Pre-existing betatron motion and spin flipping with RF fields in storage rings

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Abstract. It seems to be a common perception that pre-existing, non-spin-resonant, vertical, betatron motion in storage rings has little influence on spin-flipping with external, resonant, radio-frequency, magnetic fields. While this is often the case, care is needed, and useful insights are obtained by studying the geometry of the invariant spin field (ISF). I illustrate this by numerical means for protons in the COSY ring and I suggest a test that could be carried out at COSY.

1. Introduction

In recent years a series of careful measurements have been carried out at the COSY ring in Jülich to check the effect of local radio-frequency (RF) magnet fields on proton and deuteron spins [1, 2, 3, 4]. For example, it was demonstrated that proton and deuteron spins could be flipped in COSY by sweeping the frequency of the RF field across the natural spin-precession frequency. Moreover, the so-called resonance strengths were measured by applying the Froissart-Stora formula [5]. However, for the case of an oscillating radial field, generated by a so-called RF dipole (RFD), this field generates forced vertical motion in the rest of the ring. Then the magnetic fields in the quadrupoles and dipoles augment the effect of the field of the RFD itself and can contribute to the resonance strength. These contributions are easily calculated, and once they are included, the measured resonance strengths agree well with expectation [6, 7, 8, 9]. Contrary to claims in [1, 2] and elsewhere, there is nothing unexpected [6, 7, 8, 9].

As is often the case for such measurements, it was assumed that the pre-existing betatron motion would have no influence on the measurements since the spin motion was not in resonance with the vertical betatron motion. Since the amplitude of the pre-existing vertical betatron motion was small this was a safe assumption. Moreover measurements of resonance strengths with uncooled and cooled beams showed no dependence on the emittances. However, if the amplitude is large enough, the pre-existing vertical betatron motion should indeed have a noticeable influence on spin flipping, and in a way that illustrates the desirability of rigorous mathematical concepts and the matching computational tools for handling subtleties of the spin motion. For example, spin flipping by varying the frequency of RF magnetic fields involves time-dependent parameters. But as we have emphasised earlier, much can be learnt by studying the so-called invariant spin field (ISF) of the system with *fixed* parameters [10, 11].

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The measurements in [1, 2, 3, 4] have shown that COSY is an ideal machine for tests of the spin dynamics of protons and deuterons. In particular it might be possible to measure the influence of large-amplitude, pre-existing, vertical betatron motion on spin flipping.

This paper, which is a short summary of a part of an extensive paper in preparation [9], then has three aims:

- to remind the reader that pre-existing betatron motion has the potential for significant effects on spin flipping with RF fields, and to illustrate this by simulations,
- to demonstrate and emphasise how recourse to precise mathematical concepts and theorems, and to accompanying computational tools, can shine light into apparently complicated spin motion and "deconstruct" it into simpler parts,
- to suggest how it might be possible to measure the so-called amplitude-dependent spin tune for large-amplitude, vertical betatron motion using the COSY ring.

Due to constraints on space I must assume that the reader is very familiar with the standard terminology of this field [10, 12, 13, 14]:

