Introduction to Spin Polarisation in Accelerators and Storage Rings

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The aims and approach for the course

Give an introduction to the theory, phenomenology and practice of spin polarisation in accelerator physics.

Thereby to present a tool kit of concepts and principles as a basis for critical evaluation of existing and proposed systems.

While avoiding overwhelming detail by appealing, where necessary or appropriate, to heuristic arguments.

Minimal mathematics -- leaving room for physical pictures.

Topics

0. Introductory comments.
1. Spin and its motion – basics – SI units!
2. The stability of spin motion in rings.
3. The effect of synchrotron radiation I -self polarisation
4. The effect of synchrotron radiation II -- depolarisation.
5. Depolarisation: the invariant spin field.
6. A Fokker-Planck equation for the polarisation density.
7. Polarimetry in high energy electron rings.
8. Acceleration of pre-polarised electrons.
9. Polarised protons at high energy.
10. Kinetic polarisation of electrons.

Literature

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3. M. Berglund, DESY-THESIS-2001-044 .
4. G. H. Hoffstaetter, Habilitation, "High Energy Polarized Proton Beams: a Modern View", Springer, coming soon.
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9. Z. Huang Thesis SLAC-R-527 ,1998 (also papers with R. Ruth).
10. A.A. Sokolov & I.M. Ternov ``Synchrotron Radiation", 1968,1986.
11. A.I. Akhiezer, "High Energy Electrodynamics in Matter".
12. J. Mehra,"The Golden age of Theoretical Physics", Ch 17.
13. S-I Tomonaga, "The Story of Spin" .
14. and lots more………

Self polarisation of electrons -some history

The history of radiative spin polarization
in storage rings.

Table 1:

Name	Year	Max. Polarization	Energy
VEPP	1970	80% vert	0.65 GeV
ACO	1970	90% vert	0.53 GeV
VEPP–2M	1974	90% vert	0.65 GeV
VEPP–3	1976	80% vert	2 GeV
SPEAR	1975	90% vert	3.7 GeV
VEPP–4	1982	80% vert	5 GeV
CESR	1983	30% vert	5 GeV
DORIS	1983	80% vert	5 GeV
PETRA	1982	70% vert	16.5 GeV
TRISTAN	1990	~~70%~~? vert	29 GeV
LEP	1993	57% vert	47 GeV
HERA	1993	60% vert	26.7 GeV
HERA	1994	70% long	27.5 GeV
LEP	1999	7% vert	60 GeV

Polarised protons: history, hopes and future

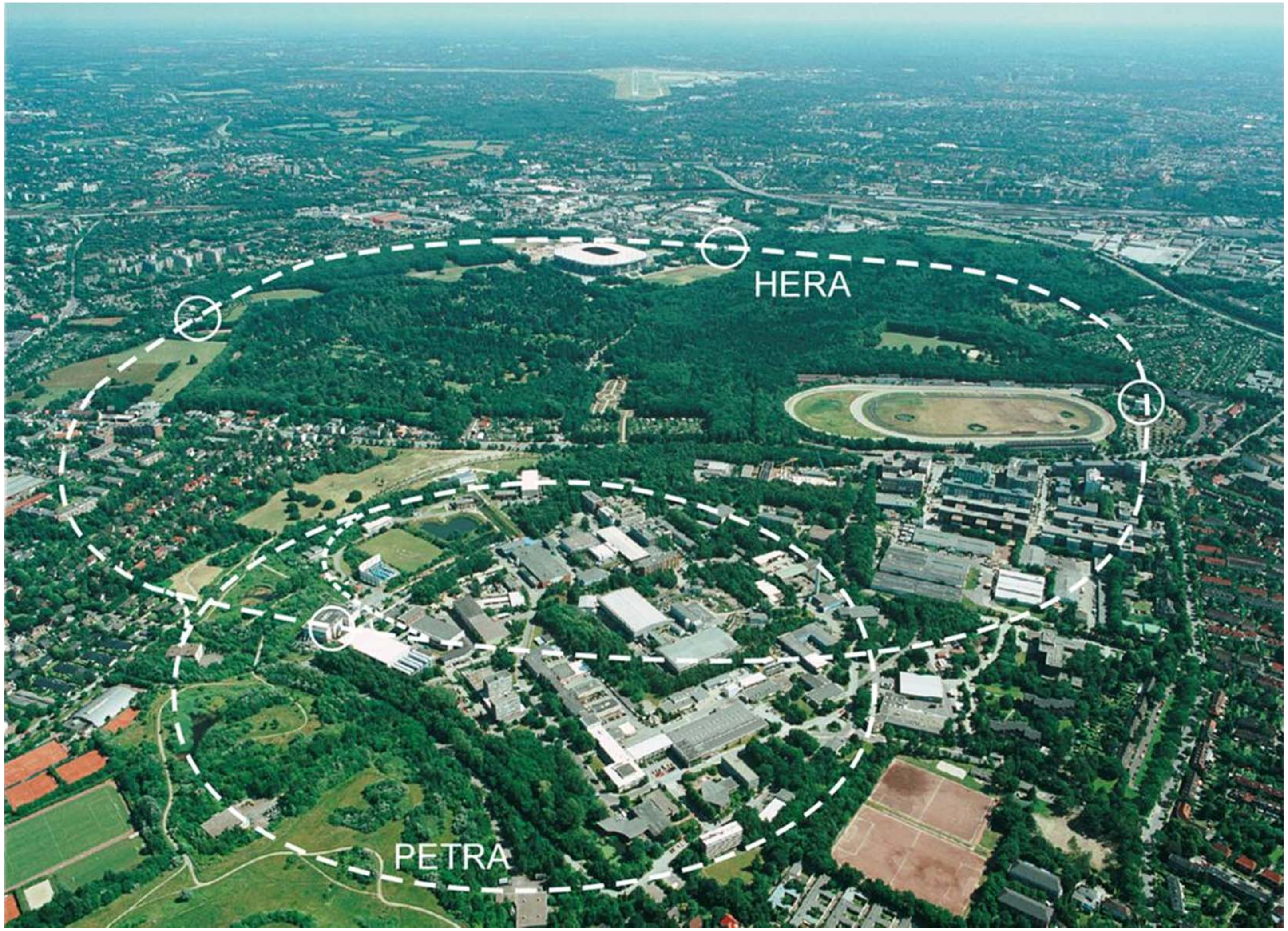
The quest for high energy polarised protons

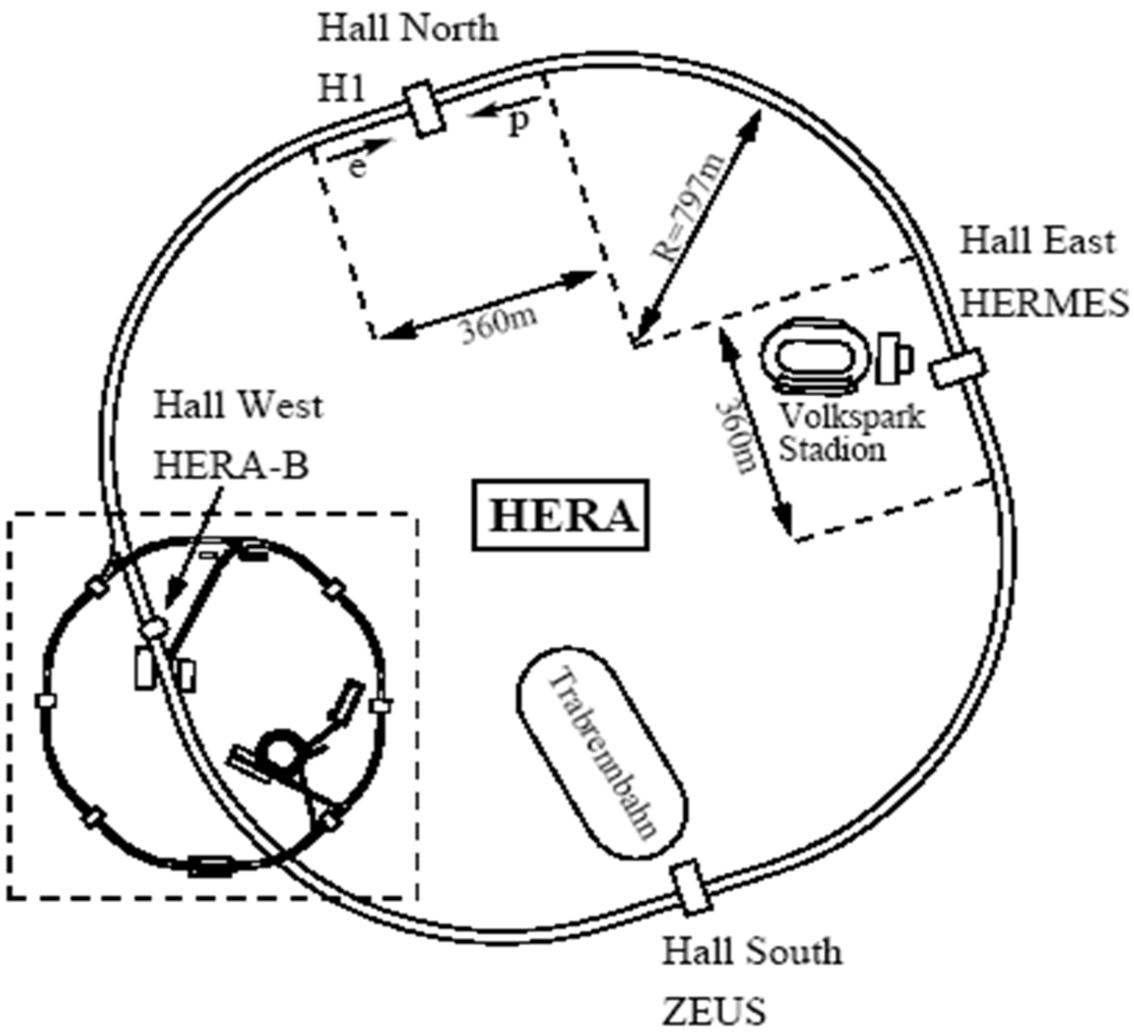
Name	Energy
ZGS	12 GeV
KEK PS	12 GeV
AGS	26 GeV
Saturn II	3 GeV
IUCF	1 GeV
PSI cycl.	0.6 GeV
TRIUMF cycl.	0.5 GeV
LAMPF	0.8 GeV
COSY	3.65 GeV
RHIC	205 GeV

High energy accelerators whose polarised proton capabilities have been analysed

Name	Energy
FNAL Main Injector	120 GeV
FNAL Tevatron	900 GeV
LISS	20 GeV
RHIC	250 GeV
HERA	920 GeV
EPIC	32 GeV
VLHC Booster	3000 GeV
HESR(GSI)	15 GeV
JPARC	50 GeV

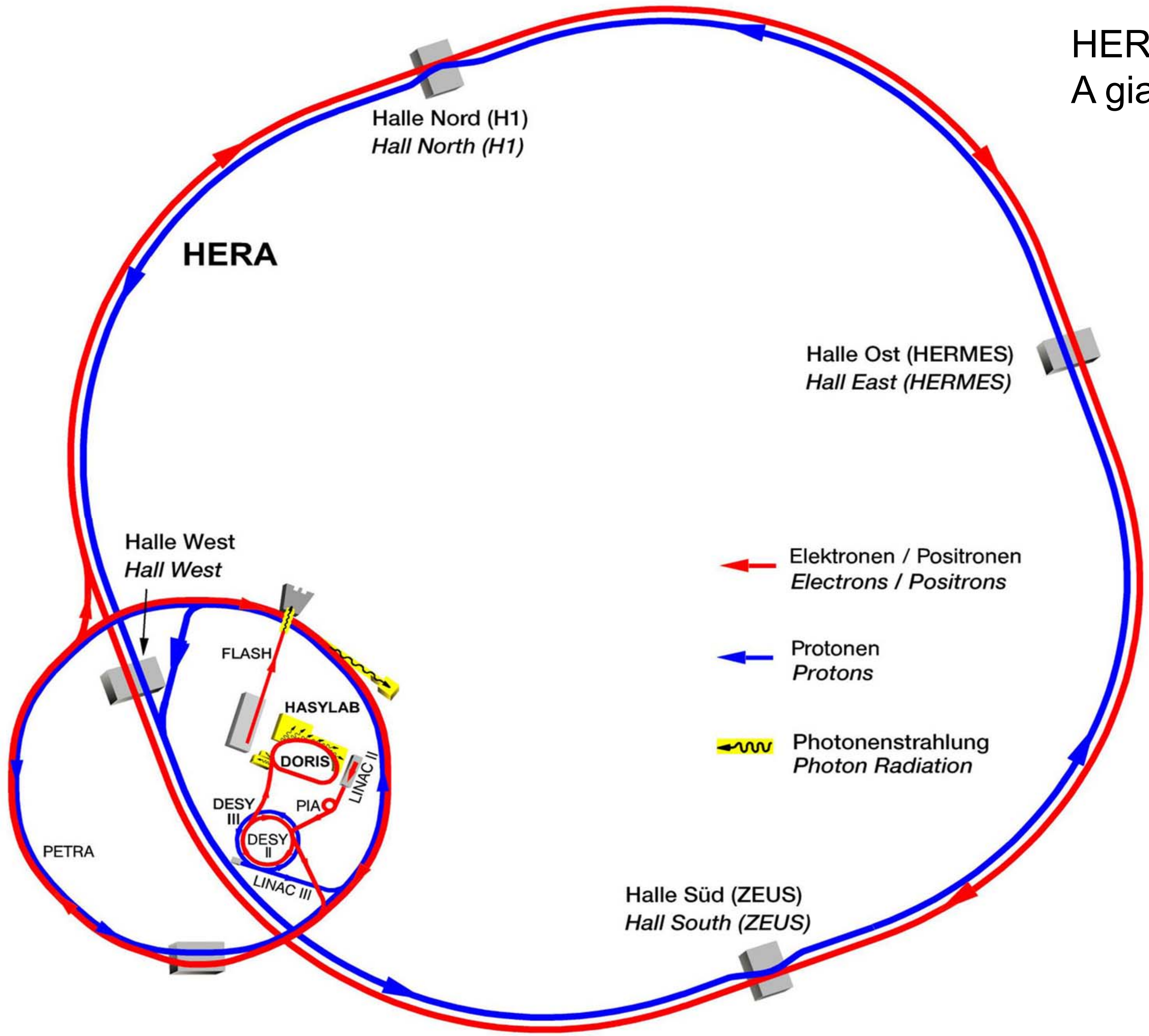
HERA





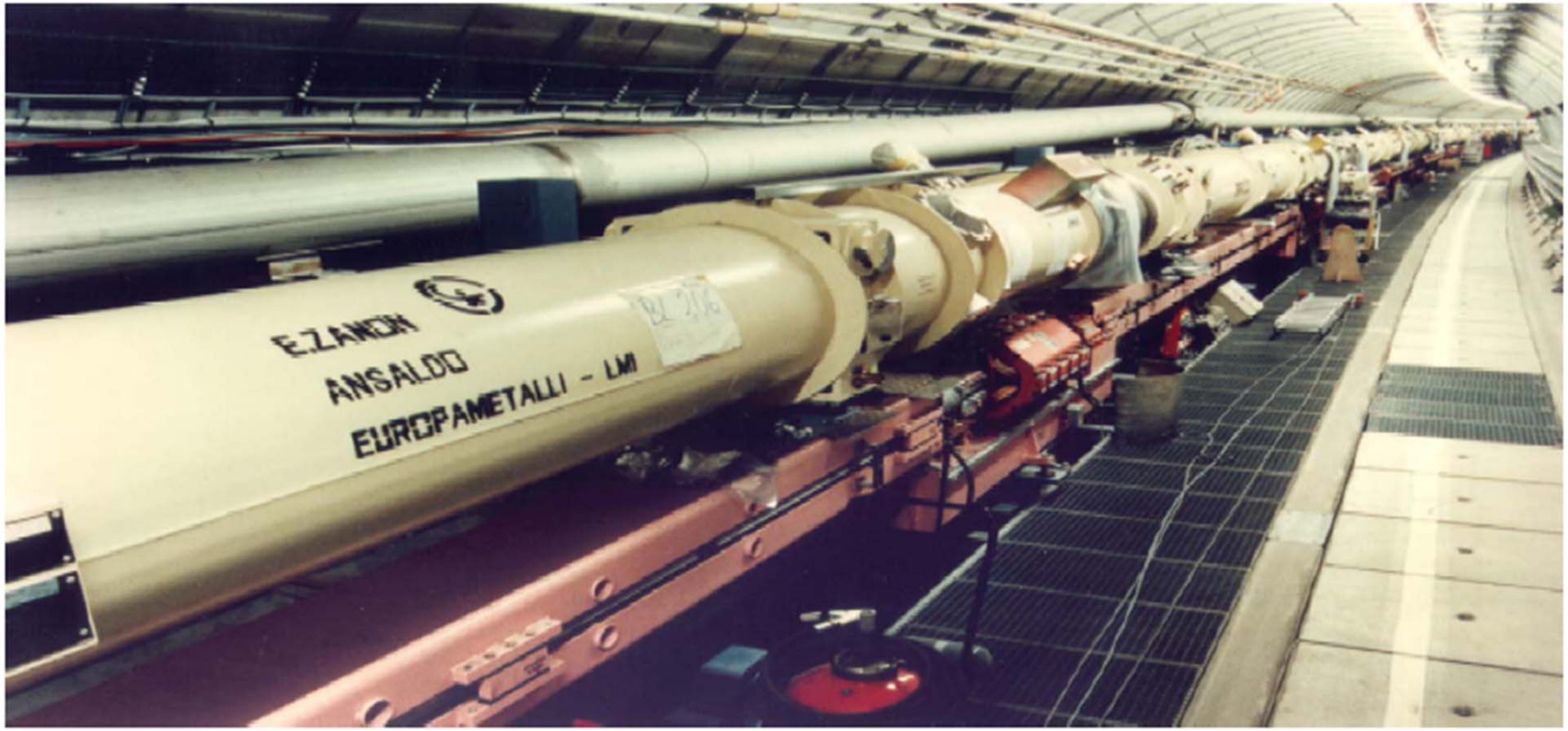
HERA:
820—920 GeV protons
27.5 GeV electrons/positrons.

HERA:
A giant quantum machine.

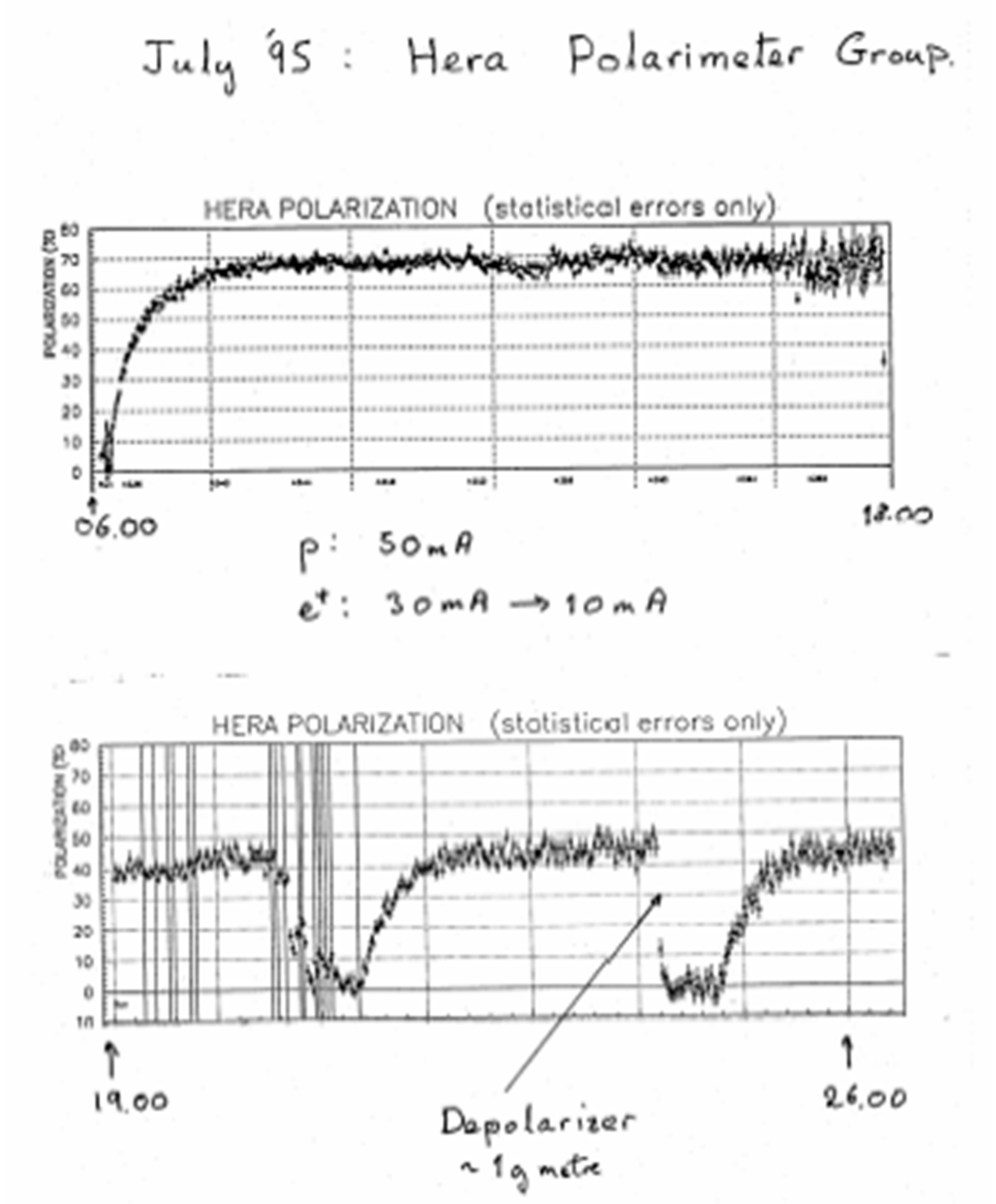


History

- First ideas: 1981.
- PEAS workshop: 1982.
- Spin rotator designs: Buon and Steffen, and Barber et al. 1982—1985
  ===> dipole variable energy and geometry "Mini Rotators" chosen.
- Spin matching for 4 pairs of rotators with a civilised optic demonstrated; 1985 (Barber).
- "Transverse" polarimeter designed and installed: 1988—1991.
- First measurements 1991: $\approx$ 8 percent vertical polarisation.
- Better ring alignment and "harmonic bumps" 1992: over 60 percent.
- Winter shutdown 1993—1994: installed one pair of rotators for HERMES in the East.
- May 1994: first run with rotators: immediate success!
- 1994 — September 2000: routinely running with over 50 percent even with high proton current making beam–beam interaction in N and S.
- But Nov. 1996: high proton currents ===> first indications of beam–beam effect on electron/positron polarisation.
- September 2000 — summer 2001: installation for the Upgrade.



Stolen from Bernhard Holzer



HERA was the first and only high energy electron storage ring to provide longitudinal polarisation -- and at three interaction points!

DESY TELEGRAMM

vom 3. März 2003

Alle drei HERA-Rotatoren in Betrieb: 50% Polarisation!

Großer Erfolg für HERA II kurz vor Beginn des *Shutdowns*

All Three HERA Rotators Running: 50% Polarization!

Great success for HERA II shortly before shutdown

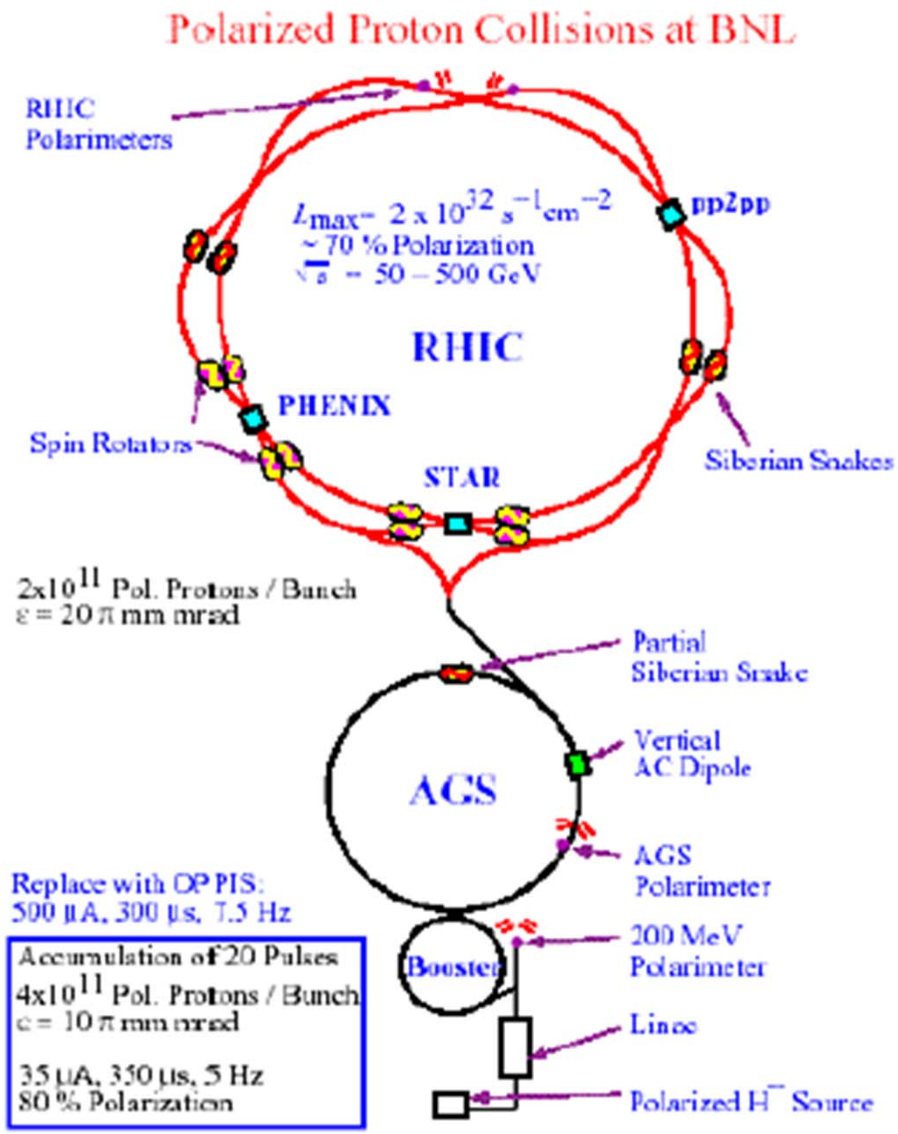
In der vergangenen Woche gelang es der HERA-Crew, alle drei Spin-Rotatorpaare nach einer sehr kurzen Optimierungsphase gleichzeitig in Betrieb zu nehmen und sehr schnell einen Polarisationsgrad der Positronen von etwa 50% zu erzielen. Bei diesem Prozess wurden die Spins, das heißt die Drehachsen der Teilchen des in HERA II gespeicherten Positronenstrahls, vor den Wechselwirkungszonen von HERMES, H1 und ZEUS in die Flugrichtung und auf der anderen Detektorseite wieder zurück geklappt - mehrere Stunden lang, jeweils 47000-mal in der Sekunde. Im Norden (H1) und Süden (ZEUS), wo die Rotatormagnete erst 2001 eingebaut worden sind, konnten sogar bei eingeschalteten Detektor-Magneten polarisierte Positronen mit Protonen zur Kollision gebracht werden - das ist nicht nur bei HERA das erste Mal, sondern auch weltweit. Alle vier HERA-Experimente konnten mit den drei eingeschalteten Rotatorpaaren Ereignisse messen. Nach dem *Shutdown* wird sich für die beiden Kollisionsexperimente durch die Polarisation ein aufregendes neues Experimentierprogramm eröffnen.

Am 3. März, um 7 Uhr wurden die beiden Beschleuniger von HERA II für die geplante Betriebsunterbrechung herunter gefahren. In diesem 18-wöchigen *Shutdown* sollen technische Arbeiten am Vakuumsystem von HERA II sowie an einigen Komponenten der Detektoren H1 und ZEUS ausgeführt werden, um die unerwartet hohen Untergrundraten, die nach dem Umbau der beiden Wechselwirkungszonen aufgetreten sind, zu reduzieren.

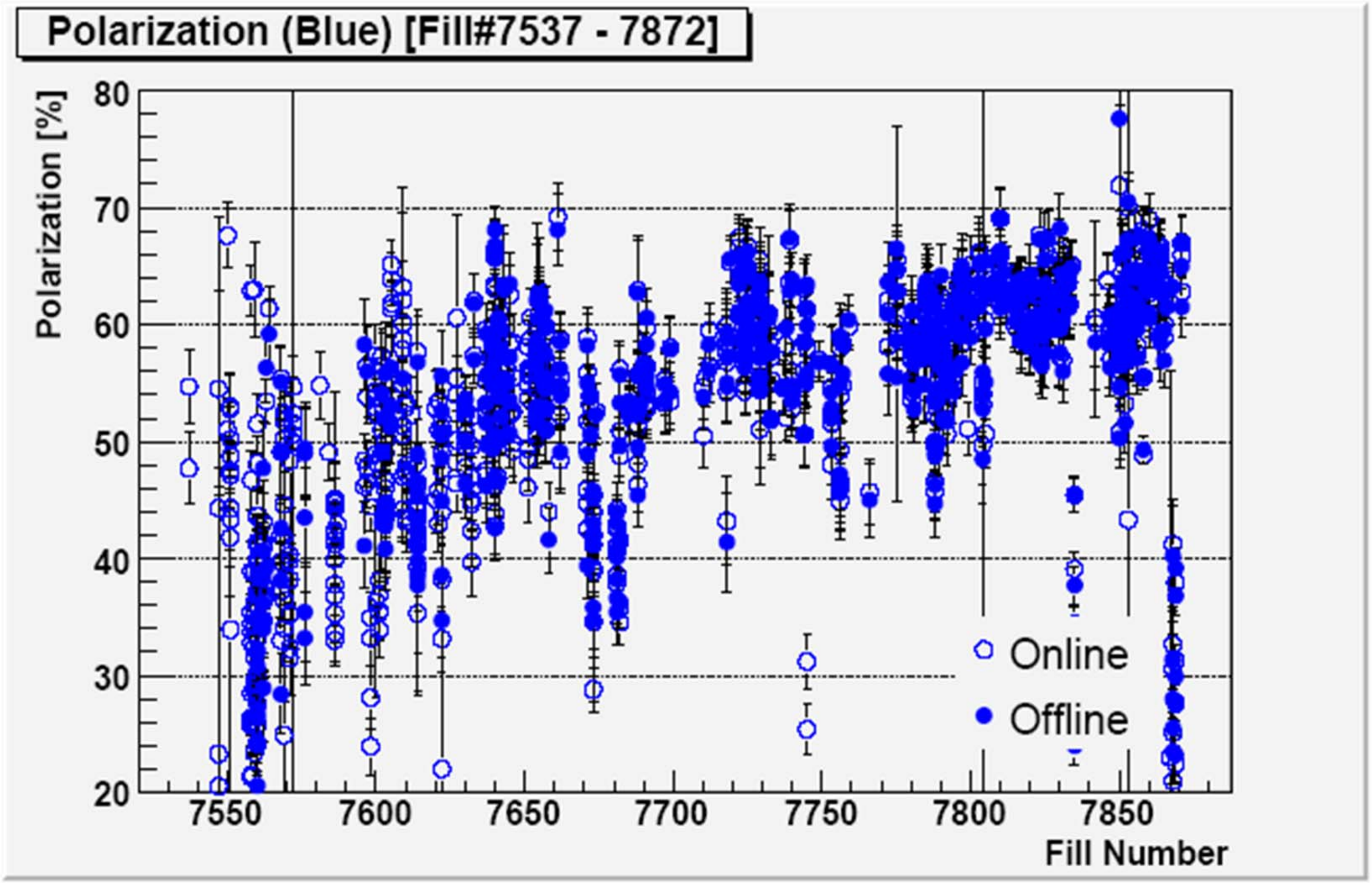
Last week, after a very short optimization phase, the HERA crew succeeded in putting all three spin rotators pairs simultaneously into operation and very quickly achieved a polarization degree of the positrons of approx. 50%. In this process, the spins, i.e. the "rotational axes" of the positrons stored in HERA II were flipped into the direction of the beam in front of the interaction regions of HERMES, H1 and ZEUS and back on the other side of the detectors - for many hours, 47000 times per second. In the north (H1) and south (ZEUS), where the rotator magnets were built in only in 2001, polarized positrons were brought into collision with protons even with activated detector magnets. This is not only the first time for HERA but also worldwide. All four HERA experiments were able to take data with activated rotator pairs. After the shutdown, polarization will open up an exciting new experimental program for both collision experiments.

On March 3, at 7 a.m. both accelerators of HERA II were turned off for the scheduled shutdown. In this 18-weeks period, technical work will be performed on the vacuum system of HERA II and also on some components of the H1 and ZEUS detectors, in order to reduce the unexpectedly high background rates which emerged after the upgrade of both interaction regions.

Herausgegeben von DESY/PR, Aushang bis 14.3.2003


Polarized Proton Collisions at BNL
RHIC Polarimeters
$L_{max} = 2 \times 10^{32}\ s^{-1} cm^{-2}$
~ 70 % Polarization
$\sqrt{s}$ = 50 – 500 GeV
pp2pp
RHIC
PHENIX
Spin Rotators
STAR
Siberian Snakes
2x10^11 Pol. Protons / Bunch
ε = 20 π mm mrad
Partial Siberian Snake
Vertical AC Dipole
AGS
AGS Polarimeter
Replace with OPPIS:
500 µA, 300 µs, 7.5 Hz
Accumulation of 20 Pulses
4x10^11 Pol. Protons / Bunch
ε = 10 π mm mrad
35 µA, 350 µs, 5 Hz
80 % Polarization
Booster
200 MeV Polarimeter
Linac
Polarized H⁻ Source

Polarisation at RHIC 2006 –up to 100GeV



Topic 1

Spin and its motion - basics - SI units!(*)

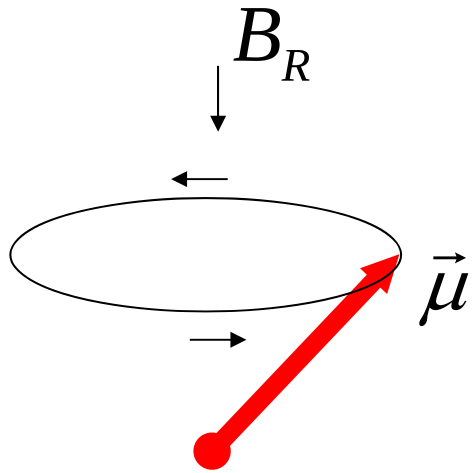
*: Except that $\hbar = 1.3 \times 10^{-37}$ horsepower sec^{2}

The standard textbook introduction

(The charge *e* is –ve for electrons, +ve for positrons)

A particle at rest in a classical picture:

$$\frac{d\vec{S}}{d\tau} = \vec{\Gamma} = \vec{\mu} \times \vec{B}_R \qquad \text{But } \vec{\mu} = \underbrace{\left(\frac{g_s e}{2m}\right)}_{\textit{Gyromagnetic ratio}} \vec{S} = \left(\frac{g_s \mu_B}{\hbar}\right) \vec{S} \text{ !!}$$



$$\Rightarrow \quad \frac{d\vec{S}}{d\tau} = \frac{g_s e}{2m} \vec{S} \times \vec{B}_R$$

i.e precession around $\vec{B}_R$ at fixed $\vec{S}.\vec{B}_R$

$$\equiv \vec{\Omega} \times \vec{S} \;\Rightarrow \vec{\Omega} = -\frac{g_s e}{2m} \vec{B}_R$$

<u>Larmor</u> frequency

Goudsmit +Uhlenbeck

Stern-Gerlach energy: $\Delta E_{SG} = -\vec{\mu}.\vec{B}_R = \vec{\Omega}.\vec{S}$

Stern-Gerlach experiment: For $e^{\pm}$: $g \equiv g_s \approx 2$

Spin is a 3-vector defined in the rest frame and this is the object needed for calculating scattering cross-sections.

But in accelerators we have an ensemble of particles with various energies moving on various trajectories in the laboratory. At each instant, the particles have different rest frames and these rest frames are non-inertial when the trajectories are curved as in magnetic fields. Moreover, the magnets are fixed in the laboratory.

So we need a way to express rest-frame spin motion in a common coordinate system and to replace time as the independent variable by the distance around the ring.

A reminder:

Consider a 3-vector $\vec{\Gamma} = \Gamma_1 \hat{e}_1 + \Gamma_2 \hat{e}_2 + \Gamma_3 \hat{e}_3$

where $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is an orthonormal rotating coordinate system with

$$\frac{d\hat{e}_1}{dt} = \vec{\omega}_R \times \hat{e}_1 , \frac{d\hat{e}_2}{dt} = \vec{\omega}_R \times \hat{e}_2 , \frac{d\hat{e}_3}{dt} = \vec{\omega}_R \times \hat{e}_3$$

If $(\Gamma_1, \Gamma_2, \Gamma_3)$ are constant then $\frac{d\vec{\Gamma}}{dt} = (\dot{\hat{e}}_1 \Gamma_1 + \dot{\hat{e}}_2 \Gamma_2 + \dot{\hat{e}}_3 \Gamma_3) = \vec{\omega}_R \times \vec{\Gamma}$

Otherwise
$$\frac{d\vec{\Gamma}}{dt} = (\hat{e}_1 \dot{\Gamma}_1 + \hat{e}_2 \dot{\Gamma}_2 + \hat{e}_3 \dot{\Gamma}_3) + (\dot{\hat{e}}_1 \Gamma_1 + \dot{\hat{e}}_2 \Gamma_2 + \dot{\hat{e}}_3 \Gamma_3)$$

$$\Rightarrow \frac{d\vec{\Gamma}}{dt} = (\hat{e}_1 \dot{\Gamma}_1 + \hat{e}_2 \dot{\Gamma}_2 + \hat{e}_3 \dot{\Gamma}_3) + \vec{\omega}_R \times \vec{\Gamma}$$

Equivalently $\left.\frac{d\vec{\Gamma}}{dt}\right|_{\text{lab}} = \left.\frac{d\vec{\Gamma}}{dt}\right|_{\substack{\text{in rot}\\ \text{frame}}} + \vec{\omega}_R \times \vec{\Gamma}$ or $\left.\frac{d}{dt}\right|_{\text{lab}} = \left.\frac{d}{dt}\right|_{\substack{\text{in rot}\\ \text{frame}}} + \vec{\omega}_R \times$

Goldstein: Classical Mechanics?

A particle moving on a curved trajectory: Thomas precession!

Case 1: $\vec{\mu} \neq \vec{0}, e = 0$, e.g., neutrons $\Longrightarrow$

Motion is a straight line in the lab. using $d\tau = dt/\gamma$

$$\frac{d\vec{S}}{dt} = \frac{\vec{\mu} \times \vec{B}_R}{\gamma} = \vec{\mu} \times \left(\vec{B} - \frac{\beta \hat{t} \times \vec{E}}{c} + \frac{1-\gamma}{\gamma} (\hat{t} . \vec{B}) \hat{t} \right)$$

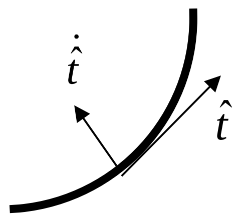
$$= \vec{\mu} \times \left(\vec{B}_\perp - \frac{\vec{\beta} \times \vec{E}}{c} + \frac{\vec{B}_{//}}{\gamma} \right)$$

Case 2: $\vec{\mu} = \vec{0}, e \neq 0$, no torque in rest frame but the Lorentz force causes the orbit direction to rotate at a rate

$$\vec{\omega}_O = -\dot{\hat{t}} \times \hat{t} \; ; \qquad \dot{\hat{t}} = \frac{e}{m\gamma} \left(\frac{\vec{E} - (\vec{E}.\hat{t})\hat{t}}{c\beta} + \hat{t} \times \vec{B} \right) \qquad \Rightarrow \qquad \vec{\omega}_O = \frac{-e}{m\gamma} \left(\vec{B}_\perp + \frac{\vec{E} \times \hat{t}}{c\beta} \right)$$

Might expect w.r.t. lab: recall $: \frac{d}{dt} \Rightarrow \frac{d}{dt} + \vec{\omega} \times$

$$\frac{d\vec{S}}{dt} = \vec{\omega}_O \times \vec{S} = \frac{e}{m\gamma} \vec{S} \times \left(\vec{B}_\perp + \frac{\vec{E} \times \hat{t}}{c\beta} \right)$$



Case 3: $\vec{\mu} \neq \vec{0},\, e \neq 0$, e.g., electrons, protons

Might expect w.r.t. lab:

$$\frac{d\,\vec{S}}{d\,t} = \frac{g_s e}{2\,m}\vec{S} \times \left(\vec{B}_{\perp} - \frac{\beta\,\hat{t} \times \vec{E}}{c} + \frac{\vec{B}_{//}}{\gamma} \right) \quad + \quad \frac{e}{m\,\gamma}\vec{S} \times \left(\vec{B}_{\perp} + \frac{\vec{E} \times \hat{t}}{c\,\beta} \right)$$

ONLY A PART OF THE STORY!

W.r.t. the set of instantaneous inertial frames we should use :

$$\vec{\omega}_O = (\gamma - 1)\dot{\hat{t}} \times \hat{t} \;\; !!$$

This whole piece is called the Thomas precession rate. There is an additional rotation w.r.t. these frames when successive boosts in different directions are composed. See also ``Wigner rotation'' and Fermi-Walker transport.

Really need:

$$\frac{d\,\vec{S}}{d\,t} = \frac{g_s e}{2\,m}\vec{S} \times \left(\vec{B}_{\perp} - \frac{\beta\,\hat{t} \times \vec{E}}{c} + \frac{\vec{B}_{//}}{\gamma} \right) \quad - \quad \frac{e(\gamma - 1)}{m\,\gamma}\vec{S} \times \left(\vec{B}_{\perp} + \frac{\vec{E} \times \hat{t}}{c\,\beta} \right)$$

No EDM allowed here, otherwise extra terms due to electric field in the rest frame

$$\frac{d\vec{S}}{dt} = \frac{e}{m}\vec{S} \times \left((\frac{g}{2} - 1 + \frac{1}{\gamma})\vec{B} - (\frac{g}{2} - 1)\frac{\gamma\beta^2}{1+\gamma}(\hat{t}.\vec{B})\hat{t} - (\frac{g}{2} - \frac{\gamma}{1+\gamma})\frac{\beta\,\hat{t}\times\vec{E}}{c} \right)$$

Use: $a = \dfrac{g-2}{2} \quad (= G)$

$$\frac{d\vec{S}}{dt} = \frac{e}{m\gamma}\vec{S} \times \left((1 + a\gamma)\vec{B} - \frac{a\gamma^2\beta^2}{1+\gamma}(\hat{t}.\vec{B})\hat{t} - (a\gamma + \frac{\gamma}{1+\gamma})\frac{\beta\,\hat{t}\times\vec{E}}{c} \right)$$

$$\frac{d\vec{S}}{dt} = \frac{e}{m\gamma}\vec{S} \times \left((1 + a\gamma)\vec{B} - a(\gamma - 1)(\hat{t}.\vec{B})\hat{t} - (a\gamma + \frac{\gamma}{1+\gamma})\frac{\beta\,\hat{t}\times\vec{E}}{c} \right)$$

Several forms. It’s very easy to get things wrong. Note that $G \neq g$!!!!!!!

