Beh II – 18.
Ideas for the beh II–26 →

Because, by Theorem 7.1b, every $\Xi(\phi_0)$ is nonempty, our argumentation leading to (E.14) can be repeated for arbitrary ξ, i.e. (E.14) holds for every $\xi \in \mathbb{R}^d$. Thus under the assumption that $a(U_{\phi_0}, \lambda)$ is continuous in ϕ_0, we have shown that all $\Lambda\Big(M(\cdot;\phi)\Big)$ are equal.

Thus it remains to be proven that this assumption is true. If λ is not in Y_ω then, by Corollary A.2c, $a(U_{\phi_0}, \lambda) = 0$ so that in this case the continuity in ϕ_0 is obvious. We thus have to consider only the case when $\lambda \in Y_\omega$, i.e. $\lambda = m \cdot \omega + m_0$ with $m \in \mathbb{Z}^d, m_0 \in \mathbb{Z}$. We first define, for every ϕ_0, the continuous and 2π–periodic function $g(\cdot;\phi_0) : \mathbb{R}^d \to SO(3)$ by $g(\phi;\phi_0) := v(0, \phi_0 + \phi)$. We obtain, for all ϕ_0, that

$$a(U_{\phi_0}, m\cdot\omega + m_0) = a(U_{\phi_0}, m\cdot\omega) = a\Big(g(2\pi\omega\cdot;\phi_0), m\cdot\omega\Big) = g_m(\cdot;\phi_0)$$
$$= \exp(im\cdot\phi_0) g_m(\cdot;0) , \qquad \text{(E.15)}$$

where in the third and fifth equations we used Lemma A.1c and where $g_m(\cdot;\phi_0)$ denotes the m–th Fourier coefficient of $g(\cdot;\phi_0)$. By (E.15), $a(U_{\phi_0}, m\cdot\omega + m_0)$ is continuous in ϕ_0 which completes the proof. Note that (E.15) implies that $a(U_{\phi_0}, m\cdot\omega + m_0) = \exp(im\cdot\phi_0) a(U_0, m\cdot\omega + m_0)$. □

F Example 3

One objective of this appendix will be to investigate, if the spin–orbit system $(\mathfrak{a}^\epsilon, 1/2)$ has n–turn invariant spin fields and n–turn invariant frame fields. Note that if $\mathcal{S}$ denotes an ISF then (recall (3.4) and Remark 1 in Sec. 3), $\mathcal{S}(0,\cdot)$ satisfies the eigenproblem $M^\epsilon(2;\cdot)\mathcal{S}(0,\cdot) = \mathcal{S}(0,\cdot)$, where M^ϵ denotes the spin transfer matrix of the spin–orbit system $(\mathfrak{a}^\epsilon, 1/2)$. Thus, computing the 2–turn spin transfer matrix will be our first aim. Other objectives of this section are the quasiperiodicity of the solutions of (2.3) and stroboscopic sequences of polarization fields.

F.1

To meet the first objective we first conclude from (2.21) and (2.22) that

$$\mathfrak{a}^\epsilon(\cdot + \pi) = \begin{pmatrix} 1-2c^2 & -2bc & 2ac \\ -2bc & 1-2b^2 & 2ab \\ -2ac & -2ab & 2a^2-1 \end{pmatrix} . \qquad \text{(F.1)}$$

From (2.21),(2.10),(F.1) it follows that the 2–turn spin transfer matrix reads as

$$M^\epsilon(2;\cdot) = \mathfrak{a}^\epsilon(\cdot+\pi)\mathfrak{a}^\epsilon = \begin{pmatrix} 1-8c^2+8c^4 & 4bc(1-2c^2) & 4ac(1-2c^2) \\ -4bc(1-2c^2) & 1-8b^2c^2 & -8abc^2 \\ -4ac(1-2c^2) & -8abc^2 & 1-8a^2c^2 \end{pmatrix} . \qquad \text{(F.2)}$$

As mentioned above, the eigenproblem

$$M^\epsilon(2;\phi)f(\phi) = f(\phi) , \qquad \text{(F.3)}$$

is essential for our objective so that we have to find a nonzero solution $f(\phi)$. By (F.2) the skew–symmetric part of $M^\epsilon(2;\,\cdot)$ reads as

$$\Big(M^\epsilon(2;\,\cdot) - (M^\epsilon)^T(2;\,\cdot)\Big)/2 = \begin{pmatrix} 0 & 4bc(1-2c^2) & 4ac(1-2c^2) \\ -4bc(1-2c^2) & 0 & 0 \\ -4ac(1-2c^2) & 0 & 0 \end{pmatrix}, \tag{F.4}$$

so that the eigenproblem (F.3) is satisfied by

$$f \equiv \Big(0, 4ac(1-2c^2), -4bc(1-2c^2)\Big) . \tag{F.5}$$

Multiplying this solution by a scalar gives another solution of (F.3) - in particular $f(\phi) \equiv h(\phi)$ satisfies (F.3), if h denotes the continuous and 2π–periodic function $h : \mathbb{R} \to \mathbb{R}^3$, defined by

$$h := (0, ab, -b^2) . \tag{F.6}$$

Note that, by (2.23) and (F.6),

$$|h| = |b|\sqrt{1-c^2} . \tag{F.7}$$

Thus h is not $\mathbb{S}^2$–valued so that h still is not convenient (the same holds for the r.h.s. of (F.5)). We therefore will normalize h and we observe that, if $|h(\phi)| \neq 0$, then $f(\phi) \equiv h(\phi)/|h(\phi)|$ satisfies the eigenproblem (F.3) and that

$$\begin{aligned} n(\phi) &:= \frac{h(\phi)}{|h(\phi)|} \\ &= \frac{\cos(\pi\epsilon/2)}{|\cos(\pi\epsilon/2)|\sqrt{1-\sin^2(\pi\epsilon/2)\cos^2(\phi)}}\Big(0, \sin(\pi\epsilon/2)\sin(\phi), -\cos(\pi\epsilon/2)\Big) . \end{aligned} \tag{F.8}$$

It is obvious that, for all ϕ, $n(\phi)$ is well–defined and satisfies

$$M^\epsilon(2;\phi)n(\phi) = n(\phi) , \tag{F.9}$$

if $|\sin(\pi\epsilon/2)|$ neither equals 0 or 1, i.e. if ϵ is not an integer. It is thus convenient to meet our objective separately for two cases, which we denote by Case 1 (ϵ is not an integer) and by Case 2 (ϵ is an integer).