- The ***T-BMT equation*** for the motion of the rest-frame spin expectation value $\vec{S}$ (the "spin") at ring position s: $d\vec{S}/ds = \vec{\Omega}(z,s) \times \vec{S}$ with $\vec{\Omega}(z,s) = \vec{\Omega}_0(s) + \vec{\omega}(z,s)$ where $z = (x,\ p_x,\ y,\ p_y, \sigma, \delta)$ and where $\vec{\omega}(0,s) = \vec{0}$.
- The ***invariant spin field*** (ISF): $\hat{n}(z, s+C) = \hat{n}(z,s)$ whereby $\hat{n}(z,s)$ obeys the T-BMT equation on a trajectory and C is the ring circumference.
- The 1-turn-periodic orthonormal coordinate system of the SLIM formalism $(\hat{l}(s), \hat{n}_0(s), \hat{m}(s))$ [12], whereby $\hat{n}_0(s) = \hat{n}(0,s)$ and $d\hat{n}_0(s)/ds = \vec{\Omega}_0(s) \times \hat{n}_0(s)$.
- The gyromagnetic anomaly $a = (g-2)/2$ and the ***spin tune*** on the closed orbit (CO), ν_0: in a perfectly flat ring with no solenoids, $\nu_0 = a\gamma$ where γ is the Lorentz factor.
- The ***uniform invariant frame field*** (u-IFF): $(\hat{u}_1(z,s),\ \hat{n}(z,s),\ \hat{u}_2(z,s))$ whereby $\hat{u}_1(z,s+C) = \hat{u}_1(z,s),\ \hat{u}_2(z,s+C) = \hat{u}_2(z,s)$.
- The ***equivalence class of amplitude-dependent spin tunes*** (ADST).
- The ***single resonance model*** (SRM) [13, 10] for the ISF and ADST: a single Fourier harmonic of $\vec{\omega}(z,s) \cdot (\hat{l}(s) - i\hat{m}(s))$ has dominant control of spin motion near the conditions $\nu_0 = \kappa \equiv k_0 \pm Q_{\rm I},\quad k_0 \pm Q_{\rm II},\quad k_0 \pm Q_{\rm III}$ or $\nu_0 = \kappa^{\rm rf} \equiv k_0 \pm Q_{\rm rf}$ for integers k_0, where $Q_{\rm I}, Q_{\rm II}, Q_{\rm III}$ are orbital tunes (e.g., $Q_{\rm x}, Q_{\rm y}, Q_{\rm s}$), and $Q_{\rm rf}$ is the tune of the RF field.
- The usual notation for the SRM. E.g: $\delta^{\rm v} = \nu_0 - \kappa^{\rm v}$ ($\delta^{\rm rf} = \nu_0 - \kappa^{\rm rf}$), the (complex) resonance strength $\epsilon^{\rm v}_\kappa$ ($\epsilon^{\rm rf}_\kappa$) and the phase, ϕ_y ($\phi_{\rm rf}$), of the vertical betatron motion or RF field.
- The simple version of the ***Froissart–Stora formula*** (F-S) [5] for survival of vertical polarisation when $\delta^{\rm v}$ or $\delta^{\rm rf}$ crosses 0 at a constant rate.

The super(sub)script rf will appertain to the RF field and the super(sub)scripts v or y will appertain to the vertical betatron motion. The brackets [..] will denote the fractional part of a number.

The resonance strengths for this paper have been calculated with the SLIM formalism [12, 15] in the way described in [16, 17, 9]. They can be obtained by an eigen-analysis or by spin-orbit tracking. The SLIM formalism was designed for providing, in a systematic way, first estimates of equilibrium polarisation in $e^{\pm}$ storage rings. It can handle coupling in the optics, vertical bends, spin rotators and misalignments etc. The formalism describes spin motion in the reference frame $(\hat{l}(s), \hat{n}_0(s), \hat{m}(s))$. Since the equilibrium polarisation of a beam is along $\hat{n}_0$, no matter how much $\hat{n}_0$ is tilted from the vertical, this is a natural choice. Resonance strengths in the SLIM formalism therefore also appertain to spin motion in the $(\hat{l}(s), \hat{n}_0(s), \hat{m}(s))$ frame. It is also clear that this formalism for the resonance strengths automatically handles motion in the 6-D phase space, transverse coupling and vertical bends etc., and any resulting tilt of $\hat{n}_0$ from the vertical. Of course, it delivers the correct orbital tunes in the presence of transverse coupling and the correct CO spin tune ν_0 in the presence of elements which tilt $\hat{n}_0$ from the vertical. Thus it obviates the need for handling coupling with complicated transformations such as those in [18]. Since the equilibrium polarisation is along $\hat{n}_0$, it would be inappropriate to apply the F-S formula naively to spin flipping w.r.t. the vertical axis if $\hat{n}_0$ were tilted. Instead, the F-S formula should then be applied to spin flip w.r.t. $\hat{n}_0$. Then it delivers estimates of $(\vec{S}\cdot\hat{n}_0)^{\rm final}/|\vec{S}|$.

The calculation of the ISF and the simulation of spin flipping requires the use of full 3-D spin motion. This is handled by an extension of the SLIM formalism that goes beyond the linearisation of spin motion contained in the original formalism, and the ISF is calculated within the $(\hat{l}(s), \hat{n}_0(s), \hat{m}(s))$ frame [9].