This is the Thomas-Bargmann-Telegdi-Michel (T-BMT) equation. It expresses the motion of rest-frame spin w.r.t. lab axes, in terms of lab. time and fields.

Many derivations, e.g., Thomas 1927 (Phil. Mag.) to solve a problem with atomic spectra,e.g.,

$$\Delta E_{SG} = -\vec{\mu}.\vec{B} \Rightarrow -\frac{1}{2}\vec{\mu}.\vec{B} \approx 10^{-4} \times \text{orbital} \quad \text{energy}$$

Frenkel 1927,
BMT 1959 (PRL) (without reference to Thomas)

Without Thomas precession, fine structure spectra, $``g = 2"$ anomalous Zeeman effect. are inconsistent. .

Analogous extra precessions appear in many places, e.g., polarised light. .

Literature:
Jackson, Hagedorn, Sard, Panofsky+Philips(?), Møller, Bell (CERN75-11)

Formal covariant approach: ``symbolic” and for completeness.

Define a 4-spin $\Sigma^{\mu} = (\Sigma_0, \vec{\Sigma})$ which reduces to $(0, \vec{S})$ in the rest frame.

The 4-velocity is $V^{\mu} \equiv (\gamma c, \gamma \vec{\upsilon})$ and $V^{\mu}\Sigma_{\mu} = 0$

The Lorentz equation is $$\frac{dV^{\mu}}{d\tau} \equiv \frac{e}{m} F^{\mu\nu} V_{\nu}$$

With the field strength tensor $$F^{\mu\nu} = \frac{1}{c}\begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -cB_3 & cB_2 \\ E_2 & cB_3 & 0 & -cB_1 \\ E_3 & -cB_2 & cB_1 & 0 \end{bmatrix}$$

Then by searching for a most general covariant form and matching to the known rest-frame EOSM and the using the LE for $dV^{\mu}/d\tau$

$$\frac{d\Sigma}{d\tau} = -\frac{e}{m}\left(\frac{g}{2}\Sigma F - \frac{g-2}{2c^2} V(\Sigma F V)\right)$$ BMT

The Lorentz transform for $(\Sigma_0, \vec{\Sigma}) \Leftrightarrow (0, \vec{S})$ delivers the T-BMT equation.!!

Thomas

A sketch of the quantum mechanical picture

The Dirac equation:
for no em fields

$$(\gamma_\mu \partial_\mu + \kappa)\Phi = 0 \qquad \partial_k = \partial / \partial x_k , \qquad \partial_4 = \frac{1}{ic}\partial / \partial t$$

4X4 matrices

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}$$

4 comp wave function

$$\kappa = \frac{mc}{\hbar} I$$

Spin operators $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$

Careful with the symbol 'γ'

$$\sigma_1 = -i\gamma_2\gamma_3 , \sigma_2 = -i\gamma_3\gamma_1 , \sigma_3 = -i\gamma_1\gamma_2 \qquad [\sigma_1, \sigma_2] = 2i\hbar\sigma_3 \ \text{etc. cyclic}$$

Ansatz for plane wave along ``3'': $\Phi = u\, e^{\frac{i}{\hbar}(p_3 x_3 - Et)}$ u is a 4 comp column matrix

$$E = +c\sqrt{p_3^2 + m^2c^2} ; \quad \sigma_3 u^I = u^I ; \quad \sigma_3 u^{II} = -u^{II}$$

$$E = -c\sqrt{p_3^2 + m^2c^2} ; \quad \sigma_3 u^{III} = u^{III} ; \quad \sigma_3 u^{IV} = -u^{IV}$$

i.e., 4 solutions for *u* :
2 energies, 2 spin states.

The Dirac equation: with em fields: minimal coupling

$$[\gamma_\mu(\partial_\mu - \frac{ie}{\hbar}A_\mu) + \kappa]\Phi = 0$$

vector potential.

Operate from the left with:
Clean up and take the low +ve energy limit. Only B fields

$$\gamma_\mu(\partial_\mu - \frac{ie}{\hbar}A_\mu) - \kappa$$

At low energy:
Schrödinger eqn including the ``Pauli term''.

$$\left(\frac{1}{2m}\vec{P}^2 - \frac{e\hbar}{2m}\vec{\sigma}.\vec{B}\right)\Phi = (E - mc^2)\Phi$$

where. $P_k = \frac{\hbar}{i}\frac{\partial}{\partial x_k} - eA_k$

$$\Delta E_{SG} = -\vec{\mu}.\vec{B}_R$$

Stern-Gerlach term in the Hamiltonian.

$$H_{SG} = -\frac{e}{m}\left(\frac{\hbar}{2}\vec{\sigma}\right).\vec{B} = -\frac{g_s e}{2m}\vec{S}.\vec{B}$$

Heisenberg EOM (for operators)

$$\frac{d\vec{S}}{dt} = \frac{i}{\hbar}\left[H_{SG}, \vec{S}\right]; \quad \text{with } \left[S_1, S_2\right] = i\hbar S_3 \ \text{etc. cyclic}$$

$$\Rightarrow \frac{d\vec{S}}{dt} = \frac{e}{m}\vec{S}\times\vec{B} \Rightarrow \frac{d\langle\vec{S}\rangle}{dt} = \frac{e}{m}\langle\vec{S}\rangle\times\vec{B}$$

So, in the classical picture applied to fermions, we were talking about $\langle\vec{S}\rangle$
--the expectation value obeys a classical equation.

So far each spin state needed a 4 component wave function.

However, for each energy(+/-), two components should suffice!

So, try to find a a canonical (unitary) transformation to achieve this:

$$\Phi \Rightarrow \hat{\Phi} = e^{iD}\Phi \,; \quad H \Rightarrow \hat{H} = e^{iD} H e^{-iD}$$

In the absence of em fields we need a hermitian

$$D = D(m, c, \gamma_i, p_i) \text{ with } \quad p_i = \frac{\hbar}{i}\frac{\partial}{\partial x_i}$$

Then with $\hat{\Phi} = \hat{\Phi}_+ + \hat{\Phi}_-$ we have $\hat{H}\hat{\Phi}_+ = |E|\,\hat{\Phi}_+ \,; \quad \hat{H}\hat{\Phi}_- = -|E|\,\hat{\Phi}_-$

where $\hat{H} = \gamma_4 c\sqrt{p^2 + m^2c^2}$

Moreover one can choose a representation with $\hat{\Phi}_+ = \begin{pmatrix} \hat{\Phi}_{+1/2} \\ \hat{\Phi}_{-1/2} \\ 0 \\ 0 \end{pmatrix}; \; \hat{\Phi}_- = \begin{pmatrix} 0 \\ 0 \\ \hat{\Phi}_{+1/2} \\ \hat{\Phi}_{-1/2} \end{pmatrix}$ and $\hat{\sigma}_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$

Usual 2X2

Now we effectively have just 2-components for each (+/-) energy.

This is a Foldy-Wouthuysen transformation (F-W 1950).

At very low energy $\hat{\Phi}_{-} << \hat{\Phi}_{+}$

Now turn on the em field and glue a term into the DE to account for fact that g ≠ 2.

$$-\frac{e(g-2)\hbar}{4m}\left(\gamma_4(\hat{\sigma}.\vec{B})+(\vec{\gamma}.\frac{\vec{E}}{c})\right) \quad \text{``Pauli term'', 1941 (``standard'' repn.)}$$

This then gets much messier but a transformation to a 2-component formalism can still be found. For example there is a 2-component formalism with time dependent fields with

$$\hat{H} = c\sqrt{(\vec{p}-e\vec{A})^2+m^2c^2} + e\phi - \frac{\hbar}{2}\vec{\Omega}.\vec{\sigma} \ + O(\hbar^2.....) \ \text{where}$$

$$\vec{\Omega} = \frac{e}{m\gamma}\left((1+a\gamma)\vec{B} - \frac{a\gamma^2}{1+\gamma}(\vec{\beta}.\vec{B})\vec{\beta} - (a\gamma+\frac{\gamma}{1+\gamma})\frac{\vec{\beta}\times\vec{E}}{c}\right) \ !!!!$$

This is usually called the Derbenev-Kondratenko Hamiltonian.

To include radiation: $\hat{H} \Rightarrow \hat{H}_{\text{class}} + \hat{H}_{\text{rad}} \ : \ \hat{H}_{\text{rad}} = e(\varphi_{\text{rad}} - \vec{\beta}.\vec{A}_{\text{rad}}) + \frac{\hbar}{2}\vec{\Omega}_{\text{rad}}.\vec{\sigma}$

$$\Rightarrow \frac{d\vec{S}}{dt} = \frac{e}{m}\,\vec{S}\times\vec{\Omega} \;\Rightarrow \frac{d\langle\vec{S}\rangle}{dt} = \frac{e}{m}\langle\vec{S}\rangle\times\vec{\Omega}$$

So, for fermions, the T-BMT eqn. can be obtained from the Dirac eqn and an appropriate F-W transformation taken to *O(ħ)* !

OR

Use the T-BMT eqn to provide the interpretation of the meaning of $\vec{S}$ in the D-K Hamiltonian.

Foldy + Wouthusen(1950), Pryce, Blount, Case (1953), Mendlowitz +Case, Fradkin + Good, Suttorp + DeGroot, K. Heinemann, Pursey, Derbenev + Kondratenko etc……….

In any case we have a classical EOM for $\langle\vec{S}\rangle$

Atomic physics with the F-W representation gives the correct spectra automatically.

The T-BMT eqn applies to the $\langle \vec{S} \rangle$ of all particles with spin: $e^{\pm}, \mu^{\pm}, p^{\pm}, d, {}^{3}He$ ....

For deuterons (spin-1) $\langle \vec{S} \rangle$ is not the full story: ``tensor polarisation”

From now on I’ll use the symbol $\vec{S}$ to mean $\langle \vec{S} \rangle$!!!!

We will call this the ``spin”

Back to the T-BMT !

$$\frac{d\vec{S}}{dt} = \frac{e}{m\gamma}\vec{S}\times\left((1+a\gamma)\vec{B} - a(\gamma-1)(\hat{t}.\vec{B})\hat{t} - (a\gamma+\frac{\gamma}{1+\gamma})\frac{\beta\,\hat{t}\times\vec{E}}{c}\right)$$

$$\vec{\omega}_O = -\dot{\hat{t}}\times\hat{t}\,; \qquad \dot{\hat{t}} = \frac{e}{m\gamma}\left(\frac{\vec{E}-(\vec{E}.\hat{t})\hat{t}}{c\beta} + \hat{t}\times\vec{B}\right) \qquad \Rightarrow \qquad \vec{\omega}_O = \frac{-e}{m\gamma}\left(\vec{B}_\perp + \frac{\vec{E}\times\hat{t}}{c\beta}\right)$$

$$\frac{d\vec{S}}{dt} = \vec{\tilde{\Omega}}_s\times\vec{S} \quad \text{with}$$

$$\vec{\tilde{\Omega}}_s = -\vec{\Omega} = -\frac{e}{m\gamma}\left((1+a\gamma)\vec{B} - a(\gamma-1)(\hat{t}.\vec{B})\hat{t} - (a\gamma+\frac{\gamma}{1+\gamma})\frac{\beta\,\hat{t}\times\vec{E}}{c}\right)$$

Then w.r.t. the particle's orbit we have $\frac{d\vec{S}}{dt} = (\vec{\tilde{\Omega}}_s - \vec{\omega}_0)\times\vec{S}$

$$\text{with} \quad \vec{\tilde{\Omega}}_s - \vec{\omega}_0 = -\frac{e}{m\gamma}\left(a\gamma\,\vec{B} - (a\gamma-\frac{g}{2})(\hat{t}.\vec{B})\hat{t} - (a\gamma-\frac{\gamma}{\gamma^2-1})\frac{\beta\,\hat{t}\times\vec{E}}{c}\right)$$

$$\text{or} \quad \vec{\tilde{\Omega}}_s - \vec{\omega}_0 = -\frac{e}{m\gamma}\left(a\gamma\,\vec{B}_\perp + \frac{g}{2}\vec{B}_{//} - (a\gamma-\frac{\gamma}{\gamma^2-1})\frac{\beta\,\hat{t}\times\vec{E}}{c}\right)$$

In a perfectly flat ring, with no solenoids, the design orbit is a closed planar curve which turns a total of 2π radians around the vertical dipole fields in one pass. W.r.t. this orbit, a spin precesses by $2\pi a\gamma$ radians around the vertical since $\vec{\beta} \times \vec{E} = \vec{0}$

$$\nu_0 \equiv a\gamma$$

is an example of ``the spin tune on the design orbit''.
It is the basic spin frequency of the system.
It is also the ``sensitivity parameter'' for spin.
For advanced applications the subscript is important.

$$e^{\pm} : a = \frac{(g-2)}{2} = 0.001159652 \quad .... \quad (g = 2.0023 \ ... \ )$$

$$\mu^{\pm} : a = \frac{(g-2)}{2} = 0.001165923 \quad ... \quad (g = 2.0023 \ ....)$$

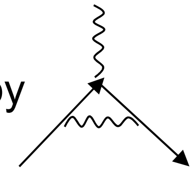
$$p^{\pm} : a = \frac{(g-2)}{2} = 1.7928474 \quad ... \quad (g = 5.58 \ ..... \ )$$

$$d^{\pm} : a = \frac{(g-2)}{2} = -0.14301 \quad .. \quad (g = 0.85699 \ ..)$$

$$^{3}He : a = \frac{(g-2)}{2} = -8.368 \ .. \quad (g = -4.184 \ ....)$$

Note that for electrons and muons $g \neq 2$!

A deviation from 2 for a fermion indicates effective internal structure.

E.g. for electrons and muons the coupling to photons is modified by  $\frac{\alpha}{2\pi} \approx \frac{1}{2\pi}\frac{1}{137}$

And very complicated higher order diagrams.

A precise measurement of $g-2$ for muons gives very precise info on limits of the Standard Model --- and the T-BMT eqn provides a chance to measure it.

$\Longrightarrow$ **The CERN and Brookhaven muon $g-2$ experiments**

A beam of decaying pions produces a fully polarised muon beam

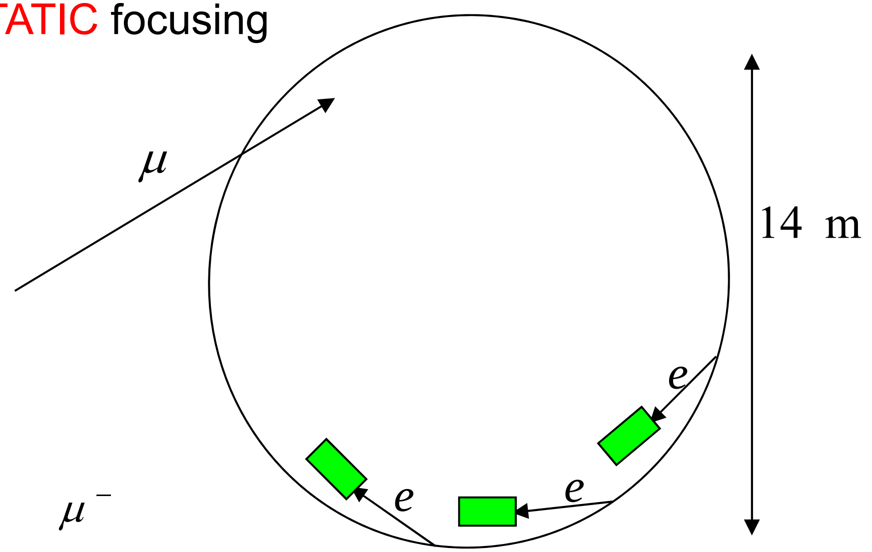
The electron direction in the decay $\mu \rightarrow e + \nu_{\mu} + \overline{\nu}_e$ is correlated to the direction of the muon's spin.

Very high quality dipole fields and ELECTROSTATIC focusing

AND run at the magic energy of 3.094 GeV.

i.e., with γ. ~ 29.4 so that $a = \dfrac{1}{\gamma^2 - 1}$

Also compare $g - 2$ f or μ^+ and μ^-



The layout: VERY schematic

E821 at BNL

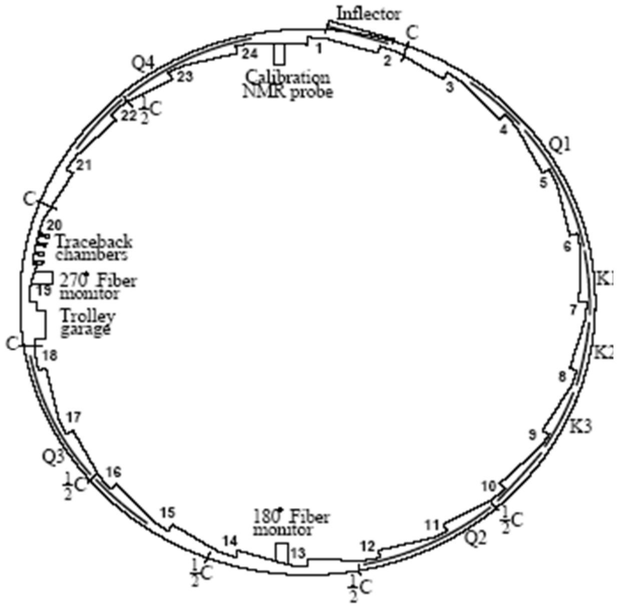


FIG. 8: The $(g-2)$ storage ring layout. The 24 numbers represent the locations of the calorimeters immediately downstream of the scalloped vacuum chamber subsections. Inside the vacuum are

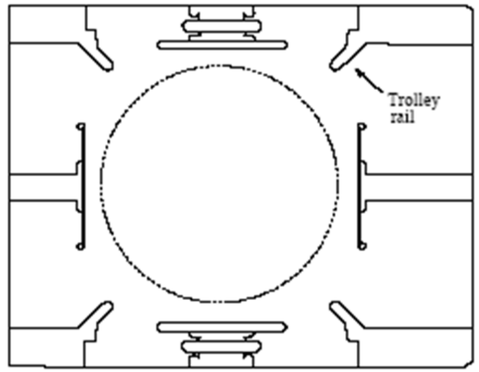


FIG. 9: Schematic view of the electrostatic quadrupoles inside the vacuum chamber. For positive muons, the top and bottom electrodes are at $\sim +24$ kV; the side electrodes are at ~ -24 kV. The

Final combined +/- result::

$$a_\mu = 11659208 \quad \underset{\text{stat}}{(5.4)}\underset{\text{syst}}{(3.3)} \times 10^{-10}$$

Now look at the EDM: extend T-BMT eqn.

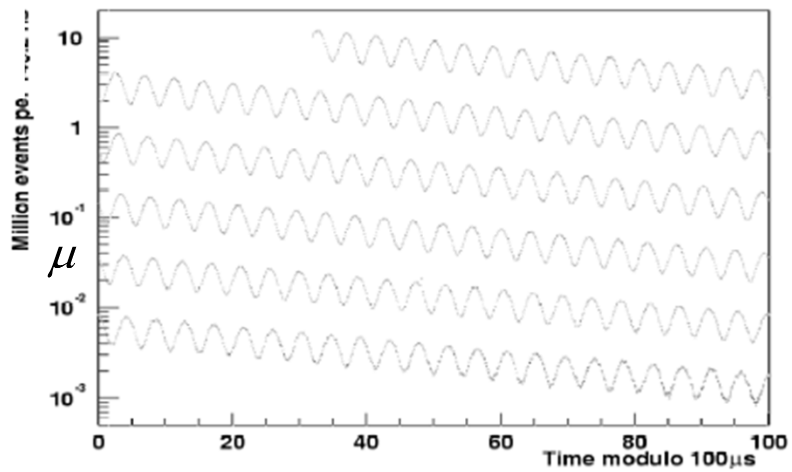


FIG. 2: Distribution of electron counts versus time for the 3.6 billion muon decays in the R01 μ^- data-taking period. The data is wrapped around modulo 100 μs.

Topic 2

A first look at the stability of spin motion

Stability of spin motion

In all other rings $\vec{E}$ and $\vec{B}$ are not superimposed so that it suffices to calculate w.r.t. the design orbit by defining

$$\vec{\tilde{\tilde{\Omega}}} = -\frac{e}{m\gamma}\left((a\gamma + 1)\vec{B} - \vec{B}_{\text{guide}} - \frac{a\gamma^2}{1+\gamma}(\vec{\beta}.\vec{B})\vec{\beta} - (a\gamma + \frac{\gamma}{1+\gamma})\frac{\vec{\beta}\times\vec{E}}{c} \right)\frac{dt}{ds}$$

where s in the distance around the design orbit and $\vec{B}_{\text{guide}}(s)$ is the magnetic field in the dipoles defining the design orbit, and writing:

$$\frac{d\vec{S}}{ds} = \vec{\tilde{\tilde{\Omega}}} \times \vec{S}$$

Now clean up the notation again and write: $\vec{\tilde{\tilde{\Omega}}} \Rightarrow \vec{\Omega}$

So in ``machine coordinates'': i.e., w.r.t. the design orbit: $$\frac{d\vec{S}}{ds} = \vec{\Omega} \times \vec{S}$$

The magnets and cavities are fixed in space --- so this is the description that we need.

Orbit coordinates: level 1:

For particle optics/motion we calculate w.r.t. the design orbit or if there are distortions, w.r.t.. the **1-turn periodic** closed orbit.

Orbit coordinates: level 2:

And for insights and analysis we express particle coordinates

$$u \equiv (x, x', y, y', \sigma, \delta = \Delta E / E)$$

in terms of Courant-Snyder parameters and/or 1-turn eigenvectors.

$$u(s_2) = M_{6\times 6}(s_2, s_1) u(s_1)$$

$$M_{6\times 6}(s_1 + C, s_1) \upsilon_k(s_1) = e^{2\pi i Q_k} \upsilon_k(s_2) \qquad k = I, II, III, -I, -II, -III$$

$$Q_{-I} = -Q_I \;;\; \upsilon_{-I} = \upsilon_I^* \qquad \text{etc.}$$

$$u(s) = \sum_k A_k \upsilon_k(s)$$ Treat the *A*'s as coordinates in the basis υ
The motion is **quasiperiodic.**

On **resonance** with rational tunes *Q*, $(m_1 Q_1 + m_2 Q_2 + m_3 Q_3 = 0)$ the motion is periodic over some integer number of turns and it is very sensitive to small perturbations...

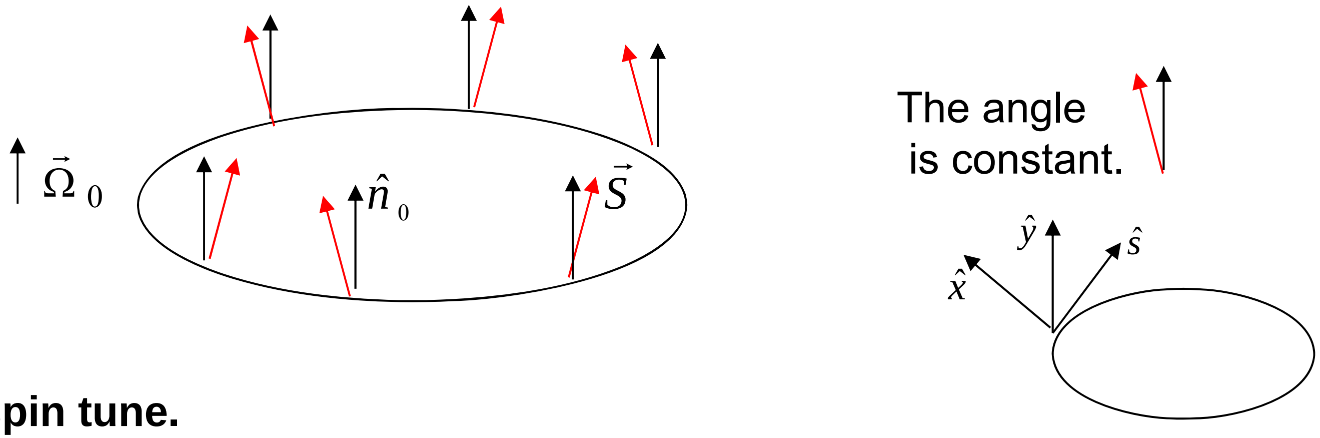
See Appendix 2 for more on orbital eigenvectors.

For particle motion we define the closed, reference orbit.

Is there a closed reference “orbit” for spin?

In a perfectly aligned flat ring with no solenoids the design orbit is flat and the particles only see a vertical $\vec{\Omega}_0 \propto \vec{B}$

A vertical solution of the T-BMT eqn. remains vertical: it is **1-turn periodic.**
It is an example of the vector $\hat{n}_0(s)$ which is the 1-turn periodic solution of the T-BMT eqn. on the design or closed orbit. Any other spin precesses around the vertical $a\gamma$ times.
If $a\gamma$ is an integer, all spins are 1-turn periodic and $\hat{n}_0(s)$ is non-unique!



$a\gamma$ is called the **spin tune.**

The vector $\hat{n}_0(s)$ lies at the centre of all serious calculations of polarisation in rings.

How do we generalise to complicated field configurations?

On the design or closed orbit use the symbol $\vec{\Omega}_0$

In general for any $\vec{\Omega}$

$$\frac{d\,\vec{S}_1}{d\,s} = \vec{\Omega} \times \vec{S}_1 \text{ and } \frac{d\,\vec{S}_2}{d\,s} = \vec{\Omega} \times \vec{S}_2 \text{ then } \vec{S}_1.\vec{S}_2 = const\ .$$

so that the length of a spin is constant. Then on the design or closed orbit we write

$$\vec{S}(s_1) = R_{3\times 3}(s_1, s_0)\ \vec{S}(s_0) \qquad \text{with } \vec{S} =: \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} \qquad R(s_n, s_0) = R(s_n, s_{n-1}).....\ R(s_1, s_0)$$

where R is an orthogonal matrix: $R^T = R^{-1} \Rightarrow R^T R = I$ and $\text{Det}\,(R) = 1$

To transport a spin for 1 turn along the design orbit: we use $R(s_0 + C, s_0)$

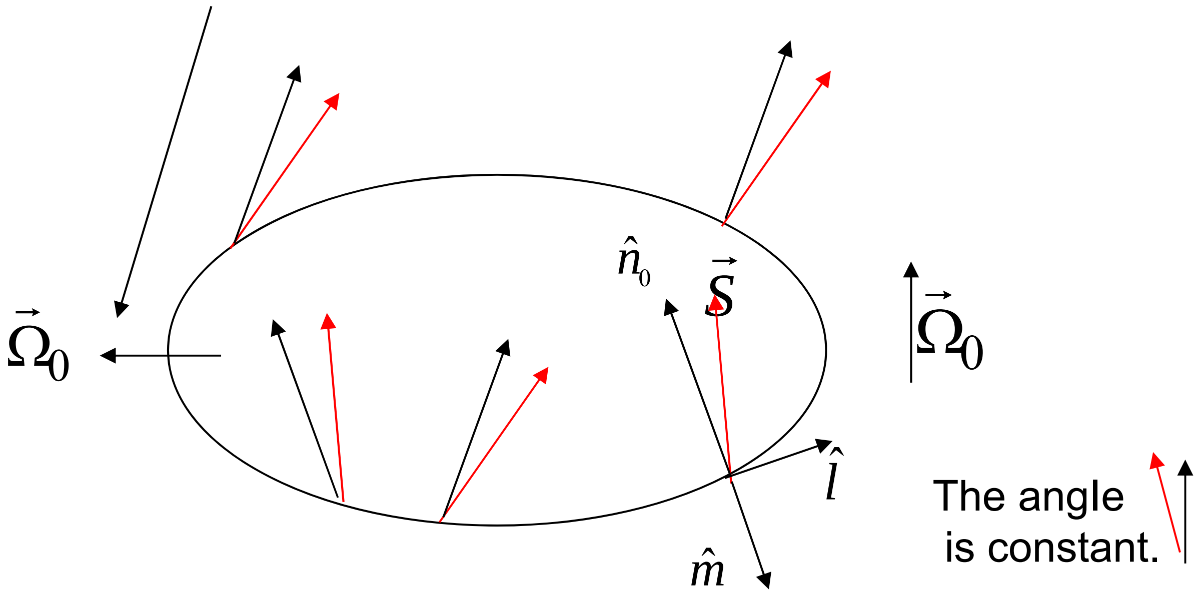
The eigenvalues of an orthogonal matrix take the form $1,\ e^{2\pi i \nu_0},\ e^{-2\pi i \nu_0}$

There is always at least 1 real eigenvector $\hat{n}_0(s_0)$:

$$\hat{n}_0(s_0 + C) = R(s_0 + C, s_0)\hat{n}_0(s_0) = \hat{n}_0(s_0)$$

It corresponds to the eigenvalue of 1. It is the direction of the effective axis of rotation over 1 turn.

Consider a local perturbation – a piece of radial field. Then $\hat{n}_0(s)$ cannot be vertical everywhere



$\hat{n}_0(s)$ is periodic by definition and construction but $\vec{S}$ is usually non-periodic.

An arbitrary spin moving on the central orbit rotates around $\hat{n}_0(s)$ by $2\pi\nu_0$ radians per turn

This ν_0 is the generalisation of the concept of **spin tune** to non-flat rings – such as HERA with its spin rotators and/or rings with solenoids.

Since the fields on the closed orbit are 1-turn periodic $\hat{n}_0(s_2) = R(s_2, s_1)\hat{n}_0(s_1)$

So $\hat{n}_0(s)$ obeys the T-BMT eqn. for all *s*.

$\hat{n}_0(s)$ is the periodic solution to the T-BMT eqn on the closed orbit. A spin set parallel to $\hat{n}_0(s)$ returns to the same direction turn by turn!!

$\hat{n}_0(s)$ is the **stable spin direction** on the closed orbit.

$\hat{n}_0(s)$ is the starting point for any respectable calculation in a ring!

$\hat{n}_0(s)$ is the analogue for spin of the periodic closed orbit.

An arbitrary spin moving on the closed orbit rotates around $\hat{n}_0(s)$ by $2\pi\nu_0$ per turn

In general $\boxed{\nu_0 \neq a\gamma}$

If ν_0 is an integer, $R(s_0 + C, s_0)$ is a unit matrix and $\hat{n}_0(s)$ is arbitrary!

Can only extract the fractional part $[\nu_0]$ of ν_0 from $e^{2\pi i \nu_0}$

If ν_0 not an integer the other 2 eigenvectors are complex and conjugate pairs.

Write them as $\hat{m}_0(s) \pm i\hat{l}_0(s)$ for $e^{\pm 2\pi i \nu_0}$ $\quad$ Can prove $: \hat{l}_0 \perp \hat{n}_0 \, , \hat{m}_0 \perp \hat{n}_0 \, , \hat{l}_0 \perp \hat{m}_0$

The set $(\hat{n}_0(s), \hat{m}_0(s), \hat{l}_0(s))$ with $\hat{n}_0(s) = \hat{m}_0(s) \times \hat{l}_0(s)$ gives a right-handed local coordinate system (**dreibein)**

If ν_0 not an integer $\hat{m}_0$ and $\hat{l}_0$ are not 1-turn periodic.

Construct the **periodic** vectors $\hat{m}$ and $\hat{l}$ by winding back $\hat{m}_0$ and $\hat{l}_0$ uniformly.

$$\hat{m}(s) + i\hat{l}(s) = e^{-i2\pi\nu_0 (s - s_0)/C} \left(\hat{m}_0(0) + i\hat{l}_0(0)\right)$$

$$\Rightarrow \frac{d(\hat{m}(s) + i\hat{l}(s))}{ds} = (\vec{\Omega}_0 + \frac{2\pi\nu_0}{C}\hat{n}_0) \times (\hat{m}(s) + i\hat{l}(s)) \quad \text{Also recall} : \frac{d}{dt} \Rightarrow \frac{d}{dt} + \vec{\omega} \times$$

See Appendix 1

We now have a **periodic** r.h. coordinate system $(\hat{n}_0(s), \hat{m}(s), \hat{l}(s))$ for spin.

That there is periodic coordinate system in which spins precess uniformly is a manifestation of the Floquet theorem for solutions of differential equations with periodic coefficients.

$\hat{n}_0(s)$ is 1-turn periodic – if viewed stroboscopically turn-by turn it is a constant vector. Any other solution of the T-BMT equation on the closed orbit would exhibit a harmonic at ν_0 under a discrete Fourier transform.

See the Barber, Ellison and Heinemann,
PRST-AB 7 (12), 124002 (2004) for details (Sect 3).

Many other periodic coordinate systems $\hat{n}_0, \hat{m}, \hat{l}$ can be constructed with

E.g. with $\hat{m} = \dfrac{\hat{x} \times \hat{n}_0}{|\hat{x} \times \hat{n}_0|}$, $\hat{l} = \hat{n}_0 \times \hat{m}$ but the precession is not uniform.

An arbitrary spin moving on the design rotates around $\hat{n}_0(s)$ by $2\pi\nu_0$ per turn

The subscript 0 on $\hat{n}_0(s)$ is **essential.**

If ν_0 is an integer, $R(s_0 + C, s_0)$ is a unit matrix and $\hat{n}_0(s)$ is arbitrary!

In a perfectly aligned flat ring with no solenoids $\nu_0 = a\gamma$

For electrons we have $\nu_0 = a\gamma = \frac{E}{0.440652} \frac{GeV}{GeV}$

For electrons in a perfectly aligned flat ring with no solenoids, spin motion on the design orbit tends to become unstable at intervals of 440.652 MeV where small perturbations have an overwhelming effect in defining the eigenvectors of an originally unit matrix!

$\Longrightarrow$ **``Integer resonances"**

If ν_0 is very near to an integer, a spin is almost in resonance with a perturbing field and the kicks to the spin tend to build up from turn to turn so that the spin motion appears to be very complicated. However, it is still simple, being a precession around $\hat{n}_0(s)$

Instead of doing **multi-turn** tracking, note that all the basic information is contained in $\hat{n}_0(s)$ which is obtained from a **1-turn** map and which will be strongly tilted from the vertical while the angle between the spin and . $\hat{n}_0(s)$ is constant.

Recall use of eigenvectors and C-S parameters for orbital motion.

Q: Why are electric quads. used in g-2 experiments instead of magnetic quads?

For protons $$\nu_0 = a\gamma \approx \frac{E}{0.523}\frac{GeV}{GeV}$$

For muons $$\nu_0 = a\gamma \approx \frac{E}{90}$$

For deuterons $$\nu_0 = a\gamma \approx \frac{E}{13}$$

Note that in a transverse field the rate of precession w.r.t. the design orbit depends just on $\frac{B}{m}$

The factor γ appears because rings have a fixed size(!) so that the $B \propto \gamma$

Summary on stability.

$\hat{n}_0(s)$: the (usually) unique **periodic** solution of the T-BMT eqn. on the (distorted) closed orbit.

arbitrary spin on the C.O. precesses $2\pi\nu_0$ radians per turn around $\hat{n}_0(s)$

In a perfectly aligned flat ring with no solenoids:
$\hat{n}_0(s)$ is vertical, $\nu_0 = a\gamma$,

in a misaligned ring
$\hat{n}_0(s)$ is tilted, $\nu_0 = a\gamma + \text{some deviation}$

if ν_0 is an integer, $\hat{n}_0(s)$ is arbitrary. Near an integer $\hat{n}_0(s)$ is strongly tilted from the design direction,

ν_0 is called the **spin tune** on the C.O.

attach T-BMT solutions $\hat{m}_0(\text{s})$ and $\hat{l}_0(s)$ to get an orthonormal coordinate system $(\hat{n}_0(s), \hat{m}_0(s), \hat{l}_0(s))$ at each point on the C.O.

$\hat{m}_0(s)$ and $\hat{l}_0(s)$ are usually not periodic.

construct the wound-back periodic versions $\hat{m}(s)$ and $\hat{l}(s)$

We now have a **periodic** r.h. coordinate system $(\hat{n}_0(s), \hat{m}(s), \hat{l}(s))$ for spin.

$\hat{m}(s)$ and $\hat{l}(s)$ don't obey the T-BMT eqn.

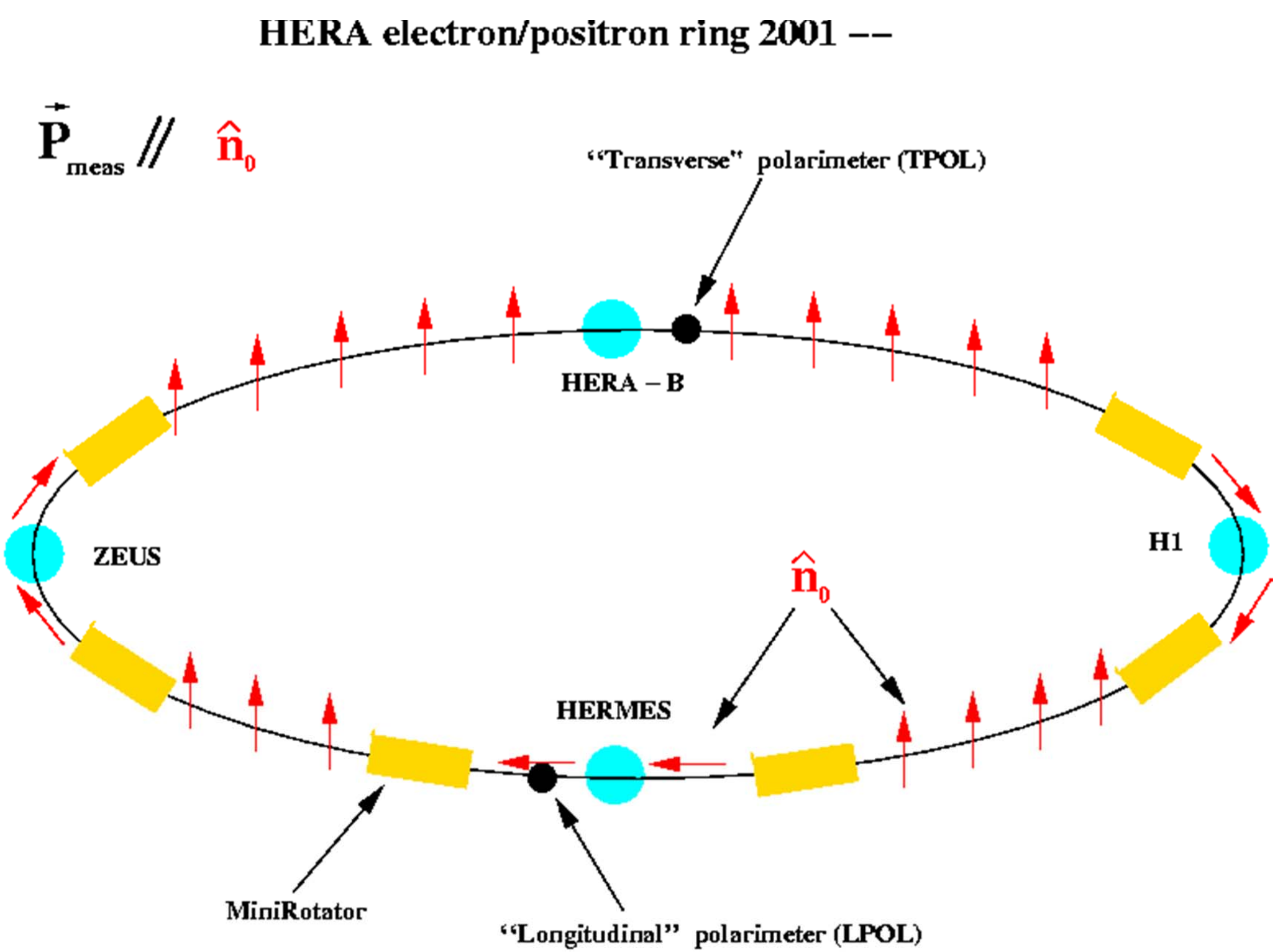
everything needed can be extracted from an eigenanalysis of the 1-turn spin transfer matrix *R.*

Spin motion often looks complicated in machine coordinates but it looks

simple in this $(\hat{n}_0(s), \hat{m}(s), \hat{l}(s))$ coordinate system. It's a simple precession

at a constant rate around $\hat{n}_0(s)$ with a fixed angle of tilt away from $\hat{n}_0(s)$.

Any complications of the spin motion are absorbed in complications in the motion *of* $(\hat{n}_0(s), \hat{m}(s), \hat{l}(s))$ *!!*



HERA electron/positron ring 2001 --
$\vec{P}_{meas}$ // $\hat{n}_0$
"Transverse" polarimeter (TPOL)
HERA – B
ZEUS
H1
$\hat{n}_0$
HERMES
MiniRotator
"Longitudinal" polarimeter (LPOL)

Topic 3

The effect of synchrotron radiation I

-- self polarisation

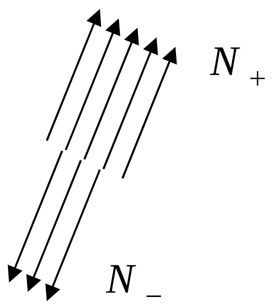
The definition of polarisation

$$\vec{P} = \frac{1}{|\vec{S}|} \frac{\sum_{i=1}^{1=N} \vec{S}_i}{N}$$

Just an average of normalised expectation values – applicable both to pure and mixed states.

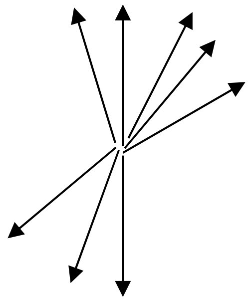
Obviously $|\vec{P}| \leq 1$

Example 1: a mixed fermion state consisting of N_+ (N_-) spins pointing up (down) along some direction:



$$P_1 = \frac{N_+ - N_-}{N_+ + N_-}$$ along the common direction.

Example 2: a mixed state consisting of spins pointing in many directions.



$\vec{P}_2$

For fermions: if $\vec{P}_1 = \vec{P}_2$ the two states are **completely indistinguishable** w.r.t. any observations.

The polarisation and the T-BMT equation are linear in the spins.
So the polarisation for spins in an infinitisimal volume of phase space around an orbit obeys the T-BMT equation.

For fermions: if $\vec{P}_1 = \vec{P}_2$ the two states are **completely indistinguishable** w.r.t. any observations.

The spin density matrix: $\rho_{1/2} = \frac{1}{2}\{I_{2\times 2} + \vec{P}.\vec{\sigma}\}$

For $\frac{|\langle \vec{S} \rangle|}{\hbar} > \frac{1}{2}$ the polarisation still obeys the T-BMT equation but does not give a full specification of the mixed state.

E.g. deuterons, extending the model of Ex. 1 with N_+, N_-, N_0

various combinations can all give the same **vector** polarisation $|\vec{P}| = \frac{N_+ - N_-}{N_+ + N_- + N_0}$

In polarised sources and targets, resolve the ambiguity using the **alignment**

$$A = 1 - 3\frac{N_0}{N_+ + N_- + N_0}$$

So, for this configuration, just the direction of the vector polarisation, *P* and *A* are needed to specify the mixed state.