We begin with Case 1. Then ϵ is not an integer so that n, defined by (F.8), is a continuous and 2π–periodic function $n : \mathbb{R} \to \mathbb{S}^2$ which satisfies the eigenproblem (F.9). Note that if $\mathcal{S}$ is a spin field such that $M^\epsilon(2;\,\cdot)\mathcal{S}(0,\,\cdot) = \mathcal{S}(0,\,\cdot)$, then, by (3.4), $\mathcal{S}$ is a 2–turn invariant spin field. It follows that the spin field $\mathcal{S}$, fixed by $\mathcal{S}(0,\,\cdot) := n$, is a 2–turn invariant spin field. Because the first component of $\mathcal{S}$ vanishes, one observes that v, defined by

$$v := [e^1 \times \mathcal{S}, e^1, \mathcal{S}] , \tag{F.10}$$

is a 2–turn invariant frame field which will be useful lateron.

THE FOLLOWING TRICK HAS SHORTENED THE TREATMENT OF THE 2-SNAKE MODEL BY ONE PAGE IN COMPARISON WITH FEBRUARY

Let $f : \mathbb{R} \to \mathbb{S}^2$ be a continuous and 2π–periodic function which satisfies (F.3) for all ϕ. Defining $M := \{\phi \in \mathbb{R} : M^\epsilon(2;\phi) = I\}$, we observe by (F.9) that, for $\phi \in \mathbb{R} \setminus M$, $|f(\phi) \cdot n(\phi)| = 1$. Note that, by (F.2), $M = \{\phi \in \mathbb{R} : c(\phi)(c^2(\phi) - 1) = 0\}$ so that M consists only of isolated points (recall that this means that each point of M is contained in an open interval which contains no other point of M). Because $|f(\phi) \cdot n(\phi)|$ is continuous in ϕ, it follows that $|f(\phi) \cdot n(\phi)| = 1$ holds for every ϕ. Thus either $f = n$ or $f = -n$ so that $\mathcal{S}$ and $-\mathcal{S}$ are the only 2–turn invariant spin fields. We conclude from (2.21),(3.1) and (F.8) that

$$\mathcal{S}(1, \cdot + \pi) = \mathfrak{a}^\epsilon \mathcal{S}(0, \cdot) = -\mathcal{S}(0, \cdot + \pi) \ . \tag{F.11}$$

Eq. (F.11) implies that neither $\mathcal{S}$ nor $-\mathcal{S}$ is an ISF. We thus have shown that none of the 2–turn invariant spin fields is an ISF. We conclude that $(\mathfrak{a}^\epsilon, 1/2)$ has no ISF.

We now come to Case 2. Then ϵ is an integer so that, by (2.21), the 1–turn spin transfer matrix obtains the simple form $\mathfrak{a}^\epsilon = \begin{pmatrix} 1-2c^2 & 0 & 2ac \\ 0 & 1 & 0 \\ -2ac & 0 & 2a^2-1 \end{pmatrix}$, which implies that e^2 is an ISF. Thus $[e^3, e^1, e^2]$ is an IFF.

This completes the study of the existence problem of the ISF and IFF and we can summarize the above by the following

Proposition F.1 a) The spin–orbit system $(\mathfrak{a}^\epsilon, 1/2)$ has an ISF, iff ϵ is an integer. If ϵ is an integer, then an IFF exists.
b) The spin–orbit system $(\mathfrak{a}^\epsilon, 1/2)$ has, for every value of ϵ, a 2–turn invariant frame field.
□

Remark:

(1) We now review Case 1 from a general point of view by considering a spin–orbit system $(\mathfrak{a}, \omega)$ on orbital resonance for which $d = 1$, i.e. $\omega = p/q$, where $q > 0, p$ denote integers (note that the spin–orbit system $(\mathfrak{a}^\epsilon, 1/2)$ is a special case). Recalling from (2.11) that

$$\mathfrak{a} M(q; \cdot)\mathfrak{a}^T = M(q; \cdot + 2\pi\omega) \ , \tag{F.12}$$

we obtain that $\mathfrak{a}^T \mathcal{S}(0, \cdot + 2\pi\omega)$ satisfies the eigenproblem

$$M(q; \cdot)\mathfrak{a}^T \mathcal{S}(0, \cdot + 2\pi\omega) = \mathfrak{a}^T M(q; \cdot + 2\pi\omega)\mathcal{S}(0, \cdot + 2\pi\omega) = \mathfrak{a}^T \mathcal{S}(q, \cdot + 2\pi\omega(q+1)) \\ = \mathfrak{a}^T \mathcal{S}(0, \cdot + 2\pi\omega) \ , \tag{F.13}$$

if $\mathcal{S}$ is a q–turn invariant spin field. Eq. (F.13) implies that the spin field $\tilde{\mathcal{S}}$, fixed by $\tilde{\mathcal{S}}(0, \cdot) := \mathfrak{a}^T \mathcal{S}(0, \cdot + 2\pi\omega)$, is a q–turn invariant spin field so that, if $\mathcal{S}$ and $-\mathcal{S}$ are the only q–turn invariant spin fields, then $\tilde{\mathcal{S}}$ is either equal to $\mathcal{S}$ or $-\mathcal{S}$, i.e. a ξ exists in $\{-1, 1\}$ such that

$$\mathfrak{a}\mathcal{S}(0, \cdot) = \xi \mathcal{S}(0, \cdot + 2\pi\omega) \ . \tag{F.14}$$

This is the situation of Case 1 and it is therefore no coincidence that the r.h.s. of (F.11) has the form $\xi\mathcal{S}(0, \cdot + 2\pi\omega)$.

F.2

We now will meet the second objective, which is to investigate the quasiperiodicity of the solutions of (2.3) and related questions. As in the previous section, we consider Case 1 and Case 2 separately.

We begin with Case 1, where ϵ is not an integer and we first of all will search for an ω–quasiperiodic SPF.