2. Simulations

2.1. The ISF

I begin by considering protons at 2.1 GeV/c in the COSY ring running with the betatron tunes $Q_x = 3.5436$ and $Q_y = 3.5801$ which were among those used for the measurements reported in [1]. I allow no misalignments so that $\hat{n}_0(s)$ is vertical. There is no synchrotron motion. At 2.1 GeV/c the spin tune on the CO is $\nu_0 = a\gamma = 4.3950$. Then spins can be flipped when the tune $Q_{\rm rf}$ of the RFD is swept through 0.605. The total resonance strength $|\epsilon_\kappa^{\rm rf}|$ for the RFD and the ring together is 47.2×10^{-6}. The observed efficacy of the F-S formula for describing spin flip implies that close to the resonance, the spin motion is well described by the so-called single resonance model (SRM). This can be checked by looking at the components of the ISF as a function of $[\phi_{\rm rf}/2\pi]$. For this study, the ISF is calculated by stroboscopic averaging [13]. So far, this is the most general model-independent way to obtain the ISF. There is no need for the ISF to be close to $\hat{n}_0$.

Thus figure 1 shows the components of the vector $\hat{n}^{\rm rf}(\phi_{\rm rf})$ of the ISF in the $(\hat{l}, \hat{n}_0, \hat{m})$ frame and just before the RFD for $|\delta^{\rm rf}| = |\epsilon_\kappa^{\rm rf}|$ where $\delta^{\rm rf} \equiv \nu_0 - \kappa^{\rm rf}$ with $\kappa^{\rm rf} = 5 - Q_{\rm rf}$. The horizontal axis is $[\phi_{\rm rf}/2\pi]$. As expected from the SRM, the components are single-valued functions of $[\phi_{\rm rf}/2\pi]$ and the component along $\hat{n}_0$ is $1/\sqrt{2}$, namely $|\delta^{\rm rf}|/\Lambda^{\rm rf}$ with $\Lambda^{\rm rf} = \sqrt{(\delta^{\rm rf})^2 + |\epsilon_\kappa^{\rm rf}|^2}$. The other two components oscillate sinusoidally with $[\phi_{\rm rf}/2\pi]$. Curves (not shown) expected for the SRM, are obtained at other values of $\delta^{\rm rf}$, indicating that the spin motion is well described by the SRM.

For the spin-flipping experiments in COSY, the 3$-\sigma$ vertical emittance of the pre-existing betatron motion was about 3π mm.mrad before electron cooling and about 0.3π after cooling and the measured resonance strengths were consistent with the predictions of the F-S formula in both cases. I now add pre-existing vertical motion for protons *on* the 18π mm.mrad ellipse. Then the ISF in the $(\hat{l}, \hat{n}_0, \hat{m})$ frame and just before the RFD is as in figure 2. Such a large emittance would probably not be attainable in reality, but it is allowed in a simulation. We now see that the vector $\hat{n}^{\rm tot}(\phi_{\rm rf}, \phi_y)$ for the ISF is very different from that in figure 1. Of course, for a large vertical emittance, some change is to be expected [19], but here the $\hat{n}_0$-component of the ISF has not only risen to about 0.934, but apart from a small ripple due to dependence on ϕ_y, it is still independent of $[\phi_{\rm rf}/2\pi]$. However, the other two components occupy broad bands, indicating their dependence on ϕ_y. Nevertheless, this is symptomatic of spin motion's still being approximately described by a SRM, but being further from resonance than in figure 1. This is indeed the case as explained below. Since the ISF is now very different from that for the SRM with the RFD running alone, one does not expect the F-S formula to apply in its original form. Simulations with a spin $\vec{S}$ set initially parallel either to $\hat{n}_0$ or to $\hat{n}^{\rm v}$ (below) confirm this.

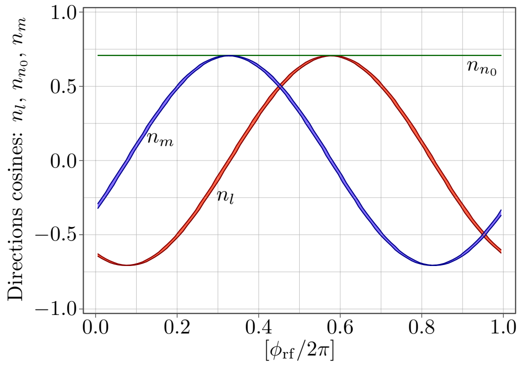

Figure 1. The components of the ISF due to the RFD with $|\delta^{\rm rf}| = |\epsilon_\kappa^{\rm rf}|$.