The spin density matrix:, with $\vec{\sigma}_{3\times 3}$

$$\rho_1 = \frac{1}{3}\{I_{3\times 3} + \frac{3}{2}\vec{P}.\vec{\sigma} + \sqrt{\frac{3}{2}}P_{ij}(\sigma_i\sigma_j + \sigma_j\sigma_i)\}$$

So, in general **tensor** polarisation components P_{ij} must be considered too.
A total of 8 parameters for deuterons.

(15 parameters for spin 3/2)

For these lecture we will only look at **electrons** (positrons) and **protons.**

Some history of the theory of self polarisation

Korovina, Loskutov and Ternov , 1962: first indications from theory that self polarisation might be possible . Dirac eqn. in uniform magnetic field with photon emission.
Sokolov and Ternov, 1964.

Baier and Orlov , 1964, first realisation that sync. rad could also depolarise.

Baier, Katkov, Strakhovenko 1968-1970, a semiclassical treatment for practicable calculations. but no depolarisation.

Derbenev, Kondratenko, 1971-1975, a unified way to marry polarisation and depolarisation.

Years of confusion.

Chao 1981 introduced the SLIM formalism , enabling some practical calculations

Barber, 1982------ detailed work on spin matching

Yokoya, 1986, explained in the ``West'' what the “n axis” in the D-K papers mean.

Mane 1987, rederived the D-K formula for the polarisation from a new point of view.

Barber 1994, adopted a more transparent view of the “n axis” and renamed it to reflect its significance..

Barber , Heinemann 1994 ------- at last, a Fokker-Planck eqn for spin and an explanation of the meaning of another D-K paper..

The basic S-T results: motion in a plane perpendicular to a magnetic field.
S-T book 1968 page 55 onwards (Laguerre polynom. wave functions and TDPT).

Transition probabilities per unit time for $\uparrow$ to $\downarrow$ (p_+) and $\downarrow$ to $\uparrow$ (p_-)

$$p_{\pm} = \frac{5\sqrt{3}}{16}\frac{c\, r_e\, \lambda\!\!\!^{-}_c\, \gamma^5}{|\rho^3|}\left(1 \pm \frac{8}{5\sqrt{3}}\right)$$

$r_0 = \frac{e^2}{mc^2}$: classical electron radius $\approx 2.81 \times 10^{-15}$ m

$\lambda\!\!\!^{-}_c = \frac{\hbar}{mc}$: reduced Compton wavelength $\approx 3.86 \times 10^{-13}$ m

$$\frac{dN_+}{dt} = N_- p_- - N_+ p_+ \qquad N = N_+ + N_- = const \;.\Rightarrow \frac{dN_-}{dt} = -\frac{dN_+}{dt}$$

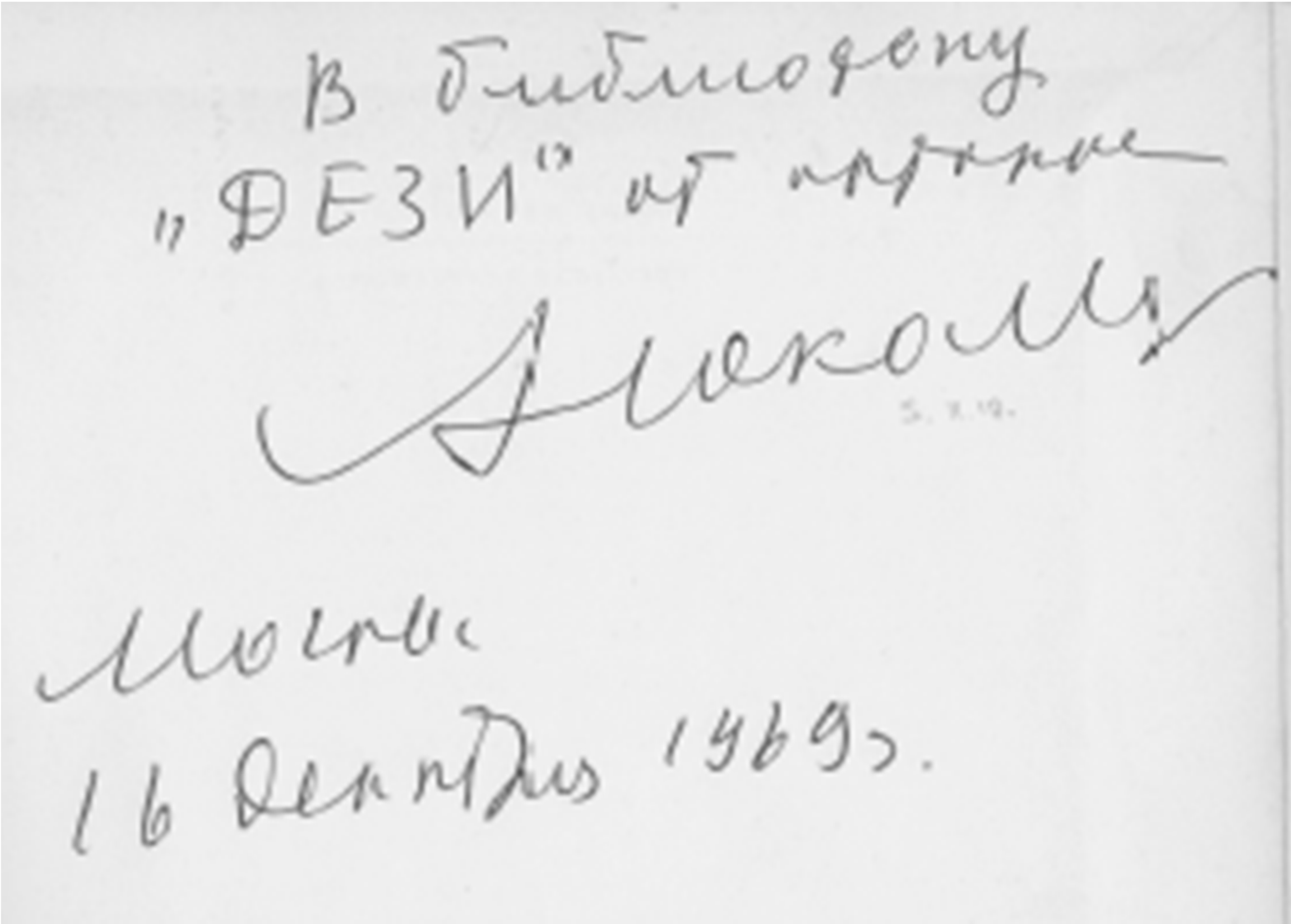
$$P = \frac{N_+ - N_-}{N} \Rightarrow N_+ = \frac{1}{2}(1+P)\,,\; N_- = \frac{1}{2}(1-P)$$

$$\frac{dP}{dt} = \frac{\dot{N}_+ - \dot{N}_-}{N} = \underbrace{(p_- - p_+)}_{\text{Inhom}} - \underbrace{(p_- + p_+)P}_{\text{+ve hom.}}$$

$\frac{dP}{dt} = 0$ $\qquad\qquad$ $\hbar \Leftrightarrow$ QM

$$\Rightarrow P(t) = P_\infty\left(1 - e^{-t/\tau_{ST}}\right) + P(0)e^{-t/\tau_{ST}} \qquad P_\infty = \frac{8}{5\sqrt{3}} = 0.9238 \qquad \tau_{ST}^{-1} = \frac{5\sqrt{3}}{8}\frac{c\lambda\!\!\!^{-}_c r_e \gamma^5}{|\rho^3|}$$

$\hbar$ is small $\Rightarrow \tau_{ST}$ is minutes to hours in typical rings. But note the γ^5


В библиотеку
„ДЕЗИ" от автора
Москва
16 декабря 1969 г.

From the S-T book

where n_{10} and n_{20} are the values n_1 and n_2 at the initial moment ($t = 0$). For the relaxation time τ we get the following expression:

$$\tau = (w_{12} + w_{21})^{-1} = \left[\frac{5\sqrt{3}}{8} \frac{\hbar}{m_0 c R} \left(\frac{E}{m_0 c^2}\right)^5 \frac{e_0^2}{m_0 c R^2}\right]^{-1}$$
$$= \left[\frac{5\sqrt{3}}{8} \frac{m_0 c e_0^2}{\hbar^2} \left(\frac{E}{m_0 c^2}\right)^2 \left(\frac{H}{H_0}\right)^3\right]^{-1}, \tag{7.22}$$

where $H_0 = m_0^2 c^3/e_0 \hbar = 4{\cdot}41 \times 10^{13}$ Oe.

In particular, putting $H = 10^4$ Oe and $E = 1$ BeV we have $\tau \approx 1$ h.

For $t \gg \tau$ the ratio n_1/n_2 tends to the limit

$$\frac{n_1}{n_2} = \frac{w_{21}}{w_{12}} = \frac{15 + 8\sqrt{3}}{15 - 8\sqrt{3}}, \tag{7.23}$$

which does not depend on the initial distribution of the electrons over states of transverse polarization.

From (7.23) follows that after a time interval exceeding the relaxation time ($t \gg \tau$) as a result of the radiation 96% of the electrons will have a spin oriented against the magnetic field.

Note that the effect of transverse polarization, which has been considered here as a result of the radiation, may be of interest in the investigation of the motion of electrons in storage rings. Of course, the conditions of the motion of an electron in a real storage ring are much more complicated compared with the conditions in the case of the motion in a constant and homogeneous magnetic field which we are treating here.

Therefore, the obtained estimation of the magnitude of effect (7.23) may prove to be too optimistic.

Synchrotron radiation – qualitative –from M.Sands, SLAC121, 1970!

Circular motion in a uniform magnetic field. Classical radiation theory gives:

$$P_\gamma = \int_0^\infty \mathscr{P}(\omega)\, d\omega .$$

$$\mathscr{P}(\omega) = \frac{P_\gamma}{\omega_c}\, S\!\left(\frac{\omega}{\omega_c}\right)$$

$$\omega_c = \frac{3}{2}\frac{c\gamma^3}{\rho}$$

$$u_c = \hbar\omega_c = \frac{3}{2}\frac{\hbar c\gamma^3}{\rho}$$

$$S(\xi) = \frac{9\sqrt{3}}{8\pi}\,\xi\int_\xi^\infty K_{5/3}(\bar{\xi})\, d\bar{\xi}$$

$$\int_0^\infty S(\xi)d\xi = 1$$

$$F(\xi) = \frac{1}{\xi}S(\xi)$$

$$u = \hbar\omega,$$

$$n(u) = \frac{P_\gamma}{u_c^2}\, F\!\left(\frac{u}{u_c}\right)$$

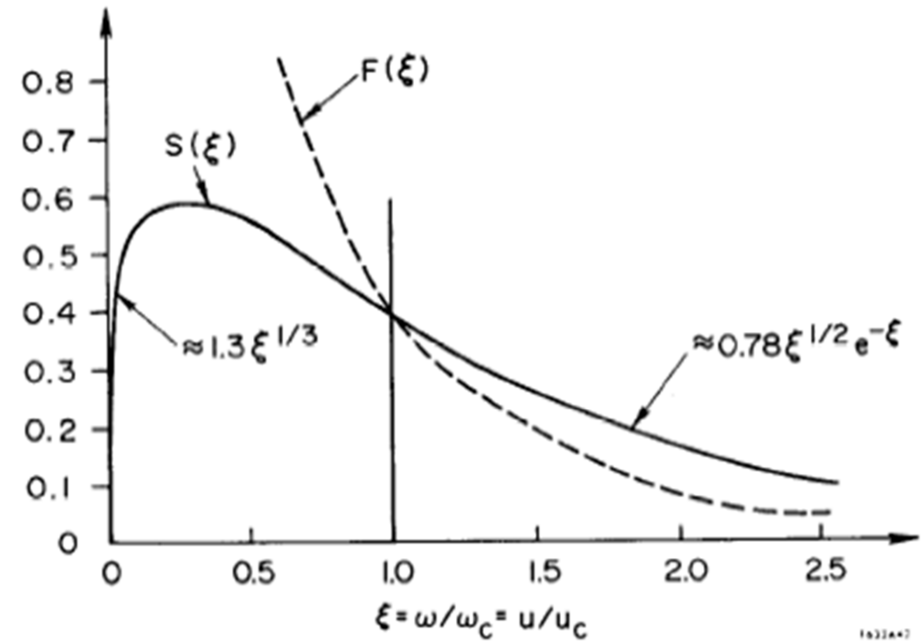
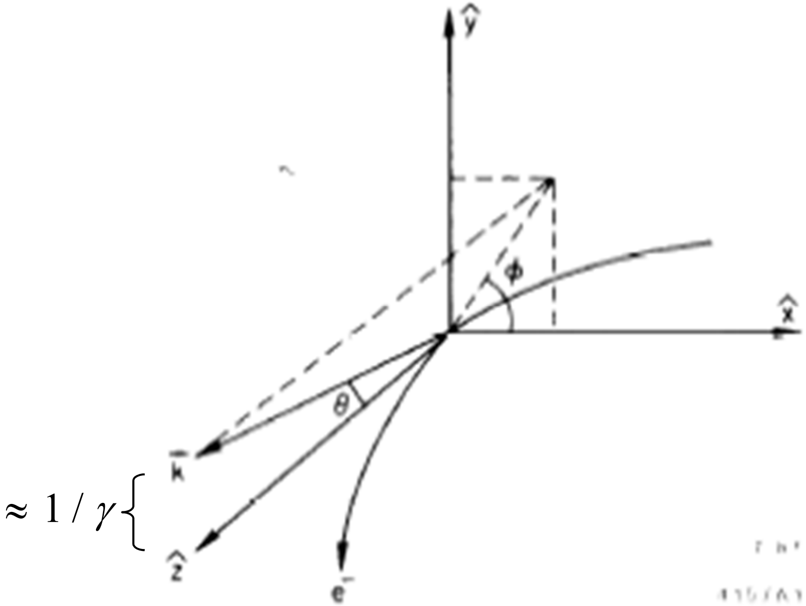


FIG. 42--Normalized power spectrum S and photon number spectrum F of synchrotron radiation.

$$P_\gamma = \frac{cC_\gamma}{2\pi}\frac{E^4}{\rho^2} \qquad C_\gamma = \frac{4\pi}{3}\frac{r_e}{(mc^2)^3} = 8.85\times 10^{-5}\ \text{meter-GeV}^{-3} \qquad P_\gamma = \frac{e^2c^3}{2\pi}C_\gamma E^2B^2$$

For our purposes it suffices to know that the SR photons are emitted within an angle of a few $1/\gamma$ w.r.t. from the particle direction.



Chao 1981.

Fig. 1 Relative orientations of the coordinate system, the electron trajectory and the wave vector $\vec{k}$ of an emitted synchrotron photon. The bending magnetic field is in the $\hat{y}$ direction. The polar angle θ is defined relative to $\hat{z}$.

From the S-T book

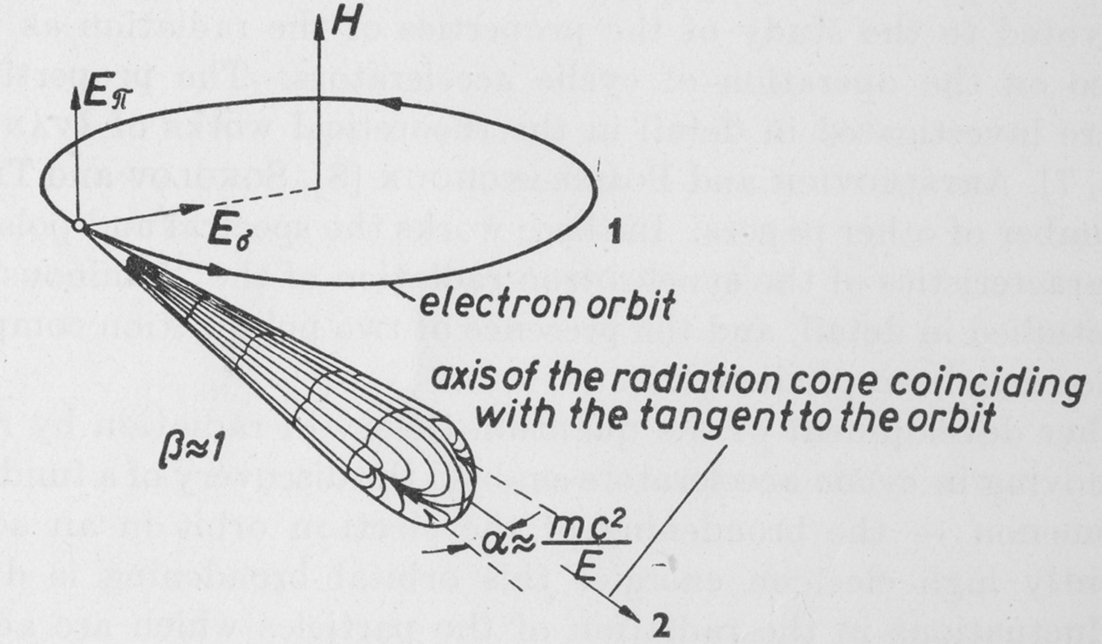


Fig. 5.1

Synchrotron radiation – the semiclassical approach.

The S-T prediction is fine as a starting point but one needs a more convenient formalism which is more in contact with practical rings and which can exploit approximations at high Energy (γ). It doesn't seem sensible to try to apply the Dirac eqn. to a full storage ring. – especially since the polarisation is often not be parallel to the dipole fields.

Instead we want to extract the essentials of the calculation of the spin-flip process and recombine them at will.

For this we note that the ``critical photon energy'' $E_c = \frac{3}{2}\hbar\omega_0\gamma^3 = \frac{3}{2}\hbar\frac{c}{\rho}\gamma^3$ is very small compared to the beam energy so that the orbital motion can be treated as almost classical during photon emission -- after all, a classical calculation is OK for the synch. radiation.

But spin must be handled with QM. So we need a creative approach that combines the best of all worlds.

⇒ The (semi-classical) formalisms of

Schwinger 1954,
Baier, Katkov and Starkhovenko 1968-1970
Derbenev and Kondratenko 1971-1972
Jackson 1976

A Time for Faith

Or think like physicists!

First order time dependent perturbation theory:

Transition amplitude $\Rightarrow T_{\mathrm{fi}} = \frac{1}{i\hbar}\int_{-\infty}^{+\infty} \langle f(t) | H_{\mathrm{rad}}(t) | i(t) \rangle \, dt$ Just QM, no 2nd quantisation

$$\hat{H} \Rightarrow \hat{H}_{\mathrm{class}} + \hat{H}_{\mathrm{rad}} \; : \quad \hat{H}_{\mathrm{rad}} = e(\varphi_{\mathrm{rad}} - \vec{\beta}.\vec{A}_{\mathrm{rad}}) + \frac{\hbar}{2}\vec{\Omega}_{\mathrm{rad}}.\vec{\sigma} \quad \equiv \hat{H}^{\mathrm{orb}}_{\mathrm{rad}} + \hat{H}^{\mathrm{spin}}_{\mathrm{rad}}$$

Instead of $\propto \vec{\mu}.\vec{B}_{\mathrm{rad}}$

Choose the radiation (Coulomb, transverse) gauge: $\varphi_{\mathrm{rad}} = 0 \, ; \quad \vec{\nabla}.\vec{A}_{\mathrm{rad}} = 0$

$$\vec{A}_{\mathrm{rad}} \propto \vec{\varepsilon}\, e^{-i\vec{k}.\vec{r} + i\omega t} + c.c. \Rightarrow \vec{\varepsilon}.\vec{k} = 0 \, , \quad \vec{E}_{\mathrm{rad}} = -i\omega\vec{A}_{\mathrm{rad}} + c.c \, , \quad \vec{B}_{\mathrm{rad}} = \frac{1}{c}\hat{k} \times \vec{E}_{\mathrm{rad}} + c.c.$$

Power: $\frac{d^2\Pi}{d\omega \, d\Omega} = \frac{\hbar\omega^3}{(2\pi)^4 c^2 \rho} \sum_{\vec{\varepsilon}} | T_{\mathrm{fi}} |^2 \, , \quad \text{Corres. rate} : \frac{1}{\hbar\omega}\frac{d^2\Pi}{d\omega \, d\Omega}$

$$| T_{\mathrm{fi}} |^2 = | T^{\mathrm{orb}}_{\mathrm{fi}} + T^{\mathrm{spin}}_{\mathrm{fi}} |^2$$

$T^{\mathrm{spin}}_{\mathrm{fi}}$ contains an extra ħ

The resulting expressions are extremely complicated and require skill and physical insight for extracting recognisable and useful expressions: e.g. Schwinger, Baier+Katkov, Jackson

Decompose at various levels keeping leading orders, ignoring/keeping commutators and $\frac{1}{\gamma^N}$
Stay as classical as possible.

1) Ignore spin and ignore commutators $\Rightarrow$ the classical power and spectrum Π_{cl}

2) Ignore spin and keep dominant terms from commutators $\Rightarrow$ the 1st order Schwinger correction (recoil)

$$\Pi_{\mathrm{cl}} \Rightarrow \Pi_{\mathrm{cl}} \left(1 - \frac{55\sqrt{3}}{16} \frac{\lambda\!\!\!^{-}_c}{\rho} \gamma^2 \right) = \Pi_{\mathrm{cl}} \left(1 - \frac{55\sqrt{3}}{16} \frac{\hbar \omega_0}{E} \gamma^3 \right); \quad \omega_0 = \frac{c}{\rho}; \ \omega_c = \frac{3}{2} \omega_0 \gamma^3$$

Care! For some: $\omega_c = 3\omega_0 \gamma^3$

3) Now include non-flip spin: e.g.,B field along $\hat{y}$; spin along $\hat{\zeta}$

$$\Pi_{\mathrm{nf}} = \Pi_{\mathrm{cl}} \left(1 - \left[\frac{55\sqrt{3}}{16} + \frac{3}{2} \hat{\zeta} . \hat{y} \right] \frac{\hbar \omega_0}{E} \gamma^3 + O\left(\frac{E_c^2}{E^2} \right) \right); \quad E_c = \frac{3}{2} \hbar \omega_0 \gamma^3$$

``Spin light'' for measuring long. pol.

4) Now spin flip: $\Pi_{\mathrm{flip}}^{\uparrow\downarrow} = \Pi_{\mathrm{cl}}\, 3 \left(\frac{\hbar \omega_0}{E} \gamma^3 \right)^2 \left(1 \pm \frac{35}{64} \sqrt{3} \right) \quad \Rightarrow \quad \frac{\Pi_{\mathrm{flip}}^{\uparrow\downarrow}}{\Pi_{\mathrm{nf}}} = O\left(\left(\frac{\hbar \omega_0}{E} \gamma^3 \right)^2 \right)$

Observations:

a) This (usual) simple formula uses $g = 2 \Rightarrow a = 0$
This suffices and allows massive simplification but need $a \neq 0$ for classical precession..

b) Spin flip is weak $\frac{\Pi_{\text{flip}}^{\uparrow\downarrow}}{\Pi_{\text{nf}}} = O\left(\left(\frac{\hbar\omega_0}{E}\gamma^3\right)^2\right) = O(10^{-11})$ at 27.5 GeV at HERA

c) Use Heisenberg EOM for evolution of $\vec{\sigma}$ operators and orbit:: connects to classical motion.

d) The dominant contributions to integrals are from a time interval (overlap time) $\frac{\rho}{c\gamma} = \frac{1}{\omega_0\gamma}$

The spin precession rate is $(a\gamma + 1)\omega_0 \Rightarrow \Delta\theta_{\text{spin}} \approx a + \frac{1}{\gamma}$

At HERA (27.5 GeV) $\Rightarrow \gamma \approx 5.4 \times 10^4 \Rightarrow \Delta\theta_{\text{spin}} \approx 1$ mrad

So there was no sloppiness in the meaning of $\vec{\zeta}$

$L_{\text{dipole}} \approx 10$ m ; $E_c \approx 78$ KeV ; $E_{\text{loss}} \approx 80$ MeV/turn at HERA

But we need flip **rates,** now just power. Recall for spins along the field of S-T

For electrons $$p_{\pm} = \frac{5\sqrt{3}}{16}\frac{c\, r_e\, \lambdabar_c\, \gamma^5}{|\rho^3|}\left(1 \pm \frac{8}{5\sqrt{3}}\right) = \frac{1}{2\tau_{ST}}\left(1 \pm \frac{8}{5\sqrt{3}}\right)$$ Sokolov+Ternov

Generalises to $$p_{\pm} = \frac{5\sqrt{3}}{16}\frac{c\, r_e\, \lambdabar_c\, \gamma^5}{|\rho^3|}\left(1 - \frac{2}{9}(\hat{\zeta}.\hat{\upsilon})^2 + \frac{8}{5\sqrt{3}}\hat{\zeta}.\hat{b}\right)$$ Baier+Katkov 1968

$\hat{b}$ = mag. field direction ; $\hat{\zeta}$ = initial spin direction; $\hat{v}$ = particle direction

Also OK in solenoids because of $\frac{1}{|\rho^3|}$ For positrons use $-\frac{8}{5\sqrt{3}}$

Recall $$\frac{dP}{dt} = (p_- - p_+) - (p_- + p_+)P \Rightarrow P_\infty = \frac{p_- - p_+}{p_- + p_+}; \quad \tau_{ST}^{-1} = p_- + p_+$$

$$\text{-------}\rightarrow P_\infty = \frac{8}{5\sqrt{3}}\frac{\oint ds \dfrac{(\hat{\zeta}.\hat{b})}{|\rho^3|}}{\oint ds \dfrac{\left(1 - \frac{2}{9}(\hat{\zeta}.\hat{\upsilon})^2\right)}{|\rho^3|}}$$

But what is $\hat{\zeta}$ in a ring with spin rotators and/or misalignments? And for which particle?

Try to work with polarisation directly to handle all particles at once

$\Longrightarrow$ Baier, Katkov and Strakhovenko: 1969

For electrons: drives $\vec{P}$ against $\vec{B}$

$$\frac{d\vec{P}}{dt} = \frac{d\langle\vec{\sigma}\rangle}{dt} = \frac{ds}{dt}\vec{\Omega}\times\vec{P} - \frac{1}{\tau_0}\left(\vec{P} - \frac{2}{9}\hat{\upsilon}(\vec{P}.\hat{\upsilon}) - \frac{8}{5\sqrt{3}}\hat{b}\right)$$

$$\mathrm{T\text{-}BMT}: a = \frac{g-2}{2} \neq 0$$

S-T/B-K $a = \dfrac{g-2}{2} \Rightarrow 0$

The BKS calculation is *ab initio* and goes to 2nd order to get the T-BMT term automatically with the ``correct'' value of $g-2$

$$\tau_0^{-1}(s) = \frac{5\sqrt{3}}{8}\frac{c\lambda\!\!\!^{-}{}_c r_e \gamma^5}{|\rho(s)^3|}$$

Before trying to exploit this……

Why not 100 percent ?

Recall: $\Delta E_{SG} = -\vec{\mu}.\vec{B}_R \Rightarrow \vec{\Omega}.\vec{S}$

So perhaps the spins just want to fall into the lowest S-G. energy state.

WRONG! – but a useful memory device for electrons/positrons.

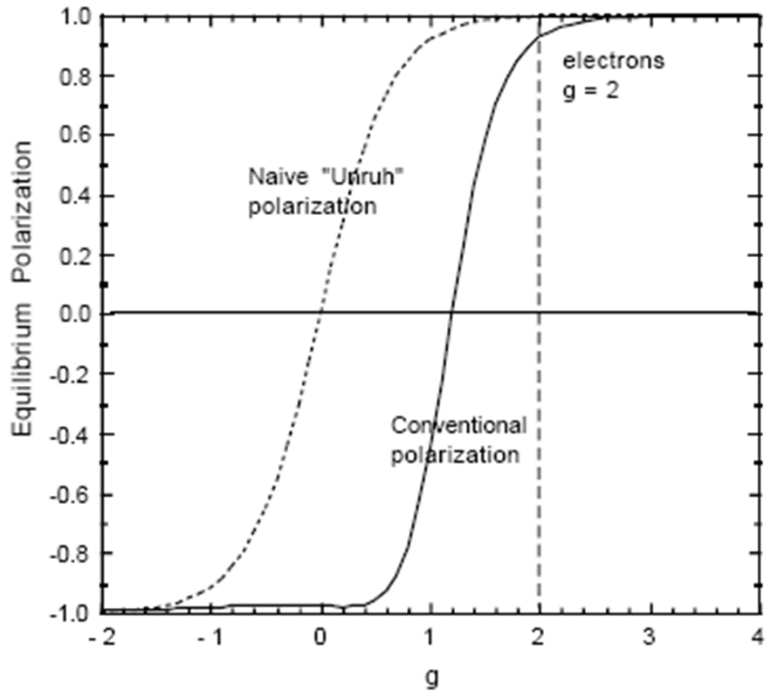


Figure 1. Equilibrium electron polarization in storage rings. Solid curve, conventional result[3], dashed curve, P = tanh(πg/2), the naive BL result[2]. For the range, 0 < g < 1.2, the effective temperature T_{eff} is negative (gP < 0).

Derbenev and Kondratenko 1973
Jackson 1976 and 1998.

Keep $\frac{g-2}{2}$ in the pure spin flip calculations. 92.38 % is not special.

If you are dying to know….

$$\frac{\tau_o(a)}{\tau_o} = \Big[\left(1 + \frac{41}{45}a - \frac{23}{18}a^2 - \frac{8}{15}a^3 + \frac{14}{15}a^4\right) e^{-\sqrt{12}|a|}$$

$$- \frac{8}{5\sqrt{3}} \frac{a}{|a|} \left(1 + \frac{11}{12}a - \frac{17}{12}a^2 - \frac{13}{24}a^3 + a^4\right) e^{-\sqrt{12}|a|}$$

$$+ \frac{8}{5\sqrt{3}} \frac{a}{|a|} \left(1 + \frac{14}{3}a + 8a^2 + \frac{23}{3}a^3 + \frac{10}{3}a^4 + \frac{2}{3}a^5\right)\Big]^{-1}$$

$$P_o(a) = -\frac{8}{5\sqrt{3}} \frac{\tau_o(a)}{\tau_o} \left(1 + \frac{14}{3}a + 8a^2 + \frac{23}{3}a^3 + \frac{10}{3}a^4 + \frac{2}{3}a^5\right)$$

A spin is not an isolated system. Consider a uniform magnetic field. The angular frequency of motion is $\omega_0 = \frac{c}{\rho}$ and $\omega_c = \frac{3}{2}\omega_0\gamma^3$

Energy separation of orbital levels: $\Delta E_{\text{orb}} = \hbar\omega_0$ and $E_c = \frac{3}{2}\hbar\omega_0\gamma^3 \Rightarrow$

Emission of a ``critical photon'' causes a shift of $O(\gamma^3)$ levels but this is still a very small relative shift since the level number $\gamma m c^2/(\hbar c/\rho) = \gamma\rho/\lambdabar_c \approx 10^{20}$ (HERA)

---agrees of course with $E_c/E \approx 10^{-6}$ so that the orbit is hardly effected, as advertised.

The S-G energy is $\Delta E_{\text{SG}} \approx \frac{eB}{m\gamma}\frac{\hbar}{2}(a\gamma + 1) = \omega_0\frac{\hbar}{2}(a\gamma + 1) \Rightarrow \frac{\Delta E_{\text{SG}}^{\text{flip}}}{\Delta \mathrm{E}_{\text{orb}}} \approx (a\gamma + 1)$

$\approx 10^{-8}$ eV at HERA

$\times\, O(\gamma^2)$?
Jackson

So, if it were to occur, a spontaneous spin flip (up or down) could conserve energy if the orbit were to jump by $a\gamma + 1$ levels. So ``upper'' and ``lower'' spin levels have little meaning. There is very much more to this system than meets the eye at first.

Note also the huge disparity between the S-G and E_c energy scales. And spin-flip photons tend to have higher energies than non-flip photons.

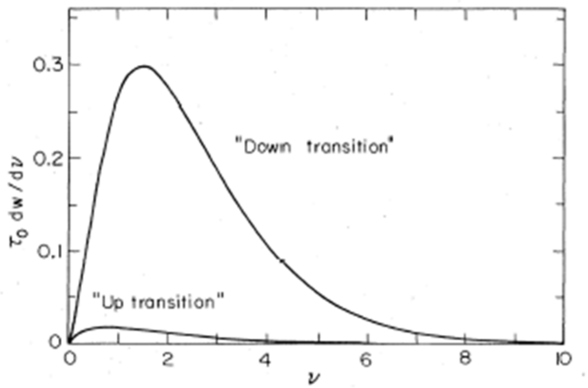


FIG. 7. Normalized frequency spectra $\tau_0 dw/d\nu$ for the *number of photons* emitted per unit interval in the dimensionless frequency variable $\nu = 2\omega/3\gamma^3\omega_0$. The dominant "down" transition corresponds to a spin-flip from $\cos\theta_0 = +1$ to $\cos\theta_0 = -1$ (spin finally in the direction opposite to $\hat{\beta}\times\dot{\hat{\beta}}$). The small "up" transition is in the reverse direction.

From Jackson 1976

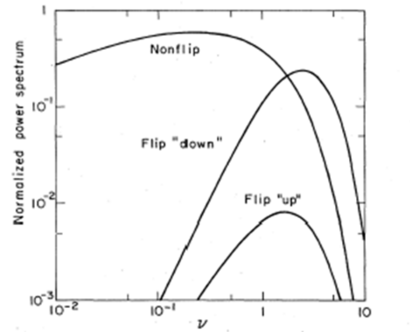


FIG. 8. Log–log plot of separately normalized ordinary (nonflip) and spin-flip *power* frequency spectra as functions of the dimensionless variable $\nu = 2\omega/3\gamma^3\omega_0$. The actual spin-flip power is much smaller than the ordinary power provided $\gamma \ll \gamma_c$ [see Eq. (3) or (71)]. At low frequencies ($\nu \ll 1$), the nonflip distribution varies as $\nu^{1/3}$, while the spin-flip distributions vary as $\nu^{7/3}$. At high frequencies ($\nu \gg 1$) all spectra vanish exponentially (times different powers).

That we have significant spin-orbit coupling is not unexpected because of Thomas precession

which effectively shifts $\frac{g}{2} \Rightarrow \frac{g}{2} - 1 + \frac{1}{\gamma}$ See the shift in the D-K/Jackson plot too..

Overall, it's no surprise that 100% is not attainable . But there is no simple heuristic explanation for the number ``92.38%''.

Note that a simple 2-level picture does give a characteristic time with about the right size and a γ^{-5} dependence.

Note that there is nothing magic about using magnetic fields.

Equivalent transverse electric fields (i.e., which give the same curvature) would also be OK:

It’s the geometry of the orbit that counts in the integrals.

Real rings have spin rotators and solenoids and misalignments – so the magnetic fields on the closed orbit are not always vertical. Then which way does the polarisation point and how large is it?

If we believe that the polarisation reaches equilibrium both magnitude and direction then at equilibrium it can only point along $\hat{n}_0(s)$ and it then points in the same direction at the end of each turn.

What does the BKS eqn. say? Sit on the closed orbit. $\vec{\Omega}_0(s)$ refers to the closed orbit

$$\frac{d\vec{P}}{dt} = \frac{ds}{dt}\vec{\Omega}_0 \times \vec{P} - \frac{1}{\tau_0}\left(\vec{P} - \frac{2}{9}\hat{\upsilon}(\vec{P}.\hat{\upsilon}) - \frac{8}{5\sqrt{3}}\hat{b} \right)$$

$$\Rightarrow \frac{d\vec{P}}{ds} = \vec{\Omega}_0 \times \vec{P} - \frac{1}{\tau_0}\underbrace{\frac{dt}{ds}}_{1/c}\left(\vec{P} - \frac{2}{9}\hat{\upsilon}(\vec{P}.\hat{\upsilon}) - \frac{8}{5\sqrt{3}}\hat{b} \right)$$

Look at it in the $(\hat{n}_0, \hat{m}, \hat{l})$ frame. $\Rightarrow \vec{P} = P_n\hat{n}_0 + P_m\hat{m} + P_l\hat{l}$

Where does the polarisation end up?

$$\frac{d\vec{P}}{ds} = \vec{\Omega}_0 \times \vec{P} - [ST] = \hat{n}_0 \dot{P}_{\mathrm{n}} + \hat{m}\dot{P}_{\mathrm{m}} + \hat{l}\dot{P}_{\mathrm{l}} + P_{\mathrm{n}}\dot{\hat{n}}_0 + P_{\mathrm{m}}\dot{\hat{m}} + P_{\mathrm{l}}\dot{\hat{l}}$$

$$\frac{d\vec{P}}{ds} = \vec{\Omega}_0 \times \vec{P} - [ST] = \hat{n}_0 \dot{P}_{\mathrm{n}} + \hat{m}\dot{P}_{\mathrm{m}} + \hat{l}\dot{P}_{\mathrm{l}} + (\vec{\Omega}_0 + \frac{2\pi\nu_0}{C}\hat{n}_0) \times \vec{P}$$

$$\hat{n}_0 \dot{P}_{\mathrm{n}} + \hat{m}\dot{P}_{\mathrm{m}} + \hat{l}\dot{P}_{\mathrm{l}} = -\frac{2\pi\nu_0}{C}\hat{n}_0 \times \vec{P} - [ST] \qquad (\text{Also recall}: \quad \frac{d}{dt} \Rightarrow \frac{d}{dt} + \vec{\omega}\times)$$

$$\frac{dP_{\mathrm{n}}}{ds} = \qquad -\frac{1}{c\tau_0}\left(P_{\mathrm{n}} - \frac{2}{9}(\hat{n}_0.\hat{\upsilon})(\vec{P}.\hat{\upsilon}) - \frac{8}{5\sqrt{3}}\hat{n}_0.\hat{b}\right)$$

$$\frac{dP_{\mathrm{m}}}{ds} = \frac{2\pi\nu_0}{C}P_{\mathrm{l}} - \frac{1}{c\tau_0}\left(P_{\mathrm{m}} - \frac{2}{9}(\hat{m}.\hat{\upsilon})(\vec{P}.\hat{\upsilon}) - \frac{8}{5\sqrt{3}}\hat{m}.\hat{b}\right)$$

$$\frac{dP_{\mathrm{l}}}{ds} = -\frac{2\pi\nu_0}{C}P_{\mathrm{m}} - \frac{1}{c\tau_0}\left(P_{\mathrm{l}} - \frac{2}{9}(\hat{l}.\hat{\upsilon})(\vec{P}.\hat{\upsilon}) - \frac{8}{5\sqrt{3}}\hat{l}.\hat{b}\right)$$

Intuitively expect on P_{n} to survive.

$$\frac{dP_{\mathrm{n}}}{ds} = -\frac{1}{c\,\tau_0}\left(P_{\mathrm{n}}[\underbrace{1 - \frac{2}{9}(\hat{n}_0.\hat{\upsilon})(\hat{n}_0.\hat{\upsilon})]}_{\text{periodic}} - \underbrace{\frac{8}{5\sqrt{3}}\hat{n}_0.\hat{b}}_{\text{periodic}} - \frac{2}{9}(\hat{n}_0.\hat{\upsilon})[P_{\mathrm{m}}(\hat{m}.\hat{\upsilon}) + P_{\mathrm{l}}(l.\hat{\upsilon})] \right)$$

$$\frac{dP_{\mathrm{m}}}{ds} = \frac{2\pi\nu_0}{C}P_{\mathrm{l}} - \frac{1}{c\,\tau_0}\left(P_{\mathrm{m}}[1 - \frac{2}{9}(\hat{m}.\hat{\upsilon})^2] - \frac{8}{5\sqrt{3}}\hat{m}.\hat{b} - \frac{2}{9}(\hat{m}.\hat{\upsilon})[P_{\mathrm{l}}(\hat{l}.\hat{\upsilon}) + P_{\mathrm{n}}(n_0.\hat{\upsilon})] \right)$$

$$\frac{dP_{\mathrm{l}}}{ds} = -\frac{2\pi\nu_0}{C}P_{\mathrm{m}} - \frac{1}{c\,\tau_0}\left(P_{\mathrm{l}}[1 - \frac{2}{9}(\hat{l}.\hat{\upsilon})^2] - \frac{8}{5\sqrt{3}}\hat{l}.\hat{b} - \frac{2}{9}(\hat{l}.\hat{\upsilon})[P_{\mathrm{m}}(\hat{m}.\hat{\upsilon}) + P_{\mathrm{n}}(n_0.\hat{\upsilon})] \right)$$

$$\left(c\,\tau_0(s)\right)^{-1} = c\,\frac{5\sqrt{3}}{8}\frac{c\lambda\!\!\!^{-}{}_c r_e \gamma^5}{|\rho(s)^3|} \approx O(3 \times 10^{-12}\ \mathrm{m}^{-1})$$ also periodic.

$$\frac{2\pi\nu_0}{C} \approx O(6 \times 10^{-2}\ \mathrm{m}^{-1})$$

$$\frac{dP_{\mathrm{n}}}{ds} = -\frac{1}{c\,\tau_0}\left(P_{\mathrm{n}}[\underbrace{1-\frac{2}{9}(\hat{n}_0.\hat{\upsilon})^2}_{\text{periodic}}] - \underbrace{\frac{8}{5\sqrt{3}}\hat{n}_0.\hat{b}}_{\text{periodic}} - \frac{2}{9}(\hat{n}_0.\hat{\upsilon})\ \underbrace{[P_{\mathrm{m}}(\hat{m}.\hat{\upsilon}) + P_{\mathrm{l}}(l.\hat{\upsilon})]}_{\text{Can oscill. at rate } 2\pi\nu_0/C} \right)$$

The periodic terms contribute the same amount on each turn but the oscillating terms (if P_{m} and P_{l} non-zero), tend to average away – so forget them!. If they were zero they will stay zero so drop them.

Such tricks can be put on a respectable basis by appealing to "averaging techniques" of the theory of dynamical systems:

Invented/introduced by the Russian schools: Bogoluibov, Krylov, Mitropolski…….

A huge literature: 100's of theorems exploiting time scales in multi-frequency systems.

Basically, the high frequency terms in a DE should have much less influence than (hide behind) the low frequency terms.

So one can often bury the embarrassing parts that are otherwise hard to treat.
– see later too.

Since $(c\,\tau_{\mathrm{ST}}(s))^{-1}$ is very small, P_{n} hardly changes over thousands of turns.
So take 1-turn averages around the ring of the r.h.s.

