega)\Big)/f(\phi) = 0\,,
\end{aligned}
$$

where in the third equation we used (4.3). Thus $\mathcal{S}^3$ is an ISF and $\mathcal{S}^1 \cdot \mathcal{S}^3 = 0$. We therefore define the function $v : \mathbb{Z} \times \mathbb{R}^d \to SO(3)$ by $ve^1 := \mathcal{S}^3 \times \mathcal{S}^1, ve^2 := \mathcal{S}^3, ve^3 := \mathcal{S}^1$ and observe that v is a uniform IFF and fulfills (8.5) with $\nu = 0$. Thus, by Remark 3, the spin–orbit system is well–tuned and 0 is a spin tune. The system is therefore on spin–orbit resonance.
□

Remark:

(4) Let ω be off orbital resonance and let the spin–orbit system be well–tuned and off spin–orbit resonance. Then picking any ϕ_0, we find that $\Xi(\phi_0) \neq \emptyset$ so that, by Theorem 8.1b, a uniform IFF exists. In particular, an ISF $\mathcal{S}$ exists. Using the proof of Theorem 8.1c, it follows that $\mathcal{S}$ and $-\mathcal{S}$ are the only ISF's. In other words, the ISF is unique up to a sign, thus confirming the corresponding finding of [1] in the flow formalism. This behavior was predicted earlier in [?].

(5) Remark 2 shows how an IFF supplies, for every ϕ_0, a generator of a ω–quasiperiodic SPF starting at ϕ_0. Most interestingly, off orbital resonance, the converse holds by Theorem 8.1a. Analogously, Remark 3 shows how a uniform IFF supplies, for every ϕ_0, a generator of a ω–quasiperiodic UPF starting at ϕ_0 and that, off orbital resonance, the converse holds by Theorem 8.1b. In particular, off orbital resonance, a uniform IFF exists iff the system is well tuned. The above mentioned converses could not be proved with the flow formalism of [1]. Theorems 4.1 and 8.1 have in common that they only make assumptions about a *single* ϕ_0 and draw conclusions valid for *every* ϕ_0. Note also that the assumptions of all three parts of Theorem 8.1 are stronger than the assumptions of Theorem 4.1.

(6) By Theorem 8.1b and off orbital resonance, $\Xi(\phi_0)$ is independent of ϕ_0. In particular, off orbital resonance, ill-tuning is impossible if a nonempty $\Xi(\phi_0)$ exists. It also follows from Theorem 7.1c and Theorem 8.1b that, off orbital resonance, Ξ cannot have uncountably many elements.

(7) Note that Theorem 8.1 has no counterpart in [1]

□

We call a frame field v an "n–turn invariant frame field" if, for all integers k, $v(k+n, *) = v(k, *)$. This concept is especially useful if $\omega \in \mathbb{Q}^d$ and we apply it in Appendix F. Note that the third column of an n–turn invariant frame field is an n–turn invariant spin field (recall Remark 1 in Sec. 4). Note also that an n–turn invariant frame field is an $(-n)$–turn invariant frame field. Of course, an IFF is an 1–turn invariant frame field and every 1–turn invariant frame field is an IFF. For arbitrary integer n, every IFF is an n–turn invariant frame field.

8.2

Now that we have the concept of the IFF, we can explain a practical algorithm for getting the spin tune.

As explained in [1, 30, 18, 9, 10, 11, 8], the spin tune is obtained in the spin-orbit tracking code code SPRINT, by logging the turn–to–turn spin phase advance w.r.t. the so–called pseudo–u_1, u_2 axes and then averaging those phase advances to get the average phase advance per turn. In the language of this paper this corresponds to the following operations.