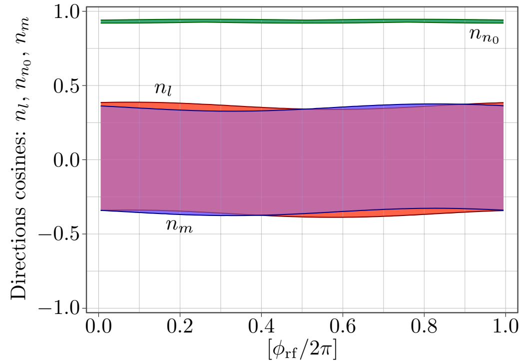

Figure 2. The components of the ISF due to the pre-existing vertical betatron motion and the RFD with $|\delta^{\rm rf}| = |\epsilon_\kappa^{\rm rf}|$.

2.2. The amplitude-dependent spin tune: ADST

The idea that the system is further from resonance, suggests that the essence of the matter can be captured in the notion of a "ground state", having a characteristic spin tune and being probed by the small magnetic fields from the RFD and the forced vertical motion. Then from this perspective, in the absence of pre-existing betatron motion, the ground state is for particle motion on the CO with the spin tune ν_0. On the other hand, the inclusion of pre-existing vertical betatron motion changes the ground-state so that the spin tune relevant for studying resonance should surely be the amplitude-dependent spin tune for that betatron motion. In fact, it should be that member of the equivalence class of ADST's that reduces to ν_0 on the CO [10, 14]. I call this the preferred ADST (p-ADST) and I denote it by $\nu(J_y)$ where J_y is the orbital action for the vertical motion. In fact the necessity of taking into account a shifted spin tune in analysing experiments on spin precession has already been recognised [20, 21, 22]. However, as explained in [10] some literature on spin tunes contains definitions which, as well as being unhelpful, also impede proper understanding. Yet with appropriate mathematical and numerical tools there is no need for misconceptions. I shall illustrate that here using ideas from rigorous earlier work [10] together with our ability to calculate the ISF for this system.

In this spirit, my next step is to explore the notion of a ground state by calculating the ISF for the vertical betatron motion. For $Q_y = 3.5801$ the vertical motion is in resonance with the spin motion at 2.1143 GeV/c when $\nu_0 = 4.4199 = \kappa^{\rm v} \equiv 8 - Q_y$. Then using the SLIM formalism [9], one finds that the resonance strength $|\epsilon_\kappa^{\rm v}|$ is 0.00958. At 2.1 GeV/c, $\delta^{\rm v} = \nu_0 - \kappa^{\rm v} = -0.0249$. The ISF in the $(\hat{l}, \hat{n}_0, \hat{m})$ frame and just before the RFD at 2.1 GeV/c is shown figure 3. The $\hat{n}_0$-component, $\hat{n}^{\rm v} \cdot \hat{n}_0$, of $\hat{n}^{\rm v}$ is 0.934 at all $[\phi_y/2\pi]$. This is symptomatic of the SRM. In the SRM, $|\hat{n}^{\rm v} \cdot \hat{n}_0|$ is given by $|\delta^{\rm v}|/\Lambda^{\rm v}$ with $\Lambda^{\rm v} = \sqrt{(\delta^{\rm v})^2 + |\epsilon_\kappa^{\rm v}|^2}$. From this we obtain an $|\epsilon_\kappa^{\rm v}|$ in perfect agreement with the value obtained directly, and with this value of $|\epsilon_\kappa^{\rm v}|$ it is found that $|\hat{n}^{\rm v} \cdot \hat{n}_0|$ is given by $|\delta^{\rm v}|/\Lambda^{\rm v}$ at other values of $|\delta^{\rm v}|$. The ISF for the pre-existing, vertical, betatron motion is therefore well described by the SRM. This should come as no surprise since the F-S formula has long been successful for describing the polarisation surviving during acceleration across spin-orbit resonances.