Write $$(c\,\tau_0(s))^{-1} = A\,\frac{1}{|\,\rho(s)^3\,|}$$

$$\Rightarrow \frac{dP_{\mathrm{n}}}{ds} \approx -\frac{A}{C}\left(P_{\mathrm{n}} \oint ds\, \frac{1-\frac{2}{9}(\hat{n}_0.\hat{\upsilon})^2}{|\,\rho^3\,|} - \oint ds\, \frac{\frac{8}{5\sqrt{3}}\,\hat{n}_0.\hat{b}}{|\,\rho^3\,|} \right)$$

$$\Rightarrow P_{\mathrm{BK}} \equiv P_{\mathrm{n}}(s=\infty) = -\frac{8}{5\sqrt{3}}\,\frac{\oint ds\, \frac{\hat{n}_0.\hat{b}}{|\,\rho^3\,|}}{\oint ds\, \frac{1-\frac{2}{9}(\hat{n}_0.\hat{\upsilon})^2}{|\,\rho^3\,|}}$$

$$\tau_{\mathrm{BK}}^{-1} = \frac{5\sqrt{3}}{8}\,\frac{c\,\lambda\!\!\!^{-}{}_c\, r_e \gamma^5}{C} \oint ds\, \frac{1-\frac{2}{9}(\hat{n}_0.\hat{\upsilon})^2}{|\,\rho^3\,|}$$

$$\Rightarrow P_n(t) = P_n(t=\infty)(1-e^{-t/\tau_{\mathrm{BK}}}) + P_n(0)\,e^{-t/\tau_{\mathrm{BK}}}$$

$$\frac{dP_m}{ds} = \frac{2\pi\nu_0}{C} P_l - \frac{1}{c\tau_0}\left(P_m[1 - \frac{2}{9}(\hat{m}.\hat{\upsilon})^2] - \underbrace{\frac{8}{5\sqrt{3}}}_{\text{periodic}} \hat{m}.\hat{b} - \frac{2}{9}(\hat{m}.\hat{\upsilon})[\overbrace{P_l(\hat{l}.\hat{\upsilon})}^{\text{can oscill.}} + P_n \underbrace{(n_0.\hat{\upsilon})}_{\text{periodic}}] \right)$$

rotation .

$$\frac{dP_l}{ds} = -\frac{2\pi\nu_0}{C} P_m - \frac{1}{c\tau_0}\left(P_l[1 - \frac{2}{9}(\hat{l}.\hat{\upsilon})^2] - \frac{8}{5\sqrt{3}} \hat{l}.\hat{b} - \frac{2}{9}(\hat{l}.\hat{\upsilon})[P_m(\hat{m}.\hat{\upsilon}) + P_n(n_0.\hat{\upsilon})] \right)$$

The extra, fast rotation should have a dramatic effect. Difficult for the S-T to get a grip.

A very rough estimate: $$\frac{d(P_m + iP_l)}{ds} = (-\frac{i\,2\pi\nu_0}{C} - \frac{E}{|R_1^3|})\left(P_m + iP_l\right) + \frac{F}{|R_2^3|}$$

with constants (averages) $|R_1^3|, |R_2^3| \approx O(|\rho^3|$ and $\frac{E}{|R_1^3|}, \frac{F}{|R_2^3|} \approx O\left((c\tau_0)^{-1}\right)$

$$\sqrt{(P_m^2 + P_l^2)_{s\to\infty}} \approx O\left(\frac{\frac{F}{|R_2^3|}}{\frac{2\pi\nu_0}{C}}\right) \approx 10^{-10}$$

So, on a time scale comparable to τ_0 the components of the polarisation perp. to $\hat{n}_0$ essentially vanish and the polarisation finishes closely parallel to $\hat{n}_0$
The time average of these components is very small much earlier.

Note the explicit appearance of $2\pi\nu_0$ in the estimate

There is a tiny remaining ripple on due to the unevenness of the synch. rad. around the ring – compare $\tau_0(s)$ with *C/c.*

The radiative polarisation settles down parallel to $\hat{n}_0$!!!!!!

Confirmed by simulation using a practical artificial τ_0

An example of how to get to the answer that one's intuition said was there!
One should always try to see an answer before trying to prove it!

If P_m and P_l are zero at the start, many of the tricky terms vanish anyway. .

If $\hat{n}_0$ is vertical everywhere it also simplifies.. .

If ν_0 is very close to an integer, the averages won't vanish and the direction will be sensitive to detail. In any case $\hat{n}_0$ will be unstable too.

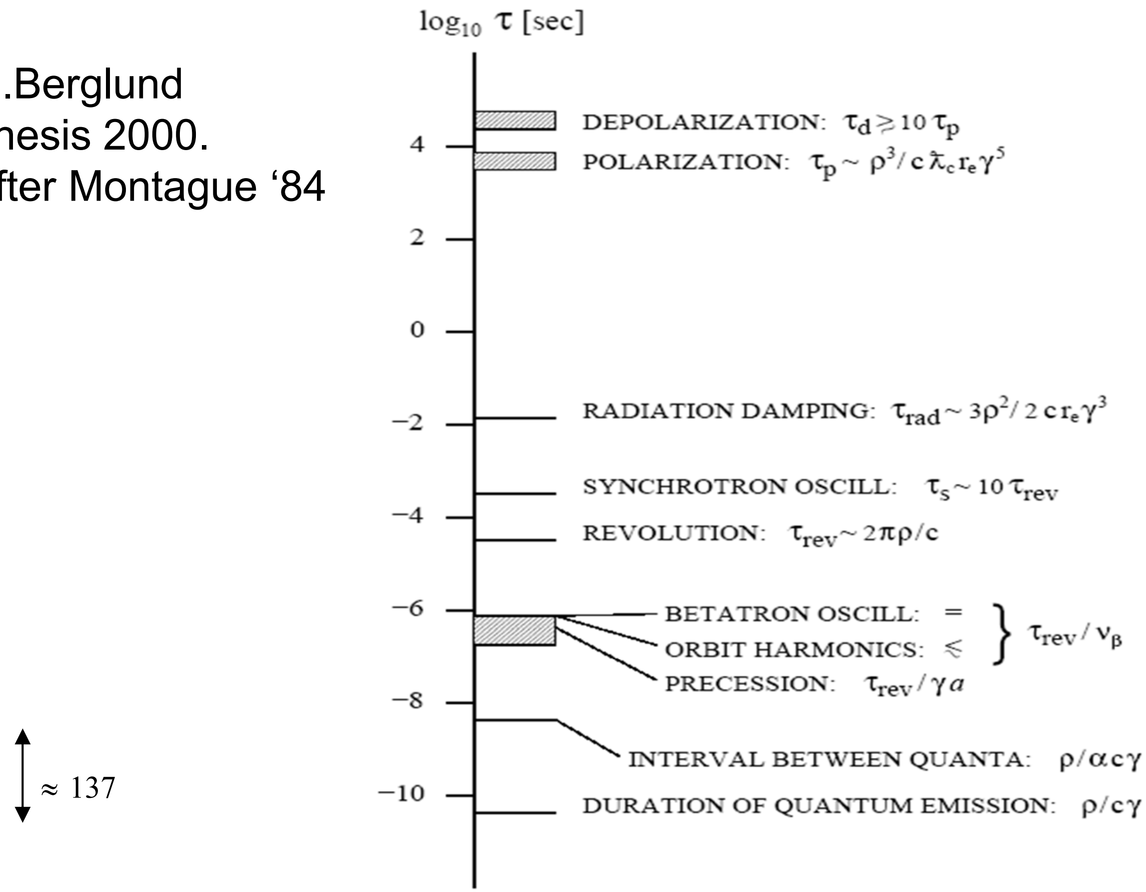
A crude example of averaging techniques for getting rid of the embarrassing bits that one can't deal with otherwise.
Thousands of theorems exploiting time scales in multi-frequency systems. Bogoluibov, Krylov, Mitropolsky…..et al ad infinitum.

Choosing a periodic uniform precession frame: several examples in proton spin dynamics, e.g., for adiabatic spin invariants.

Alternative attack: $\vec{P} = \vec{\xi}\, e^{-s/(c\tau_0)}$ and solve for $\vec{\xi}$?? Or use a 3X3 matrix DE?

Characteristic times of processes

M.Berglund
Thesis 2000.
After Montague '84



$$\left(\alpha = \frac{r_e}{\lambda\!\!\!^{-}_c} \right)$$

Figure B.1: Characteristic time scales in a typical 25 GeV electron storage ring [Mo84]. Legend: ρ = bending radius, $\lambda\!\!\!^{-}_c$ = Compton wavelength, r_e = classical electron radius, ν_β = betatron tune, α = fine structure constant, a = gyromagnetic anomaly. Although it is desirable that $\tau_d \geq 10\,\tau_p$, this is difficult to achieve in practice.

Summary: $$P_{\mathrm{BK}} = -\frac{8}{5\sqrt{3}} \frac{\oint ds \, \frac{\hat{n}_0 . \hat{b}}{|\rho^3|}}{\oint ds \, \frac{1 - \frac{2}{9}(\hat{n}_0 . \hat{\upsilon})^2}{|\rho^3|}}$$ and it's parallel to $\hat{n}_0$

⟹ To get high polarisation arrange that $\hat{n}_0$ is parallel to $\hat{b}$ in as many places with high curvature as possible.

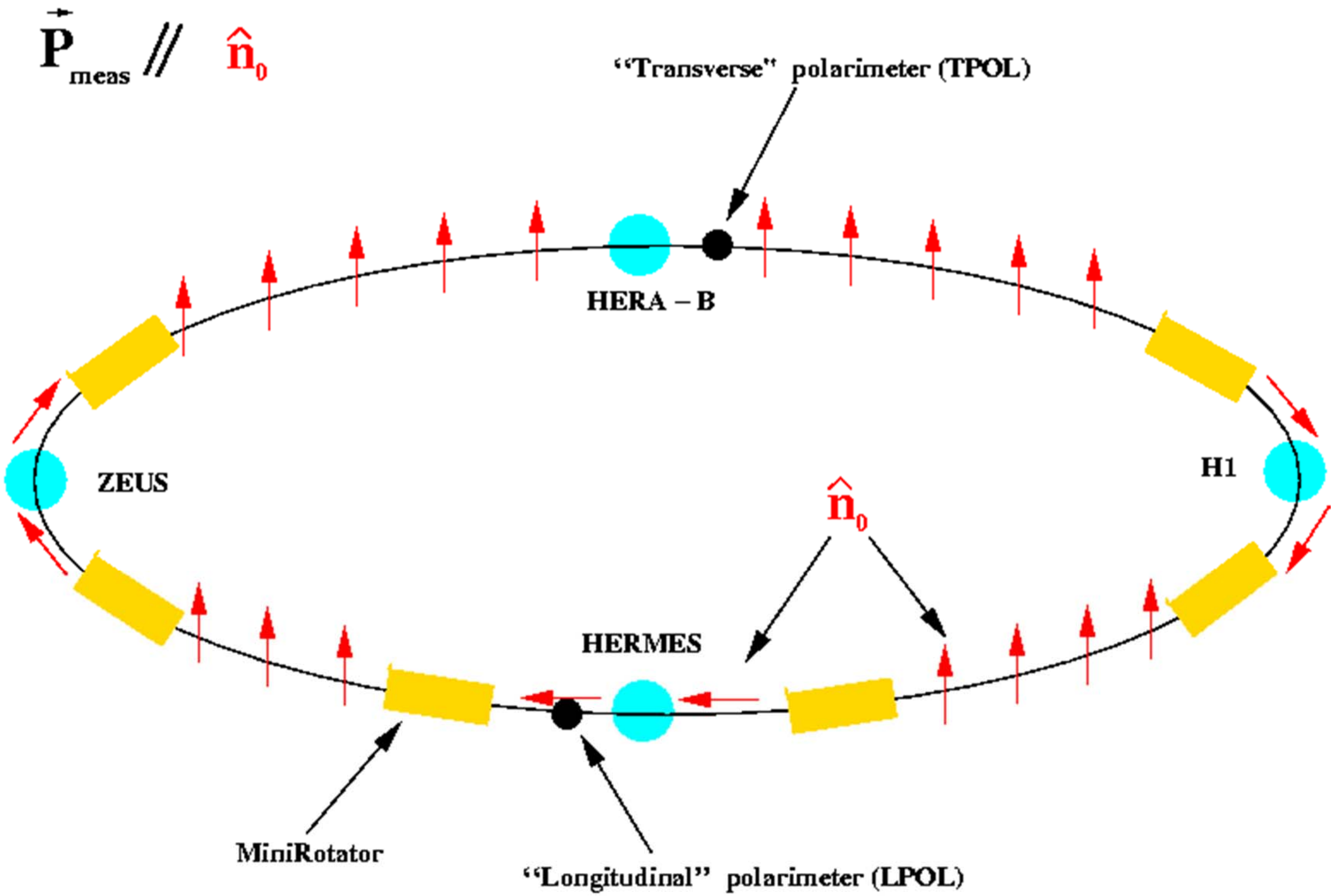
If longitudinal polarisation is required at IP's , arrange that $\hat{n}_0$ is longitudinal at the IP's.

It's all about designing $\hat{n}_0$

e.g., HERA ⟹

Note that polarising mechanisms that don't conserve particles, don't have exponential time dependence, e.g., ``spin filters''.

HERA electron/positron ring 2001 --



Rotators: the T-BMT equation.

$$\frac{d\vec{S}}{ds} = \vec{\Omega} \times \vec{S}$$

- Dipole rotators: in transverse fields:

$$\delta\theta_{spin} = a\gamma \cdot \delta\theta_{orbit}$$

$a = (g-2)/2$ where g is the electron g factor.

At 27.5 GeV $a\gamma = 62.5$

===>Suitable for high energy

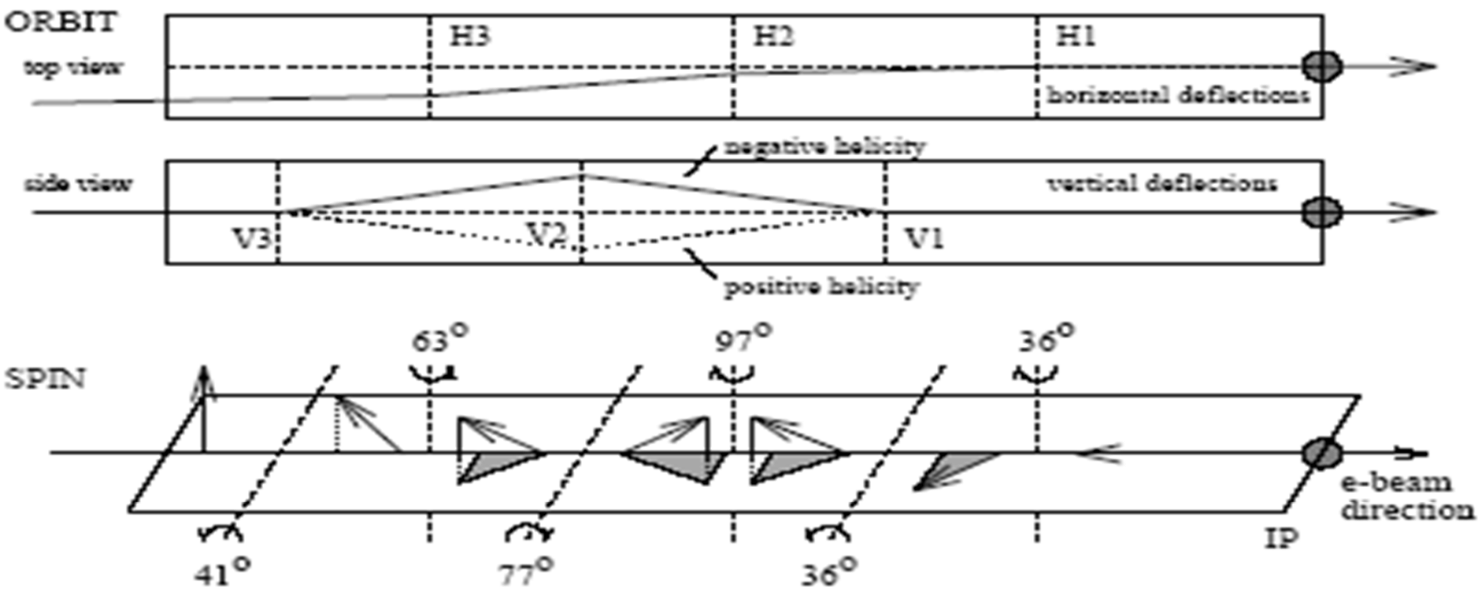
- Solenoid rotators:

$$\delta\theta_{spin} \propto \frac{B_{\mathrm{long}}}{E}$$

===>Only suitable for low energy.

. Also nontrivial spin–orbit coupling.

HERA MiniRotator: Buon + Steffen



56 m ("short") → no quads.

27 – 39 GeV, both helicities, variable geometry

Makes ``spin matching'' much easier.

Dipole spin rotators

- need $\vec{P}_{meas} \parallel \vec{B}$ in most of ring to drive Sokolov–Ternov.
- The natural polarization is vertical.
- But need $\vec{P}_{meas}$ longitudinal at IP.
- ===> Rotate $\vec{P}_{meas}(\hat{n}_0)$
  - Vertical $\rightarrow$ longitudinal just before an IP.
  - Longitudinal $\rightarrow$ vertical just after an IP.
- Recall: $\delta\theta_{spin} = a\gamma \cdot \delta\theta_{orbit}$
  Finite rotations do not commute
  ===>Use string of interleaved vertical and horizontal bends.

===> For HERA: MiniRotator by Buon and Steffen.

$$P_{\mathrm{BK}} = -\frac{8}{5\sqrt{3}} \frac{\oint ds \dfrac{\hat{n}_0 . \hat{b}}{|\rho^3|}}{\oint ds \dfrac{1 - \frac{2}{9}(\hat{n}_0 . \hat{v})^2}{|\rho^3|}}$$

At HERA the maximum attainable S-T poln is reduced from 92.38% by about 3% per rotator pair.
Current maximum is 83%.

M. Berglund Thesis 2000.

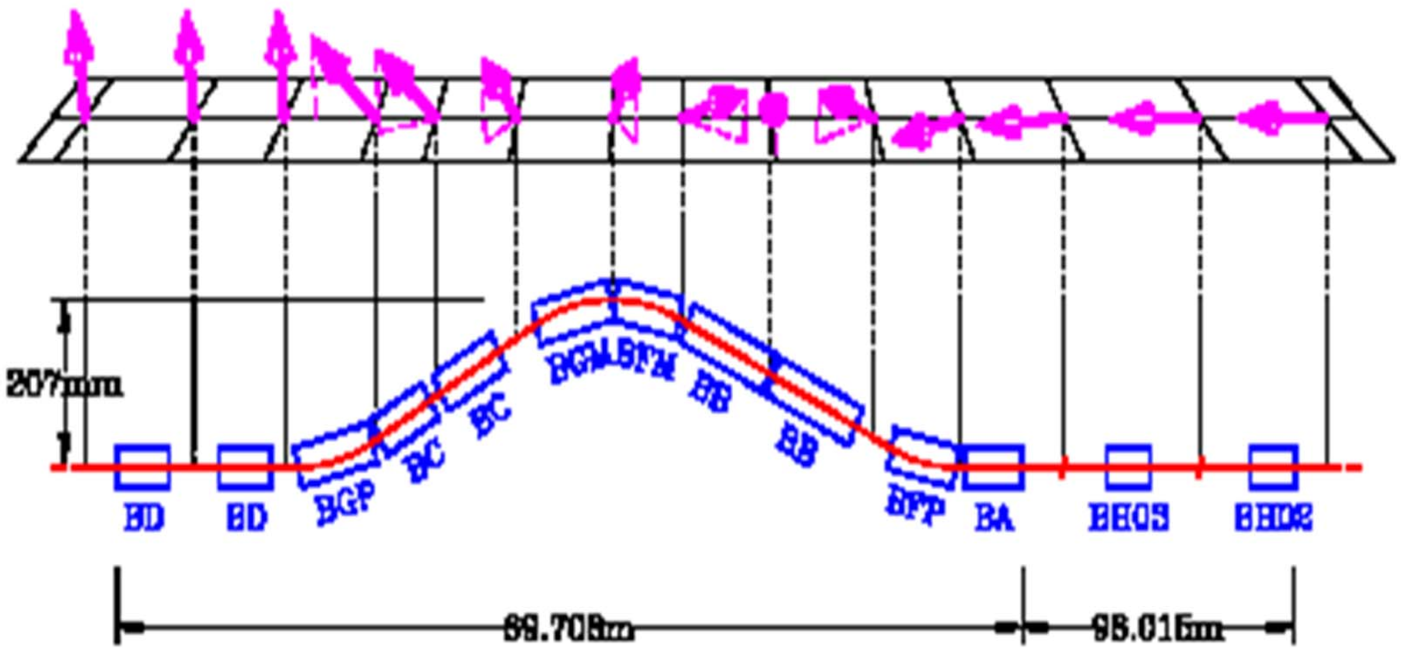


Figure 4.1: The left half of the rotator pair in the East. The magnets BH03 and BH02 are not an integral part of the rotator, but are needed to complete the spin rotation into the longitudinal direction. On the opposite side of the IP a similar magnet arrangement with reversed radial fields (BF and BG magnets) brings the polarization back to the vertical direction. Courtesy of M. Wendt.

Chief features of the MiniRotator

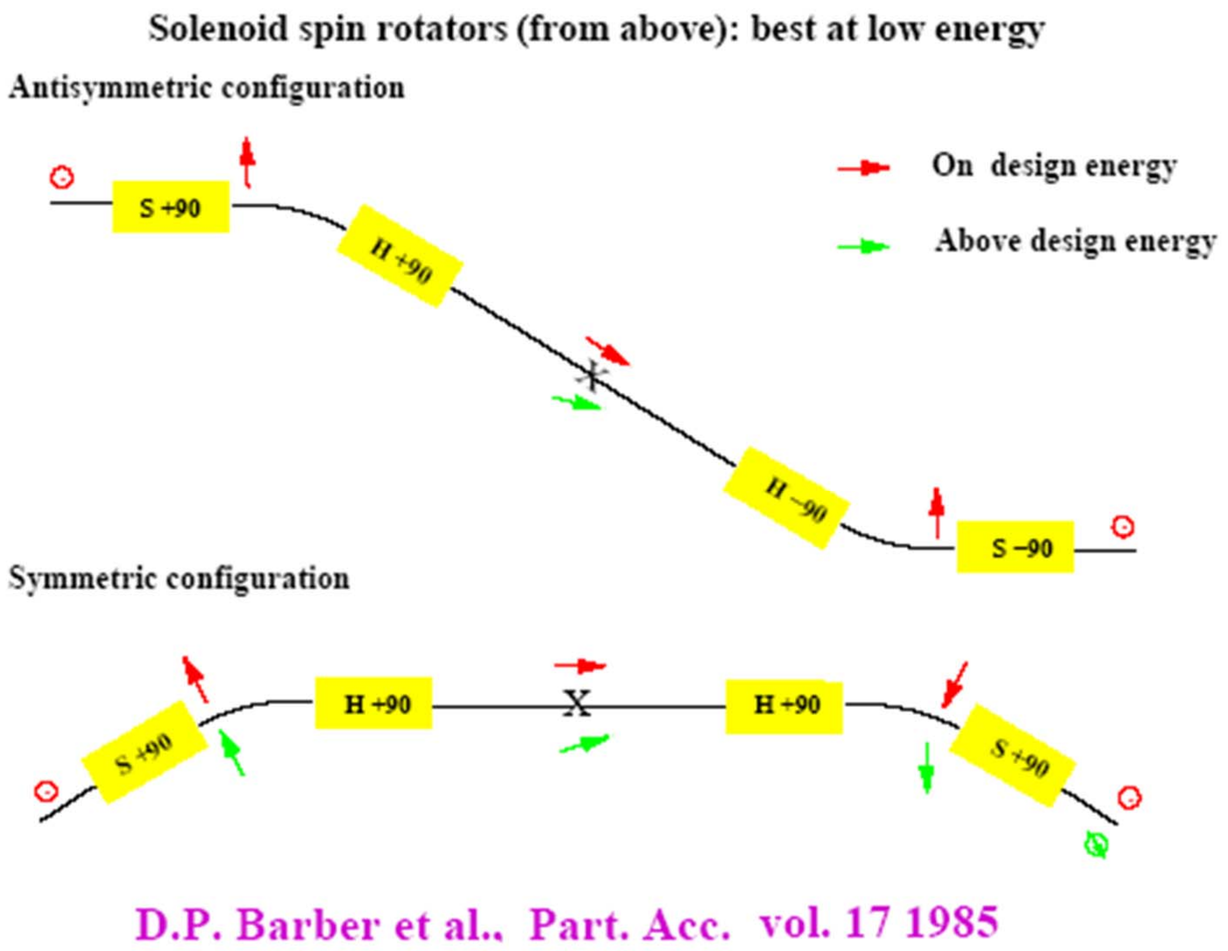
- Length limited to 56 metre: no quads needed, but $\delta P \propto L^{-2}$,
- Variable energy: 27–39 GeV, variable geometry.
- Symmetric w.r.t. IP in hor. plane. Antisymmetric in vert. plane.
- Maximum polarization: $P_{39} > P_{27}$, P_{BK} maximized.
- +ve and -ve helicity: reverse vertical bends.
- Ends fixed: middle section must me movable.
- Vertical orbit excursion small: 20 cm.
- Total vertical bend is zero.
- Total horizontal bend non–zero: include in arc to save space.
- Spin tune shift: $\delta\nu_{spin}$ small.


NR.24
NR.23
NR.21

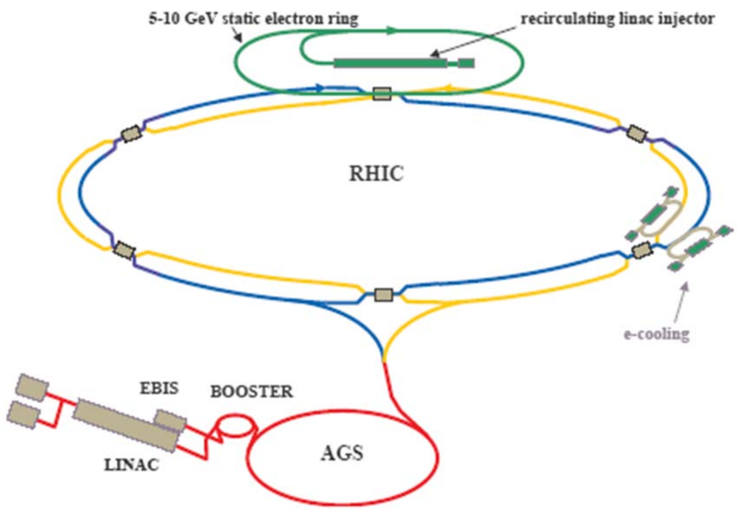
NR.24
NR.23
NR.21

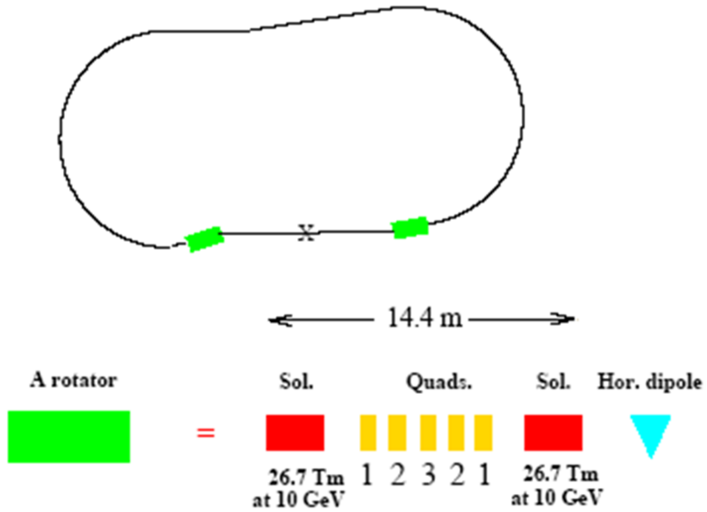
NR24
NR23

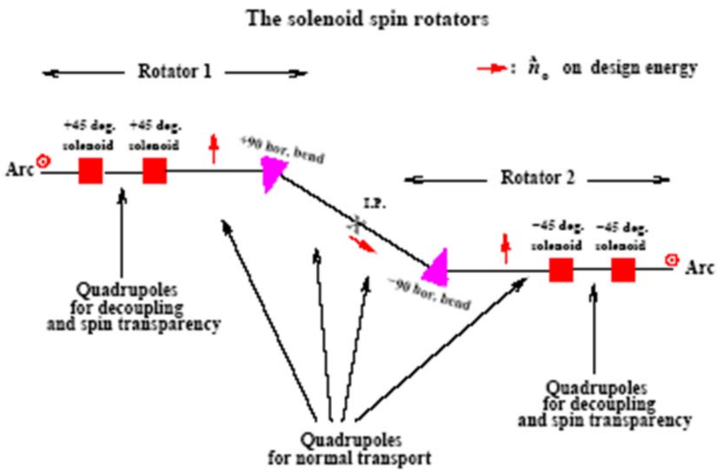
tesla

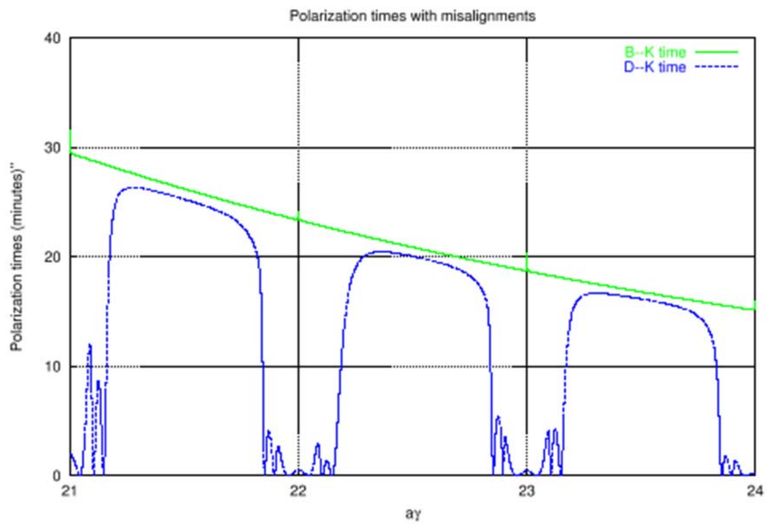

Solenoid spin rotators (from above): best at low energy
Antisymmetric configuration
S +90
H +90
H −90
S −90
On design energy
Above design energy
Symmetric configuration
S +90
H +90
H +90
S +90
D.P. Barber et al., Part. Acc. vol. 17 1985

eRHIC   Maximum attainable S-T polarisation: about 84%









Is this the ultimate free lunch?

Recall $$\frac{\Pi_{\text{flip}}^{\uparrow\downarrow}}{\Pi_{\text{nf}}} = O\left(\left(\frac{\hbar\omega_0}{E}\gamma^3\right)^2\right) = O(10^{-11}) \quad \text{at } 27.5 \text{ GeV} \quad \text{at HERA}$$

So what’s all the other (non-flip) synchrotron radiation doing?

It’s depolarising the beam!!!!! Aaaaaaaaaaaaaaggggggggghhhhhhhhhhhh

Topic 4

The effect of synchrotron radiation II

-- depolarisation.

Why? --- for the same reason that electron (positron) beams have no memory!

namely that the energy loss caused by synchrotron radiation leads to damping effects and the discrete content (photons) of the radiation leads to irreversible excitation of the orbits.

The joint effect is that the particle distribution reaches an equilibrium state which depends only of the geometrical and optical state of the ring.

If the beam is disturbed, it returns to its equilibrium over a few damping times and forgets the disturbance.

A damping time usually corresponds to a several 100 turns: an electron beam is a weakly dissipative system.

Time scales!

Depolarisation has three origins:

Spin –orbit coupling (T-BMT equation)

Inhomogeneous fields (e.g. quadrupoles)

Noise due to synchrotron radiation.

A single photon emission results in only a tiny disturbance to a spin but there is a very large number of photons!

A proton beam emits essentially no synchrotron radiation and in the absence of perturbations like wake fields, the proton trajectories can be predicted using (reversible) Hamiltonian mechanics. Then the dimensions of proton bunches depend only on the volume of phase space occupied at the source, the gain in energy by acceleration and on the Courant-Snyder parameters.

See Bernhard Holzer's material on the Liouville theorem and adiabatic damping.

Depolarisation can occur for protons but noise is not a contributer.

For protons, the problem lies with acceleration.

Depolarisation 1: the SLIM algorithm (A.W. Chao NIM, 1981).

Orbit coordinates: level 2 (w.r.t. the closed orbit.)

$$u \equiv (x, x', y, y', \sigma, \delta = \Delta E / E) \qquad u(s_2) = M_{6\times 6}(s_2, s_1) u(s_1)$$

$$M_{6\times 6}(s_1 + C, s_1)\upsilon_k(s_1) = e^{-2\pi i Q_k}\upsilon_k(s_2) \qquad k = I, II, III, -I, -II, -III$$

$$Q_{-I} = -Q_I \ ; \upsilon_{-I} = \upsilon_I^{*} \quad \text{etc.} \qquad u(s) = \sum_k A_k \upsilon_k(s)$$

$M(s_2, s_1)$ represents a canonical transformation. $\Rightarrow M^{\mathrm{T}} J M = J$

(Recall $R^{\mathrm{T}} R = I$ for $SO(3)$)

with $$J = \begin{bmatrix} j_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & j_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} & j_{2\times 2} \end{bmatrix} ; \quad j_{2\times 2} = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}$$

Orthogonallity of eigenvectors:: e.g., $(\vec{\upsilon}_k^{*})^{\mathrm{T}} J \vec{\upsilon}_k = i$ etc.

The eigenvectors and eigenvalues encode everything about linearised motion

See Appendix 2 for more on orbital eigenvectors.

Look at spin motion in the $(\hat{n}_0 \,,\, \hat{m}_0 \,,\, \hat{l}_0)$ frame and then in the $(\hat{n}_0 \,,\, \hat{m} \,,\, \hat{l})$ frame

Write $\vec{\Omega}(u\,;s) = \vec{\Omega}_0(s) + \vec{\omega}(u\,;s)$; $\quad$ (Define $\vec{\omega} = \vec{\Omega} - \Omega_0$)

$\vec{\omega}(s;u)$ embodies the effect of synchro-betatron motion on spin.

By working in the $(\hat{n}_0 \,,\, \hat{m}_0 \,,\, \hat{l}_0)$ frame we, in effect, ``subtract out '' the trivial spin motion so that we can concentrate on the more tricky stuff.

$$\frac{d\vec{S}}{ds} = (\vec{\Omega}_0 + \vec{\omega}) \times \vec{S} = \hat{n}_0 \dot{\vec{S}}_{\mathrm{n}} + \hat{m}_0 \dot{S}_{\mathrm{m}_0} + \hat{l}_0 \dot{S}_{\mathrm{l}_0} + S_{\mathrm{n}} \dot{\hat{n}}_0 + S_{\mathrm{m}_0} \dot{\hat{m}}_0 + S_{\mathrm{l}_0} \dot{\hat{l}}_0$$

$$\frac{d\vec{S}}{ds} = (\vec{\Omega}_0 + \vec{\omega}) \times \vec{S} = \hat{n}_0 \dot{\vec{S}}_{\mathrm{n}} + \hat{m}_0 \dot{S}_{\mathrm{m}_0} + \hat{l}_0 \dot{S}_{\mathrm{l}_0} + \vec{\Omega}_0 \times \vec{S} \qquad (\text{Also recall} : \frac{d}{ds} \Rightarrow \frac{d}{ds} + \vec{\omega} \times)$$

$$\Rightarrow \hat{n}_0 \dot{S}_{\mathrm{n}} + \hat{m}_0 \dot{S}_{\mathrm{m}_0} + \hat{l}_0 \dot{S}_{\mathrm{l}_0} = \vec{\omega} \times (\hat{n}_0 \dot{S}_{\mathrm{n}} + \hat{m}_0 \dot{S}_{\mathrm{m}_0} + \hat{l}_0 \dot{S}_{\mathrm{l}_0})$$

Change notation to fit convention $S_{\mathrm{n}} \Rightarrow \gamma_0 \;;\; S_{\mathrm{m}_0} \Rightarrow \alpha_0 \;;\; S_{\mathrm{l}_0} \Rightarrow \beta_0 \qquad \gamma_0 = \sqrt{1 - \alpha_0^2 - \beta_0^2}$

$$\Rightarrow \frac{d}{ds}\begin{pmatrix} \gamma_0 \\ \alpha_0 \\ \beta_0 \end{pmatrix} = \begin{pmatrix} 0 & -\hat{l}_0.\vec{\omega} & \hat{m}_0.\vec{\omega} \\ \hat{l}_0.\vec{\omega} & 0 & -\hat{n}_0.\vec{\omega} \\ -\hat{m}_0.\vec{\omega} & \hat{n}_0.\vec{\omega} & 0 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \alpha_0 \\ \beta_0 \end{pmatrix}$$

i.e., as usual, the coefficient matrix for rotations is antisymmetric
$\longrightarrow$ orthogonal transport matrices..

Consider very small α_0 and $\beta_0 : \alpha_0^2 + \beta_0^2 << 1 \Rightarrow \gamma_0 \approx 1$ and treat $\vec{\omega}$ as a small perturbation.

Then $\vec{S} \approx \hat{n}_0 + \alpha_0 \hat{m}_0 + \beta_0 \hat{l}_0$ i.e., α_0 and β_0 are 2 small angles of tilt from : $\hat{n}_0$

Then at 1st order: $\dfrac{d\alpha_0}{ds} = \vec{\omega}.\hat{l}_0 \quad \dfrac{d\beta_0}{ds} = -\vec{\omega}.\hat{m}_0$

A first hint of spin-orbit resonances --- and trouble!

Write

$$\vec{\omega}(s).\{\hat{m}_0(s) + i\hat{l}_0(s)\} = \sum_{j,k} C^{+}_{jk} e^{-i2\pi(j-\nu_0+Q_k)s/C} + \sum_{j,k} C^{-}_{jk} e^{-i2\pi(j-\nu_0-Q_k)s/C}$$

At a (first order) resonance, i.e., if $\nu_0 = j \pm Q_k$

$$\int \vec{\omega}(s).\{\hat{m}_0(s) + i\hat{l}_0(s)\}\, ds$$

increases indefinitely at a rate proportional to its C_{jk}

This would take $\alpha_0^2 + \beta_0^2$ way out of the range of the approximation but we already see a mechanism for large excursions of spins away from $\hat{n}_0$

These are then amplified by the noise of photon emission.

We prefer a periodic coordinate system: $\hat{m}(s) + i\,\hat{l}(s) = e^{-i\psi_0(s)}\left[\hat{m}_0(s) + i\,\hat{l}_0(s)\right]$ $\psi_0(s+C) - \psi_0(s) = 2\pi\nu_0$

$$\Rightarrow \frac{d\alpha}{ds} = \vec{\omega}.\hat{l} + \beta\psi_0' \qquad \frac{d\beta}{ds} = -\vec{\omega}.\hat{m} - \alpha\psi_0' \qquad \frac{d}{ds}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \underbrace{\begin{bmatrix}0 & \psi_0'\\ -\psi_0' & 0\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}}_{\text{Homogeneous.}} - \vec{\omega}.\underbrace{\begin{bmatrix}\hat{l}\\ -\hat{m}\end{bmatrix}}_{\substack{\text{Inhom.}\\ \text{(driving)}}}$$

We impose **3** linearisations

The orbital motion (use transfer matrices)

The dependence of $\vec{\omega}(u;s)$ on u in the T-BMT eqn.(*)

The spin motion: small $(\alpha, \beta) \Rightarrow$ simple (α', β')

*: e.g., in a quad $\vec{\omega}(u;s) = (a\gamma+1)k(s)\,x\,\hat{y} - (a\gamma+1)k(s)\,y\,\hat{x}$ $k =$ quad strength.

$$\Rightarrow \vec{\omega}.\hat{l} = (a\gamma+1)k(s)\,x\,l_y - (a\gamma+1)k(s)\,y\,l_x \quad \text{Analogue for } \vec{\omega}.\hat{m}$$

In general write

$$\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_s\end{pmatrix} = F_{3\times 6}(s)\begin{pmatrix}x\\ x'\\ y\\ y'\\ \sigma\\ \delta\end{pmatrix}$$

$\Rightarrow \frac{d\alpha}{ds}$ and $\frac{d\beta}{ds}$ are functions of *s* and linear functions of *u just as , e.g., x' = -k(s)x*

So now work with the spin-orbit vectors (u, α, β) !

The linearised orbital eqns can be integrated to give 6x6 transport matrices.

Now expand to 8x8 matrices:.

No Stern-Gerlach

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ \sigma \\ \delta \\ \alpha \\ \beta \end{pmatrix}(s_1) = \begin{bmatrix} M_{6\times 6} & 0_{6\times 2} \\ G_{2\times 6} & D_{2\times 2} \end{bmatrix}(s_1, s_0)\begin{pmatrix} x \\ x' \\ y \\ y' \\ \sigma \\ \delta \\ \alpha \\ \beta \end{pmatrix}(s_0) \qquad D(s_0 + C, s_0) = \begin{bmatrix} \cos 2\pi\nu_0 & \sin 2\pi\nu_0 \\ -\sin 2\pi\nu_0 & \cos 2\pi\nu_0 \end{bmatrix}$$

The spin-orbit coupling matrix *G* is obtained by joint integration of the EOM and EOSM

The matrix *D* does the book keeping for motion in the periodic reference frame
In the $(\hat{n}_0, \hat{m}_0, \hat{l}_0)$ frame it would just be $I_{2\times 2}$

E.g. the 8x8 matrix for a quad with spin described in the $(\hat{m}_0, \hat{l}_0)$ frame:

$$
\left[\begin{array}{cc|cc|cc|cc}
C_x & S_x & & & & & & \\
\hat{C}_x & \hat{S}_x & & & & & & \\
\hline
 & & C_y & S_y & & & & \\
 & & \hat{C}_y & \hat{S}_y & & & & \\
\hline
 & & & & 1 & & & \\
 & & & & & 1 & & \\
\hline
(1+a\gamma)\hat{C}_x l_{0y} & (1+a\gamma)(\hat{S}_x - 1) l_{0y} & -(1+a\gamma)\hat{C}_y l_{0x} & -(1+a\gamma)(\hat{S}_y - 1) l_{0x} & 0 & 0 & 1 & 0 \\
-(1+a\gamma)\hat{C}_x m_{0y} & -(1+a\gamma)(\hat{S}_x - 1) m_{0y} & (1+a\gamma)\hat{C}_y m_{0x} & (1+a\gamma)(\hat{S}_y - 1) m_{0x} & 0 & 0 & 0 & 1
\end{array}\right]
$$

This simply expresses the expectation $\Delta\vartheta_{\text{spin}} = (1 + a\gamma)\Delta\vartheta_{\text{orb}}$
So we could have written *G* by using intuition and without integrating!
Barber et al., Particle Accel. 1985.

By subsequently transforming to the periodic reference frame $(\vec{S} \approx \hat{n}_0 + \alpha\,\hat{m} + \beta\,\hat{l})$ we ensure that *G* is 1-turn periodic.
Then we can do an eigen-analysis of the whole 8x8 matrix.

NOTE THE FACTORS $(1 + a\gamma)$

Again: the matrix *G* just expresses the standard expectations – but in a well organised form.

Examine in some detail:

$$
\left[\begin{array}{cc|cc|cc|cc}
C_x & S_x & & & & & & \\
\hat{C}_x & \hat{S}_x & & & & & & \\
\hline
 & & C_y & S_y & & & & \\
 & & \hat{C}_y & \hat{S}_y & & & & \\
\hline
 & & & & 1 & & & \\
 & & & & & 1 & & \\
\hline
(1+a\gamma)\hat{C}_x l_{0y} & (1+a\gamma)(\hat{S}_x-1)l_{0y} & -(1+a\gamma)\hat{C}_y l_{0x} & -(1+a\gamma)(\hat{S}_y-1)l_{0x} & 0 & 0 & 1 & 0 \\
-(1+a\gamma)\hat{C}_x m_{0y} & -(1+a\gamma)(\hat{S}_x-1)m_{0y} & (1+a\gamma)\hat{C}_y m_{0x} & (1+a\gamma)(\hat{S}_y-1)m_{0x} & 0 & 0 & 0 & 1
\end{array}\right]
$$

Vertical quadrupole fields due to radial betatron motion: cause spins to tilt left/right
If $\hat{n}_0$ is vertical (m_{0y}, l_{0y}) vanish. .
So no effect as expected for vertical spins in vertical fields.
If $\hat{n}_0$ is horizontal, (m_{0y}, l_{0y}) are *O(1).*
If $\hat{n}_0$ is tilted due to misalignments, there is some effect spin precession around the vertical quad fields.