IT NOW PAYS OFF THAT, IN THE NEW DESIGN OF THE PAPER, FRAME FIELDS (AS WELL AS SPIN FIELDS) ARE EXPLICITLY TIME DEPENDENT SO THAT THE 2-TURN IFF v 'AUTOMATICALLY' IS A GENERATOR OF A SPF. THE LATTER PROPERTY WAS NOT TRUE IN THE OLD DESIGN AND, ACCORDINGLY, THIS FURTHER SHORTENS THE TREATMENT OF EXAMPLE 3 IN COMPARISON WITH FEBRUARY

We consider the 2–turn invariant frame field v given by (F.10) where the 2–turn invariant spin field $\mathcal{S}$ is fixed by the condition $\mathcal{S}(0,\cdot) = n$ with n given by (F.8). Note that $\mathcal{S}$ and v were already used in the previous section. Because v is a frame field, it follows from Remark 1 in Sec. 6 that U_{ϕ_0}, defined by $U_{\phi_0}(n) := v(n, \phi_0 + \pi n)$, is a SPF starting at ϕ_0. Because v is a 2–turn invariant frame field we have for arbitrary integer n that

$$\begin{aligned} U_{\phi_0}(2n) &= v(2n, \phi_0 + 2\pi n) = v(0, \phi_0) = U_{\phi_0}(0)\,, \\ U_{\phi_0}(2n+1) &= v(2n+1, \phi_0 + \pi(2n+1)) = v(1, \phi_0 + \pi) = U_{\phi_0}(1)\,. \end{aligned} \tag{F.15}$$

Also we have by (F.10),(F.11) that

$$\begin{aligned} v(1, \phi_0 + \pi) &= [e^1 \times \mathcal{S}(1, \phi_0 + \pi), e^1, \mathcal{S}(1, \phi_0 + \pi)] \\ &= [-e^1 \times \mathcal{S}(0, \phi_0 + \pi), e^1, -\mathcal{S}(0, \phi_0 + \pi)] \\ &= [e^1 \times \mathcal{S}(0, \phi_0 + \pi), e^1, \mathcal{S}(0, \phi_0 + \pi)] \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ &= v(0, \phi_0 + \pi) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} =: v(0, \phi_0 + \pi)\hat{\mathcal{J}}\,. \end{aligned} \tag{F.16}$$

It follows from (F.15),(F.16) that for all n

$$U_{\phi_0}(n) = v(0, \phi_0 + \pi n) \begin{pmatrix} \cos(\pi n) & 0 & -\sin(\pi n) \\ 0 & 1 & 0 \\ \sin(\pi n) & 0 & \cos(\pi n) \end{pmatrix} = v(0, \phi_0 + \pi n)\hat{\mathcal{J}}^n\,, \tag{F.17}$$

so that U_{ϕ_0} is generated by u_{ϕ_0}, defined by $u_{\phi_0}(\phi) := v(0, \phi_0 + \phi) \begin{pmatrix} \cos(\phi) & 0 & -\sin(\phi) \\ 0 & 1 & 0 \\ \sin(\phi) & 0 & \cos(\phi) \end{pmatrix}$,
i.e. $U_{\phi_0}(n) = u_{\phi_0}(\pi n)$. Thus U_{ϕ_0} is an ω–quasiperiodic SPF starting at ϕ_0. We now check if this SPF is a UPF. To apply Remark 1 in Sec. 5, we compute, for every ϕ_0 and every integer n,

$$\begin{aligned} & U^T_{\phi_0}(n+1)\mathfrak{a}^\epsilon(\phi_0 + \pi n)U_{\phi_0}(n) \\ &\quad = \hat{\mathcal{J}}^{n+1} v^T(0, \phi_0 + \pi(n+1))\mathfrak{a}^\epsilon(\phi_0 + \pi n) v(0, \phi_0 + \pi n)\hat{\mathcal{J}}^n\,, \end{aligned} \tag{F.18}$$

where we used (F.17). We also have

$$
\begin{aligned}
& v^T(0,\cdot+\pi)\mathfrak{a}^\epsilon v(0,\cdot) \\
&= \begin{pmatrix} 0 & -n_3 & -n_2 \\ 1 & 0 & 0 \\ 0 & -n_2 & n_3 \end{pmatrix} \begin{pmatrix} 1-2c^2 & 2bc & 2ac \\ 2bc & 1-2b^2 & -2ab \\ -2ac & 2ab & 2a^2-1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -n_3 & 0 & n_2 \\ n_2 & 0 & n_3 \end{pmatrix} \\
&= \begin{pmatrix} 2c^2-1 & \frac{\sqrt{2}bc\sqrt{1-c}}{|\cos(\pi\epsilon/2)|} & 0 \\ \frac{\sqrt{2}bc\sqrt{1-c}}{|\cos(\pi\epsilon/2)|} & 1-2c^2 & 0 \\ 0 & 0 & -1 \end{pmatrix} ,
\end{aligned}
\tag{F.19}
$$

where in the second equation we used (2.22). From (F.18) and (F.19) it follows, for every ϕ_0 and every integer n, that

$$
\begin{aligned}
& U^T_{\phi_0}(n+1)\mathfrak{a}^\epsilon(\phi_0+\pi n)U_{\phi_0}(n) \\
&= \hat{\mathcal{J}}^{n+1} \begin{pmatrix} 2c^2(\phi_0+\pi n)-1 & \frac{\sqrt{2}b(\phi_0+\pi n)c(\phi_0+\pi n)\sqrt{1-c(\phi_0+\pi n)}}{|\cos(\pi\epsilon/2)|} & 0 \\ \frac{\sqrt{2}b(\phi_0+\pi n)c(\phi_0+\pi n)\sqrt{1-c(\phi_0+\pi n)}}{|\cos(\pi\epsilon/2)|} & 1-2c^2(\phi_0+\pi n) & 0 \\ 0 & 0 & -1 \end{pmatrix} \hat{\mathcal{J}}^n \\
&= \hat{\mathcal{J}}^{n+1} \begin{pmatrix} 2c^2(\phi_0)-1 & (-1)^n\frac{\sqrt{2}b(\phi_0)c(\phi_0)\sqrt{1-c(\phi_0)}}{|\cos(\pi\epsilon/2)|} & 0 \\ (-1)^n\frac{\sqrt{2}b(\phi_0)c(\phi_0)\sqrt{1-c(\phi_0)}}{|\cos(\pi\epsilon/2)|} & 1-2c^2(\phi_0) & 0 \\ 0 & 0 & -1 \end{pmatrix} \hat{\mathcal{J}}^n \\
&= \hat{\mathcal{J}}^{n+1} \begin{pmatrix} -2c^2(\phi_0)+1 & (-1)^n\frac{\sqrt{2}b(\phi_0)c(\phi_0)\sqrt{1-c(\phi_0)}}{|\cos(\pi\epsilon/2)|} & 0 \\ -(-1)^n\frac{\sqrt{2}b(\phi_0)c(\phi_0)\sqrt{1-c(\phi_0)}}{|\cos(\pi\epsilon/2)|} & 1-2c^2(\phi_0) & 0 \\ 0 & 0 & 1 \end{pmatrix} \hat{\mathcal{J}}^{n+1} \\
&= \begin{pmatrix} -2c^2(\phi_0)+1 & \frac{\sqrt{2}b(\phi_0)c(\phi_0)\sqrt{1-c(\phi_0)}}{|\cos(\pi\epsilon/2)|} & 0 \\ -\frac{\sqrt{2}b(\phi_0)c(\phi_0)\sqrt{1-c(\phi_0)}}{|\cos(\pi\epsilon/2)|} & 1-2c^2(\phi_0) & 0 \\ 0 & 0 & 1 \end{pmatrix} ,
\end{aligned}
\tag{F.20}
$$