One begins by computing, as pointed out in Section 4.3, an ISF $\mathcal{S}$ by stroboscopic averaging and from that, by simple orthonormalization, an IFF v whose third column is $\mathcal{S}$. Then u_1, u_2 are just the unit vectors corresponding to columns one and two of v. The remaining steps in SPRINT, effectively consist of putting $t := v(0, *)$, and applying equation (3.1) to compute λ_p. This yields the sequence $F(0), F(1), \dots$ where F is the ω–quasiperiodic function generated by λ_p via the prescription $F(n) = \lambda_p(2\pi n\omega)$. By applying Lemma A.1c and (3.4), one observes that, off orbital resonance, $2\pi\nu = a(F, 0)$. The spin tune ν is then obtained by noting that for sufficiently large T

$$2\pi\nu = a(F, 0) \approx a_T(F, 0) \ . \tag{8.7}$$

Note that $a_T(F, 0)$ is obtained by simply tracking along orbits to accumulate the F's over a sufficient number T of turns. In applications $T \approx 1000$ usually suffices.

8.3 The Time–Averaged Polarization

In [1, Sec. I] we discussed the significance of the ISF for calculating the maximum equilibrium polarization and, citing [18, 30], the time–averaged polarization on single tori and for the whole phase space. We also referred to the large oscillations in Figure 9 in [17]. See [5] also. In that case all spins on the torus are set initially parallel to the ISF on the closed orbit [1], so that the initial polarization is 1. Then the polarization repeatedly returns to this value over a long time period although the quantity of interest to experimenters, namely the

time average, is much less than 1. Of course, the time–averaged polarization depends on the initial spin distribution. We are then interested in finding the initial spin distribution which maximizes the time–averaged polarization and since maps provide a natural tool, we now look at it within our formalism [1]. It will suffice for our demonstration to consider motion on a fixed torus, i.e. to calculate at fixed J and to "view" the polarization stroboscopically at times n where we imagine a particle detector to be located.

We begin by introducing the polarization at time n and phases ϕ on the torus. We call this the *local polarization* $P_{loc}(n,\phi)$. It is the average over the spin degrees of freedom of normalized spins at (n,ϕ) and it obeys the T–BMT equation at (n,ϕ) since the T–BMT equation is linear is spin. Of course, $|P| \leq 1$. Thus P_{loc} is a polarization field. The *polarization* $P(n)$ on the torus is then

$$P(n) := \int \rho(n,\phi) P_{loc}(n,\phi) d\phi \,, \tag{8.8}$$

where $\rho(n,\phi)$ is the normalized phase space density. We take $\rho(n,*)$ to be continuous and 2π–periodic and since a phase space density is constant along an orbit we have

$$\rho(n+1,*) = \rho(n,* - 2\pi\omega) \,. \tag{8.9}$$

Since we are only interested in the case of stable beams, we only consider the case where ρ is n–independent. Then, off orbital resonance and recalling (8.9) and Corollary A.2e, ρ must be independent of ϕ. Since it is normalized we have $\rho = (1/2\pi)^d$ so that in this case (8.8) simplifies to

$$P(n) = \frac{1}{(2\pi)^d} \int d\phi P_{loc}(n,\phi) \,. \tag{8.10}$$

In this situation, as the following theorem shows, one can make an important statement about the long time average of the polarization.

Theorem 8.2 *Let ω be off orbital resonance and let a uniform IFF exist. Let the spin–orbit system be off spin–orbit resonance. Thus (recall Remark 4) a ISF S exists which is unique up to a sign. Then, $(1/N)\sum_{n=0}^{N-1} P(n)$ converges as $N \to \infty$ and the Hoffstaetter-Vogt inequality [30, 18] holds, i.e.,*

$$|\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} P(n)| \leq \frac{1}{(2\pi)^d} |\int d\phi S(0,\phi)| \,,$$

where P is defined by (8.10).

[1] We point out in this context that the quantity P_{fm} defined in equation NNN in [6] and subsequently plotted in figures N1 and N2, is usually of no relevance for quantifying the performance of a storage ring or for the analysis of data from particle collisions. However, it does give the time–averaged polarization on a torus for the special case underlying fig N3 in [7] where the angle between $\mathbf{n}_0$ and the ISF $\mathbf{n}$ is independent of orbital phase. We also point out that the words "maximum long term polarization" erroneously cited in [6] as appearing in [4] and then used to justify an irrelevant and misguided claim, are to be found nowhere in [4].