Radial quadrupole fields due to vertical betatron motion cause spins to tilt forward and backward. If $\hat{n}_0$ is vertical (m_{0x}, l_{0x}) are horizontal *O(1).*

In quads:
no sync. terms
at this order.

Now apply the methods for calculating orbital emittances of electron beams to spin as well

The ``DESY approach' (Mais + Ripken): a stochastic calculus which puts text book accounts of the effects of synchrotron radiation on a formal basis and thereby exposes more detail than is found in text books and shows the dangers of those treatments.
(See also A.W. Chao J.App.Phys. 1979, Kolomensky and Lebedev, S+T 1968)

Express the energy loss by sync. rad. in the form:

$P(s) = P_{\text{class}}(s) + P_{\text{stoch}}(s)$ where $P_{\text{stoch}}(s)$ results from a series of +ve & -ve kicks which average to zero.

$$\frac{du}{ds} = (\underbrace{B(s)}_{\text{symplectic motion}} + \underbrace{\delta B(s)}_{\text{damping with sync. rad.}})\, u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \varsigma \end{bmatrix} \leftarrow \text{Noise from photon emission } \delta\text{–function-like kicks}$$

Model the energy kicks from photon emission as ``Gaussian white noise" :

$$\langle \varsigma(s) \rangle = 0 \qquad \langle \varsigma(s_1)\varsigma(s_2) \rangle = \Xi(s_1)\delta(s_1 - s_2) \qquad \Xi(s) = \frac{55}{24\sqrt{3}} \frac{r_e \lambda\!\!\!^{-}{}_c}{|\rho^3(s)|} \gamma^5$$

$$\text{For a quad}: B = \begin{bmatrix} 1 & 0 & & & & \\ -k & 1 & & & & \\ & & 1 & 0 & & \\ & & +k & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

A solution for *u* depends on the history of the kicks.
The only definite objects are averages over all possible histories of kicks
i.e. ``stochastic averages''
We take these averages to represent averages over the ensemble of electrons:
each electron has its own stochastic history (Wiener process).

For the origin of ``stochastic'' see:

http://smccd.net/accounts/goth/MainPages/wordphys.htm

-- and then wonder why we use the word.

Recall: in the absence of sync. rad. $u(s) = \sum_k A_k \upsilon_k(s)$

If an electron gets an energy kick $\Delta\delta$ at s_0, the A_k change: $\Delta A_k = i\upsilon^*_{5k}(s)\,\Delta\delta \quad \forall\ k$

$$\Rightarrow \frac{\partial A_k}{\partial \delta} = i\upsilon^*_{5k}(s)$$

Include damping too:
and avoid orb. res.
(needed averaging!).

$$\frac{d\,A_k}{d\,s} = A_k(-\alpha_k - i2\pi\,(\delta Q_k)) + i\,\Xi(s)\,\upsilon^*_{k5}(s)\,\varsigma(s)$$

damping constant for mode k

real tiny tune shift

Finally after some foot work, the emittances:

$$\langle |A_k(s)|^2\rangle = \text{constant} = \frac{55}{24\sqrt{3}} r_e \lambda\!\!\!^{-}_c \gamma^5 \frac{1}{2\alpha_k} \oint_s d\tilde{s}\ \frac{1}{|\rho^3(\tilde{s})|} |\upsilon_{k5}(\tilde{s})|^2$$

If α_k is very small.

The system can be fully coupled. $|\upsilon_{k5}(s)|^2$ Is the analogue of function ``*H*" usually seen when using beta functions and dispersions.

Beam sizes: $$\langle u_m(s)\,u_n(s)\rangle = 2 \sum_{k=I,II,III} \langle |A_k|^2\rangle\, \mathrm{Re}[\,\upsilon_{km}(s)\upsilon^*_{kn}(s)]$$

Can get to the beta-dispersion picture by a canonical transformation and then in :
simple cases the usual *H* function appears: Barber + Ripken: Handbook of ASE

The damping times $\tau_k = \dfrac{C}{c\,\alpha_k}$ usually corresponds to many 100's of turns. Time scales!

To keep an emittance $\langle | A_k(s) |^2 \rangle$ small, keep its $\upsilon^*_{k5}(s)$ small where $\dfrac{1}{|\rho^3|}$ is large.
E.g. damping rings.

$$\frac{\partial A_k}{\partial \delta} = i\,\upsilon^*_{5k}(s) \Rightarrow \upsilon^*_{5k}(s)$$ encodes the sensitivity of the amplitude k to energy kicks

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ \sigma \\ \delta \\ \alpha \\ \beta \end{pmatrix}(s_1) = \begin{bmatrix} M_{6\times 6} & 0_{6\times 2} \\ G_{2\times 6} & D_{2\times 2} \end{bmatrix}(s_1, s_0)\begin{pmatrix} x \\ x' \\ y \\ y' \\ \sigma \\ \delta \\ \alpha \\ \beta \end{pmatrix}(s_0)$$

The spin $\vec{S} \approx \hat{n}_0 + \alpha\,\hat{m} + \beta\,\hat{l}$ is coupled to the orbit via the matrix G and we expect

that as a result of the noise in the orbital motion, α and β execute some kind of a random walk.

Then with $\sqrt{1-\alpha^2-\beta^2} \approx 1 - \frac{1}{2}(\alpha^2+\beta^2)$ we want to find

$$\tau^{-1}_{\text{dep}} = -\frac{1}{P_n}\frac{dP_n}{dt} \approx \frac{d}{dt}\{\frac{1}{2}(\sigma^2_\alpha + \sigma^2_\beta)\} = \frac{d}{dt}\;\frac{1}{2}\{\langle\alpha^2\rangle + \langle\beta^2\rangle\}$$

Extend the orbital eigenvectors to include spin

$$\vec{q}_k(s_0) = \begin{pmatrix} \vec{v}_k(s_0) \\ \vec{w}_k(s_0) \end{pmatrix}, \quad \vec{q}_{-k}(s_0) = [\vec{q}_k(s_0)]^* \qquad \vec{q}_k(s_0) = \begin{pmatrix} \vec{0}_6(s_0) \\ \vec{w}_k(s_0) \end{pmatrix}, \quad \vec{q}_{-k}(s_0) = [\vec{q}_k(s_0)]^*$$

$$\text{for } k = I,\ II,\ III\ ; \qquad\qquad \text{for } k = IV$$

No back reaction of the spin on the orbit (no S-G) so:

$$\vec{w}_k(s_0) = -\left[\mathbf{D}(s_0 + C, s_0) - \hat{\lambda}_k\right]^{-1} \mathbf{G}(s_0 + C, s_0)\vec{v}_k(s_0)$$

$$\text{for } k = I,\ II,\ III\ ;$$

$$\vec{w}_{IV}(s_0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i\psi_{spin}(s_0)}$$

$$\text{for } k = IV$$

Resonance denominators!
Trouble if for any mode *k*:

$\nu_0 \approx$ integer $\pm\, Q_k$

unless $G\,\upsilon_k = 0$

⇒ SPIN MATCHING!

Extend the stochastic calculus used for the orbit to the spin.

Introduce the periodic ``diffusion vector'' $\vec{d}(s) \equiv \underbrace{d_\alpha(s)\hat{m}(s) + d_\beta(s)\hat{l}(s)}_{\text{periodic}}$ perp.to $\hat{n}_0$

Where $$\begin{bmatrix} d_\alpha \\ d_\beta \end{bmatrix}(s) = -2\ \mathrm{Im} \sum_{k=I,II,III} \upsilon_{k5}^*(s) \begin{bmatrix} w_{k\alpha} \\ w_{k\beta} \end{bmatrix}$$

This characterises the strength of random walk of α and β :

Note that the linearisation allows a decomposition into the effects of each mode.

$$\tau_{\mathrm{dep}}^{-1} = -\frac{1}{P_n}\frac{dP_n}{dt} = \frac{d}{dt}\frac{1}{2}\{\langle\alpha^2\rangle + \langle\beta^2\rangle\} = \frac{c}{C}\frac{55\sqrt{3}}{144} r_e \lambda\!\!\!^{-}{}_c \gamma^5 \oint ds \frac{(d_\alpha^2(s) + d_\beta^2(s))}{\underbrace{|\rho^3(s)|}_{\text{orbit excitation}}}$$

Although there is a damping mechanism for electrons, it is this mechanism that contributes to depolarisation!.

Resonances

Spins are passengers/spectators subject to interleaved kicks from quadrupoles and precessions in the dipoles.

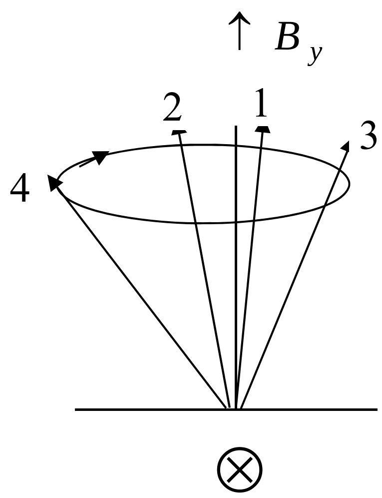
If their natural precession motion is coherent with the kicks from the synchro-betatron motion, there can be a resonant and very strong accumulation of the disturbance to the spins.

If α and β diffuse due to noise, there can be resonant enhancement of the diffusion:
This is automatically contained in the

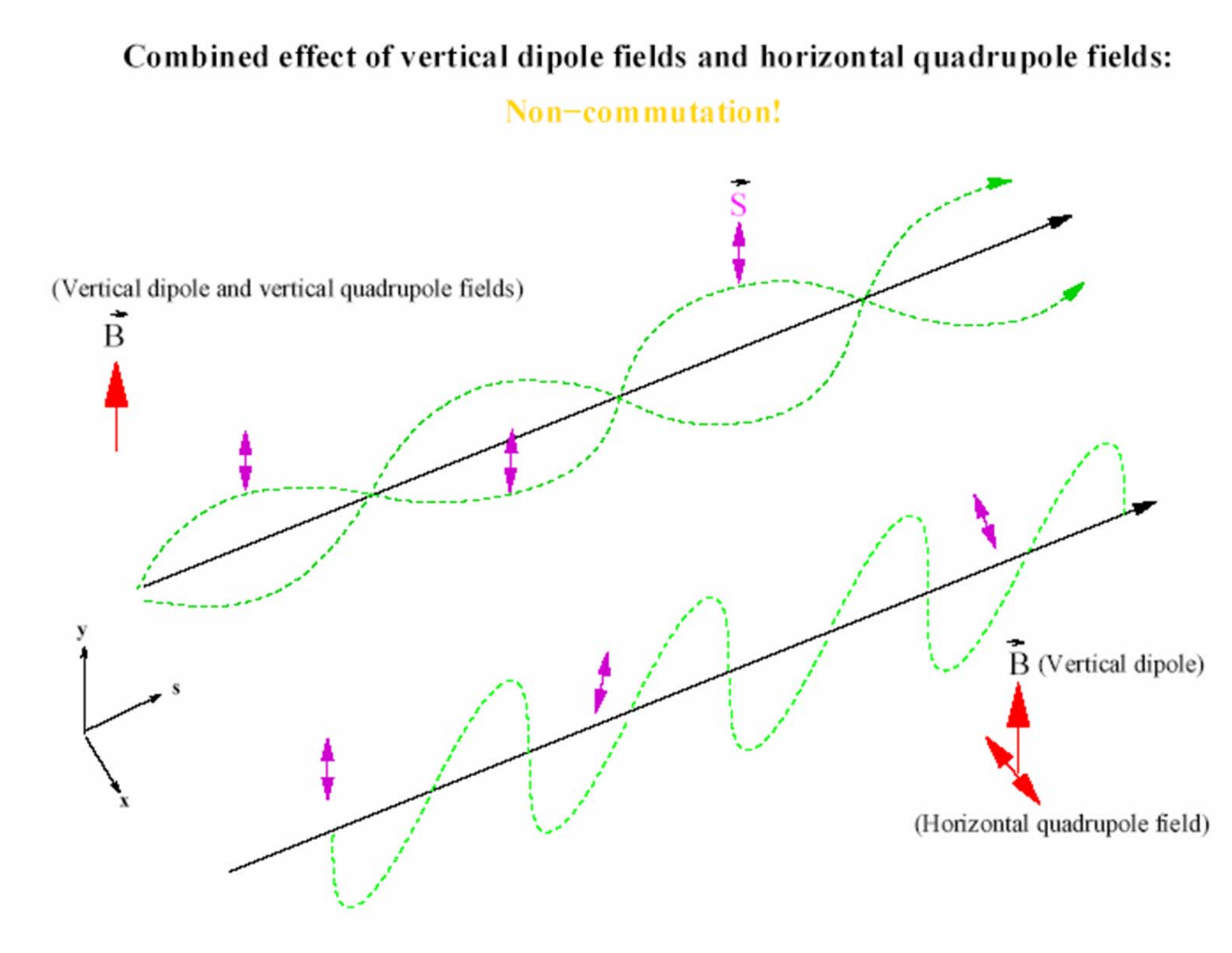
$$\begin{bmatrix} d_\alpha \\ d_\beta \end{bmatrix}(s) = -2 \;\mathrm{Im} \sum_{k=I,II,III} \upsilon^*_{k5}(s) \begin{bmatrix} w_{k\alpha} \\ w_{k\beta} \end{bmatrix}$$

A totally implausible model: 1 quad, $Q_y = \frac{1}{2}, \nu_0 = \frac{1}{2}$

At resonance the condition $\alpha_0^2 + \beta_0^2 << 1$
is violated badly, but there is no
polarisation by then any way.



Field oscillating at Q_y =1/2
into screen.


Combined effect of vertical dipole fields and horizontal quadrupole fields:
Non−commutation!
$\vec{S}$
(Vertical dipole and vertical quadrupole fields)
$\vec{B}$
y
s
x
$\vec{B}$ (Vertical dipole)
(Horizontal quadrupole field)

Add the diffusion in incoherently with the S-T (B-K)

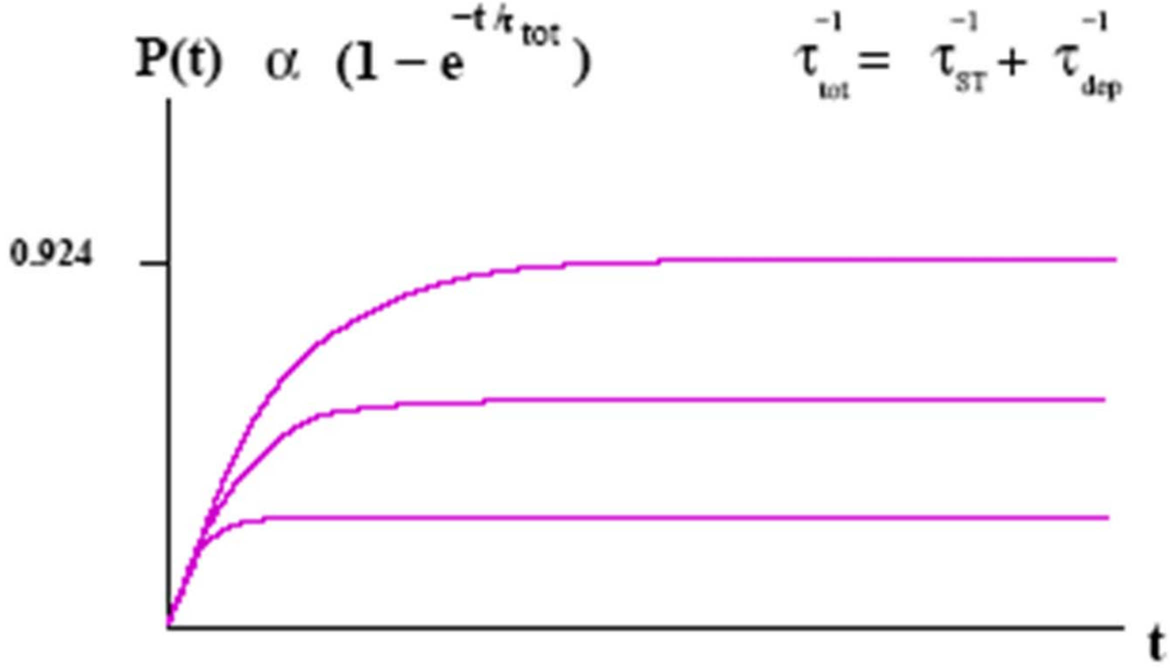
$$\frac{dP_{\mathrm{n}}}{dt} = -\left(\frac{5\sqrt{3}}{8}\frac{cr_e\lambda\!\!\!^{-}_c\gamma^5}{C}\right)\left(P_{\mathrm{n}}\oint ds\left(\frac{1-\frac{2}{9}(\hat{n}_0.\hat{\upsilon})^2+\frac{11}{18}\vec{d}^{\,2}}{|\rho^3|}\right)-\oint ds\left(\frac{8}{5\sqrt{3}}\frac{\hat{n}_0.\hat{b}}{|\rho^3|}\right)\right)$$

$$\tau_{\mathrm{tot}}^{-1} = \frac{5\sqrt{3}}{8}\frac{c\lambda\!\!\!^{-}_c r_e\gamma^5}{C}\oint ds\,\frac{1-\frac{2}{9}(\hat{n}_0.\hat{\upsilon})^2+\frac{11}{18}\vec{d}^{\,2}}{|\rho^3|} = \tau_{\mathrm{BK}}^{-1}+\tau_{\mathrm{dep}}^{-1}$$

$$\Rightarrow P_{\mathrm{n}}(s=\infty) = -\frac{8}{5\sqrt{3}}\frac{\oint ds\,\frac{\hat{n}_0.\hat{b}}{|\rho^3|}}{\oint ds\,\frac{1-\frac{2}{9}(\hat{n}_0.\hat{\upsilon})^2+\frac{11}{18}\vec{d}^{\,2}}{|\rho^3|}} = P_{\mathrm{BK}}\frac{\tau_{\mathrm{tot}}}{\tau_{\mathrm{BK}}}$$

Useful for polarimeter calibration

$$\tau_{\mathrm{tot}} = \frac{\tau_{\mathrm{BK}}\,\tau_{\mathrm{dep}}}{\tau_{\mathrm{BK}}+\tau_{\mathrm{dep}}}$$


$P(t) \;\alpha\; (1 - e^{-t/\tau_{tot}})$
$\tau_{tot}^{-1} = \tau_{ST}^{-1} + \tau_{dep}^{-1}$
0.924
t

This is formalism provides a fast practical algorithm for getting a first look at the potential polarisation in a ring. It can handle arbitrary coupling and misalignments

Although some pieces were glued together, .things can be put on a more rigorous basis.

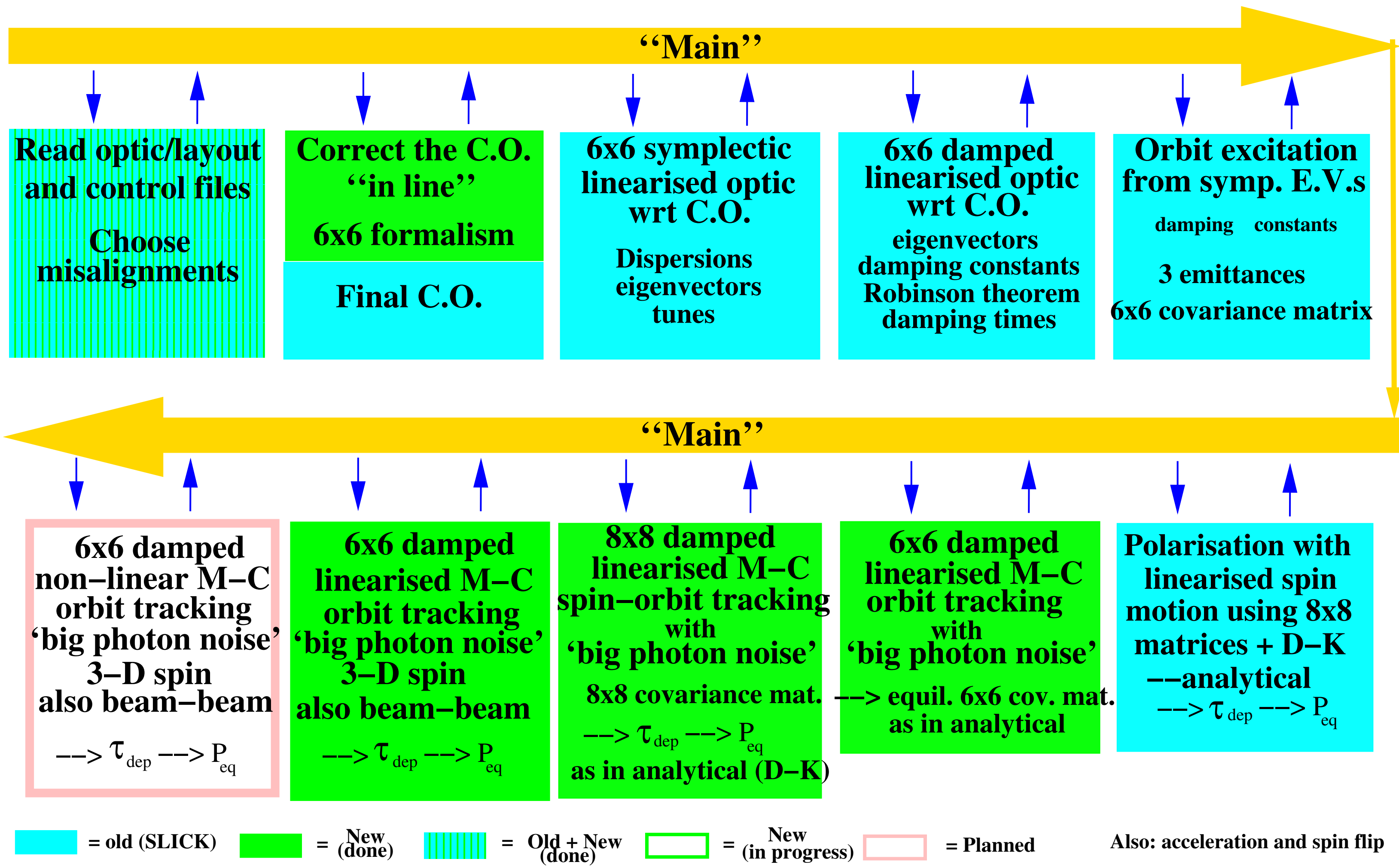
Note that the damping constants do not appear in the depolarisation rate and that the particle distribution reaches equilibrium whereas, in the absence of the S-T effect, the spin distribution does not reach equilibrium: there is no damping effect for spin. Moreover, the damping mechanism for electrons contributes to depolarisation!.
This is a consequence of the adiabatic invariance of a special spin gadget coming later.

The original thin lens code SLIM of Chao was upgraded to thick lenses in 1982 (by me): SLICK using the formalism in various Mais-Ripken papers.

The 8x8 matrices can be used to carry out spin-orbit tracking within a Monte-Carlo setting: SLICKTRACK

By extension to 9x9 matrices, the linearisation of the spin motion is avoided and extra depolarising effects exposed: See later.

The structure of SLICKTRACK



The Monte-Carlo facilities in SLICKTRACK are perfect for looking at the ILC damping rings.

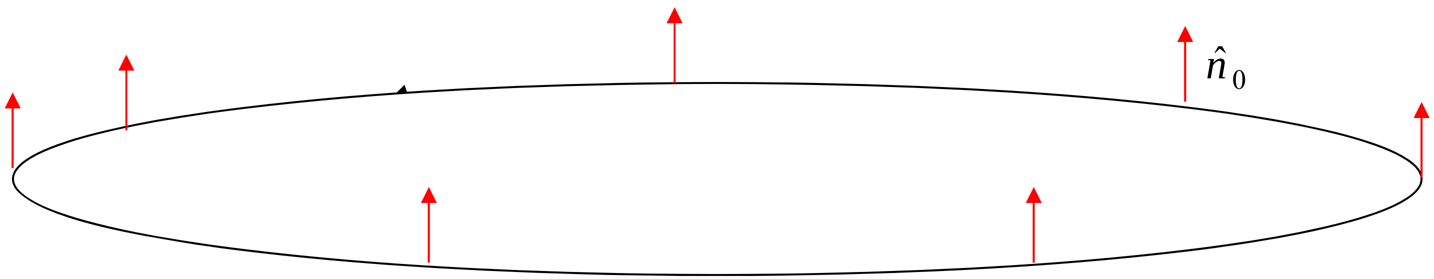
If $\frac{11}{18}\vec{d}^{\,2}$ is close to 1, we are in trouble. Why should it be large and how can it be made small?

⇨ The formalism provides the polarisation and the solution, namely analyse

$$\begin{bmatrix} d_\alpha \\ d_\beta \end{bmatrix}(s) = -2\ \mathrm{Im} \sum_{k=I,II,III} \upsilon^*_{k5}(s) \begin{bmatrix} w_{k\alpha} \\ w_{k\beta} \end{bmatrix}$$

at positions where $\frac{1}{|\rho^3|}$ is large, i.e. at places where the orbit excitation could be strong.

An example: consider a perfectly aligned ring with no solenoids, no skew quads., no vertical bends in the $(\hat{m}_0, \hat{l}_0)$ frame (i.e. set : $\psi_0 = 0$)



$$G_{\text{quad}} = \left[\begin{array}{cc|cc|cc} (1+a\gamma)\hat{C}_x l_{0y} & (1+a\gamma)(\hat{S}_x - 1) l_{0y} & -(1+a\gamma)\hat{C}_y l_{0x} & -(1+a\gamma)(\hat{S}_y - 1) l_{0x} & 0 & 0 \\ -(1+a\gamma)\hat{C}_x m_{0y} & -(1+a\gamma)(\hat{S}_x - 1) m_{0y} & (1+a\gamma)\hat{C}_y m_{0x} & (1+a\gamma)(\hat{S}_y - 1) m_{0x} & 0 & 0 \end{array}\right]$$

So no transverse coupling and no vertical dispersion and $\hat{n}_0$ is vertical so that $\hat{m}$ and $\hat{l}$ are horizontal.

$$M_{8\times 8}(s+C,s) \text{ has the structure } \left[\begin{array}{cc|cc|cc|cc}
* & * & & & * & * & & \\
* & * & & & * & * & & \\
\hline
& & * & * & & & & \\
& & * & * & & & & \\
\hline
* & * & & & * & * & & \\
* & * & & & * & * & & \\
\hline
& & * & * & & & D_{11} & D_{12} \\
& & * & * & & & D_{21} & D_{22}
\end{array}\right]$$

Barber et al.,
Part. Acc 1985.

The orbit eigenvectors have the structure:

$$\begin{array}{ccc}
\begin{bmatrix} * \\ * \\ 0 \\ 0 \\ * \\ * \end{bmatrix} &
\begin{bmatrix} 0 \\ 0 \\ * \\ * \\ 0 \\ 0 \end{bmatrix} &
\begin{bmatrix} * \\ * \\ 0 \\ 0 \\ * \\ * \end{bmatrix} \\
\mathrm{I}\ (x) & \mathrm{II}\ (y) & \mathrm{III}\ (s)
\end{array}$$

$$\begin{array}{l} w_x(s) \propto G(s+C,s)\ \upsilon_x(s) \\ w_s(s) \propto G(s+C,s)\ \upsilon_s(s) \end{array} \quad \text{vanish ;}$$

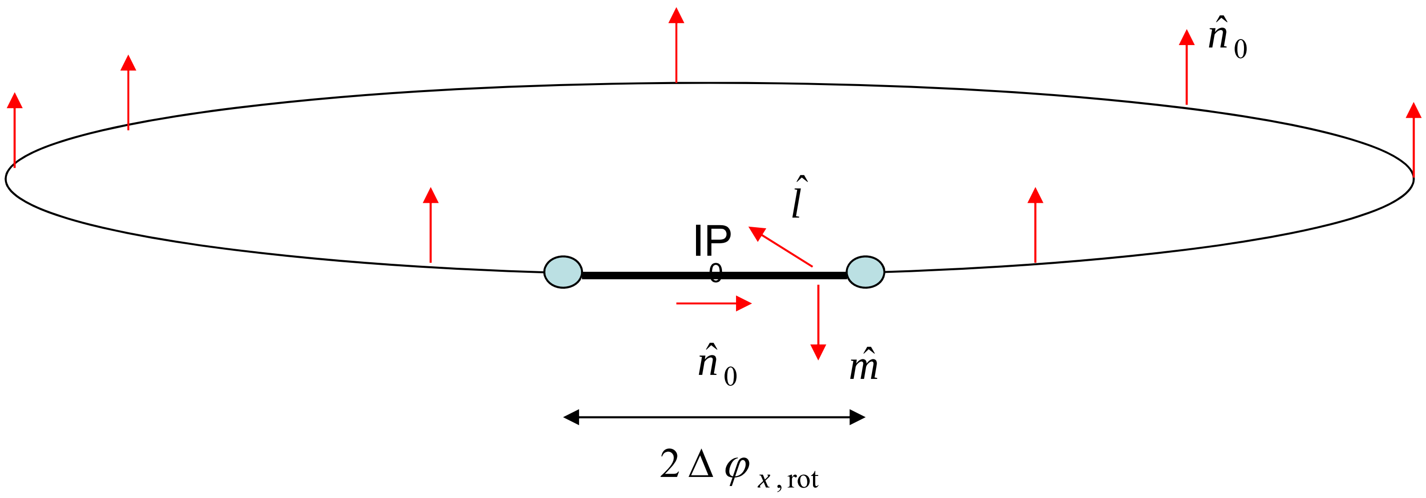
$$w_y(s) \propto G(s+C,s)\ \upsilon_y(s) \neq 0 \ \text{ but } \ \upsilon_{y5}^* w_y = 0$$

No depolarisation!   Obvious! because there is no vertical beam size (excitation) and vertical spins oscillating sideways only see vertical fields in the quads and are therefore not disturbed.

So the formalism gives back the expected answer and we could have got there without it.

Now switch on idealised ``point like” rotators framing a straight section. These rotators are supposed to rotate the spin from vertical to longitudinal without disturbing trajectories.

$\Rightarrow\ \hat{n}_0$ is longitudinal and at least one of $\hat{m}_0$ and $\hat{l}_0$ has a vertical component.



$$G_{\mathrm{quad}} = \left[\begin{array}{cc|cc|cc} (1+a\gamma)\hat{C}_x l_{0y} & (1+a\gamma)(\hat{S}_x - 1) l_{0y} & -(1+a\gamma)\hat{C}_y l_{0x} & -(1+a\gamma)(\hat{S}_y - 1) l_{0x} & 0 & 0 \\ -(1+a\gamma)\hat{C}_x m_{0y} & -(1+a\gamma)(\hat{S}_x - 1) m_{0y} & (1+a\gamma)\hat{C}_y m_{0x} & (1+a\gamma)(\hat{S}_y - 1) m_{0x} & 0 & 0 \end{array}\right]$$

(Again $\psi_0 = 0$)

$\Rightarrow G_{\substack{\text{between}\\ \text{rotators}}} \neq 0 \Rightarrow G(s+C, s) \neq 0$ seen from arc --- where the $\dfrac{1}{|\rho^3|}$ is large

potential big problems..

The solution: --- bring the first two columns of $G_{\substack{\text{between}\\ \text{rotators}}} \Rightarrow 0$ t o zero by adjusting the horizontal optic between the rotators.

The physical picture: $\Delta \vartheta_{\substack{\text{spin}\\ \text{hor}}} = (1 + a\gamma)\Delta \vartheta_{\substack{\text{orb}\\ \text{hor}}} = (1 + a\gamma)\Delta x'$ driven by the vertical quad fields.

So make $\Delta x'=0$ for all incoming (x, x') to ensure that there is no net spin precession across the straight section. Then the noise put into the horizontal betatron motion in the arc does not feed into the spin in the straight section.

If the layout is symmetric w.r.t. the IP with $\alpha_x = -\frac{\beta'_x}{2} = 0$ at the IP, the solution is

$$\tan \Delta \varphi_{x,\text{rot}} = -\alpha_{x,\text{rot}}$$

Then $G_{\substack{\text{between}\\ \text{rotators}}} = 0 \Rightarrow G(s + C, s) = 0$ seen from arc --- where the $1/\left|\rho^3\right|$ is large.

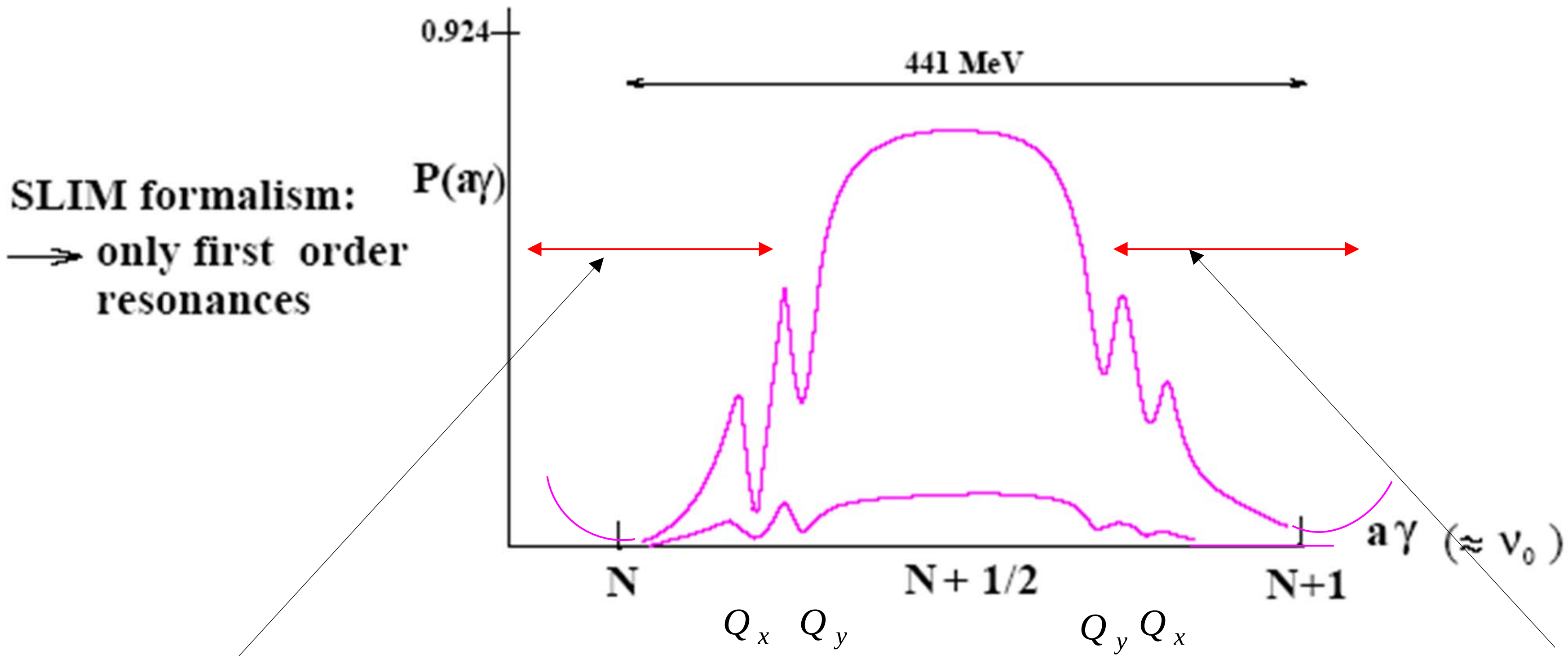
The ring is then said to be spin matched for horizontal motion and the straight section is said to be spin transparent for horizontal motion:

strong synchrobetatron spin matching (SSBSM)

Qualitative expectations with misalignments and/or a poor spin match.

Set the energy so that $\nu_0 \approx \frac{1}{2}$ to stay away from 1st order resonances.

The Q_s resonances are usually very strong – low Q_s in some integrals.



Broad (strong) overlapping
$\nu_0 = N \pm Q_s$

Broad (strong) overlapping
$\nu_0 = N + 1 \pm Q_s$

This suffices for this simple hypothetical layout.

Note that the solution is obvious and most of the maths could have been bypassed.

N.B., the spin match was attained by imposing a purely optical condition.

In general all three modes must be handled and the spin match conditions for the vertical mode are usually energy dependent and not purely optical.

The steps: get the latest fashionable optic from the friendly optics expert and then use special software (e.g., the code SPINOR) to impose as much spin matching as possible while not wiping out his/her favourite features.

The feasibility of decent spin matches can already be investigated while the design is on the table and suggestions fed back, e.g., HERA in 1984.

The formalism is well suited for computer algebra (REDUCE, MAPLE MATHEMATICA etc..)
Or just use the brute force fitting algorithms in SPINOR (say). I never managed to persuade the MAD experts to install spin matching.

Note: $\vec{d}^{\,2} \Rightarrow \dfrac{\tau^{-1}{}_{\mathrm{dep}}}{\tau^{-1}{}_{\mathrm{BK}}}$ scales at least like γ^2 .So life gets difficult at very high energy.

SPINDERELLA AND THE UGLY SISTERS
ENERGIA AND LUMINOSA

SPIN IS IN

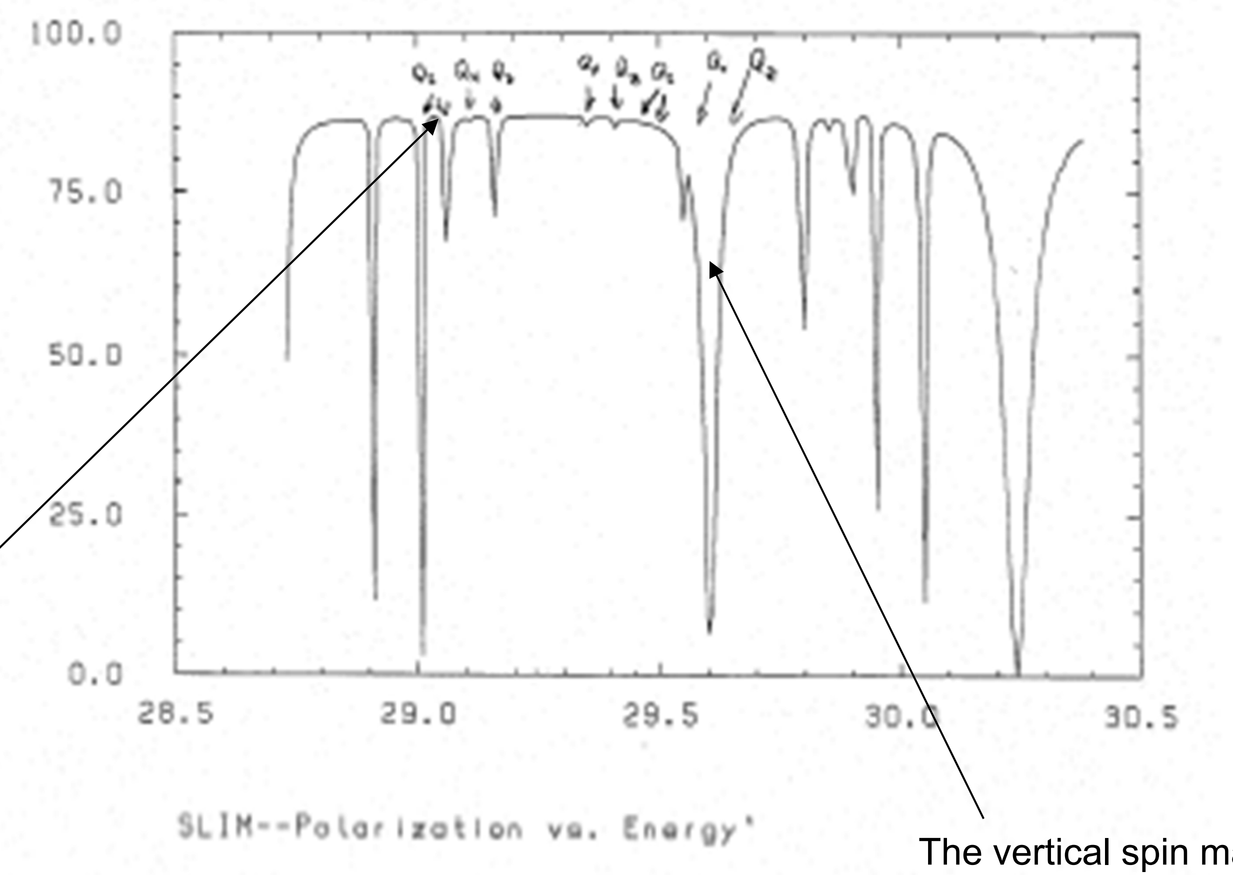
B. MONTAGUE
1980

HERA: SLIM polarisation:
3 rotator pairs, spin matched at 29.23 GeV, no distortions

First order resonances:

$\nu_0 \approx$ integer $\pm Q_k$

$\nu_0 \neq a\gamma$ shifted by rotators

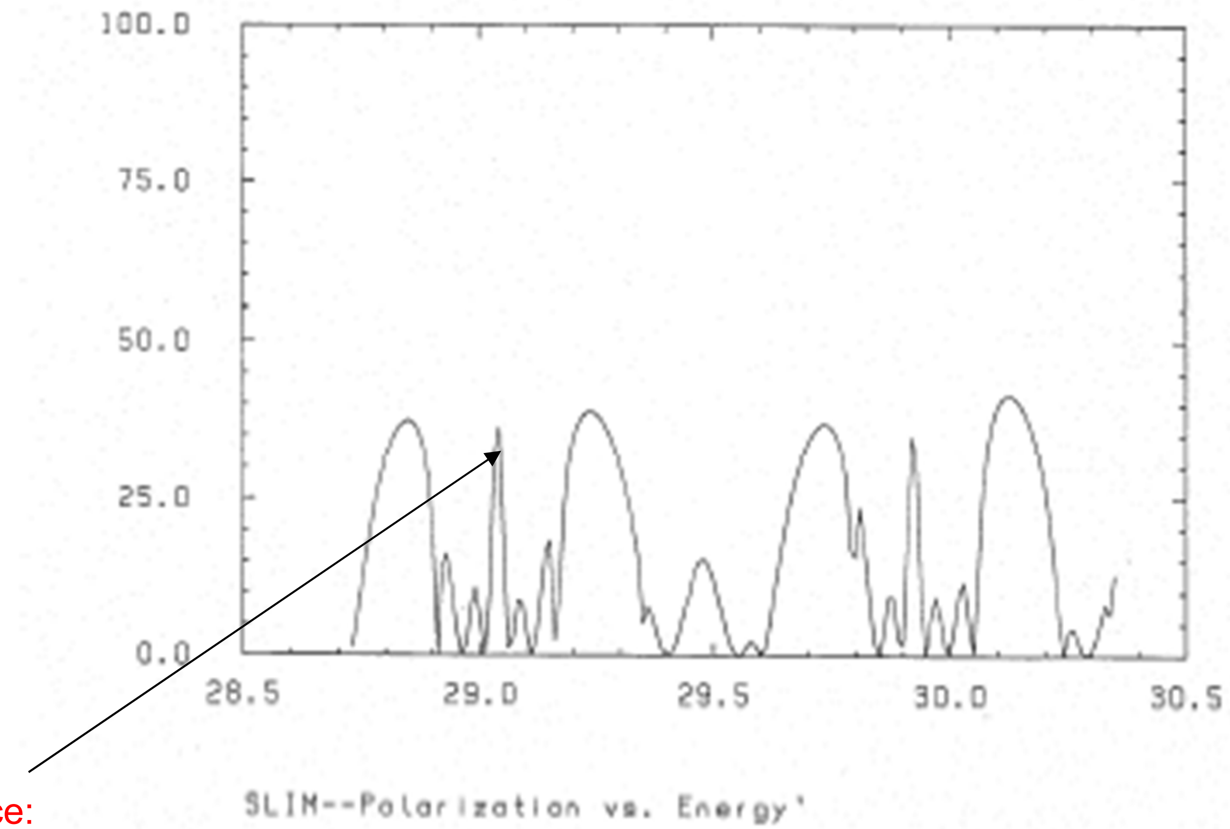


No integer resonances at 1st order. :

No “imperfection resonances”: reserved for protons

The vertical spin match is energy dependent.