where in the second equation we used (2.22). Therefore $U^T_{\phi_0}(n+1)\mathfrak{a}^\epsilon(\phi_0+\pi n)U_{\phi_0}(n)$ is independent of n so that, by Remark 1 in Sec. 5, we conclude that U_{ϕ_0} is a UPF starting at ϕ_0. Using again Remark 1 in Sec. 5, we thus get for all integers n

$$
U^T_{\phi_0}(n+1)\mathfrak{a}^\epsilon(\phi_0+\pi n)U_{\phi_0}(n) = \exp(2\pi\nu_{\phi_0}\mathcal{J}) \, , \tag{F.21}
$$

where ν_{ϕ_0} denotes the UPR corresponding to U_{ϕ_0}. From (2.22),(F.20) and (F.21) it follows that, for every ϕ_0,

$$
-1-8\sin^2(\pi\epsilon/2)\cos^2(\phi_0)\Big(\sin^2(\pi\epsilon/2)\cos^2(\phi_0)-1\Big) = 1-2c^2(\phi_0) = \cos(2\pi\nu_{\phi_0}) \tag{F.22}
$$

By (F.22) we observe that Ξ has uncountably many elements so that, by Theorem 6.1d, the spin–orbit system $(\mathfrak{a}^\epsilon, 1/2)$ has no spin frequency hence is ill-tuned.

We now come to Case 2 so that ϵ is an integer. We first assume that ϵ is an even integer. Then, by (2.21), the 1–turn spin transfer matrix obtains the simple form $\mathfrak{a}^\epsilon = \hat{\mathcal{J}}$ which implies that $w := [e^3, e^1, e^2]$ is a uniform IFF. Thus, by Remark 3 in Sec. 6, the spin–orbit system $(\mathfrak{a}^\epsilon, 1/2)$ is well–tuned. We now assume that ϵ is an odd integer. Then, by (2.21), the 1–turn spin transfer matrix obtains the simple form

$$\mathfrak{a}^\epsilon(\phi) = \begin{pmatrix} -\cos(4\phi) & 0 & -\sin(4\phi) \\ 0 & 1 & 0 \\ \sin(4\phi) & 0 & -\cos(4\phi) \end{pmatrix} , \tag{F.23}$$

which implies that w is an IFF and that

$$w^T(0, \phi + \pi)\mathfrak{a}^\epsilon(\phi)w(0, \phi) = \begin{pmatrix} -\cos(4\phi) & \sin(4\phi) & 0 \\ -\sin(4\phi) & -\cos(4\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp(-4\phi\mathcal{J}) . \tag{F.24}$$

Because w is an IFF and by (F.24) and the remarks after eq. (8.4) we have for arbitrary n, ϕ_0

$$w^T(0, \phi_0 + \pi n)M^\epsilon(n; \phi_0)w(0, \phi_0) = \exp(-4n\phi_0\mathcal{J}) . \tag{F.25}$$

Also, by Remark 2 in Sec. 6, T_{ϕ_0}, defined by $T_{\phi_0}(n) := w(0, \phi_0 + \pi n)$, is an ω–quasiperiodic SPF starting at ϕ_0. Because of (F.25) and Remark 3 in Sec. 2 the phase function μ_{ϕ_0} of T_{ϕ_0} reads as

$$\mu_{\phi_0}(n) = \exp(-4in\phi_0) =: \exp(2\pi in\kappa_{\phi_0}) , \tag{F.26}$$

where $\kappa_{\phi_0} := \lfloor -\frac{2\phi_0}{\pi} \rfloor$. We conclude (recall Remark 1 in Sec. 5) that T_{ϕ_0} is an ω–quasiperiodic UPF starting at ϕ_0 with UPR κ_{ϕ_0}. By the special form of κ_{ϕ_0} we observe that Ξ has uncountably many elements so that, by Theorem 6.1d, the spin–orbit system $(\mathfrak{a}^\epsilon, 1/2)$ has no spin frequency hence is ill–tuned.

This completes the study of well–tuning and quasiperiodicity and we can summarize the above by the following

Proposition F.2 a) The spin–orbit system $(\mathfrak{a}^\epsilon, 1/2)$ is well–tuned and has a uniform IFF if ϵ is an even integer. If ϵ is not an even integer then no spin frequency exists and in particular $(\mathfrak{a}^\epsilon, 1/2)$ is ill–tuned.
b) The spin–orbit system $(\mathfrak{a}^\epsilon, 1/2)$ has, for every value of ϵ and for every ϕ_0, an ω–quasiperiodic UPF starting at ϕ_0. Thus, for all values of ϵ and ϕ_0, $\Xi(\phi_0)$ (hence $\hat{\Xi}(\phi_0)$) is nonempty, all solutions of (2.3) are quasiperiodic and a normalized ω–quasiperiodic solution of (2.3) exists.
□

F.3 Stroboscopic sequences

In this section we study the stroboscopic sequence of an arbitrary polarization field $\mathcal{P}$ with the aim of proving Proposition F.3, stated below. For brevity we only consider the subcase of Case 1 where $0 < \epsilon < 1/2$. Recall that stroboscopic sequences are defined in Section 3.3.