With the relevance of the SRM for pre-existing, vertical betatron motion established, the p-ADST can be obtained from a simple formula, namely $\nu(J_y) = (\text{sgn}\,\delta^{\rm v})\Lambda^{\rm v} + \kappa^{\rm v}$ [13]. The shift away from ν_0 is $(\text{sgn}\,\delta^{\rm v})\Lambda^{\rm v} - \delta^{\rm v}$ which for these parameters is -0.001779. This is large compared to $|\epsilon_\kappa^{\rm rf}|$ so that the significant shift from resonance detected above is unsurprising. It corresponds to a shift *upwards* of about 2.6 kHz in the frequency of the RFD needed for resonance. This is a significant fraction of the sweeping range of 8 kHz used for spin flipping [1].

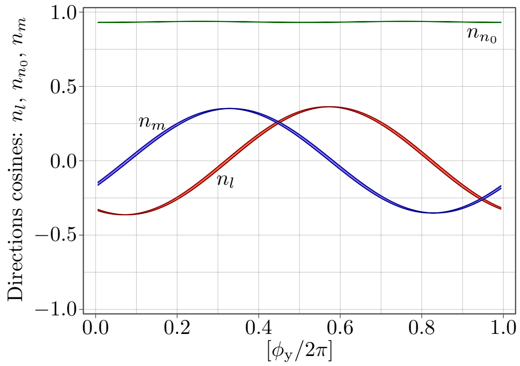

Figure 3. The components of the ISF due to the pre-existing vertical betatron motion.

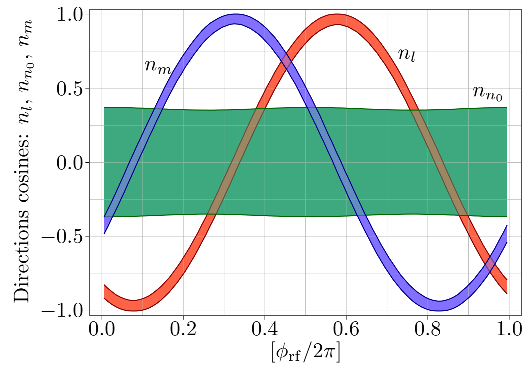

Figure 4. The components of the ISF due to the pre-existing vertical betatron motion and the RFD, with $\tilde{\delta}^{\rm rf} = 0$

The internal consistency of this picture can be checked in several ways:

- a spectral analysis, from a discrete Fourier transform of the spin motion, yields a line at $\nu(J_y)$ and lines consistent with Theorem 9.1.c in [10],
- a sweep of $Q_{\rm rf}$ to cause spin flip shows that the flipping occurs at a $Q_{\rm rf}$ corresponding to $\nu(J_y)$, not ν_0,
- the ISF vector, $\hat{n}^{\rm tot}$, for $\tilde{\delta}^{\rm rf} \equiv \nu(J_y) - \kappa^{\rm rf} = 0$ has an $\hat{n}_0$-component which oscillates around zero as in figure 4.

The ISF in figure 4 (and figure 2) should be a single-valued function of $[\phi_{\rm rf}/2\pi]$ and $[\phi_y/2\pi]$. This is confirmed in figure 5 where the $\hat{n}^{\rm tot} \cdot \hat{n}_0$ of figure 4 lies on a surface when plotted against $[\phi_{\rm rf}/2\pi]$ and $1 - [\phi_y/2\pi]$. The other two components $\hat{n}^{\rm tot} \cdot \hat{l}$ and $\hat{n}^{\rm tot} \cdot \hat{m}$ lie on surfaces too. In fact, the $\hat{n}^{\rm tot} \cdot \hat{n}_0$ in figure 5 has a simple analytical form namely: $\hat{n}^{\rm tot}(\phi_{\rm rf}, \phi_y) \cdot \hat{n}_0 = (|\epsilon^{\rm v}_\kappa|/\Lambda^{\rm v}) \cos\{[(\phi_{\rm rf} - \phi_y)/2\pi]\}$.