Proof: Let v be a uniform IFF so that (8.5) holds where $\nu \in [0, 1)$. Of course, $\mathcal{S}$, defined by $\mathcal{S} := ve^3$, is an ISF. Note that (see Remark 3) ν is a spin tune and that since the spin–orbit system is off spin–orbit resonance, ν is not in Y_ω. Thus (recall Remark 4) the ISF $\mathcal{S}$ is unique up to a sign. Then with the positive integer N, the time average P^N of P satisfies

$$P^N := \frac{1}{N}\sum_{n=0}^{N-1} P(n) = \frac{1}{(2\pi)^d}\int d\phi \frac{1}{N}\sum_{n=0}^{N-1} P_{\rm loc}(n,\phi) =: \frac{1}{(2\pi)^d}\int d\phi \mathcal{P}^N(\phi) \;, \quad (8.11)$$

where in the second equality we used (8.10). Our next task is to study the limit of P^N as $N \to \infty$. Here our strategy is the following: because each $\mathcal{P}^N$ is continuous and bounded, it is Lebesgue integrable. So we just need to show that $\mathcal{P}^N$ converges in the mean (i.e. in the L^1–norm) and pointwise as $N \to \infty$ which then implies that $\frac{1}{(2\pi)^d}\lim_{N\to\infty} \int d\phi \mathcal{P}^N(\phi) = \frac{1}{(2\pi)^d}\int d\phi \lim_{N\to\infty} \mathcal{P}^N(\phi)$. To show, first of all, that $\mathcal{P}^N$ converges pointwise we use the uniform IFF v to rotate $P_{\rm loc}$ by defining the function $\tilde{P} : \mathbb{Z}\times\mathbb{R}^d \to \mathbb{R}^3$:

$$\tilde{P}(n,\phi) := v^T(0,\phi)P_{\rm loc}(n,\phi) \;. \quad (8.12)$$

Then

$$\begin{aligned}\tilde{P}(n,\phi) &= v^T(0,\phi)P_{\rm loc}(n,\phi) = v^T(0,\phi)M(n,\phi-2\pi n\omega)P_{\rm loc}(0,\phi-2\pi n\omega)\\ &= v^T(0,\phi)M(n,\phi-2\pi n\omega)v(0,\phi-2\pi n\omega)\tilde{P}(0,\phi-2\pi n\omega)\\ &= \exp(2\pi n\nu\mathcal{J})\tilde{P}(0,\phi-2\pi n\omega)\;, \end{aligned}\quad (8.13)$$

where in the second equality we used the fact that $P_{\rm loc}$ is a polarization field and where in the fourth equality we used (8.6). Note that, by (8.13),

$$\begin{aligned}&\tilde{P}_1(n,\phi)+i\tilde{P}_2(n,\phi) = \exp(2\pi n\nu i)\Big(\tilde{P}_1(0,\phi-2\pi n\omega)+i\tilde{P}_2(0,\phi-2\pi n\omega)\Big)\;,\\ &\tilde{P}_3(n,\phi) = \tilde{P}_3(0,\phi-2\pi n\omega)\;.\end{aligned}\quad (8.14)$$

Thus $\tilde{P}_1(n,\phi)+i\tilde{P}_2(n,\phi)$ and $\tilde{P}_3(n,\phi)$ are quasiperiodic functions of n. Then, since $\tilde{P}_1$ and $\tilde{P}_2$ are real valued, each $\tilde{P}_i(*,\phi)$ is a quasiperiodic function. Therefore (recall Corollary A.2a) each $a(\tilde{P}_i(*,\phi),0)$ exists and $\frac{1}{N}\sum_{n=0}^{N-1}\Big(\tilde{P}_1(n,\phi)+i\tilde{P}_2(n,\phi)\Big)$ and $\frac{1}{N}\sum_{n=0}^{N-1}\tilde{P}_3(n,\phi)$ converge as $N\to\infty$ so that