Let N be a positive integer. Then, by (2.11), $M^\epsilon(2N+2;\,\cdot) = M^\epsilon(2;\cdot)M^\epsilon(2N;\,\cdot)$ hence, by induction in N,

$$M^\epsilon(2N;\,\cdot) = \left(M^\epsilon(2;\cdot)\right)^N . \tag{F.27}$$

We conclude from (3.1),(3.2) and (F.27) that

$$\mathcal{P}(2N;\cdot) = M^\epsilon(2N;\cdot)\mathcal{P}(0;\cdot) = (M^\epsilon(2;\cdot))^N\mathcal{P}(0;\cdot) , \tag{F.28}$$
$$\mathcal{P}(2N-1;\cdot) = \mathfrak{a}^\epsilon(\cdot+\pi)\mathcal{P}(2N-2;\cdot+\pi) . \tag{F.29}$$

We will show, among other things, that the stroboscopic average of $\mathcal{P}$ exists, i.e. that the stroboscopic sequence $\mathcal{P}^N(n,\phi)$ converges for every n and ϕ as $N\to\infty$. It is clear by Section 3.3 that this happens iff $\mathcal{P}^N(0,\phi)$ converges for every ϕ as $N\to\infty$. Before that we will show that $\mathcal{P}^{2N}(0,\phi)$ converges for every ϕ as $N\to\infty$. Note that by (F.28)

$$\mathcal{P}^{2N}(0,\phi) = \frac{1}{2N}\sum_{n=0}^{N-1}\Big(\mathcal{P}(2n;\phi)+\mathcal{P}(2n+1;\phi)\Big) , \tag{F.30}$$
$$\frac{1}{2N}\sum_{n=0}^{N-1}\mathcal{P}(2n;\phi) = \frac{1}{2N}\sum_{n=0}^{N-1}(M^\epsilon(2;\phi))^n\mathcal{P}(0;\phi) . \tag{F.31}$$

We will first show that $(1/2N)\sum_{n=0}^{N-1}\mathcal{P}(2n;\phi)$ converges for every ϕ as $N\to\infty$. It is clear that we have to compute the n-th powers of $M^\epsilon(2;\phi)$, which will be done by diagonalizing $M^\epsilon(2;\phi)$. We have, by (F.19), that

$$\mathfrak{a}^\epsilon(\phi) = v(0,\phi+\pi)M_1(\phi)v^T(0,\phi) , \tag{F.32}$$

where

$$M_1 := \begin{pmatrix} a_1 & a_2 & 0\\ a_2 & -a_1 & 0\\ 0 & 0 & -1\end{pmatrix} , \qquad a_1 := 2c^2-1 , \qquad a_2 := \frac{\sqrt{2}bc\sqrt{1-c}}{|\cos(\pi\epsilon/2)|} . \tag{F.33}$$

It follows from (F.32) that

$$\mathfrak{a}^\epsilon(\cdot+\pi) = v(0,\cdot)M_1(\cdot+\pi)v^T(0,\cdot+\pi) , \tag{F.34}$$

hence it follows from (F.2) and (F.32) and the $SO(3)$–property of v that

$$M^\epsilon(2;\cdot) = v(0,\cdot)M_1(\cdot+\pi)M_1v^T(0,\cdot) . \tag{F.35}$$

Since, by (2.22), we have

$$a_1(\phi+\pi) = a_1(\phi) , \qquad a_2(\phi+\pi) = -a_2(\phi) , \tag{F.36}$$

we obtain

$$M_1(\cdot+\pi) = \begin{pmatrix} a_1 & -a_2 & 0\\ -a_2 & -a_1 & 0\\ 0 & 0 & -1\end{pmatrix} , \tag{F.37}$$

hence

$$M_1(\cdot+\pi)M_1(\cdot) = \begin{pmatrix} a_3 & -a_4 & 0 \\ a_4 & a_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{F.38}$$

where

$$a_3 := a_1^2 - a_2^2 \, , \; a_4 := -2a_1a_2 \, . \tag{F.39}$$

Defining

$$M_2 := \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} , \qquad M_5 := a_3 - ia_4 \, , \tag{F.40}$$

we obtain that M_2 is unitary, i.e. $M_2^\dagger M_2 = I$, and that

$$M_2^\dagger M_1(\cdot+\pi)M_1(\cdot)M_2 = M_3 := \begin{pmatrix} M_5 & 0 & 0 \\ 0 & M_5^* & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{F.41}$$

so that (F.35) yields the desired diagonalization of $M^\epsilon(2;\cdot)$, i.e.

$$M^\epsilon(2;\cdot) = v(0,\cdot)M_2M_3M_2^\dagger v^T(0,\cdot) = M_4M_3M_4^\dagger \, , \tag{F.42}$$

where in the first equality we used that M_2 is unitary and where in the second equality we abbreviated

$$M_4 := v(0,\cdot)M_2 \, . \tag{F.43}$$

To complete the proof that $(1/2N)\sum_{n=0}^{N-1} \mathcal{P}(2n;\phi)$ converges we obtain from (F.41) and (F.42) that

$$(M^\epsilon(2;\cdot))^n = M_4(M_3)^nM_4^\dagger = M_4 \begin{pmatrix} (M_5)^n & 0 & 0 \\ 0 & (M_5^*)^n & 0 \\ 0 & 0 & 1 \end{pmatrix} M_4^\dagger \, , \tag{F.44}$$

hence

$$\begin{aligned} \frac{1}{2N}\sum_{n=0}^{N-1}(M^\epsilon(2;\cdot))^n &= M_4\frac{1}{2N}\sum_{n=0}^{N-1}\begin{pmatrix} (M_5)^n & 0 & 0 \\ 0 & (M_5^*)^n & 0 \\ 0 & 0 & 1 \end{pmatrix} M_4^\dagger \\ &= M_4 \begin{pmatrix} \frac{1}{2N}\sum_{n=0}^{N-1}(M_5)^n & 0 & 0 \\ 0 & \frac{1}{2N}\sum_{n=0}^{N-1}(M_5^*)^n & 0 \\ 0 & 0 & 1/2 \end{pmatrix} M_4^\dagger \, . \end{aligned} \tag{F.45}$$