2.3. "Nested" SRM's

The above simple form suggests that the ISF could be very simple when viewed in a more appropriate reference frame. Moreover, the fact that the amplitude of the variation of the $\hat{n}^{\rm v} \cdot \hat{n}_0$ is the factor $|\epsilon^{\rm v}_\kappa|/\Lambda^{\rm v}$, suggests that the frame should be associated with $\hat{n}^{\rm v}$. Furthermore the frame should reduce to the frame $(\hat{l}(s), \hat{n}_0(s), \hat{m}(s))$ at $J_y = 0$. This suggests choosing a uniform invariant frame field $(\hat{u}^{\rm v}_1(z,s),\ \hat{n}^{\rm v}(z,s),\ \hat{u}^{\rm v}_2(z,s))$ [10]. A u-IFF is a field, over the phase space, of orthonormal reference frames and it provides a local reference frame for spin at each point in phase space as a particle moves through phase space. There is an infinity of such u-IFF's, each corresponding to a single member of the countable infinity of ADST's. In each u-IFF a spin precesses around $\hat{n}$ at a uniform rate given by its corresponding ADST. Here we need the preferred u-IFF (p-u-IFF) corresponding to the p-ADST. We already have $\hat{n}^{\rm v}(\phi_y)$ just before the RFD as in figure 3. The axes $\hat{u}^{\rm v}_1(\phi_y)$ and $\hat{u}^{\rm v}_2(\phi_y)$ of the p-u-IFF can be calculated as in [13]. Then within the p-u-IFF, $\hat{n}^{\rm tot}$ takes the form in figure 6. This is indeed simple and appears to correspond to an SRM. In particular, it is found that $\hat{n}^{\rm tot} \cdot \hat{n}^{\rm v} = 0$ at all $[\phi_{\rm rf}/2\pi]$ as expected if $\nu(J_y)$ is the appropriate reference spin tune. Furthermore at other values of $\tilde{\delta}^{\rm rf}$, $\hat{n}^{\rm tot} \cdot \hat{n}^{\rm v}$ is again independent of $[\phi_{\rm rf}/2\pi]$ and the other components vary sinusoidally with $[\phi_{\rm rf}/2\pi]$ as expected for an SRM. Thus although the ISF in figure 4 looks complicated at first, it actually corresponds to an "SRM within an SRM". However, one finds, for example, that for $|\hat{n}^{\rm tot} \cdot \hat{n}^{\rm v}| = 1/\sqrt{2}$, $|\tilde{\delta}^{\rm rf}|$ must be set to $|\tilde{\epsilon}^{\rm rf}_\kappa| \equiv 0.901 \times |\epsilon^{\rm rf}_\kappa|$. So the resonance strength, $|\tilde{\epsilon}^{\rm rf}_\kappa|$, for the spin motion viewed within the p-u-IFF is less than $|\epsilon^{\rm rf}_\kappa|$. This is just as expected when $\hat{n}^{\rm v}$ is tilted away from

$\hat{n}_0$. On the other hand, the amplitude of the forced vertical motion depends on $Q_{\rm rf}$. Then the underlying $|\epsilon_\kappa^{\rm rf}|$ can change when $\tilde{\delta}^{\rm rf}$ is moved towards zero [9].

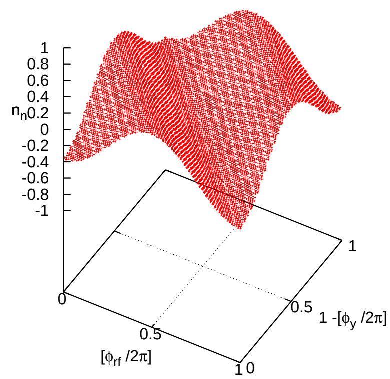

Figure 5. The $\hat{n}_0$-component of the ISF of figure 4 plotted w.r.t $[\phi_{\rm rf}/2\pi]$ and $1-[\phi_y/2\pi]$.

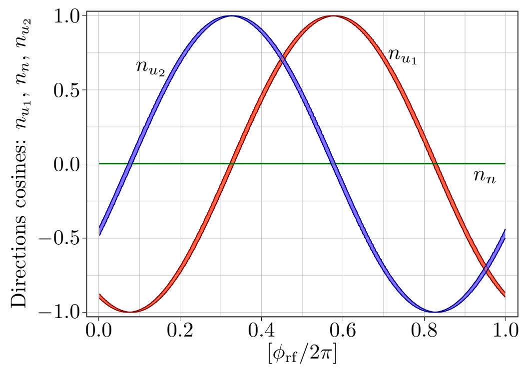

Figure 6. The ISF of figure 4 viewed within the p-u-IFF of the pre-existing vertical betatron motion.