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\Big(\tilde{P}_1(n,\phi)+i\tilde{P}_2(n,\phi)\Big) = a\Big(\tilde{P}_1(*,\phi)+i\tilde{P}_2(*,\phi),0\Big)\\ &= a\Big(\tilde{P}_1(0,\phi-2\pi\omega *)+i\tilde{P}_2(0,\phi-2\pi\omega *),-\nu\Big)\;,\end{aligned}\quad (8.15)$$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\tilde{P}_3(n,\phi) = a\Big(\tilde{P}_3(*,\phi),0\Big) = a\Big(\tilde{P}_3(0,\phi-2\pi\omega *),0\Big)\;, \quad (8.16)$$

where in the second equality of (8.15) and (8.16) we used (8.14). Because $\tilde{P}_1(0,\phi-2\pi\omega *)+i\tilde{P}_2(0,\phi-2\pi\omega *)$ is ω–quasiperiodic and because $-\nu\notin Y_\omega$ we see from Corollary A.2c that

$-\nu$ does not belong to the spectrum of $\tilde{P}_1(0, \phi - 2\pi\omega *) + i\tilde{P}_2(0, \phi - 2\pi\omega *)$. Thus by (8.15)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left(\tilde{P}_1(n, \phi) + i\tilde{P}_2(n, \phi) \right) = 0 .$$

Then, since $\tilde{P}_1$ and $\tilde{P}_2$ are real valued,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \tilde{P}_1(n, \phi) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \tilde{P}_2(n, \phi) = 0 . \tag{8.17}$$

Of course, f, defined by $f(\psi) := \tilde{P}_3(0, \phi - \psi)$ generates the ω–quasiperiodic function $\tilde{P}_3(0, \phi - 2\pi\omega *)$ via the prescription $\tilde{P}_3(0, \phi - 2\pi n\omega) = f(2\pi n\omega)$. Because ω is off orbital resonance we then have

$$\begin{aligned} a\left(\tilde{P}_3(0, \phi - 2\pi\omega *), 0 \right) &= \frac{1}{(2\pi)^d} \int d\psi f(\psi) = \frac{1}{(2\pi)^d} \int d\psi \tilde{P}_3(0, \phi - \psi) \\ &= \frac{1}{(2\pi)^d} \int d\psi \tilde{P}_3(0, \psi) , \end{aligned} \tag{8.18}$$

where in the first equation we used Lemma A.1c. From (8.16),(8.17) and (8.18) we conclude that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \tilde{P}(n, \phi) = \frac{e^3}{(2\pi)^d} \int d\psi \tilde{P}_3(0, \psi) . \tag{8.19}$$

Because of (8.12) and (8.19), $\mathcal{P}^N$ is pointwise convergent and

$$\begin{aligned} \lim_{N\to\infty} \mathcal{P}^N(\phi) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} P_{\text{loc}}(n, \phi) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} v(0, \phi) \tilde{P}(n, \phi) \\ &= v(0, \phi) \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \tilde{P}(n, \phi) = v(0, \phi) \frac{e^3}{(2\pi)^d} \int d\psi \tilde{P}_3(0, \psi) \\ &= \frac{\mathcal{S}(0, \phi)}{(2\pi)^d} \int d\psi \tilde{P}_3(0, \psi) =: c_P \mathcal{S}(0, \phi) , \end{aligned}$$

so that, for $i = 1, 2, 3$,

$$\lim_{N\to\infty} \mathcal{P}_i^N(\phi) = c_P \mathcal{S}_i(0, \phi) . \tag{8.20}$$

Having shown that $\mathcal{P}^N$ converges pointwise it is now easy to show that it converges in the mean. In fact, because the sequence of Lebesgue integrable functions $\mathcal{P}_i^N$ is bounded (due to the boundedness of $|P_{\text{loc}}|$) and because the convergence in (8.20) is pointwise we conclude by the Dominated-Convergence theorem [12, Sec. II.15] that the convergence in (8.20) is also in the mean which implies that $\lim_{N\to\infty} \int d\phi \mathcal{P}_i^N(\phi) = \int d\phi \lim_{N\to\infty} \mathcal{P}_i^N(\phi)$, i.e., by (8.20),

$$\lim_{N\to\infty} \int d\phi \mathcal{P}^N(\phi) = c_P \int d\phi \mathcal{S}(0, \phi) . \tag{8.21}$$