Note that since M_1 is in $SO(3)$ we have by (F.38) that $a_3^2 + a_4^2 = 1$ hence, by (F.40),

$$|M_5| = 1 \, . \tag{F.46}$$

Note also that by (2.22),(F.33),(F.39) and (F.40) and since a_1, a_2, a_3, a_4 are real, we have

$$\begin{aligned}
&\{\phi \in \mathbb{R} : M_5(\phi) = 1\} = \{\phi \in \mathbb{R} : a_3(\phi) = 1, a_4(\phi) = 0\} = \{\phi \in \mathbb{R} : a_1^2(\phi) = 1, a_2(\phi) = 0\} \\
&= \{\phi \in \mathbb{R} : b(\phi) = 0\} \cup \{\phi \in \mathbb{R} : c(\phi) = 0\} \cup \{\phi \in \mathbb{R} : c(\phi) = 1\} \\
&= \{\phi \in \mathbb{R} : b(\phi) = 0\} = \{\pi/2 + j\pi : j \in \mathbb{Z}\} , \qquad \text{(F.47)}
\end{aligned}$$

where in the fourth equality we used the fact that $0 < \epsilon < 1/2$. It follows from (F.47) that $M_5(\phi) = 1$ iff $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$. Thus if $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$, then

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} (M_5(\phi))^n = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} (M_5^*(\phi))^n = \frac{1}{2} , \qquad \text{(F.48)}$$

and, if $\phi \notin \{\pi/2 + j\pi : j \in \mathbb{Z}\}$,

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} (M_5(\phi))^n = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} (M_5^*(\phi))^n = 0 , \qquad \text{(F.49)}$$

where we used (F.46). We conclude from (F.45) and (F.48) that, if $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$,

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} (M^\epsilon(2;\phi))^n = \frac{I}{2} , \qquad \text{(F.50)}$$

hence, by (F.31) and if $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$,

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} \mathcal{P}(2n;\phi) = \frac{1}{2}\mathcal{P}(0;\phi) . \qquad \text{(F.51)}$$

We conclude from (F.45) and (F.49) that, if $\phi \notin \{\pi/2 + j\pi : j \in \mathbb{Z}\}$,

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} (M^\epsilon(2;\phi))^n = M_4(\phi) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} M_4^\dagger(\phi) , \qquad \text{(F.52)}$$

hence, by (F.31) and if $\phi \notin \{\pi/2 + j\pi : j \in \mathbb{Z}\}$,

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} \mathcal{P}(2n;\phi) = M_4(\phi) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} M_4^\dagger(\phi)\mathcal{P}(0;\phi) . \qquad \text{(F.53)}$$

Note that, by (F.10),(F.40) and (F.43) and since $\mathcal{S}(0;\cdot) = n(\cdot)$,

$$\begin{aligned}
&M_4 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} M_4^\dagger = v(0,\cdot)M_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} M_2^\dagger v^T(0,\cdot) \\
&= v(0,\cdot) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} v^T(0,\cdot) = [0,0,\mathcal{S}(0,\cdot)]v^T(0,\cdot) = [0,0,n(\cdot)]v^T(0,\cdot) , \qquad \text{(F.54)}
\end{aligned}$$

hence, by (F.53) and if $\phi \notin \{\pi/2 + j\pi : j \in \mathbb{Z}\}$,

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} \mathcal{P}(2n;\phi) = [0,0,\frac{n(\phi)}{2}]v^T(0,\phi)\mathcal{P}(0,\phi) = \frac{n(\phi)}{2} n(\phi)\cdot\mathcal{P}(0,\phi) \; , \tag{F.55}$$

where in the second equality we used (F.10). We thus have got: $(1/2N)\sum_{n=0}^{N-1} \mathcal{P}(2n;\phi)$ converges for every ϕ as $N \to \infty$ and the limit is given by (F.51) and (F.55).

We now will show that $(1/2N)\sum_{n=0}^{N-1} \mathcal{P}(2n+1;\phi)$ converges for every ϕ as $N \to \infty$. We have, by (F.29),

$$\begin{aligned}\frac{1}{2N} \sum_{n=0}^{N-1} \mathcal{P}(2n+1;\phi) &= \frac{1}{2N} \sum_{n=1}^{N} \mathcal{P}(2n-1;\phi) = \mathfrak{a}^\epsilon(\phi+\pi)\frac{1}{2N} \sum_{n=1}^{N} \mathcal{P}(2n-2;\phi+\pi) \\ &= \mathfrak{a}^\epsilon(\phi+\pi)\frac{1}{2N} \sum_{n=0}^{N-1} \mathcal{P}(2n;\phi+\pi) \; . \end{aligned} \tag{F.56}$$

It follows from (F.51) that, if $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$,

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} \mathcal{P}(2n;\phi+\pi) = \frac{1}{2}\mathcal{P}(0;\phi+\pi) \; , \tag{F.57}$$

hence, by (F.56) and if $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$,

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} \mathcal{P}(2n+1;\phi) = \frac{1}{2}\mathfrak{a}^\epsilon(\phi+\pi)\mathcal{P}(0;\phi+\pi) = \frac{1}{2}\hat{J}\mathcal{P}(0;\phi+\pi) \; , \tag{F.58}$$

where in the second equality we used (2.21),(2.22) and the definition of $\hat{J}$ in (F.16). If $\phi \notin \{\pi/2 + j\pi : j \in \mathbb{Z}\}$ then $\phi + \pi \notin \{\pi/2 + j\pi : j \in \mathbb{Z}\}$ and, by (F.55),

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} \mathcal{P}(2n;\phi+\pi) = \frac{n(\phi+\pi)}{2} n(\phi+\pi)\cdot\mathcal{P}(0,\phi+\pi) \; , \tag{F.59}$$

hence, by (F.56) and if $\phi \notin \{\pi/2 + j\pi : j \in \mathbb{Z}\}$,

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{2N} \sum_{n=0}^{N-1} \mathcal{P}(2n+1;\phi) &= \frac{1}{2}\mathfrak{a}^\epsilon(\phi+\pi)n(\phi+\pi)n(\phi+\pi)\cdot\mathcal{P}(0,\phi+\pi) \\ &= -\frac{1}{2}n(\phi)n(\phi+\pi)\cdot\mathcal{P}(0,\phi+\pi) \; , \end{aligned} \tag{F.60}$$

where in the second equality we used (F.11). We thus have obtained that $(1/2N)\sum_{n=0}^{N-1} \mathcal{P}(2n+1;\phi)$ converges for every ϕ as $N \to \infty$ and that the limit is given by (F.58) and (F.60).