HERA: SLIM polarisation:
3 rotator pairs, spin match broken, no distortions



Still no integer resonance:
Know the theory!!!!!
Electrons are not protons!

Spin matching conditions for HERAII. From M. Berglund Thesis 2000.

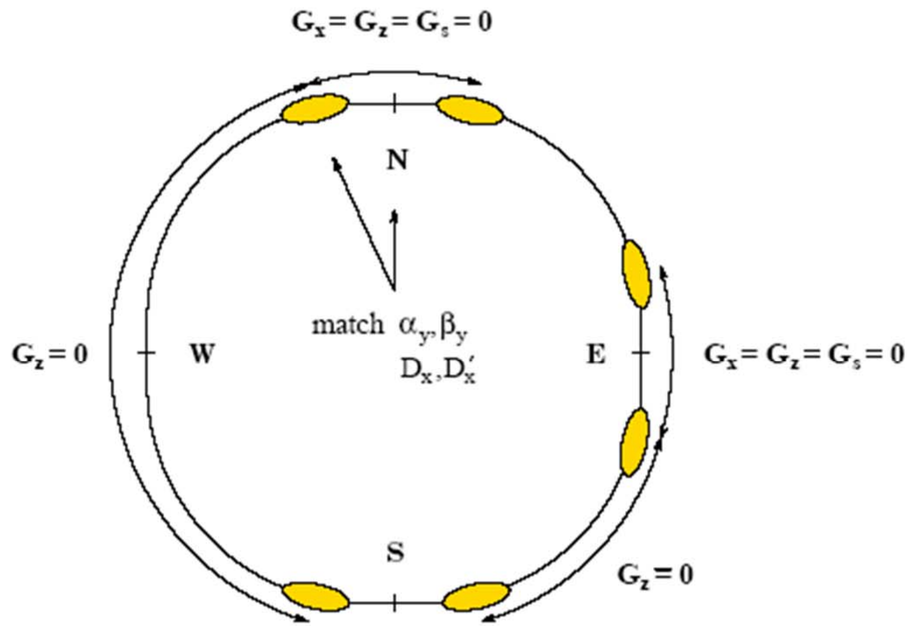
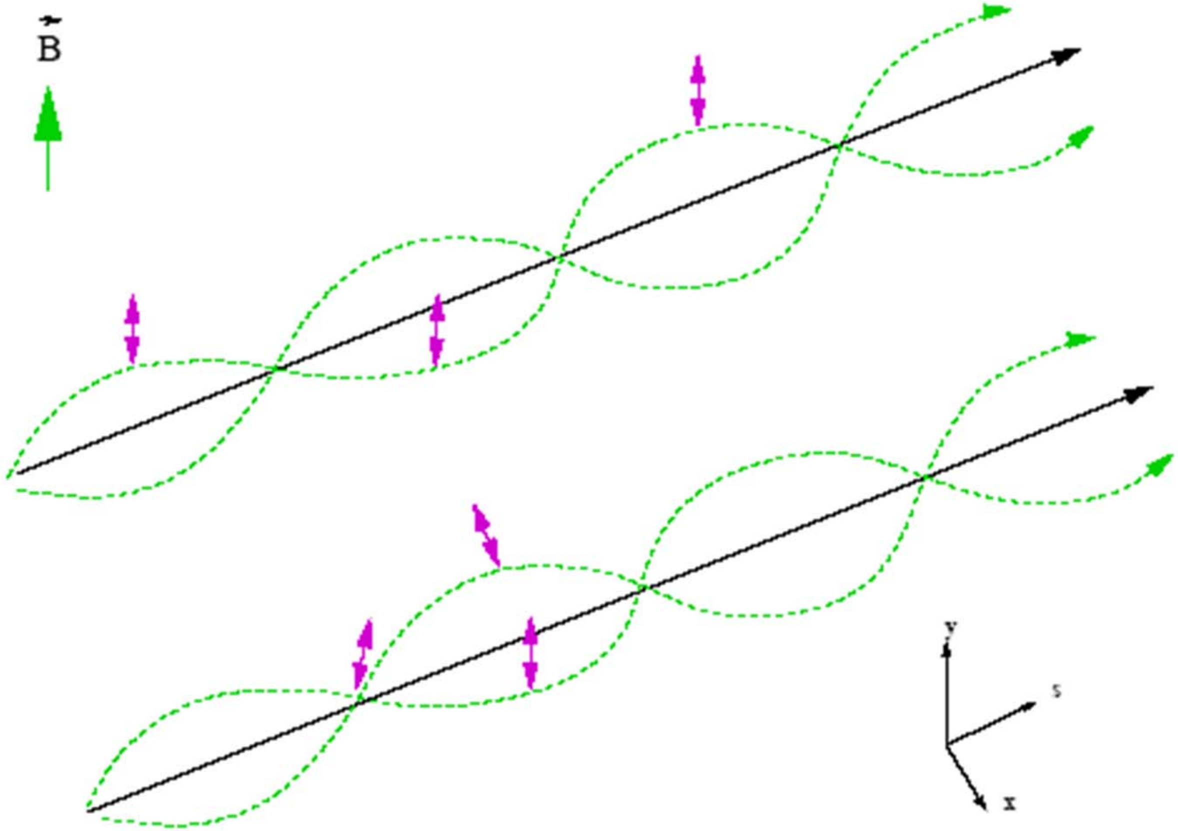


Figure 6.7: Spin matching conditions in the HERA upgrade lattice. The shaded ellipses represent the rotators.

The Courant-Snyder parameters must look good too.

Note: HERA rotators limit the S-T (B-K) polarisation. They contain no quads and are therefore essentially transparent. Therefore they don't depolarise! But they do put the spins into a vulnerable state where there is a potential for strong depolarisation.

EXAMPLE: Depolarization in arcs if $\vec{n}_0$ (s) not vertical



Misalignments are a problem too, leading to a tilted $\hat{n}_0$ in the whole ring

Then even if the strong synchrobetatron spin matching was feasible, the tilt of

$\hat{n}_0$ generates a nonzero $G(s + C, s)$ at all arc dipoles.

– very bad news because there is no way to apply strong synchrobetatron spin matching.

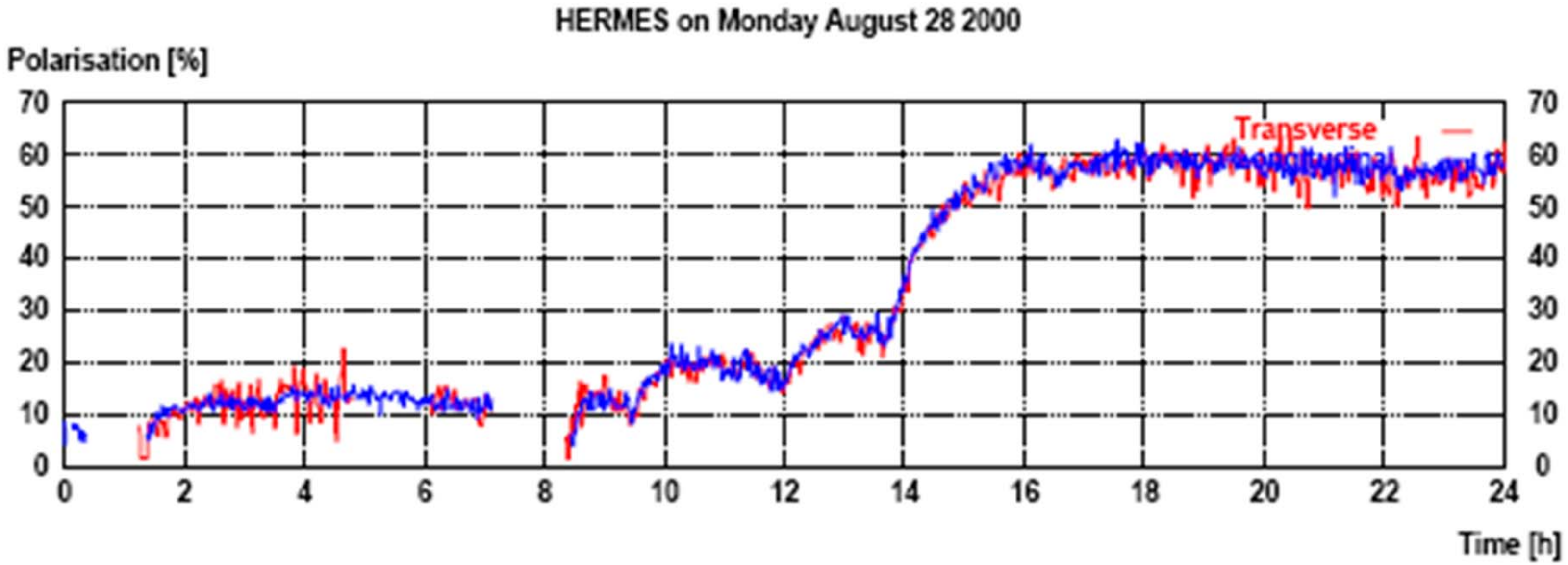
So use the polarimeter to apply an empirical closed orbit correction (based on some first order perturbation theory for the tilt of $\hat{n}_0$ and some Fourier analysis) designed to reduce the tilt.: harmonic closed orbit spin matching.

At HERA, an r.m.s. vertical closed orbit distortion of 1mm, can easily lead to an r.m.s. tilt of 30 mrad in the arcs. That is enough to pull the polarisation down to 30 %.

The HCOSM can the reduce the tilt to around 15 mrad and in HERA I, polarisations of even 70% were attained.

HCOSM is a standard faciltiy in the control system.

Tuning the harmonic bumps with the 72 degree optic.



This can last days!

M. Berglund Thesis 2000. The changes to the layout for HERAII made problems for SSBSM

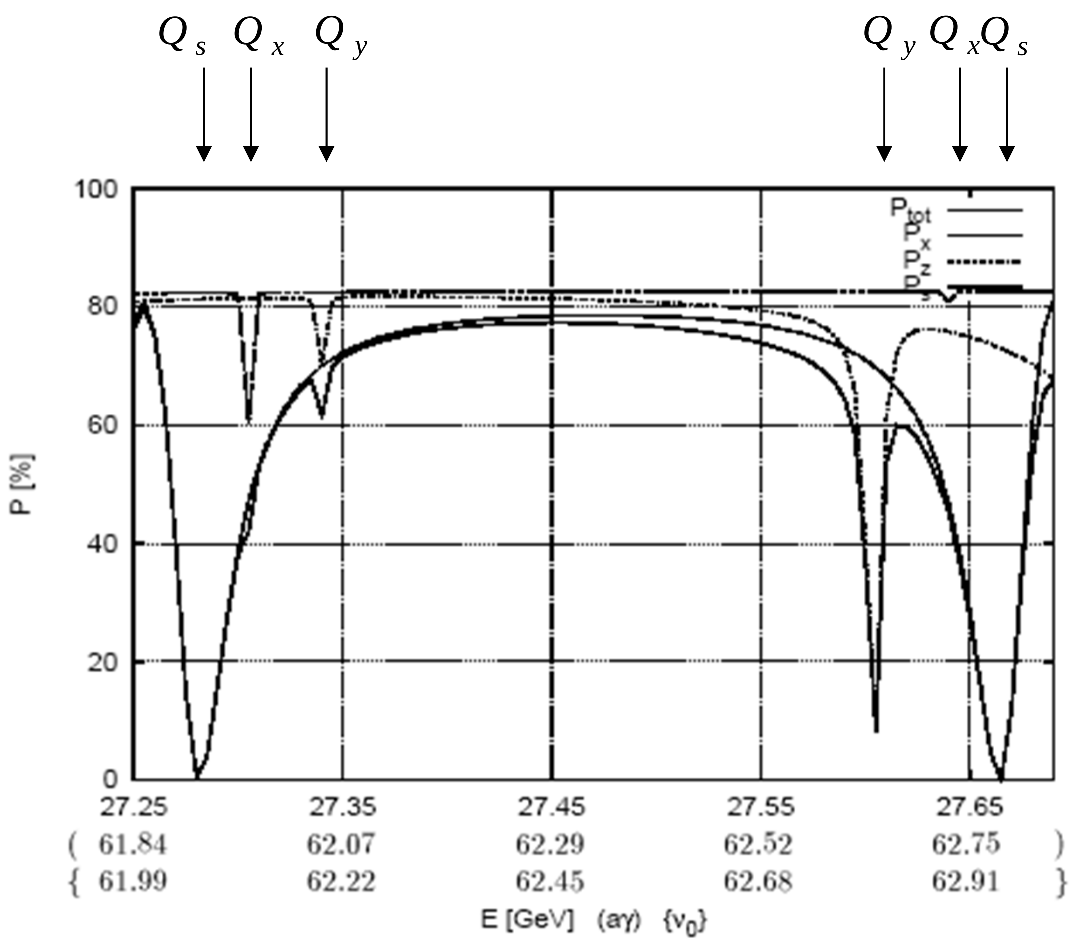


Figure 6.8: Polarization vs. energy ($a\gamma$) {spin tune} for the Rev3.2 e^- nominal lattice with 3 pairs of rotators using standard settings. No misalignments

Note the availability of curves for each mode → DIAGNOSTICS!

Even with the very best HCOSM the polarisation is down to at most 60% from a max. of 83%.

ν_0 !!!!

Tuning/optimising polarisation

(1) Set $\cancel{a\gamma}$ close to 1/2.

(2) Then choose betatron tunes intelligently, i.e. away from 1/2 but not too close to integers.

(3) Spin match the optic ("strong synchrobeta spin matching"), maintaining constraints on tunes, beam at polarimeters, beam at IP etc etc etc: use SPINOR.

(4) Flatten the orbit.

(5) Apply "harmonic closed orbit spin matching" with special vertical closed orbit bumps.

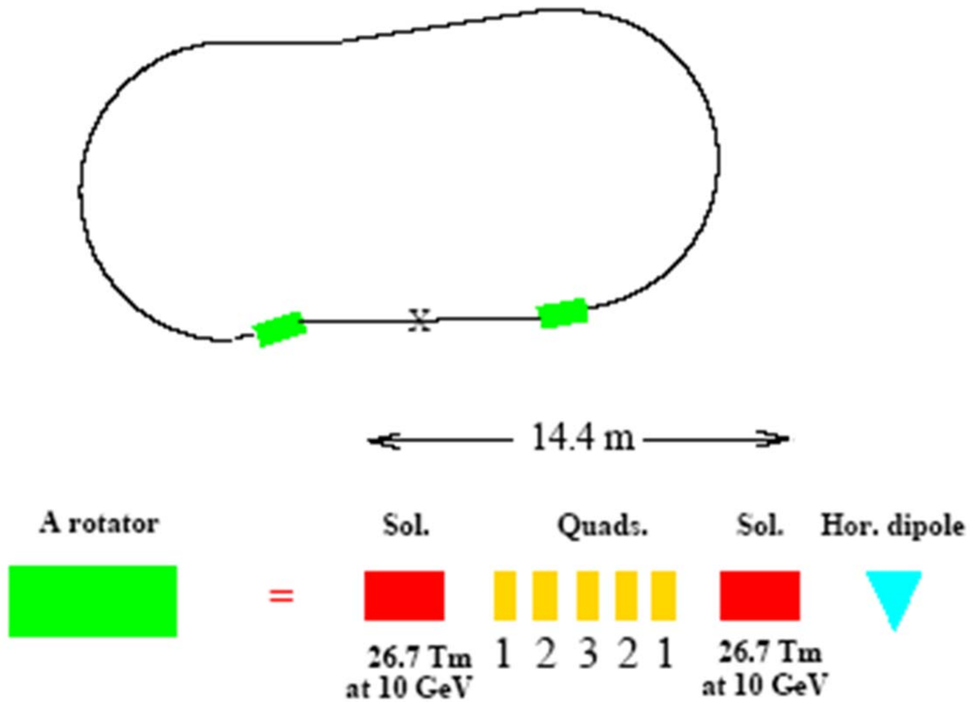
(6) Scan the energy to stay clear of synchrotron sideband resonances: $Q_s = 0.06 \rightarrow \approx 26MeV$.

(7) Play with all tunes: c.f. beam–beam.

(8) Start again at (4).

The HERA spin rotators contain no quads so that they are automatically spin transparent.

The eRHIC rotators use solenoids and the combined rotators and IR's would be spin-opaque without special quad inserts – which also kill the coupling generated .by the solenoids. Poorly designed solenoid rotators can depolarise .



Is that all? No!, for high energy it's only the beginning.

$$\frac{d}{ds}\begin{pmatrix}\gamma_0\\ \alpha_0\\ \beta_0\end{pmatrix} = \begin{pmatrix} 0 & -\hat{l}_0.\vec{\omega} & \hat{m}_0.\vec{\omega} \\ \hat{l}_0.\vec{\omega} & 0 & -\hat{n}_0.\vec{\omega} \\ -\hat{m}_0.\vec{\omega} & \hat{n}_0.\vec{\omega} & 0 \end{pmatrix}\begin{pmatrix}\gamma_0\\ \alpha_0\\ \beta_0\end{pmatrix}$$

Next level of approximation: $\gamma_0 \approx 1$ and for simplicity work with $\alpha_0, \beta_0, \hat{m}_0, \hat{l}_0$

$$\frac{d}{ds}(\beta_0 - i\alpha_0) = -\vec{\omega}.(\hat{m}_0 + i\hat{l}_0) + i\,\hat{n}_0.\vec{\omega}\,(\beta_0 - i\alpha_0)$$

Products of small terms.

$$\frac{d}{ds}\left((\beta_0 - i\alpha_0)e^{-i\int(\hat{n}_0.\vec{\omega})ds''}\right) = -\vec{\omega}.(\hat{m}_0 + i\hat{l}_0)e^{-i\int(\hat{n}_0.\vec{\omega})ds'}$$

$$\Rightarrow (\beta_0 - i\alpha_0)(s) = -e^{+i\int_{s_0}^{s}(\hat{n}_0.\vec{\omega})ds''}\int_{s_0}^{s}\vec{\omega}.(\hat{m}_0 + i\hat{l}_0)e^{-i\int_{s_0}^{s}(\hat{n}_0.\vec{\omega})ds'}ds$$

There is no convenient way to embed this in a stochastic calculus and extract results
In a universal way. $\Rightarrow$ Start again with a much more powerful approach!

But first an interlude on resonances:

$$(\beta_0 - i\alpha_0)(s) = -e^{+i\int_{s_0}^{s}(\hat{n}_0.\vec{\omega})ds''}\int_{s_0}^{s}\vec{\omega}.(\hat{m}_0 + i\hat{l}_0)e^{-i\int_{s_0}^{s}(\hat{n}_0.\vec{\omega})ds'}ds$$

Consider vertical $\hat{n}_0$ and a flat uncoupled ring with no vertical dispersion.
Switch to betatron-dispersion coordinates for convenience here:

In a quadrupole: $\vec{\omega}_{\beta_x} \propto \hat{y}\ k(a\gamma + 1)\sqrt{\beta_x}\cos(\varphi_x)$

$$\vec{\omega}_{\beta_y} \propto -\hat{x}\ k(a\gamma + 1)\sqrt{\beta_y}\cos(\varphi_y)$$

$$\vec{\omega}_{\delta} \propto \hat{y}\ k(a\gamma + 1)D_x\delta_{\max}\cos(\varphi_s)$$

$\Rightarrow \vec{\omega}_{\beta_x}.\hat{n}_0 \neq 0$ but oscillating quickly

$\vec{\omega}_{\delta}.\hat{n}_0 \neq 0$ but oscillating very slowly

$\vec{\omega}_{\beta_y}.\hat{n}_0 = 0$

Keep only the $\vec{\omega}_{\delta}.\hat{n}_0$:corresponds to a modulation of the rate of precession around $\hat{n}_0$

This gives $e^{+i\int_{s_0}^{s}(\hat{n}_0.\vec{\omega})ds} = e^{+iA\sin(2\pi Q_s s/C + \varphi_0)} = \sum_{m=-\infty}^{m=+\infty} B_m(A)e^{i2\pi m Q_s s/C}$

Bessel function gymnastics

Write $\vec{\omega}_{\beta_y} \propto -\hat{x}\, k(a\gamma+1)\sqrt{\beta_y}\,\frac{1}{2}(e^{i\varphi_y}+e^{-i\varphi_y})$

Then we can write:

$$\vec{\omega}_{\beta_y}(s).\{\hat{m}_0(s)+i\hat{l}_0(s)\} = \sum_j C_j^+ e^{-i2\pi(j-\nu_0+Q_y)s/C} + \sum_j C_j^- e^{-i2\pi(j-\nu_0-Q_y)s/C}$$

On a resonance $j-\nu_0 \pm Q_y = 0$ the corresponding term does not oscillate and then

$\int_{s_0}^{s} \vec{\omega}_{\beta_y}.(\hat{m}_0+i\hat{l}_0)ds$ varies almost linearly with s so that $|\alpha_0|$ and $|\beta_0|$ grow indefinitely.

$\Rightarrow$ a 1st order vertical betatron RESONANCE.

$\int_{s_0}^{s} \vec{\omega}_{\beta_y}.(\hat{m}_0+i\hat{l}_0)e^{-i\int_{s_0}^{s}(\hat{n}_0.\vec{\omega})ds'} ds$ gives resonances at $j-\nu_0 \pm Q_y \pm mQ_s = 0$

Synchrotron sidebands of 1st order parent vertical betatron resonance:
a standard expectation for frequency modulation.

VERY nasty at high energy

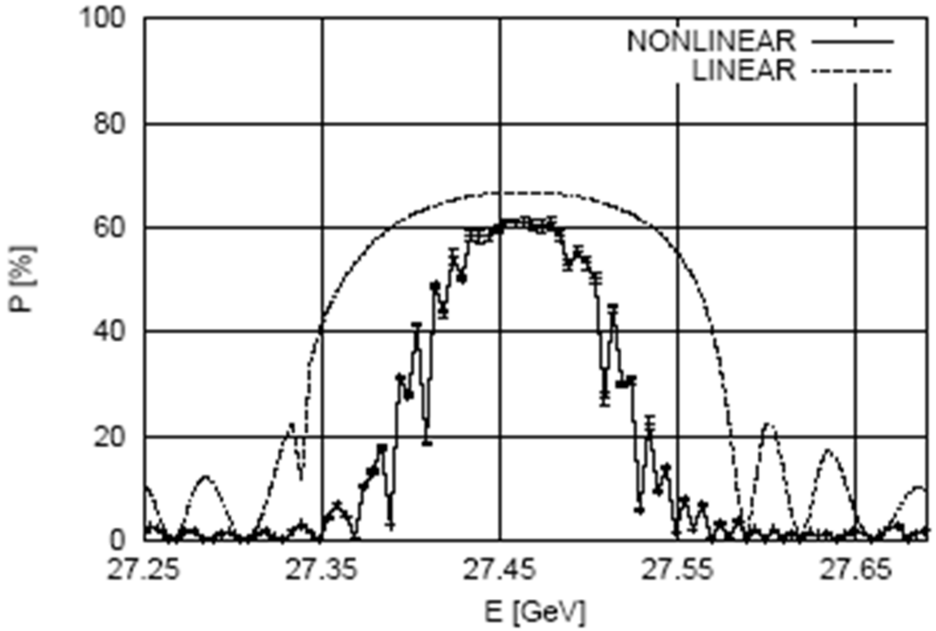
Our first example of a higher order resonance: the result of including some non-commuting 3-D spin motion.

The 3-D nature of spin motion results in high order resonances

$$\nu_0 = m_0 + m_I Q_I + m_{II} Q_{II} + m_{III} Q_{III}$$

even with linear orbital motion.!:

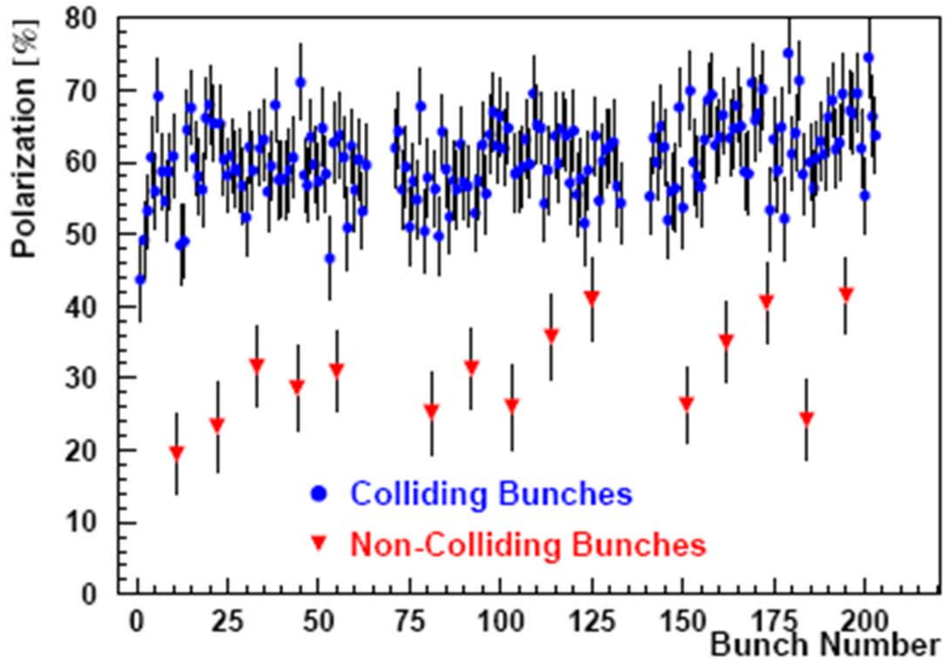
Nasty, nasty sidebands.



Polarization vs. energy for a HERA Upgrade lattice including the H1 and ZEUS solenoids: comparison of first order calculation (SLIM) and higher order calculations (SITROS).

M. Berglund, DESY–THESIS 2001–044 (2001).

Effect of beam–beam forces on polarization



Courtesy of the HERMES collaboration: Note that the polarization scale is about 10 percent too high in this figure.

Topic 5

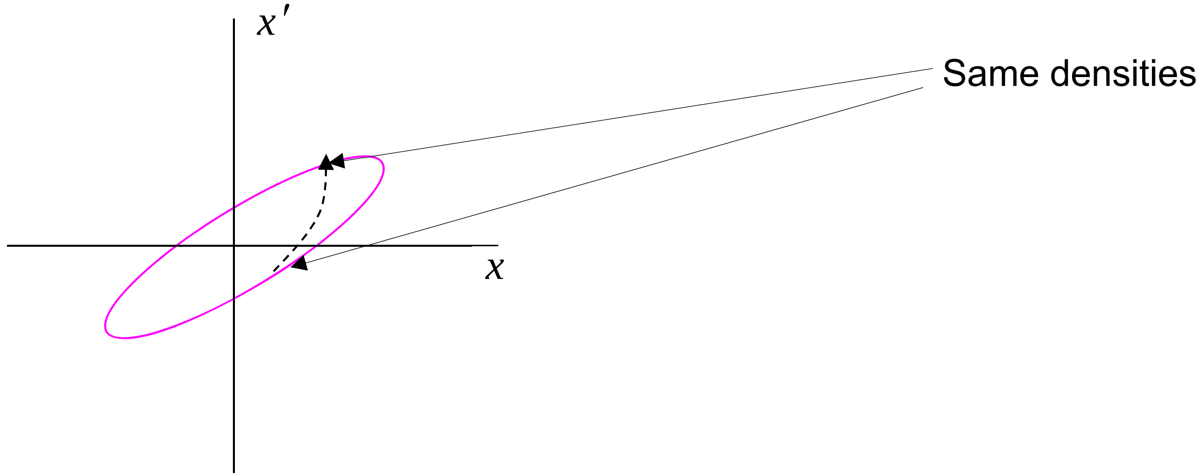
Depolarisation: the invariant spin field.

Now do it properly: the invariant spin field (ISF).

The 1-turn periodicity of the layout leads to the possibility of 1-turn periodic (``equilibrium'') phase space densities for bunches: $\rho(u;s+C)=\rho(u;s)$

During 1 turn the particles exchange places in phase space but in a way which preserve the density along each orbit (Liouville — Bernhard Holzer's course). E.g. in one phase space plane, with motion on a phase space ellipse, equilibrium corresponds to a uniform distribution around the ellipse.

(In action-angle variables, the density commutes with the Hamiltonian (Poisson bracket = 0))



In a kind world we can also get equilibrium spin distributions.

The 1-turn periodicity of the layout leads to the possibility of 1-turn periodic (``equilibrium'') local polarisation: $\vec{P}(u;s+C) = \vec{P}(u;s)$

Since the T-BMT eqn. is linear in spin, $\vec{P}$ obeys the T-BMT eqn. along orbits.

The unit vector along this special $\vec{P}(u;s)$ is denoted by $\hat{n}(u;s)$: $\hat{n}(u;s+C) = \hat{n}(u;s)$

$\hat{n}(u;s)$ obeys the T-BMT eqn. along orbits. So $\vec{S}.\hat{n}(u;s)$ is constant along an orbit.

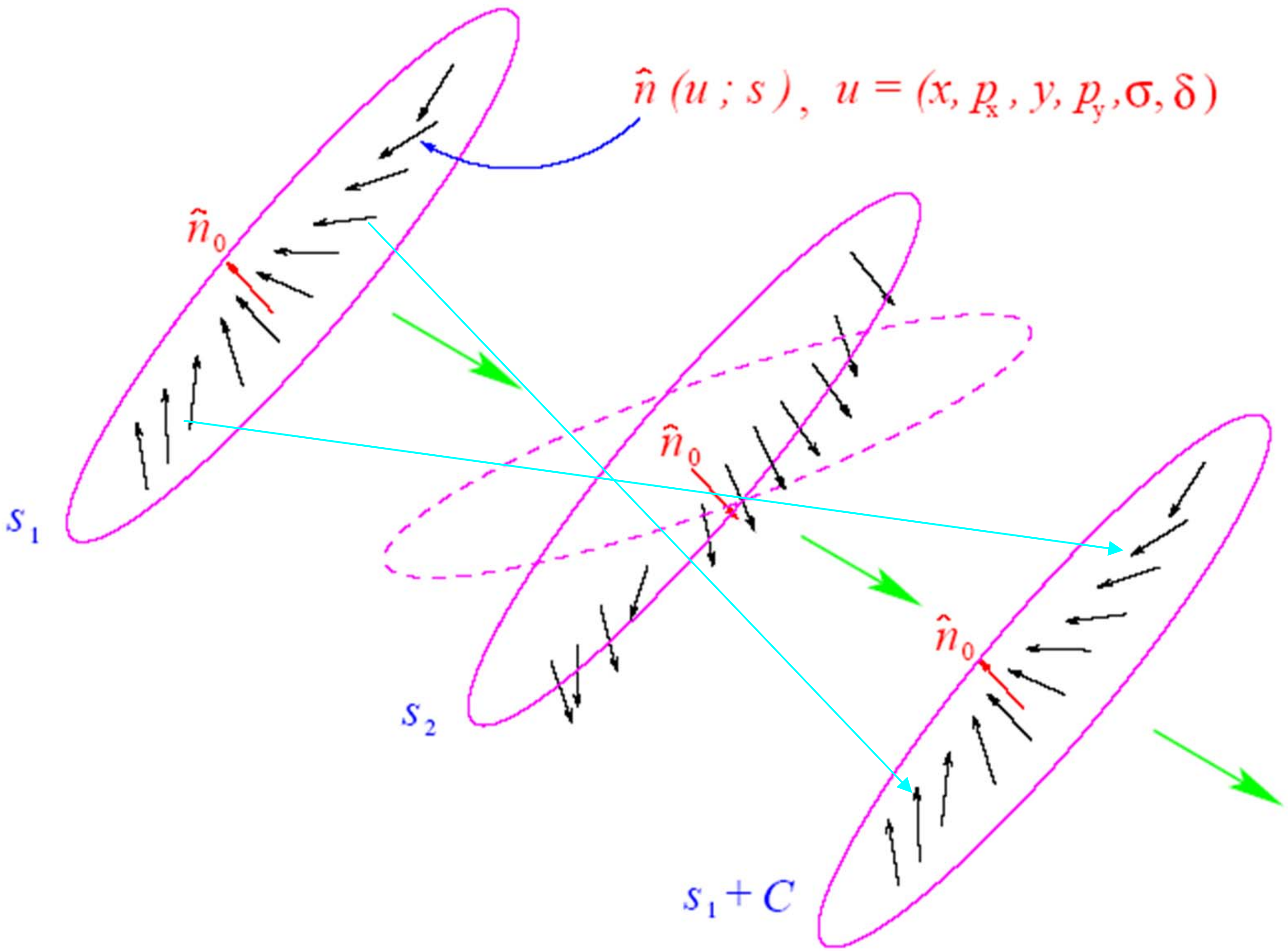
In 1 turn around the ring, not only the particles, but also the spins exchange places.

The vector $\hat{n}(u;s)$ is 3-vector field sitting on the 6-D symplectic phase space with the property that if a spin is set parallel to $\hat{n}(u;s)$, then after one turn, the spin is parallel to the $\hat{n}$ at the new point in phase space.

$$\hat{n}(M(u;s);s+C) = \hat{n}(M(u;s);s) = R(u;s)\hat{n}(u;s)$$

where *M* and *R* are the orbit and spin maps for 1-turn starting at *u* and *s*

For linear motion $M(u;s) = M_{6\times 6}(s+C,s)u$


$\hat{n}(u;s)$, $u = (x, p_x, y, p_y, \sigma, \delta)$
$\hat{n}_0$
$\hat{n}_0$
$\hat{n}_0$
s_1
s_2
$s_1 + C$

$\hat{n}(u;s)$ is a function of *u* and s. It can be calculated (in principle and if it exists!) at each *u* and s without reference to actual particles and stored in a 7-D look-up table.

It is the generalisation of $\hat{n}_0(s)$ to points off the closed orbit. Like $\hat{n}_0(s)$ it's defined without reference to radiation. $\hat{n}_0(s) = \hat{n}(0;s)$!

In contrast to a spin, and like $\hat{n}_0(s)$, it has no history. It just is.

Existence: it is a solution, with strict constraints, of the T-BMT eqn. along orbits.
If we impose smoothness, there are no proofs that it exist in general.

If it exists and is smooth then it is unique, if the system is not on spin-orbit resonance define in terms of the generalised spin tune. (Barber et. al., PRSTAB 2004)

If a spin is set initially parallel to ITS $\hat{n}(u;s)$ and a discrete Fourier transform of it orientation at a reference azimuth *s* is made, it only contains harmonics corresponding to the orbital tunes. No harmonics for the generalised spin tune.

The defining condition $\hat{n}(M(u;s);s+C) = \hat{n}(M(u;s);s) = R(u;s)\,\hat{n}(u;s)$ does not represent an eigen problem: $\hat{n}(u;s)$ doesn't return to its original *u* after 1 turn.

In contrast to $\hat{n}_0(s)$ calculating $\hat{n}(u;s)$ is a major problem.

The beam polarisation is the average over phase space at *s*

$$\langle \vec{P}(u;s)\rangle_u = \int \vec{P}(u;s)\rho(u;s)du \quad \text{with} \quad \int \rho(u;s)du = 1$$

The maximum equilibrium beam polarisation is $\langle \hat{n}(u;s)\rangle_u$ corresponding to $|\vec{P}(u;s)| = 1$

If each spin us set up parallel to ITS $\hat{n}(u;s)$ the beam polarisation is the same from turn to turn: equilibrium!

If each spin is *not* set parallel to ITS $\hat{n}(u;s)$ the beam polarisation oscillates.

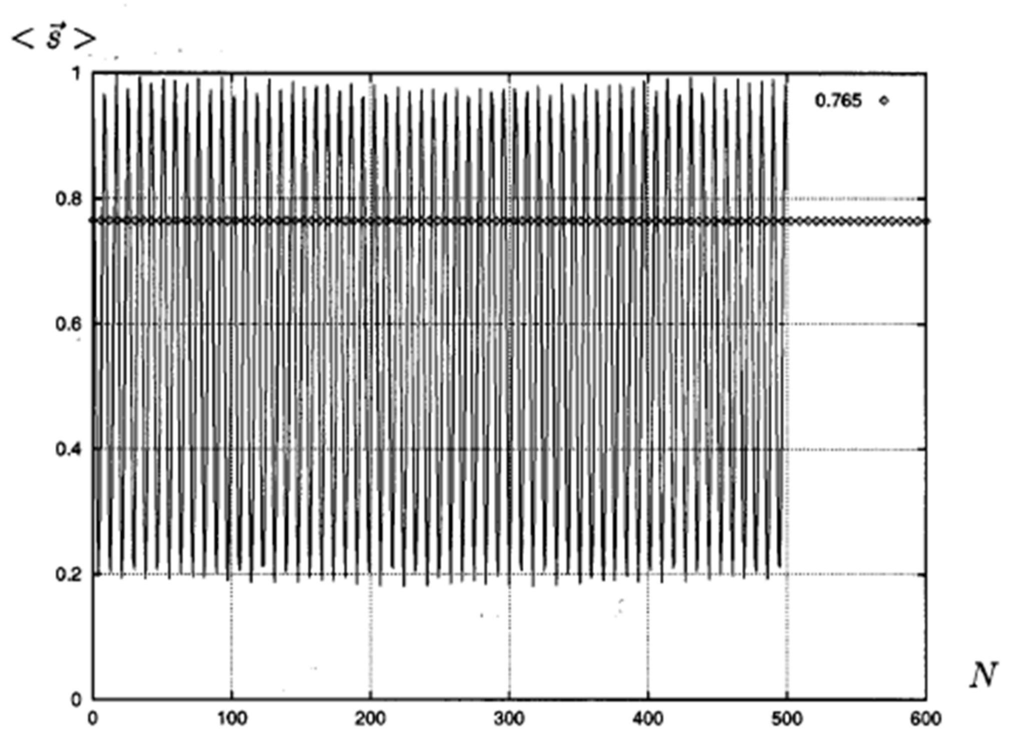
It provides allows a dreibein to be constructed at each *u* and *s* so that a generalised spin tune can be defined: spins precess around $\hat{n}(u;s)$ along orbits $\Rightarrow$ $\hat{n}(u;s)$ not $\hat{n}_0(s)$ is the appropriate precession axis..

The vector $\hat{n}(u;s)$ was introduced in 1972 by Derbenev and Kondratenko with masterly insight as a *u*-dependent quantisation axis for spin.
But most people could not understand their work or the definition of their vector. Their justification also had a weak spot.

The literature is now full of misunderstanding and false calculations, mostly associated with confusing $\hat{n}(u;s)$ with $\hat{n}_0(s)$ and then calculating $\hat{n}(u;s)$ wrongly.

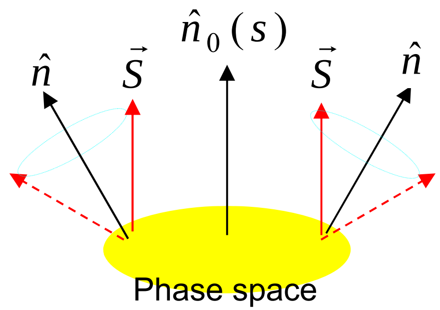
The reason for the subscript `0' on $\hat{n}_0(s)$ should now be obvious. .

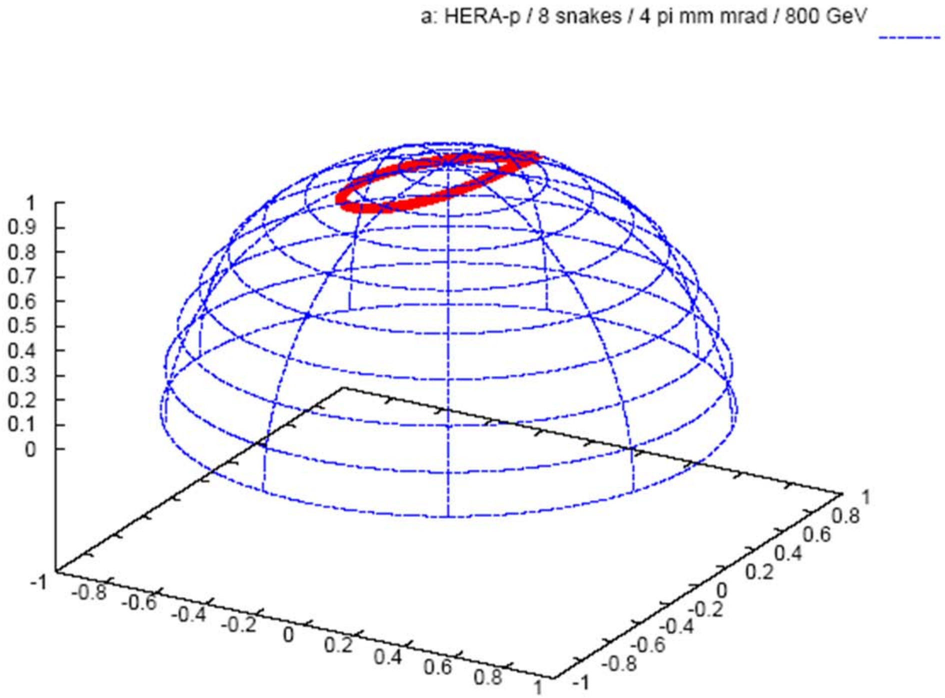
Protons at 820 GeV for HERA: Heinemann and Hoffstaetter PRE 1996.