From (8.11) and (8.21) it follows that P^N converges as $N \to \infty$ and that $\bar{P} := \lim_{N\to\infty} P^N = c_P \int d\phi \mathcal{S}(0,\phi)$, so that $|\bar{P}|$, the value of the long time average of P is given by

$$|\bar{P}| = \frac{1}{(2\pi)^d} |c_P| \, | \int d\phi \mathcal{S}(0,\phi)| \, . \tag{8.22}$$

Because the ISF $\mathcal{S}$ is unique up to a sign, the only factor on the r.h.s. of (8.22) which depends on the local polarization i.e. on the initial conditions, is $|c_P|$. We obtain $|c_P|$ from its definition:

$$c_P = \frac{1}{(2\pi)^d} \int d\phi \tilde{P}_3(0,\phi) = \frac{1}{(2\pi)^d} \int d\phi \mathcal{S}(0,\phi) \cdot P_{\rm loc}(0,\phi) \, ,$$

where in the second equation we used (8.12). Thus

$$\begin{aligned} |c_P| &= \frac{1}{(2\pi)^d} | \int d\phi \mathcal{S}(0,\phi) \cdot P_{\rm loc}(0,\phi)| \le \frac{1}{(2\pi)^d} \int d\phi |\mathcal{S}(0,\phi) \cdot P_{\rm loc}(0,\phi)| \\ &\le \frac{1}{(2\pi)^d} \int d\phi |P_{\rm loc}(0,\phi)| \le \frac{1}{(2\pi)^d} \int d\phi = 1 \, , \end{aligned} \tag{8.23}$$

where in the second inequality we used the Schwartz inequality and where in the third inequality we used the fact that $P_{\rm loc} \le 1$. Inserting (8.23) into (8.22) yields

$$|\bar{P}| \le \frac{1}{(2\pi)^d} | \int d\phi \mathcal{S}(0,\phi)| \, , \tag{8.24}$$

which completes the proof. Note that, because the ISF $\mathcal{S}$ is unique up to a sign, the r.h.s. of (8.24) is independent of the local polarization, i.e. it is independent of the initial conditions. □

Remarks:

(5) Under the conditions of Theorem 8.2, the time averaged polarization $|\lim_{N\to\infty} (1/N) \sum_{n=0}^{N-1} \frac{1}{(2\pi)^d} \int d\phi P_{\rm loc}(n,\phi)|$ is at most $| \int d\phi \frac{1}{(2\pi)^d} \mathcal{S}(0,\phi)|$ and it achieves this maximum possible value iff either $P_{\rm loc} = \mathcal{S}$ or $P_{\rm loc} = -\mathcal{S}$.
For comments on the use of this result to arrive at the maximum polarization of the whole beam see Sec. I in [1].

(6) The proof of Theorem 8.2 also shows that (see (8.20)) under the conditions of Theorem 8.2 the *stroboscopic average* $a(\mathcal{P}(*,\phi),0)$ of a polarization field $\mathcal{P}$ (in particular of a spin field) is continuous in ϕ. □

8.4 The Examples

We now try to find n–turn invariant frame fields for the examples.

Example 1 Because of (5.2) we have

$$\mathfrak{a}(\phi) = M(1;\phi) = \exp(\mathcal{J}[\phi + 2\pi\omega]) W \exp(\mathcal{J}\sigma 2\pi) W^T \exp(-\mathcal{J}\phi) \, ,$$

where W is the $SO(3)$–matrix defined for (5.2) and where in the second equality we used (5.2). Thus v, defined by $v(n,\phi) := \exp(\mathcal{J}\phi)W$, is a uniform IFF.

Example 2 Obviously I is an IFF. However, by Sec. 7.2 this spin–orbit system is ill–tuned so that (recall Remark 3) no uniform IFF exists.