The appositeness of $|\tilde{\epsilon}_\kappa^{\rm rf}| = 0.901 \times |\epsilon_\kappa^{\rm rf}|$ for describing spin motion in the p-u-IFF can be demonstrated in two other ways:

- The RF field together with the forced vertical motion causes a small change in the p-ADST and a small change in the p-u-IFF. Then by analogy with the calculation of $\nu(J_y)$ the shifted p-ADST is expected to be $\nu^{\rm tot} \equiv ({\rm sgn}\,\tilde{\delta}^{\rm rf})\tilde{\Lambda}^{\rm vrf} + \kappa^{\rm rf} = ({\rm sgn}\,\tilde{\delta}^{\rm rf})\tilde{\Lambda}^{\rm vrf} + \nu(J_y) - \tilde{\delta}^{\rm rf}$ where $\tilde{\Lambda}^{\rm vrf} = \sqrt{(\tilde{\delta}^{\rm rf})^2 + |\tilde{\epsilon}_\kappa^{\rm rf}|^2}$. This is supported by a spectral analysis of the spin motion, with a line occurring at $\nu^{\rm tot}$ together with other lines consistent with Theorem 9.1.c in [10]. Replacing $|\tilde{\epsilon}_\kappa^{\rm rf}|$ by $|\epsilon_\kappa^{\rm rf}|$ gives a poor fit. Of course, as a study of the SRM and the derivation of the F-S formula show, a shift caused by the RFD and the forced vertical motion will not cause a shift in the $Q_{\rm rf}$ needed for resonant spin flip.
- As stated above, in the presence of the large-amplitude, pre-existing, vertical betatron motion, the F-S formula does not give a good description of the final $\vec{S}\cdot\hat{n}_0$ as $\delta^{\rm rf}$ or $\tilde{\delta}^{\rm rf}$ is swept through zero. But with the establishment of the SRM in the p-u-IFF of the vertical betatron motion, we might expect the F-S formula to describe the final $\vec{S}\cdot\hat{n}^{\rm v}$ when $\tilde{\delta}^{\rm rf}$ is swept through zero after $\vec{S}$ has been set parallel to $\hat{n}^{\rm v}$ at the start. This is indeed the case providing that $|\tilde{\epsilon}_\kappa^{\rm rf}| = 0.901 \times |\epsilon_\kappa^{\rm rf}|$ is used in the F-S formula instead of $|\epsilon_\kappa^{\rm rf}|$. Use of $|\epsilon_\kappa^{\rm rf}|$ gives a poor fit.

So with three checks pointing to the same $|\tilde{\epsilon}_\kappa^{\rm rf}|$, a consistent picture has been established.

Note that the inclusion of horizontal betatron motion has an almost imperceptible influence on the ISF's, spin flipping and the p-ADST etc., and doesn't change the picture presented here. This is because the effect of the vertical quadrupole fields averages quickly away. This is discussed more fully in [9].

3. An analytical treatment

So far, the analysis of the spin motion has been numerical with the aim of emphasising and demonstrating the efficacy of precise concepts and appropriate computational tools. However, the picture of nested SRM's also emerges by expressing spins and the T-BMT equation in terms of two-component spinors and by then transforming into the p-u-IFF. The calculation is presented in [9] where I show how, for a given $\vec{\omega}(z, s)$, a resonance strength depends on the frame of reference used to calculate it. Then after accounting, in addition, for the change in $|\epsilon_\kappa^{\rm rf}|$ coming from the change in the forced vertical motion when $\tilde{\delta}^{\rm rf}$ is moved towards zero [9], one finds very good agreement with the attenuation factor of 0.901 obtained numerically. The effects on spins of a pair of nearby resonances have already been investigated in [19]. However, the potential for a reduced resonance strength and for the shift of the p-ADST away from ν_0, were obscured by the approximations used there.

4. An experimental test

We have seen that spin motion on vertical betatron trajectories is well described by the SRM and that, as a result, the p-ADST can be calculated easily when the $\epsilon_\kappa^{\rm v}$ is known. On the other hand some literature contains confusing definitions of spin tune on synchro-betatron trajectories as explained in [10]. So, given the expertise with spin manipulations at COSY, it would make sense to try to measure the p-ADST.