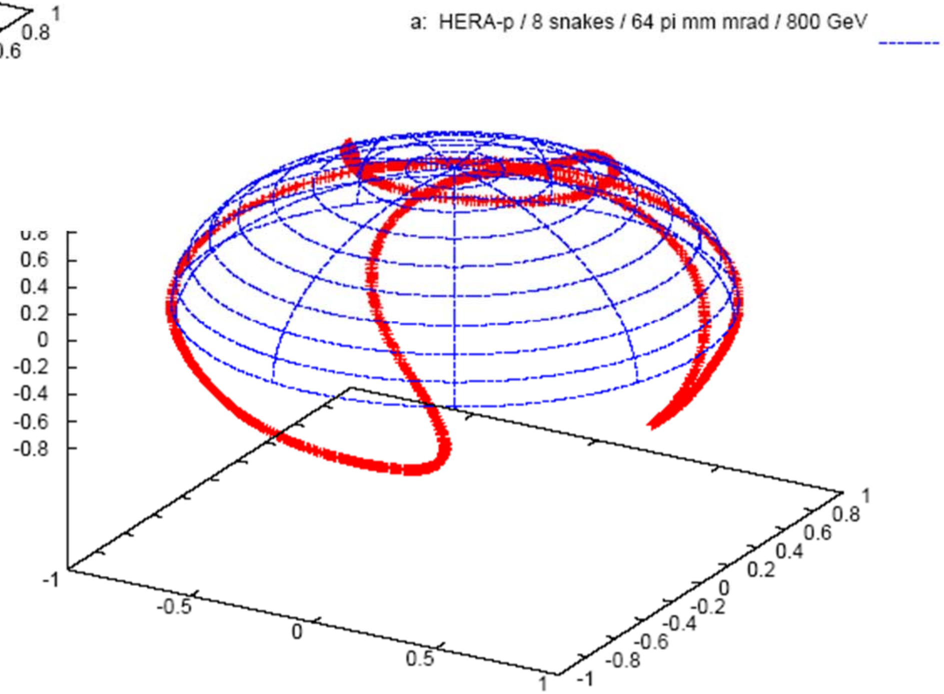


(No damping, no noise)

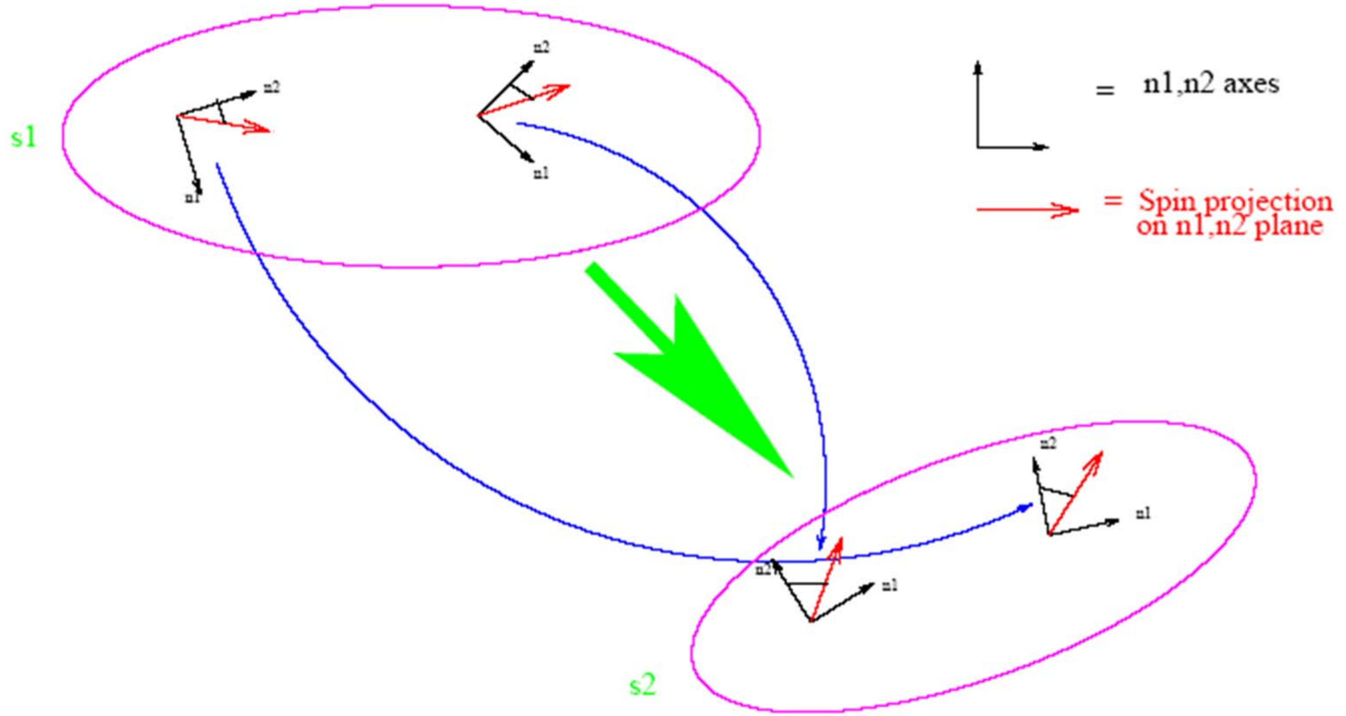
FIG. 9. Propagation of a beam that is initially completely polarized parallel to $\vec{n}_0$ leads to a fluctuating average polarization. For another beam that is initially polarized parallel to the periodic spin solution $\vec{n}$, the average polarization stays constant, in this case equal to 0.765.




a: HERA-p / 8 snakes / 4 pi mm mrad / 800 GeV
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1


a: HERA-p / 8 snakes / 64 pi mm mrad / 800 GeV
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.5
0
0.5
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1

Attaching coordinate axes to each phase space point



Spin precession rate w.r.t. n1, n2 is the same at all phase space points with same J_x , J_y , J_z .

→ Amplitude dependent spin tune! $\nu_{spin}(J)$

The DESY ``crusade'' against mysticism:

Extensive work on the definition and properties of $\hat{n}(u;s)$ avoiding the weak link in the original D-K definition.

We motivate it via the notion of equilibrium polarisation. It is now clear that there is no need to define it via a spin-orbit Hamiltonian (and therefore be forced to include the totally irrelevant Stern-Gerlach forces).

But the difficulties in calculating it remain.

It is now called the invariant spin field. (ISF)

It is the whole field that is invariant.

Back to radiating electrons

Radiative effects represent an addition to the T-BMT precesssion but the time scales for radiative effects are very much longer than the time scale of precession: the T-BMT eqn. dominates the spin motion.

So an equilibrium polarisation distribution $\vec{P}(u;s)$ (sitting in an equilbrium phase space distribution) must lie very close to $\hat{n}(u;s)$.
Also expected while the polarisation is slowly growing.

We assume that $\vec{P}(u;s)$ is in equilibrium in the sense that it is parallel to $\hat{n}(u;s)$ which is, by definition, calculated without radiation.

We want to calculate $|\vec{P}(u;s)|$

Since $\tau_0(s) >> \tau_k$ $|\vec{P}(u;s)|$ must be the same everywhere as the particles mix in phase space.

A heuristic picture:

Consider an electron at (u, s) with spin along $\hat{n}(u,s) \Rightarrow \vec{S}.\hat{n} = 1$

A photon with energy *ΔE* is emitted so that $u_6^f \Rightarrow u_6^i + \dfrac{-\Delta E}{E_0}$

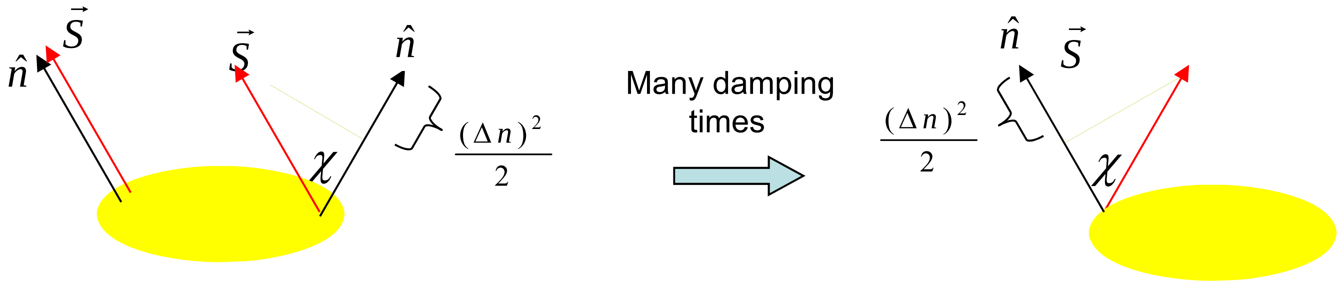
The particle is now at u^f with $\hat{n}(u^f, s) = \hat{n}(u^i, s) - \dfrac{\Delta E}{E_0}\dfrac{\partial n}{\partial \delta} \equiv \hat{n}(u^i, s) - \Delta \hat{n}$

The spin is not affected but its projection on n^f is $\vec{S}.\hat{n}^f = 1 - \dfrac{(\Delta n)^2}{2}$ (exactly)

The particle then damps back to its original position.

$\hat{n}$ can be calculated at all points along the path – not by tracking but *ab initio.,* point by point

$\vec{S}.\hat{n}$ is an adiabatic invariant so that the angle χ is preserved and $\vec{S}$ precesses around members in the sequence of $\hat{n}$'s



The overall decrement of the projection on $\hat{n}$ is $\dfrac{(\Delta n)^2}{2}$

This sequence of events is instantaneous w.r.t. τ_0

The depolarisation rate is $$\tau_{\text{dep}}^{-1} = \frac{c}{C}\frac{55\sqrt{3}}{144} r_e \lambda\!\!\!^{-}_c \gamma^5 \oint ds \left\langle \frac{\left(\frac{\partial \hat{n}}{\partial \delta}\right)^2}{|\rho^3(s)|} \right\rangle_u$$

The Derbenev-Kondratenko formula

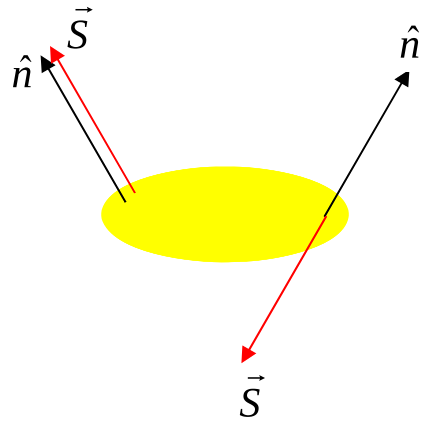
``kinetic polarisation''

$$\Rightarrow P_{\text{DK}}(s=\infty) = -\frac{8}{5\sqrt{3}} \frac{\oint ds \left\langle \frac{\hat{b}.\left(\hat{n} - \frac{\partial \hat{n}}{\partial \delta}\right)}{|\rho^3(s)|} \right\rangle_u}{\oint ds \left\langle \frac{1 - \frac{2}{9}(\hat{n}.\hat{\upsilon})^2 + \frac{11}{18}\left(\frac{\partial \hat{n}}{\partial \delta}\right)^2}{|\rho^3(s)|} \right\rangle_u}$$

The original 1973 derivation involved treating $\vec{S}.\hat{n}$ as the QM analogue of a classical integral of motion (which it isn't if the system is time dependent (cavities) – that's the weak spot) and using QM + radiation to look at the rate of variation of $\langle \vec{S}.\hat{n} \rangle$
This leads to the ``kinetic polarisation'' term too. This results from the dependence of the radiated power on the spin orientation for non-spin-flip processes. It's detection would provide massive support for the formalism.

Mane (PRA 1987) obtained the same result by considering the rates for generalised spin flip

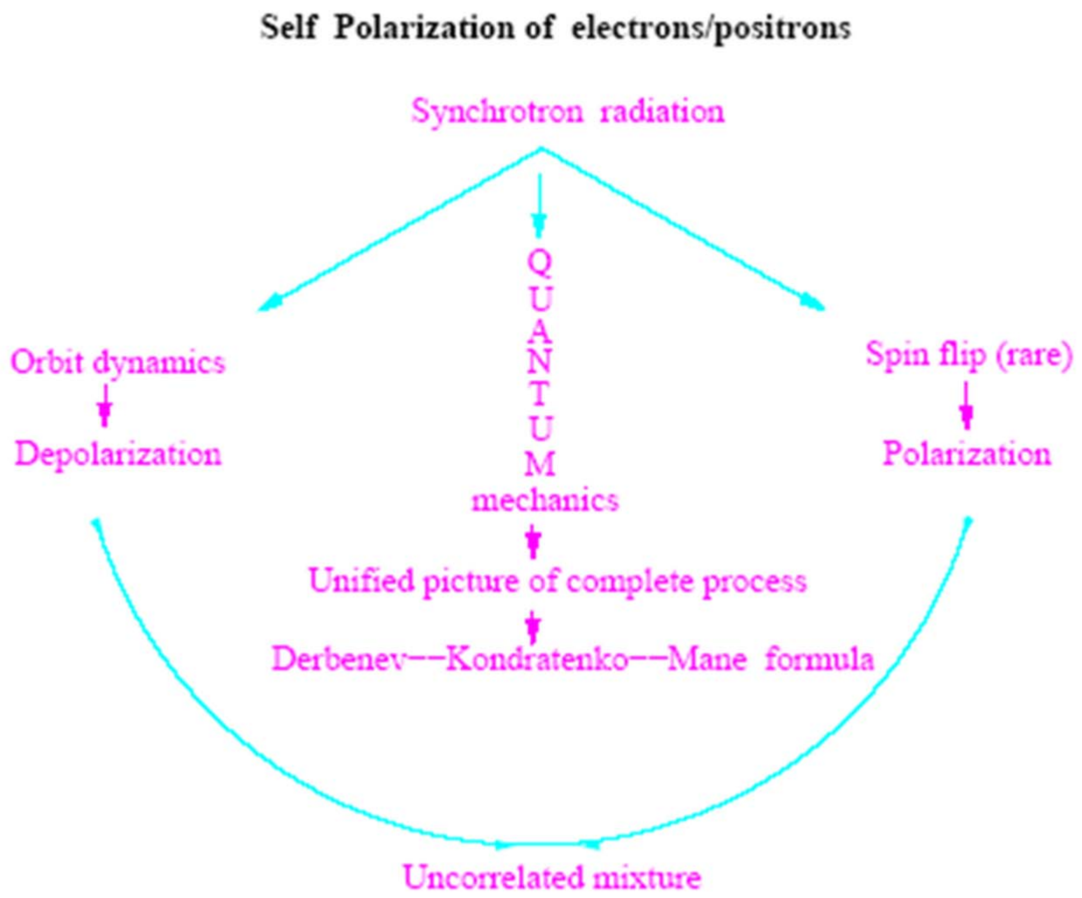
from spin along $\hat{n}(u^i, s)$ to spin along : $-\hat{n}(u^f, s) = -(\hat{n}(u^i, s) - \frac{\Delta E}{E_0}\frac{\partial n}{\partial \delta})$



and requiring that the generalised total up-to-down and down-to-up rates are equal in analogy to the original simple up-down calculation.

Moral: choose the most suitable quantisation axis.

$$\hat{n}(u^i, s) \ \text{instead of} \ \hat{n}_0$$


Self Polarization of electrons/positrons
Synchrotron radiation
QUANTUM
mechanics
Orbit dynamics
Depolarization
Spin flip (rare)
Polarization
Unified picture of complete process
Derbenev--Kondratenko--Mane formula
Uncorrelated mixture

The equilibrium beam polarisation at some position *s* is $P = P_{\text{DK}} \langle \hat{n}(u;s) \rangle_u$

The use of $\hat{n}$ and $\frac{\partial \hat{n}}{\partial \delta}$ allows all the problems associated with the non-linearity of 3-D spin motion to be packed into the geometry of the ISF!

E.g., near resonance $\nu_0 = m_0 + m_I Q_I + m_{II} Q_{II} + m_{III} Q_{III}$ he spin field opens out and is a sensitive function of *u* so that $\frac{\partial \hat{n}}{\partial \delta}$ is large, $\Rightarrow$ the polarisation is low.

For the energies of electron rings, the spread of $\hat{n}$ is small (mrads) : but if

$\sqrt{\langle (\hat{n} - \hat{n}_0)^2 \rangle}$ is about 1mrad and that is all due to the spread in δ with $\sigma_\delta \approx 10^{-3}$

then $\frac{\partial \hat{n}}{\partial \delta} \approx 1$

Since the spread is small: $P = P_{\text{DK}} \langle \hat{n}(u;s) \rangle_u \approx P_{\text{DK}} \; \hat{n}_0(s)$

In a first order calculation co-opting the SLIM formalism for other purposes,

$$\vec{u}(s) = \sum_{k=I,II,III} \{A_k \vec{v}_k(s) + A_{-k} \vec{v}_{-k}(s)\}$$

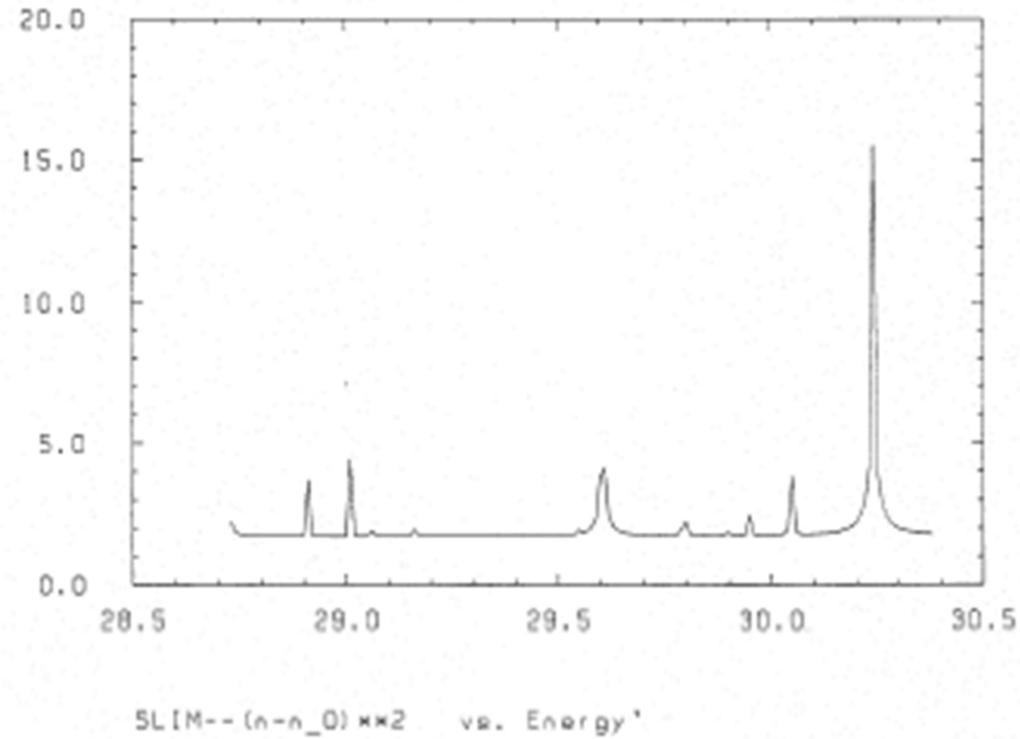
$$\hat{n}(\vec{u}; s) - \hat{n}_0(s) \equiv \begin{pmatrix} \alpha(\vec{u}; s) \\ \beta(\vec{u}; s) \end{pmatrix} = \sum_{k=I,II,III} \{A_k \vec{w}_k(s) + A_{-k} \vec{w}_{-k}(s)\}$$

$$\frac{\partial \hat{n}}{\partial \delta} \equiv i \sum_{k=I,II,III} \{v_{k5}^* \vec{w}_k - v_{k5} \vec{w}_k^*\} = -2 \,\mathrm{Im} \sum_{k=I,II,III} v_{k5}^* \vec{w}_k$$

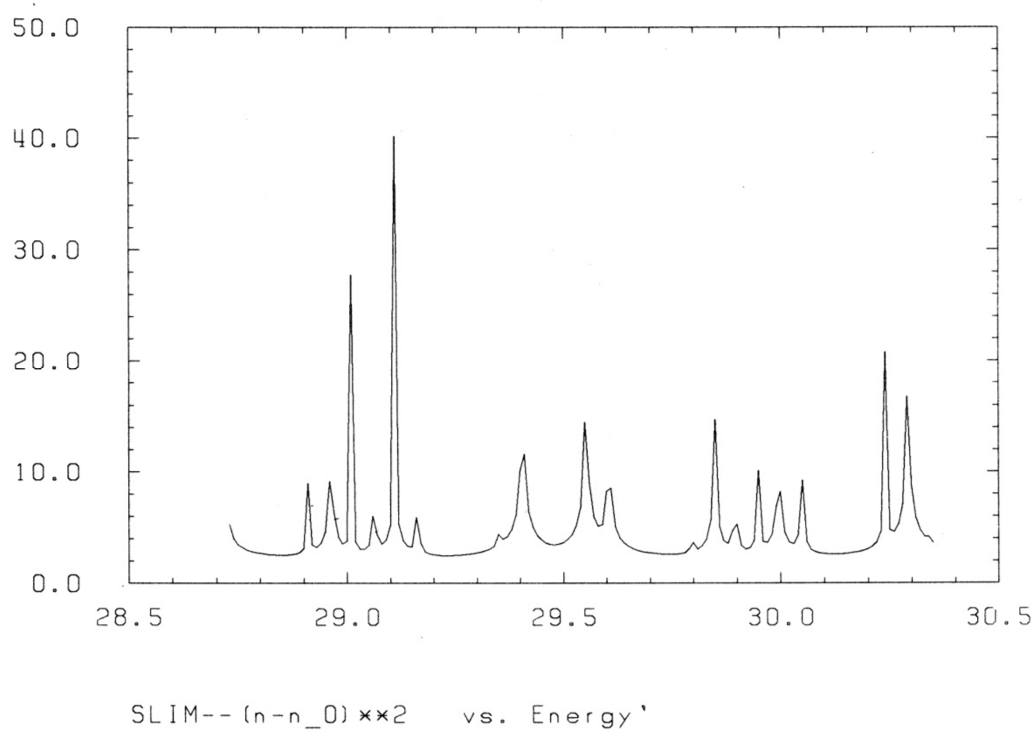
This gives exactly the same answer as the naïve SLIM approach using stochastic D.E.s recall:

$$\begin{bmatrix} d_\alpha \\ d_\beta \end{bmatrix}(s) = -2 \;\mathrm{Im} \sum_{k=I,II,III} \upsilon_{k5}^*(s) \begin{bmatrix} w_{k\alpha} \\ w_{k\beta} \end{bmatrix}$$

HERA: SLIM r.m.s angle between $\vec{n}$ and $\vec{n}_0$,
3 rotator pairs, spin matched at 29.23 GeV, no distortions



HERA: SLIM r.m.s angle between $\hat{n}$ and $\hat{n}_0$
3 rotator pairs, spin match in a straight section broken, no distortions.



The original literature (D-K) and copiers, used the notation $\gamma \frac{\partial \hat{n}}{\partial \gamma}$ for $\frac{\partial \hat{n}}{\partial \delta}$

Some uneducable people still do. This results in the misinterpretation of $\gamma \frac{\partial \hat{n}}{\partial \gamma}$

to mean $\gamma_0 \frac{\partial \hat{n}_0}{\partial \gamma_0}$ for a dependence on the machine energy $(E_0 = mc^2 \gamma_0)$

at fixed orbit length or due to motion on a dispersion orbit. Such notions have no way to include synchro-betatron motion.

Others tried to get $\hat{n}(u;s)$ as the real unit eigenvector of the 1-tune spin map on an orbit starting at $(u;s)$ but since the orbit is usually non-periodic, the eigenvector has a jump w.r.t. the eigenvector for the next turn – it doesn't obey the T-BMT equation!

Don't read anything around page 27 in Montague 1984 !!!

Tears of frustration

e.g. e-Print Archive: physics/9901038

- The S–T effect is **NOT** due to $\Delta\vec{\mu}\cdot\vec{B}$: $E_\gamma \approx$ tens of keV but S–G energy is a fraction of eV. P_{ST} does not vanish at $g = 0$ etc etc.
- The invariant spin field $\hat{n}$ is **NOT** a **"spin closed solution"**: it is analogous to the invariant torus of orbital motion.
- High order spin–orbit resonances (including synchrotron sideband resonances!) are in general **NOT** due to orbital coupling or non–linear fields, but are simply due to the non–commutative behaviour of rotations. High order resonances occur for perfectly linear orbital motion and skew quads primarily just introduce more 1st order resonances and shift the resonant spin tune as the orbital eigen–tunes shift.
- The off–closed–orbit (amplitude dependent) spin tune does **NOT** depend on the orbital phases:
  "tunes" depending on phases are meaningless.
  It is **NOT** derived from a simple eigenproblem.
- The value of P_{eq} is **NOT** due to the opening angle $|\hat{n}(\vec{u};s) - \hat{n}_0(s)|$ — in a simple picture it is due to the balance of Sokolov–Ternov effect and **diffusion**
- To paraphrase or misquote Pauli: "Some of the things in the literature are not even just wrong, they are pathological."

- Sidebands of parent first order betatron resonances: a useful approximation

$$\tau_{dep}^{-1} \propto \frac{A}{(\nu_0 \pm \nu_y)^2} \quad \rightarrow \quad \tau_{dep}^{-1} \propto \sum_{m_s=-\infty}^{\infty} \frac{A\,B(\xi;m_s)}{(\nu_0 \pm \nu_y \pm m_s \nu_s)^2}$$

  - A is an energy dependent factor
  - $B(\xi;m_s)$'s: *enhancement factors*, contain modified Bessel functions $I_{|m_s|}(\xi)$ and $I_{|m_s|+1}(\xi)$ depending on the *modulation index*

$$\xi = (\frac{a\gamma\,\sigma_\delta}{\nu_s})^2$$

===> very strong effects at high energy — dominant source of trouble.

SPEAR (SLAC) circa 1978

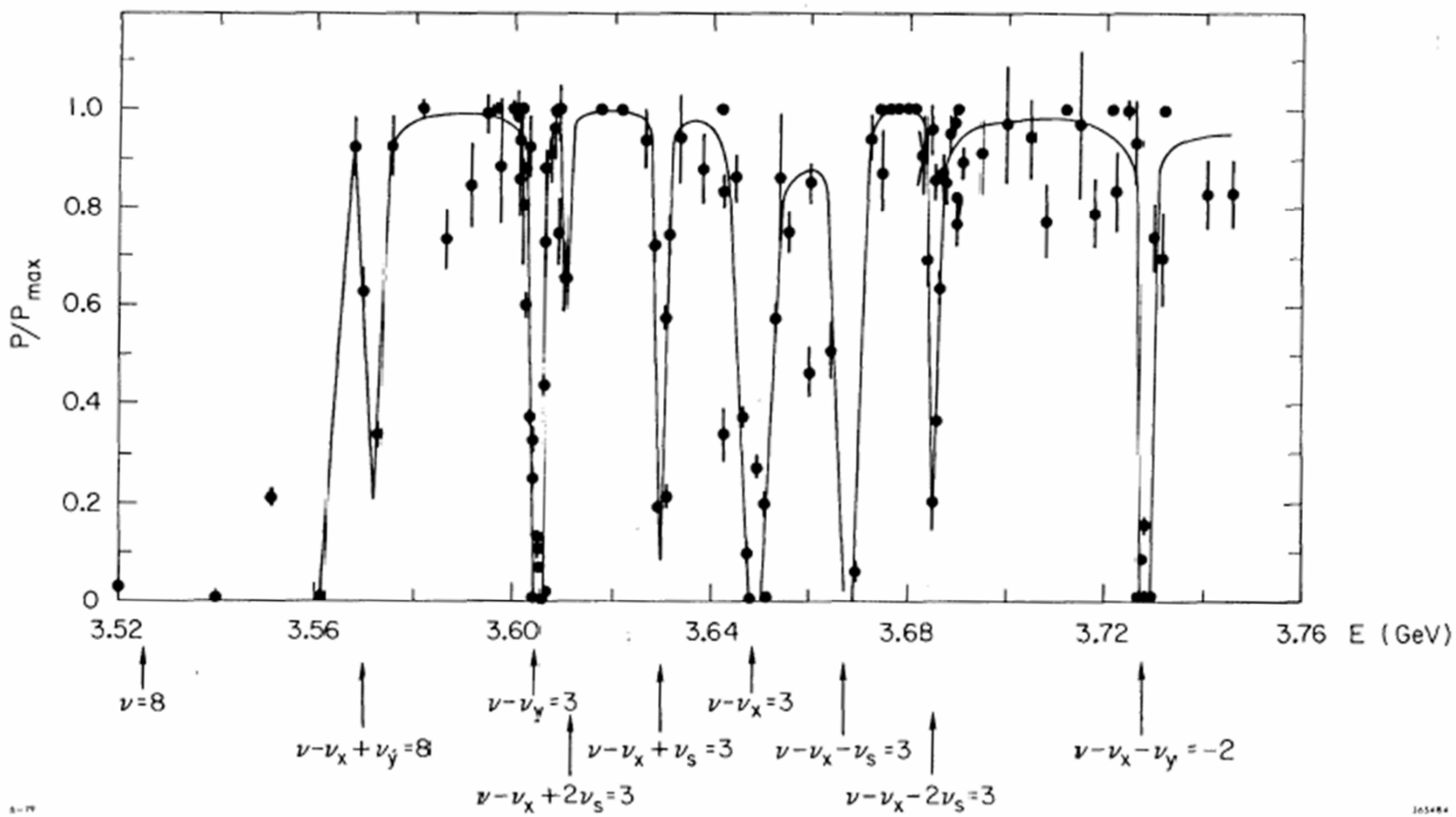
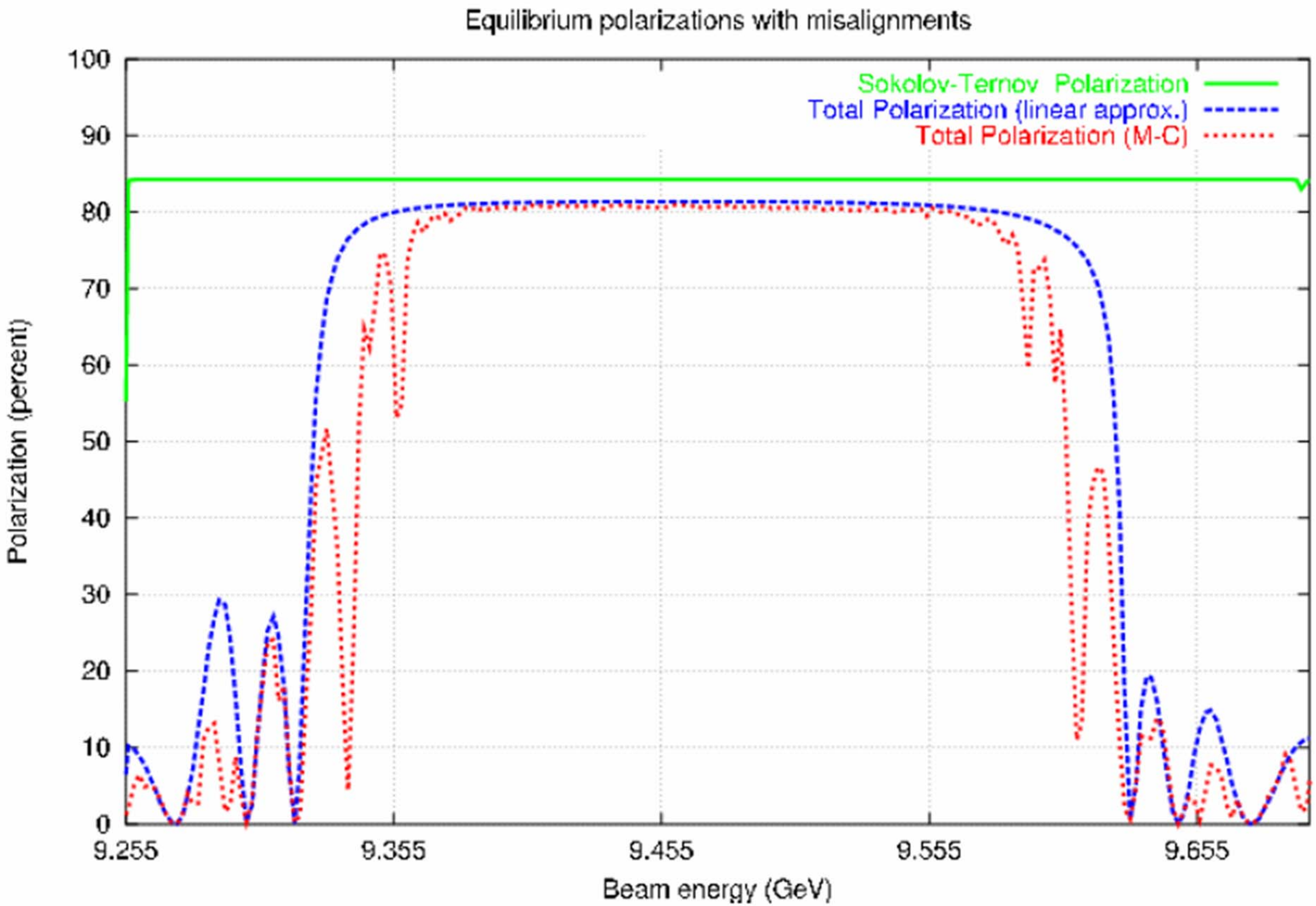
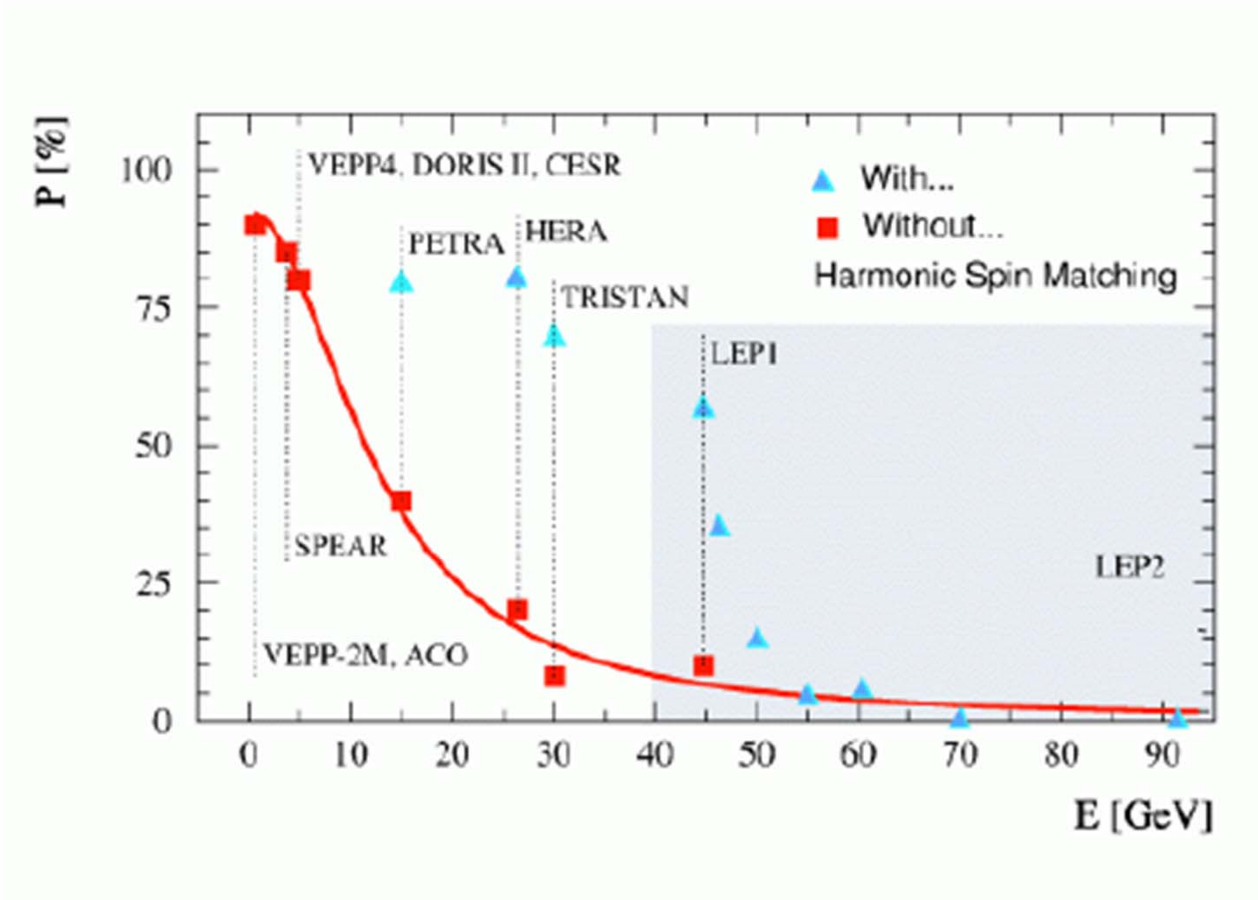


Fig. 8

eRHIC simulation



Limitations at high energy: R. Assmann et al., SPIN2000, Osaka, Japan.



It is very unlikely that TRISTAN had 70% polarisation at 29 GeV: (1990)

The machine had a large energy spread (due to the high radiative power resulting from the small radius) and uncompensated solenoids in the detectors which tilted $\hat{n}_0$ by several tens of mrads.

They claimed that the polarimeters told them that the tilt was 30 deg! That would have completely killed the polarisation at the linear level. And in any case such a tilt is inexplicable knowing the solenoid fields.

They later showed scans of polarisation versus energy which had regions of negative vertical polarisation among regions of positive polarisation!

So it can be assumed that the measurements were unreliable.

The moral:

Understand the theory at the lowest level and understand the polarimeter.

Topic 6

A Fokker-Planck equation for the polarisation density

BUT there is still no *ab initio* derivation of the D-K formula --- so far the derivation is based on plausible assumptions about time scales etc.

But there is already a formalism which does start at the beginning:

The stochastic differential equations for the orbital motion result in a

Fokker-Planck equation for the phase space density $\rho(u; s)$

See, for example, the books by Gardiner, Risken, Honerkamp…….
and the US and CERN Acc. School lectures by John Jowett (~1988).

Full 3–D spin motion

Particle transport in the presence of damping and diffusion.

Fokker–Planck equation:

$$\frac{\partial \rho}{\partial s} = \mathcal{L}_{\mathrm{FP,orb}}\ \rho$$

where with synchrotron photon emission modelled as additive noise the orbital Fokker–Planck operator can be decomposed into the form:

$$\mathcal{L}_{\mathrm{FP,orb}} = \underbrace{\mathcal{L}_{\mathrm{ham}}}_{\mathcal{L}_{\mathrm{ham}} \rightarrow \mathrm{Liouville}} + \underbrace{\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2}_{\text{damping and noise}} .$$

where the L_0, L_1, L_2 are zeroth, 1st and 2nd order partial derivatives w.r.t. the components of *u* and L_{ham} represents the Poisson bracket with the Hamiltonian.

Polarisation is not a density. It therefore has no conventional Fokker-Planck equation.

The trick is to work with the **polarisation density,** the product of the local polarisation

and the phase space density, $\vec{P}(u,s)\,\rho(u,s)$ The polarisation density is proportional

to the density of spin angular momentum.

The evolution equation for the polarisation density can be written as soon as the Fokker-Planck operator is known for the orbital density.

Self polarisation terms can then be added in by hand or by going directly to the evolution equation for the Wigner function derived by Derbenev and Koindratenko (1975).

The complete equation forms the starting point for a calculating the evolution of the polarisation for any starting condition.

Without the S–T terms, the corresponding form for the

Polarisation Density $\vec{\mathcal{P}}$:

$$\boxed{\frac{\partial \vec{\mathcal{P}}}{\partial s} = \mathcal{L}_{\mathrm{FP,orb}} \vec{\mathcal{P}} + \vec{\Omega}(\vec{u}; s) \times \vec{\mathcal{P}}}$$

Barber + Heinemann 1990's

$$\vec{P}(s) = \int d^6 u \, \vec{\mathcal{P}}(\vec{u}; s).$$

This equation:

- can be derived in a classical picture,
- is homogeneous in $\vec{\mathcal{P}}$ i.e. it's "universal",
- is valid far from spin–orbit equilibrium,
- contains the whole of depolarisation!

After including the S–T terms, this becomes (Derbenev + Kondratenko, Barber + Heinemann):

$$\underbrace{\frac{\partial \vec{\mathcal{P}}}{\partial s} = \mathcal{L}_{\rm ham}\vec{\mathcal{P}} + \vec{\Omega}(\vec{u};s)\times\vec{\mathcal{P}}}_{\equiv \text{Damping and noise free part}} + \underbrace{\mathcal{L}_0\,\vec{\mathcal{P}} + \mathcal{L}_1\,\vec{\mathcal{P}} + \mathcal{L}_2\,\vec{\mathcal{P}} + \underbrace{\frac{1}{\tau_0(\vec{u})}\left[\vec{\mathcal{P}} - \frac{2}{9}\hat{v}(\vec{\mathcal{P}}\cdot\hat{v}) + \frac{8\hat{b}(\vec{u})}{5\sqrt{3}}\rho\right]}_{\text{ST in BKS form}} + \underbrace{\text{X–terms}}_{\text{Kinetic pol.}}}_{\text{SMALL}}$$

$\Downarrow$

$\equiv$ T-BMT equation (BIG)

$\Downarrow$

Stationary state

$\Downarrow$

$\hat{n}$-axis (Invariant spin field) $\rightarrow$ DETERMINES DIRECTION

$\Downarrow$

Rate of polarisation loss $\propto$ Functional of $\hat{n}, \quad \partial_{\vec{u}}\,\hat{n}, \quad \partial^2_{\vec{u}}\,\hat{n}\ldots\ldots$ (e.g. DK formula).

$\Longrightarrow$ large near spin orbit resonances — since $\hat{n}$ is then very sensitive to $\vec{u}$.

Topic 7

Polarimetry in high energy electron rings

Various methods for measuring the polarisation of electron beam:

Compton scattering of circularly polarised photons from a laser: spin dependent scattering asymmetries.
Best at high energy where asymmetries a large.

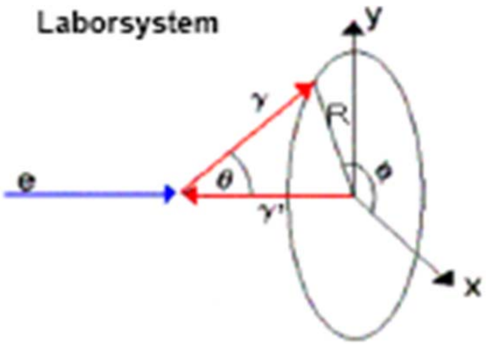
Mott scattering: spin dependent asymmetries in scattering of nuclei.
Used at polarised sources in 0-100keV range.

Møller (or Bhabba) scattering of electrons(positrons) off polarised electrons in a (magnetised) target: spin dependent asymmetries.
Used at high energy (e.g. the SLC at SLAC). Not for storage rings.

Toushek scattering: large angle particle-particle scattering from bunches in storage rings. the rate falls if the beam is polarised.
Useful at low energy in synchrotron radiation sources to detect depolarisation while measuring the beam energy via depolarisation. Also used at BINP in the VEPP2 rings.

Compton polarimetry at HERA.