Example 3 If ϵ is an even integer then a uniform IFF exists and if ϵ is an odd integer then an IFF exists but no uniform one. For every ϵ a 2–turn invariant frame field exists. These results will be demonstrated in Appendix F. Also we know, from Sec. 4.4, that no IFF exists if ϵ is not an integer.

Example 4 Defining the function $v : \mathbb{Z} \times \mathbb{R}^k \to SO(3)$ by

$$v(n,\phi) := \exp(\mathcal{J}[-N\phi + \frac{\sigma_1 \epsilon'}{2\omega(1-\cos(2\pi\omega))}(\sin(\phi) - \sin(\phi + 2\pi\omega))]) \, , \qquad (8.25)$$

where N is an integer, we see that $ve^3 = e^3$. Since (recall Section 4.4) $\mathcal{S} = e^3$ is an ISF we see that v is an IFF. It is easy to show that v satisfies (8.5) where $\nu := \lfloor \sigma_1 + N\omega \rfloor$. Thus v is a uniform IFF so that by Remark 3a the system is well–tuned and ν is a spin tune.

9 Summary and Conclusions

[SOME STUFF WHICH WE MIGHT TUCK IN HERE SOMEHOW.
Since we deal with maps we can incorporate models which would not be smooth in the azimuth of the flow formalism and we have no need to work with differentials wrt azimuth. This latter implies that we do not need the ISF and related things to be smooth in ϕ so that we are allowed to weaken the regularity in ϕ However, we maintain some regularity properties in ϕ to avoid pathalogical solutions keep Q-P. Therefore a good compromise is continuity in ϕ. Since continuity in ϕ is weaker than smoothness we see to counter–acting effects: results become weaker due to weaker regularity and results get stronger since e.g., Fourier analysis for smooth functions is simpler than for continuous functions.]

In this paper we have..........

We begin by applying the map formalism to the T–BMT equation to obtain a difference equation (2.3) which turns out to be quasiperiodic. This opens the door to the techniques of quasiperiodic functions which involve tools like generating functions and Fourier series and this in turn allows us to introduce concepts like the spin frequency, the spin tune and spin–orbit resonance. It is easily seen that the key concepts of invariant spin field and invariant frame fields play the important role as generators of quasiperiodic funnctions like spin trajectories and simple precession frames. The simple precession frames are closely connected with generalized Floquet functions. We also define stroboscopic averages of polarization fields and examine some of their properties. Since the emphasis is on the statement and proof of theorems, this paper rather mathematical. In particular great care is taken with definitions.

Theorem 4.1 shows how to obtain and invariant spin field from a quasiperiodic spin solution. Analogously, Theorem 8.1 shows how to obtain an IFF from a quasiperiodic SPF.

We define the keys sets Y_ω, $\Xi(\phi_0)$, $\hat{\Xi}(\phi_0)$ and $\Lambda\Big(M(*;\phi_0)\Big)$ of frequencies which are the subjects of most of the theorems and of the examples.

For example, in Sec. VIII.1 it is shown that off orbital resonance the system is well–tuned, i.e., has a spin tune, iff at least one $\Xi(\phi_0)$ is nonempty. The sets $\Xi(\phi_0)$ are also valuable tools for studying phase dependent effects which occur typically on orbital resonance.

We illustrate our formalism with the help of four carefully chosen examples.

Example 1 exhibits many of the typical features of non–orbital–resonant motion. In particular, and among other things, it always has an ISF and is always well–tuned. Moreover, it is even well–tune on orbital resonance.

Example 2 [?] has an ISF but no spin tune. It is only one of our four examples which has some non–quasiperiodic spin solutions. It thus provides an exception to our general expectation that systems off orbital resonance are well tuned.