The tests of spin flipping for protons in [1] involved sweeping $Q_{\rm rf}$ from 4 kHz below resonance to 4 kHz above. This is equivalent to sweeping $\delta^{\rm rf}$ over a range of about $114 \times |\epsilon_\kappa^{\rm rf}|$ so that according to the SRM, $\hat{n}^{\rm rf}$ is essentially parallel to $\hat{n}_0$ at each end of the sweep.

Measurements were made at six values of Q_y between 3.52 and 3.60, one of which was 3.58. For this interval the ratio of the total resonance strength $|\epsilon_\kappa^{\rm rf}|$ to that of the RFD alone ranged from about 13 to about 170 [2]. At the tune $Q_y = 3.5801$ of these calculations the ratio is about 42. There was no special reason for choosing this tune for the above calculations but it has sufficed for the points that I wish to make, and of course, the values of $\nu(J_y)$, ν_0, etc, given here are specific to the conditions specified and would change if the conditions changed.

However, it is clear from the formula for $\nu(J_y)$ that at a fixed $|\epsilon_\kappa^{\rm v}|$, $|\nu_0 - \nu(J_y)|$ increases as $|\delta^{\rm v}|$ decreases. But $|\epsilon_\kappa^{\rm v}|^2$ is proportional to J_y. So, to obtain a significant shift of $\nu(J_y)$ away from ν_0 but with a realistic vertical emittance, it would be better to run at $Q_y = 3.6$ where $\delta^{\rm v} = -0.005$. Then, assuming that $|\epsilon_\kappa^{\rm v}|$ is similar to that at $Q_y = 3.5801$, the SRM predicts $\nu_0 - \nu(J_y) = -0.001580$ for protons on the ellipse with an emittance of 3.6 mm.mrad. This is close to the 3-σ emittance of the uncooled beam and is therefore unexceptional. This shift in tune is comparable to the -0.001779 obtained earlier and experience with the tests of spin flipping at COSY shows that shifts of this magnitude are measurable. For this, one would use the RF longitudinal field of the RF solenoid that is already installed, and recalling figure 4, one would probe the spin-orbit system searching for the frequency that minimised the time average of the vertical polarisation. Note that a longitudinal field does not complicate matters by generating forced vertical betatron motion.

However, preparation of a suitable beam would be far from simple. The cleanest measurements would require that the protons sit *on* an ellipse in phase space. This would require cooling the beam to a vertical emittance of (say) 0.3 mm.mrad and then kicking it within one turn onto the ellipse with a specially prepared vertical kicker magnet. Then the protons would have to stay on the ellipse while subject to non-linear optical effects, scattering in the polarimeter, intrabeam scattering, space-charge forces etc, etc. Would the protons remain on the ellipse for long enough to allow measurements of the polarisation? How would the phase-space distribution be measured for checking that the particles were on the ellipse? There are several more of such questions and matters of detail [9].

In any case, since a ring is never perfectly aligned, ν_0 is never exactly equal to $a\gamma$. The measurements would therefore require finding the $Q_{\rm rf}$ of the solenoid needed to minimise the

[2] Note that the values given in Table 1 of [1] underestimate the true values by a factor of two owing to an error in the calculation of the resonance strength from the RFD alone.

time-averaged vertical polarisation for the cooled beam, and then increasing the size of the ellipse in steps to check the putative dependence of $\nu_0 - \nu(J_y)$ on the size of the ellipse.

As far as I am aware, the p-ADST has never been deliberately and directly measured while for some experiments it has been assumed that shifts in spin tunes are completely understood [20, 21, 22]. Thus for [22] the so-called pitch correction for vertical motion is considered but its calculation involves time averages without reference to the concept of the p-ADST. The pitch correction enters the SLIM formalism via column 4 of the G matrix for the vertical guide field [12, 23]. The resultant contribution to the p-ADST is therefore easily calculated using the methods of this paper. A comparison between the two approaches is made in [9].

5. Summary

This paper gives an example of how apparently complicated spin motion can be "deconstructed" into simpler parts when suitable mathematical concepts and computational tools are available. It shows how computational physics may lead to insights which might, at first, be forfeited when purely analytical methods are tried without guidance from prior numerical work.

I also suggest how the p-ADST could be measured using excellent facilities at COSY and then compared with the theory. It is clear such a measurement is on the borderline of feasibility. Nevertheless it would be of some interest to those exploiting spin precession for measurements in particle physics [20, 21, 22].

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