We can now make a first summary: $\mathcal{P}^{2N}(0,\phi)$ converges for every ϕ as $N \to \infty$ and the limit is, if $\phi \in \{\pi/2 + j\pi : j \in \mathbb{Z}\}$, given by (F.30),(F.51) and (F.58), i.e.

$$\bar{\mathcal{P}}(\frac{\pi}{2}+j\pi) := \lim_{N\to\infty} \mathcal{P}^{2N}(0,\frac{\pi}{2}+j\pi) = \frac{1}{2}\mathcal{P}(0;\frac{\pi}{2}+j\pi) + \frac{1}{2}\hat{J}\mathcal{P}(0;\frac{\pi}{2}+j\pi+\pi) \; , \tag{F.61}$$

and, if $\phi \notin \{\pi/2 + j\pi : j \in \mathbb{Z}\}$, the limit is given by (F.30),(F.55) and (F.60), i.e.

$$\bar{\mathcal{P}}(\phi) := \lim_{N\to\infty} \mathcal{P}^{2N}(0,\phi) = \frac{n(\phi)}{2} n(\phi) \cdot \mathcal{P}(0,\phi)$$
$$-\frac{1}{2} n(\phi) n(\phi+\pi) \cdot \mathcal{P}(0,\phi+\pi)\,. \tag{F.62}$$

As promised we now show that $\mathcal{P}^N(0,\phi)$ converges for every ϕ as $N \to \infty$. First of all, $\mathcal{P}(0,\cdot)$ is a bounded function since it is 2π–periodic and continuous. Thus, and by (3.2), a positive real constant a_5 exists such that, for all n and ϕ, we have $|\mathcal{P}(n,\phi)| \leq a_5$. We thus can estimate

$$|\mathcal{P}^{2n+1}(0,\phi) - \mathcal{P}^{2n}(0,\phi)| = |\frac{1}{2n+1}\sum_{k=0}^{2n} \mathcal{P}(k;\phi) - \frac{1}{2n}\sum_{k=0}^{2n-1} \mathcal{P}(k;\phi)|$$
$$= |\frac{1}{2n+1}\mathcal{P}(2n;\phi) + (\frac{1}{2n+1} - \frac{1}{2n})\sum_{k=0}^{2n-1} \mathcal{P}(k;\phi)|$$
$$\leq |\frac{1}{2n+1}\mathcal{P}(2n;\phi)| + |(\frac{1}{2n+1} - \frac{1}{2n})\sum_{k=0}^{2n-1} \mathcal{P}(k;\phi)|$$
$$\leq \frac{1}{2n+1}|\mathcal{P}(2n;\phi)| + (\frac{1}{2n} - \frac{1}{2n+1})\sum_{k=0}^{2n-1} |\mathcal{P}(k;\phi)| \leq \frac{a_5}{2n+1} + (1 - \frac{2n}{2n+1})a_5$$
$$= \frac{2a_5}{2n+1} \leq \frac{a_5}{n}\,. \tag{F.63}$$

Since, as we have shown, $\mathcal{P}^{2N}(0,\cdot)$ converges everywhere to $\bar{\mathcal{P}}(\cdot)$ as $N \to \infty$, there exists, for every $\delta > 0$ and every ϕ, an integer $N_1(\phi)$ such that, for all $n \geq N_1(\phi)$,

$$|\mathcal{P}^{2n}(0,\phi) - \bar{\mathcal{P}}(\phi)| \leq \delta \leq 2\delta\,. \tag{F.64}$$

Clearly, since a_5/n converges to zero as $n \to \infty$, there exists also an integer N_2 such that, for all $n \geq N_2$, we have $a_5/n \leq \delta$. Defining $N_3(\phi) := \max(N_1(\phi), N_2)$ we conclude that, if $n \geq N_3(\phi)$,

$$|\mathcal{P}^{2n+1}(0,\phi) - \bar{\mathcal{P}}(\phi)| \leq |\mathcal{P}^{2n+1}(0,\phi) - \mathcal{P}^{2n}(0,\phi)| + |\mathcal{P}^{2n}(0,\phi) - \bar{\mathcal{P}}(\phi)|$$
$$\leq \frac{a_5}{n} + \delta \leq 2\delta\,, \tag{F.65}$$

where in the second inequality we used (F.63) and (F.64). We conclude from (F.64) and (F.65) that, if $k \geq 2N_3(\phi)$, then

$$|\mathcal{P}^{k}(0,\phi) - \bar{\mathcal{P}}(\phi)| \leq 2\delta\,. \tag{F.66}$$

Since δ is arbitrary, we thus have shown with (F.66) that, for every ϕ, $\mathcal{P}^N(0,\phi)$ converges to $\bar{\mathcal{P}}(\phi)$ as $N \to \infty$.

We now investigate, under which conditions on $\mathcal{P}$, the limit function $\bar{\mathcal{P}}$ is continuous. Since $\mathcal{S}(0,\phi)$ and $\mathcal{P}(0,\phi)$ are continuous in ϕ we see by (F.62) that $\bar{\mathcal{P}}$ is continuous at every

ϕ which is not in $\{\pi/2+j\pi : j\in\mathbb{Z}\}$ and that $\bar{\mathcal{P}}$ converges at every $\phi\in\{\pi/2+j\pi : j\in\mathbb{Z}\}$, i.e.

$$\begin{aligned}\lim_{\phi\to\pi/2+j\pi}\bar{\mathcal{P}}(\phi) &= \lim_{\phi\to\pi/2+j\pi}\frac{n(\phi)}{2}n(\phi)\cdot\mathcal{P}(0,\phi)\\ &-\frac{1}{2}n(\phi)n(\phi+\pi)\cdot\mathcal{P}(0,\phi+\pi) = \frac{n(\pi/2+j\pi)}{2}n(\pi/2+j\pi)\cdot\mathcal{P}(0,\pi/2+j\pi)\\ &-\frac{1}{2}n(\pi/2+j\pi)n(\pi/2+j\pi+\pi)\cdot\mathcal{P}(0,\pi/2+j\pi+\pi)\,,\end{aligned}\tag{F.67}$$

where in the first equality we used (F.62). Of course, $\bar{\mathcal{P}}$ is everywhere continuous iff, at every $\phi\in\{\pi/2+j\pi : j\in\mathbb{Z}\}$, it is equal to its limit at those ϕ. In other words: $\bar{\mathcal{P}}$ is everywhere continuous iff, for every integer j,

$$\bar{\mathcal{P}}(\pi/2+j\pi) = \lim_{\phi\to\pi/2+j\pi}\bar{\mathcal{P}}(\phi)\,.\tag{F.68}$$