Example 3, in contrast to the others, has subcases parametrized by ϵ. For example, when ϵ is not an integer, there is no ISF. Thus if ϵ is not an integer, the stroboscopic average of every polarization field is either discontinuous or has zeros (EXPLAIN SOME MORE). Recall that in this paper, by definition, any polarization field, including an ISF, is not allowed to be discontinuous in the phase ϕ. As explained in detail in Sec IV, these discontinuities, which were first noticed in [4], are typical of so–called snake "resonant" orbital tunes and imply that care is needed when using this model to extract statements about the euilibrium polarization.

When ϵ is an integer, Example 3 has an ISF and, if ϵ is an integer it is also well–tuned (otherwise it is ill-tuned).

Example 4, SOMETHING LIKE: shows how modulation of the rate of spin precession by energy oscillations gives a Fourier spectrum for the spin motion which is, in fact, the equivalence class of spin tunes.

The various scenarios explored in this paper can be summarized in the Venn diagram of Fig. 1.

The meanings of the domains in Figure 1 are as follows:

- Inside the black circle: all tori, i.e. all systems considered in the paper.
- Inside the red ellipse: tori which have an ISF
- Inside the blue ellipse: tori, which at every ϕ_0 have a ω–quasiperiodic UPF (note that for those tori every spin trajectory is quasiperiodic)
- Inside the green ellipse: tori which are well–tuned
- Inside the yellow ellipse: tori which are off orbital resonance
- Inside the pink ellipse: tori with a uniform IFF

The numbered circles label specific examples, namely:

- Example 1a, which is Example 1 off orbital resonance
- Example 1b, which is Example 1 on orbital resonance

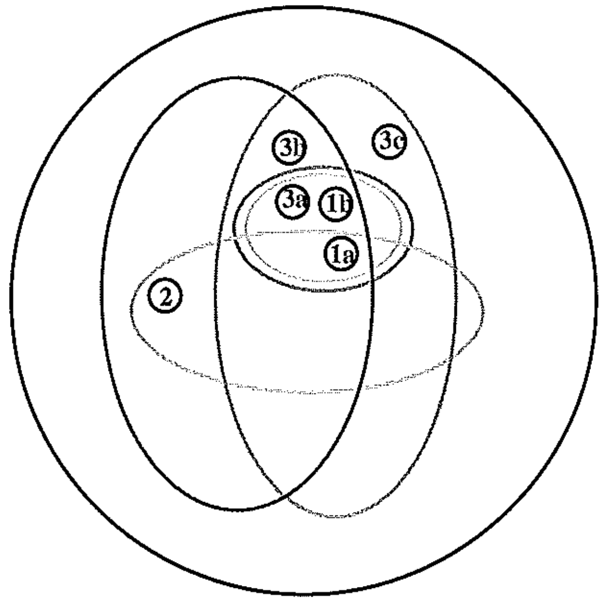

Figure 3: The logical connections between the various scenarios.

- Example 2
- Eample 3a, which is Example 3 when ϵ is an even integer
- Example 3b, which is Example 3 when ϵ is an odd integer
- Example 3c, which is Example 3 when ϵ is not an integer
- Example 4.

That the pink ellipse is contained in the green ellipse follows from Remark 3a in Sec. VIII.

By Theorem VIII.1b, the intersection of the interior of the yellow ellipse and the interior of the green ellipse equals the intersection of the interior of the yellow and the interior of the pink ellipses. Furthermore, these two identical sets are, by Theorem VIII.1b, equal to the the intersection of the interior of the yellow ellipse and the interior of the blue ellipse.

TRY TO INCLUDE A HEURISTIC INTERPRETATION OF THAT HERE: e.g. the strip between the green and the pink (inside the yellow) is an empty set. Couldn't be proved in BEHI?

The intersection of the interior of the yellow ellipse and the interior of the green ellipse, which contains example 1a, to the situation covered by Theorem VIII.1b.

TRY TO INCLUDE A HEURISTIC INTERPRETATION OF THAT HERE. MENTION RED ELLIPSE (ANOTHE EMPTY REGION)

Mention where the examples of paper 1 would fit,

All examples in [1] have a unique position in Fig 1. But we ...can't decide on the position owing to different definitions.

and perhaps comment on the simplicity wrt the VD in paper 1.