We thus compute by using (F.61) and (F.67)

$$\begin{aligned}\bar{\mathcal{P}}(\pi/2+j\pi) - \lim_{\phi\to\pi/2+j\pi:j}\bar{\mathcal{P}}(\phi) &= \frac{1}{2}\mathcal{P}(0;\pi/2+j\pi)+\frac{1}{2}\hat{J}\mathcal{P}(0;\pi/2+j\pi+\pi)\\ &-\frac{n(\pi/2+j\pi)}{2}n(\pi/2+j\pi)\cdot\mathcal{P}(0,\pi/2+j\pi)\\ &+\frac{1}{2}n(\pi/2+j\pi)n(\pi/2+j\pi+\pi)\cdot\mathcal{P}(0,\pi/2+j\pi+\pi)\,.\end{aligned}\tag{F.69}$$

Thus $\bar{\mathcal{P}}$ is everywhere continuous, iff, for every integer j, $\mathcal{P}(0,\cdot)$ solves the following linear problem for $\mathcal{P}$

$$\begin{aligned}0 &= \mathcal{P}(0;\pi/2+j\pi)+\hat{J}\mathcal{P}(0;\pi/2+j\pi+\pi)\\ &-n(\pi/2+j\pi)n(\pi/2+j\pi)\cdot\mathcal{P}(0,\pi/2+j\pi)\\ &+n(\pi/2+j\pi)n(\pi/2+j\pi+\pi)\cdot\mathcal{P}(0,\pi/2+j\pi+\pi)\,.\end{aligned}\tag{F.70}$$

We have, by (F.8) and (F.16),

$$\begin{aligned}&\hat{J}\mathcal{P}(0;\pi/2+j\pi+\pi)+n(\pi/2+j\pi)n(\pi/2+j\pi+\pi)\cdot\mathcal{P}(0,\pi/2+j\pi+\pi)\\ &=\begin{pmatrix}-1 & 0 & 0\\ 0 & \cos^2(\pi\epsilon/2) & -\sin(\pi\epsilon/2)\cos(\pi\epsilon/2)(-1)^j\\ 0 & \sin(\pi\epsilon/2)\cos(\pi\epsilon/2)(-1)^j & -\sin^2(\pi\epsilon/2)\end{pmatrix}\mathcal{P}(0,\frac{\pi}{2}+j\pi+\pi)\\ &=: M_6^j\mathcal{P}(0,\frac{\pi}{2}+j\pi+\pi)\,,\end{aligned}\tag{F.71}$$

and

$$\begin{aligned}&n(\pi/2+j\pi)n(\pi/2+j\pi)\cdot\mathcal{P}(0,\pi/2+j\pi)-\mathcal{P}(0,\pi/2+j\pi)\\ &=\begin{pmatrix}-1 & 0 & 0\\ 0 & -\cos^2(\pi\epsilon/2) & -\sin(\pi\epsilon/2)\cos(\pi\epsilon/2)(-1)^j\\ 0 & -\sin(\pi\epsilon/2)\cos(\pi\epsilon/2)(-1)^j & -\sin^2(\pi\epsilon/2)\end{pmatrix}\mathcal{P}(0,\frac{\pi}{2}+j\pi)\\ &=: M_7^j\mathcal{P}(0,\frac{\pi}{2}+j\pi)\,.\end{aligned}\tag{F.72}$$

It follows from (F.70),(F.71) and (F.72) that $\bar{\mathcal{P}}$ is everywhere continuous, iff, for every integer j,

$$M_6^j\mathcal{P}(0,\frac{\pi}{2}+j\pi+\pi) = M_7^j\mathcal{P}(0,\frac{\pi}{2}+j\pi)\ . \qquad \text{(F.73)}$$

Since

$$M_7^j = -\hat{J}M_6^{j+1}\ , \qquad \text{(F.74)}$$

we obtain by (F.73) that $\bar{\mathcal{P}}$ is everywhere continuous, iff, for every integer j,

$$M_6^j\mathcal{P}(0,\frac{\pi}{2}+j\pi+\pi) = -\hat{J}M_6^{j+1}\mathcal{P}(0,\frac{\pi}{2}+j\pi)\ . \qquad \text{(F.75)}$$

Because $\mathcal{P}(0,\phi)$ is 2π–periodic in ϕ, we conclude that (F.75) holds for every integer j iff it holds for just for $j=0$. We conclude that $\bar{\mathcal{P}}$ is everywhere continuous, iff

$$M_6^0\mathcal{P}(0,\frac{\pi}{2}+\pi) = -\hat{J}M_6^1\mathcal{P}(0,\frac{\pi}{2})\ . \qquad \text{(F.76)}$$

By (F.71), eq. (F.76) is equivalent to

$$0 = e^1\cdot\Big(\mathcal{P}(0,\frac{\pi}{2}+\pi)-\mathcal{P}(0,\frac{\pi}{2})\Big)\ , \qquad \cos(\pi\epsilon/2)e^2\cdot\Big(\mathcal{P}(0,\frac{\pi}{2}+\pi)+\mathcal{P}(0,\frac{\pi}{2})\Big)$$
$$= \sin(\pi\epsilon/2)e^3\cdot\Big(\mathcal{P}(0,\frac{\pi}{2}+\pi)-\mathcal{P}(0,\frac{\pi}{2})\Big)\ . \qquad \text{(F.77)}$$

We thus have proved:
Proposition F.3 Let $0<\epsilon<1/2$ and let $\mathcal{P}$ be a polarization field. Then the following holds.
a) For every ϕ, $\mathcal{P}^N(0,\phi)$ converges to $\bar{\mathcal{P}}(\phi)$ as $N\to\infty$ where $\bar{\mathcal{P}}(\phi)$ is defined by (F.61) and (F.62).
b) $\bar{\mathcal{P}}$ is everywhere continuous, iff (F.77) holds. □
Remark: It follows from Proposition F.3b that if $\mathcal{P}(0,\cdot)$ is a constant function then $\bar{\mathcal{P}}$ is continuous everywhere iff $0=e^2\cdot\mathcal{P}(0,\cdot)$.

Acknowledgments

We wish to thank...

Guide for the reader

Please note the following conventions used in this paper:

- Sec. 2.1: spin–orbit system, $\mathfrak{a}(\phi)$, $A(n;\phi)$, resonant, nonresonant, off orbital resonance, on orbital resonance, $SO(3)$, $\mathbb{Z}$, transpose of a matrix.