Material from the POL2000 group. http://www.desy.de/~pol2000/



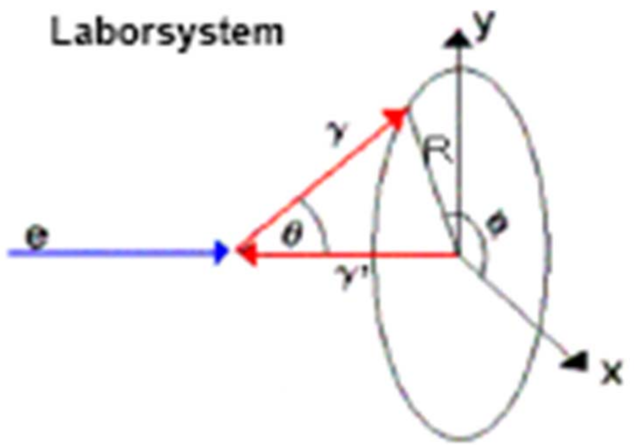
- Spin-dependent cross section for γ-e scattering

$$\frac{d^2\sigma}{dEd\phi} = \Sigma_0(E) + S_1\Sigma_1(E)\cos 2\phi + S_3\left[P_Y\Sigma_{2Y}(E)\sin\phi + P_Z\Sigma_{2Z}(E)\right]$$

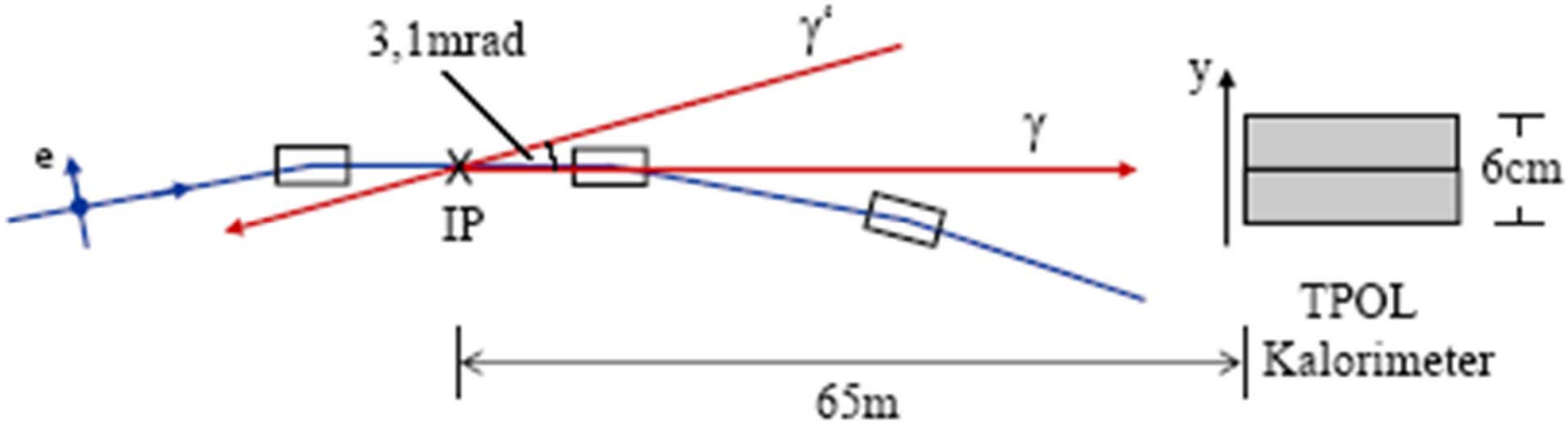
- S_1,S_3 linear and circular components of laser beam
- P_Y,P_Z transverse and longitudinal components of lepton beam polarisation
- Use asymmetry between S_3=+1 and S_3=-1 states

A simple idea leading to a technique requiring a great deal of expertise and care.

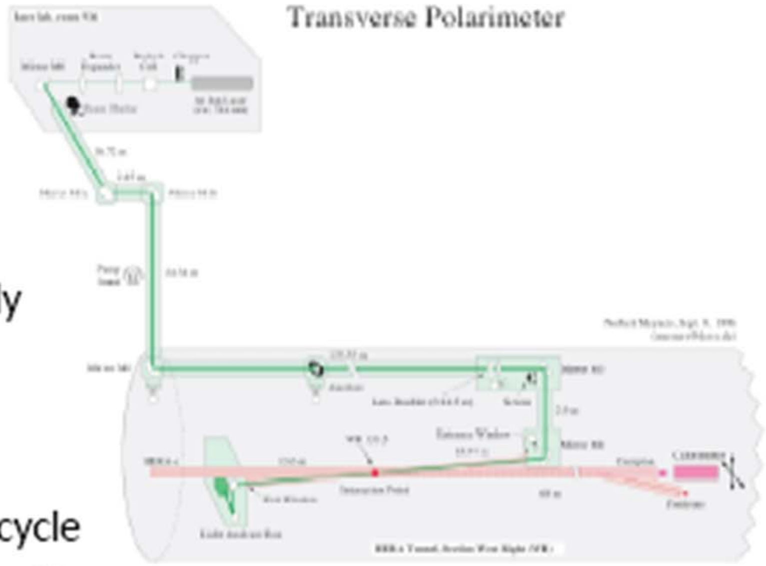
The transverse (=vertical) polarimeter. West area where $\hat{n}_0$ is vertical:

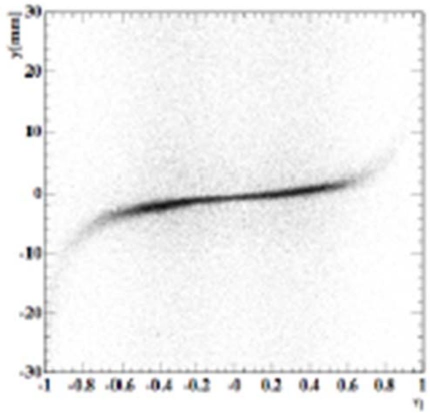


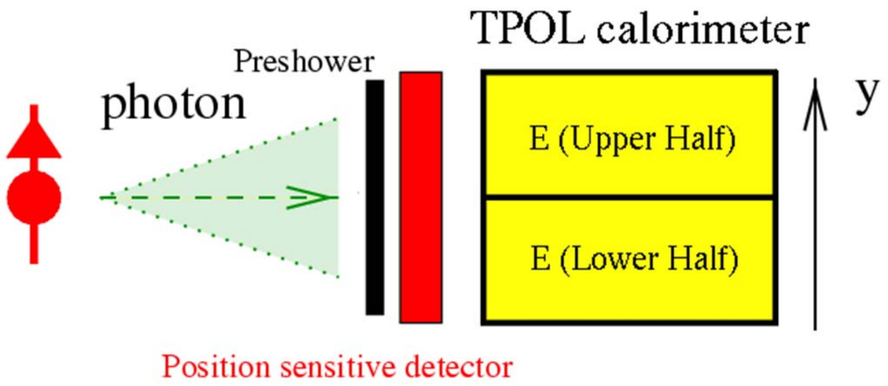
The cylindrical symmetry is broken, leading to an up-down asymmetry in the rates.



- Ar-ion 10W cw laser
- Linear polarisation >99%
- Pockels cell converts to circularly polarised
- Helicity swapped at 90 Hz
- One measurement cycle 40 secs of laser on - 20 secs laser off for background measurement
- Laser power and polarisation monitored in tunnel and ctrl room
- DAQ rate 100 kHz

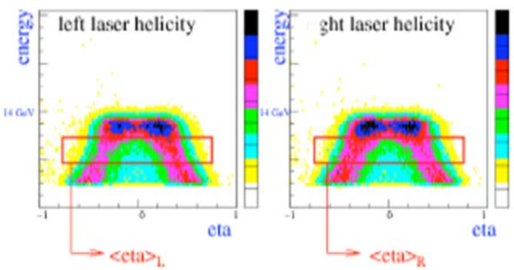






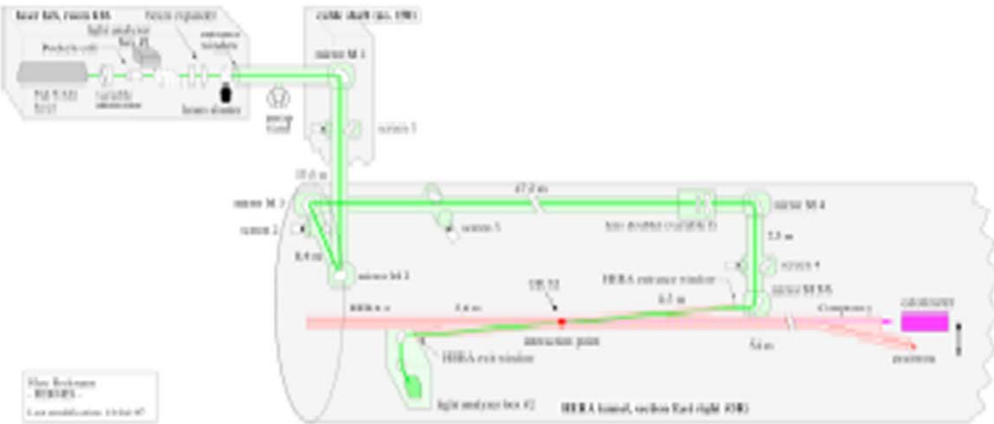
- Integrate $d^2\sigma/dE_\gamma d\eta$ over sensitive region in E_γ and η
- Consider asymmetry between laser beam helicities

$$\langle \eta_L \rangle - \langle \eta_R \rangle = 2|S_3| P_Y \Pi$$

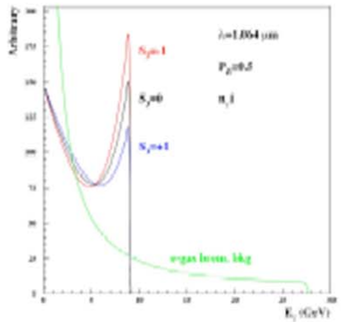


- Π is the analysing power from rise-time calibration and MC
- S_3 is measured between HERA fills to be 1 with error ±0.5%

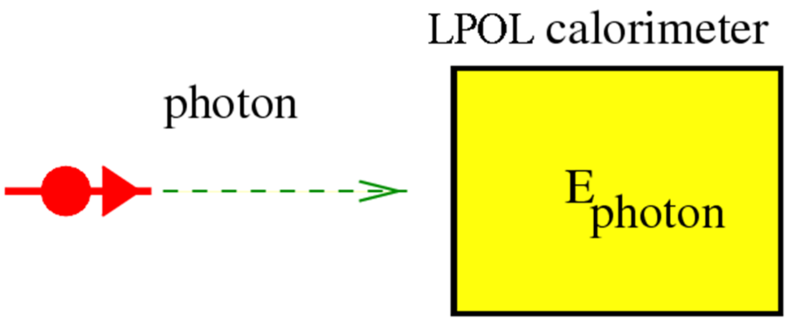
The longitudinal polarimeter: East area where $\hat{n}_0$ is longitudinal.



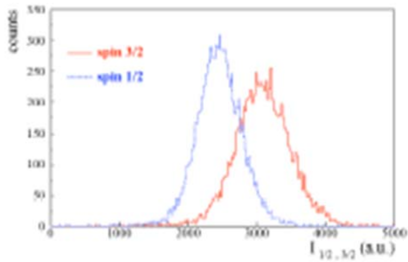
- $n_\gamma \approx 0.001$ per bunch crossing
- Can use single-photon cross section. Calculate σ from QED
- Compton edge gives energy calibration
- Large separation of LH and RH states (up to 0.6)
- But at LPOL location Bremsstrahlung background is too high
- s/b≈0.2 gives too large a statistical error (δP/P=0.01 takes 2.5 hours)
- Use for systematic studies



- Nd:YAG laser - 3ns x 100 mJ @ 100 Hz
- Pockels cell converts linear (>99%) light to circularly polarised light
- Transported to tunnel and collided with electron beam
- Detect backscattered photons in calorimeter downstream
- Laser polarisation monitored in tunnel and ctrl room

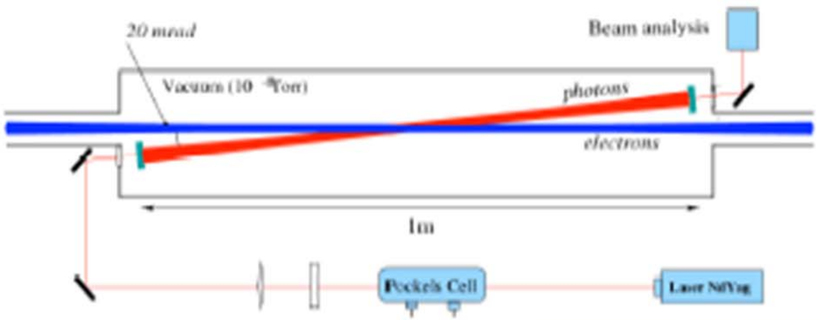


- $n_\gamma \approx 1000$ per bunch crossing
- No background problems
- No easy way to monitor calorimeter energy response (E>5 TeV!)
- High power pulsed laser but only at 100 Hz compared to HERA 10 MHz
- δP/P=0.01 in 1 minute



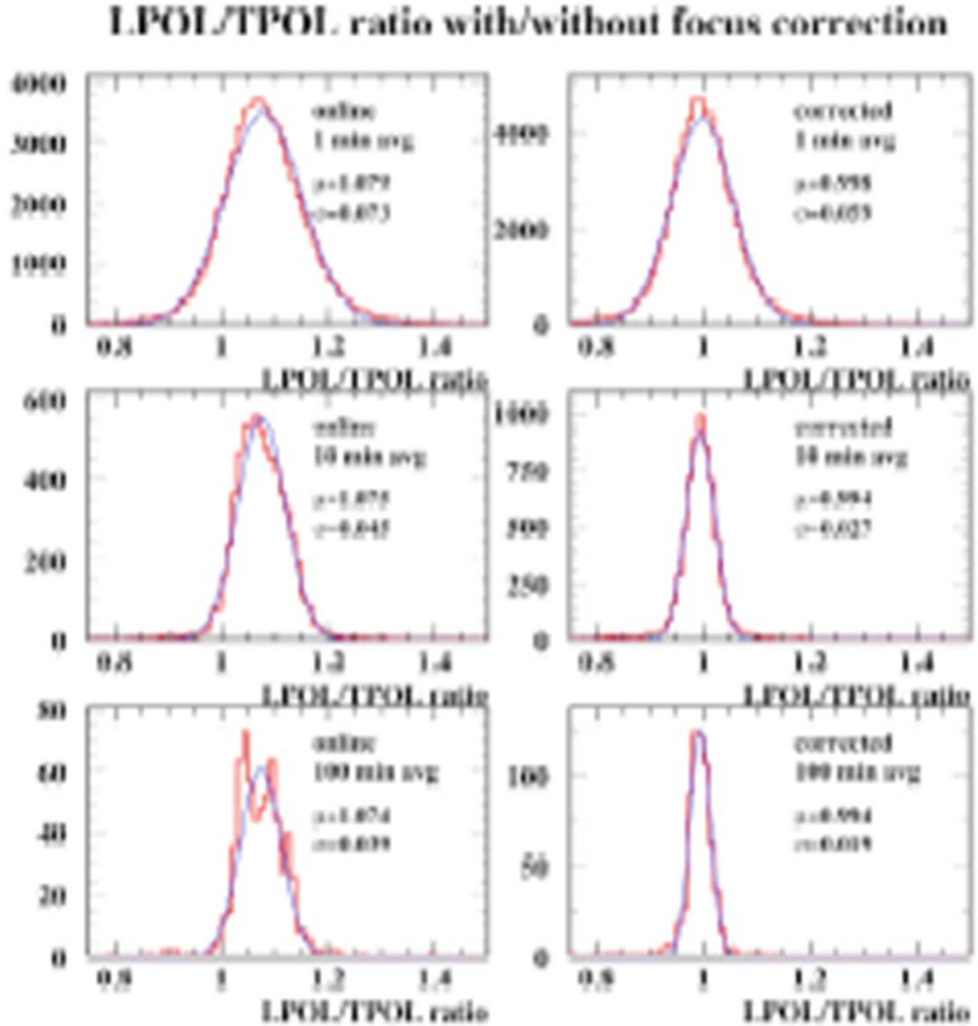
The new longitudinal polarimeter (East area) based on a Fabry-Perod cavity.

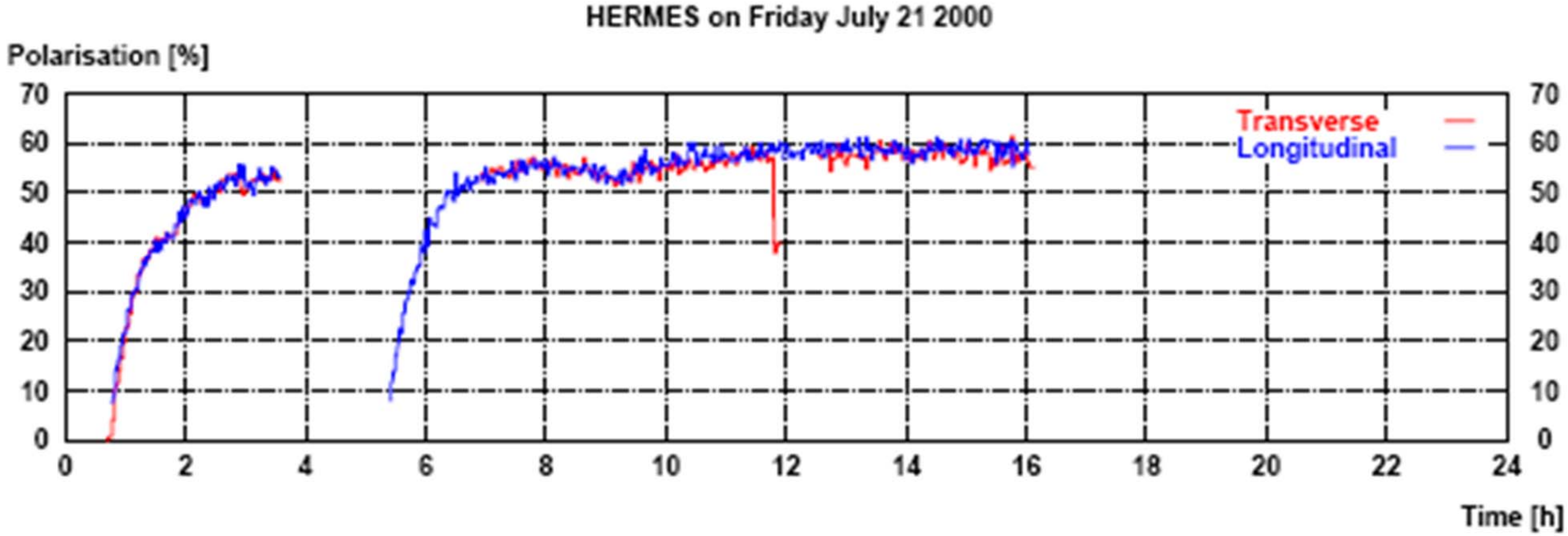
Kilowatts in the photon beam: promises enormous increase in data taking rate.



- Installed in the tunnel
- Initially laser electronics damaged by radiation but shielding improved and now able to run

A check on my argument using times scales, that the value of the polarisation should be the same at all *s.*

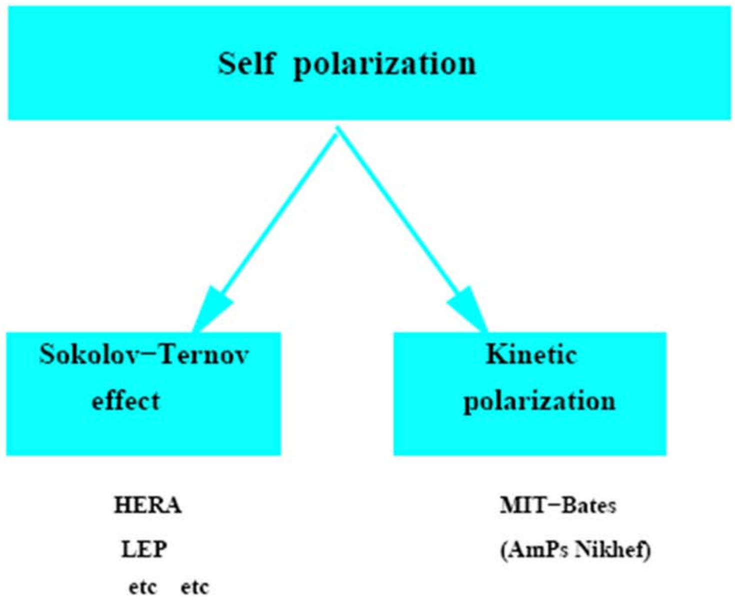


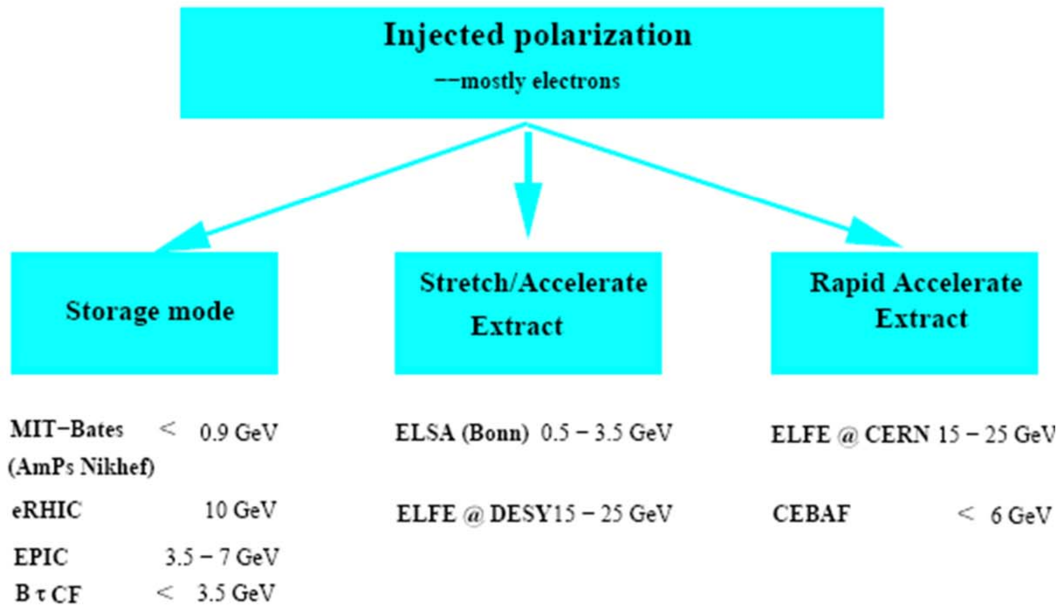


HERMES on Friday July 21 2000
Polarisation [%]
70
60
50
40
30
20
10
0
0
2
4
6
8
10
12
14
16
18
20
22
24
Time [h]
Transverse
Longitudinal

Topic 8

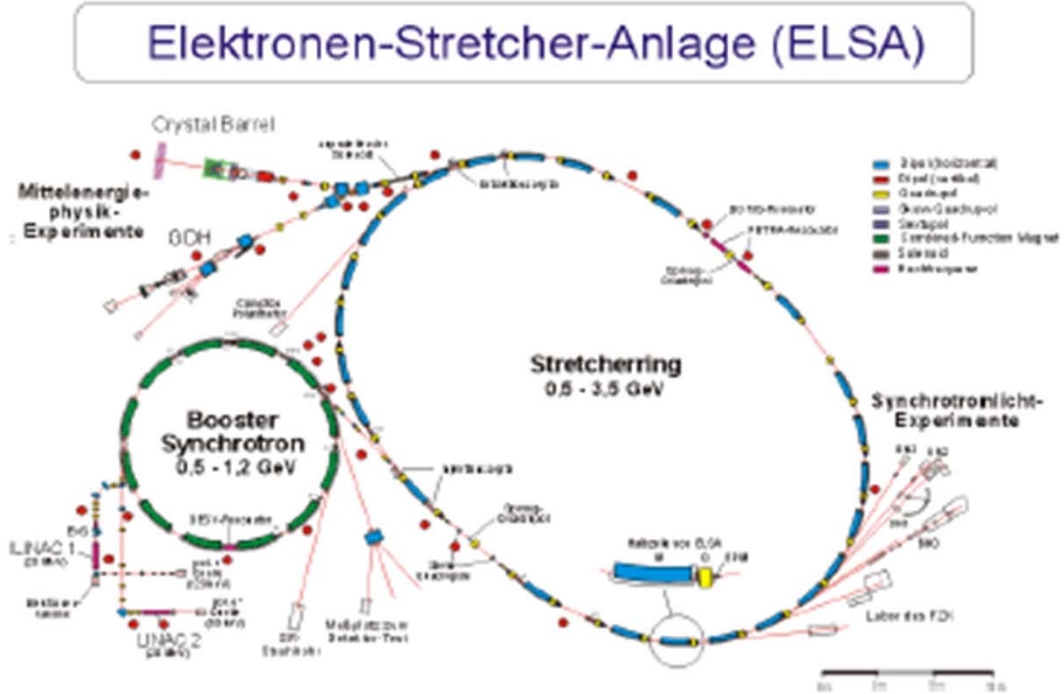
Miscellaneous on polarised electron beams


Self polarization
Sokolov−Ternov effect
Kinetic polarization
HERA
LEP
etc etc
MIT−Bates
(AmPs Nikhef)


Injected polarization
−−mostly electrons
Storage mode
Stretch/Accelerate Extract
Rapid Accelerate Extract
MIT−Bates < 0.9 GeV
(AmPs Nikhef)
eRHIC 10 GeV
EPIC 3.5 − 7 GeV
B τ CF < 3.5 GeV
ELSA (Bonn) 0.5 − 3.5 GeV
ELFE @ DESY15 − 25 GeV
ELFE @ CERN 15 − 25 GeV
CEBAF < 6 GeV

Injected polarization: stretch and accelerate

- ELSA (Univ. Bonn): 0.5 – 3.5 GeV.

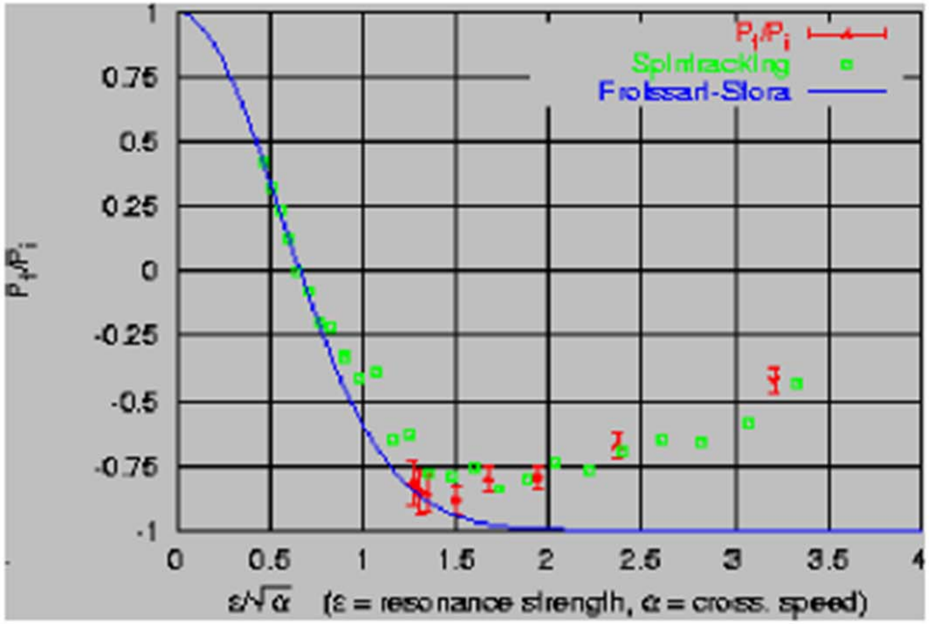


From http://www.physik.uni-bonn.de/physik/elsa99de.html

ELSA

Modification of Froisart–Stora prediction by synchrotron radiation

Crossing the imperfection resonance at 1.76 GeV



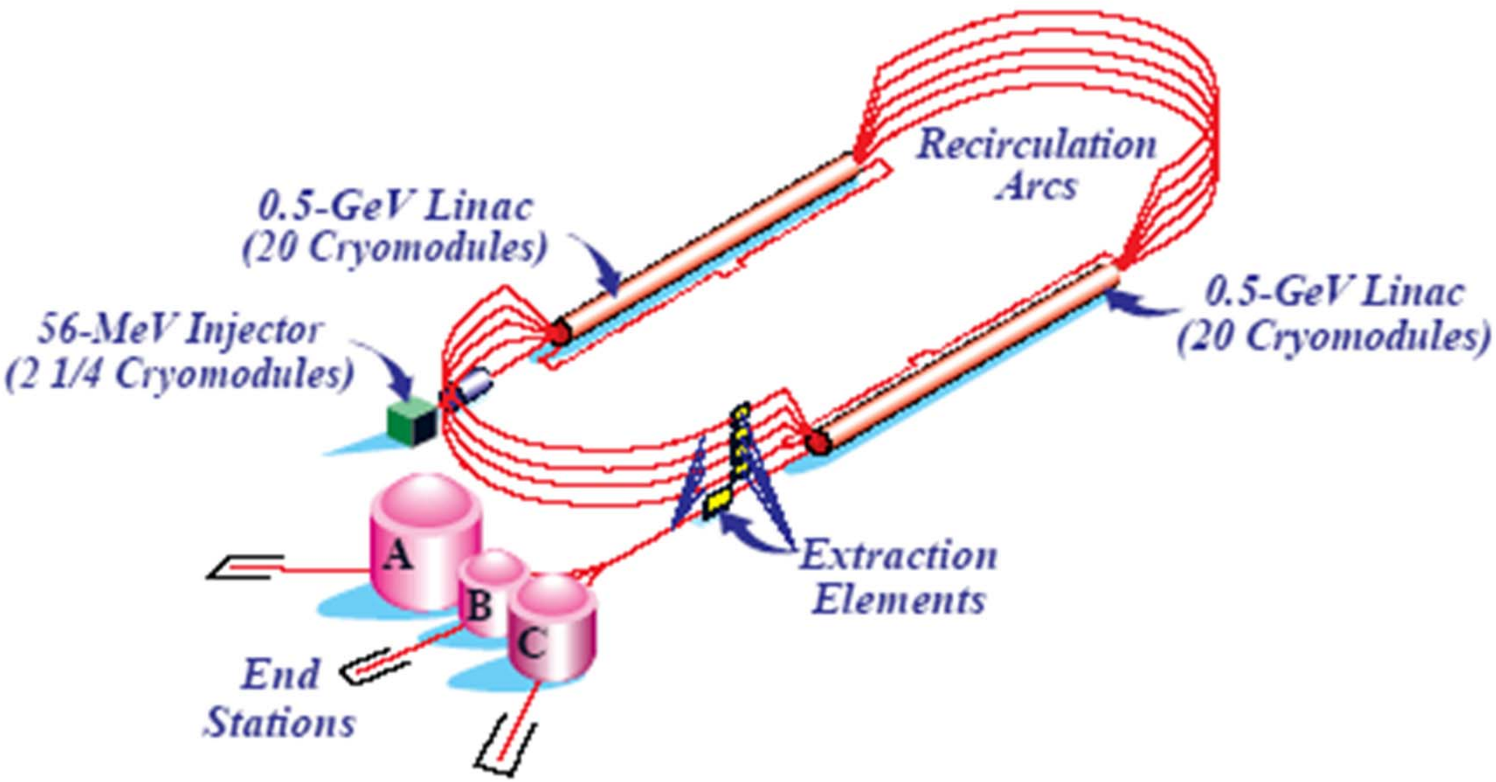
http://www-elsa.physik.uni-bonn.de/Forschungsgebiete/Polarisierte-Elektronen/pole_de.html

C. Steier et al., EPAC98

Injected polarization: rapid acceleration, then extraction.

- CEBAF at TJNAL:
  up to 5.73 GeV, up to 80 % polarization, up to 100 microamp.
  Ideas for 12 GeV: need vertical spin at 12 GeV?
- ELFE @ CERN: 25 GeV, 7 turns. Studies done with horizontal spin ===> danger of decoherence due to synchrotron motion. Why not vertical spin and a MiniRotator at full energy?

No big problems for polarization expected: energy constant in arcs, minimal effect of acceleration fields on the spin (see T–BMT), very rapid increase of energy — full energy reached in a few turns.


Recirculation
Arcs
0.5-GeV Linac
(20 Cryomodules)
0.5-GeV Linac
(20 Cryomodules)
56-MeV Injector
(2 1/4 Cryomodules)
A
B
C
Extraction
Elements
End
Stations

Topic 9

Polarised protons at high energy

The problem.

No self polarisation. $\Rightarrow$ polarised protons must be injected at low energy from a source.

Proton beams usually have equal horizontal and vertical emittances. And there are closed orbit distortions. These perturb nominally vertical spins (e.g. $\hat{n}_0$ is tilted).

$$\nu_0 \approx a\gamma$$

$$\vec{\omega}(s).\{\hat{m}_0(s) + i\hat{l}_0(s)\} = \sum_{j,k} C^+_{jk}\, e^{-i2\pi(j-\nu_0+Q_k)s/C} + \sum_{j,k} C^-_{jk}\, e^{-i2\pi(j-\nu_0-Q_k)s/C}$$

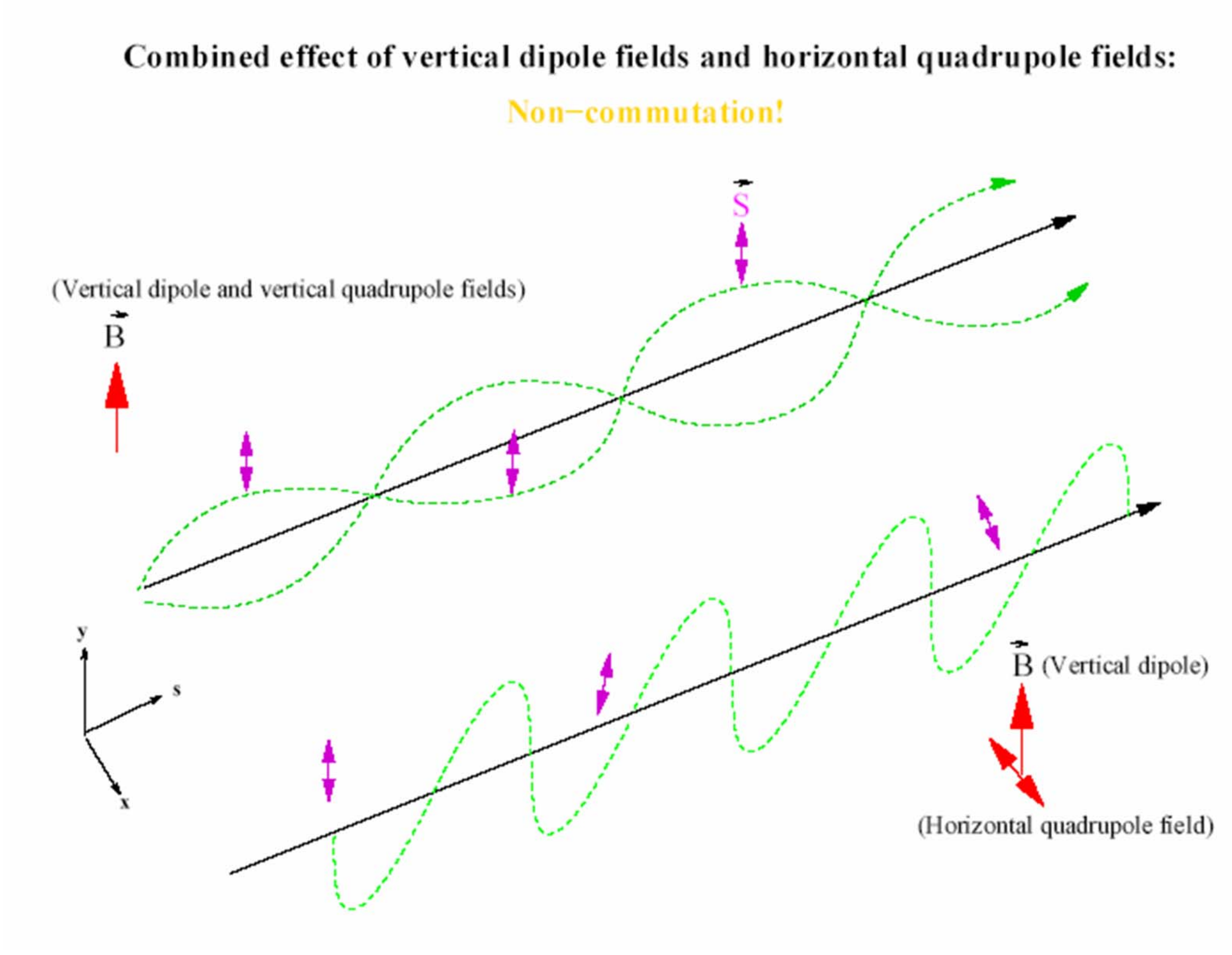
So on the way to high energy, the spins must negotiate 100's or 1000's of resonances proportional to a "resonance strength" $\varepsilon_{jk} \propto C_{jk}$

Note the contrast with electrons. Except for the T-BMT eqn. the depolarisation mechanisms are different. It often makes little sense to use common terminology.

At very high energy, the invariant spin field might be so spread out that $\langle \hat{n} \rangle$ would be small even if the beam were fully polarised at each point in phase space.

Note , with $\sigma_{\hat{n}-\hat{n}_0} \approx 60^\circ, \quad |\langle \hat{n} \rangle| \approx 0.5$

Question: how can a beam be fully polarised but the polarimeters read zero?
What do some polarimeters actually measure?


Combined effect of vertical dipole fields and horizontal quadrupole fields:
Non−commutation!
S
(Vertical dipole and vertical quadrupole fields)
B
y
s
x
B (Vertical dipole)
(Horizontal quadrupole field)

M. Vogt thesis.

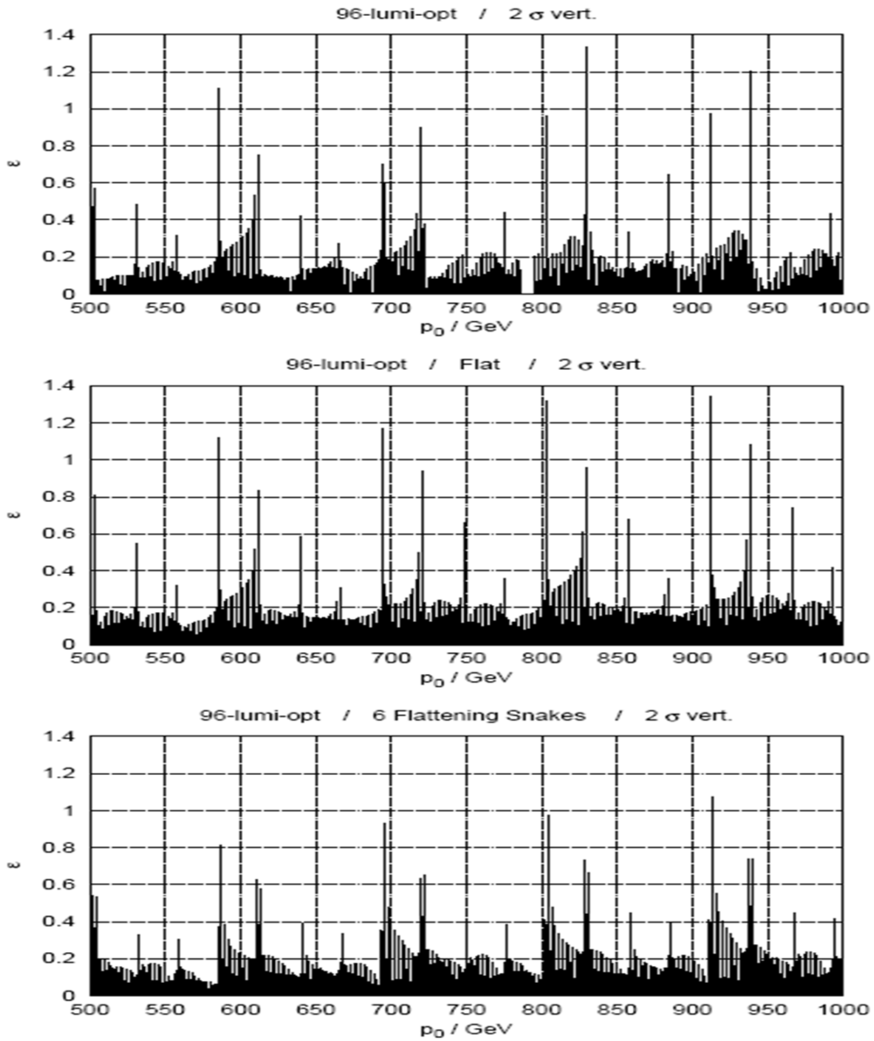
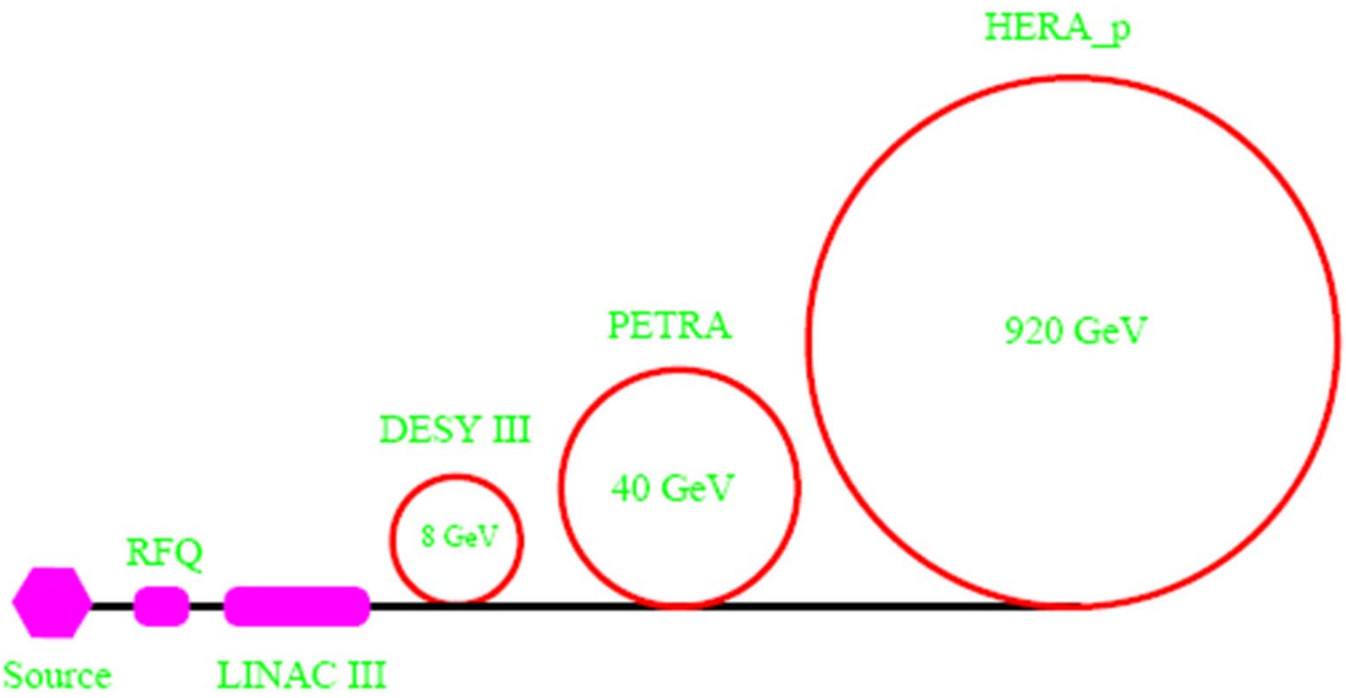
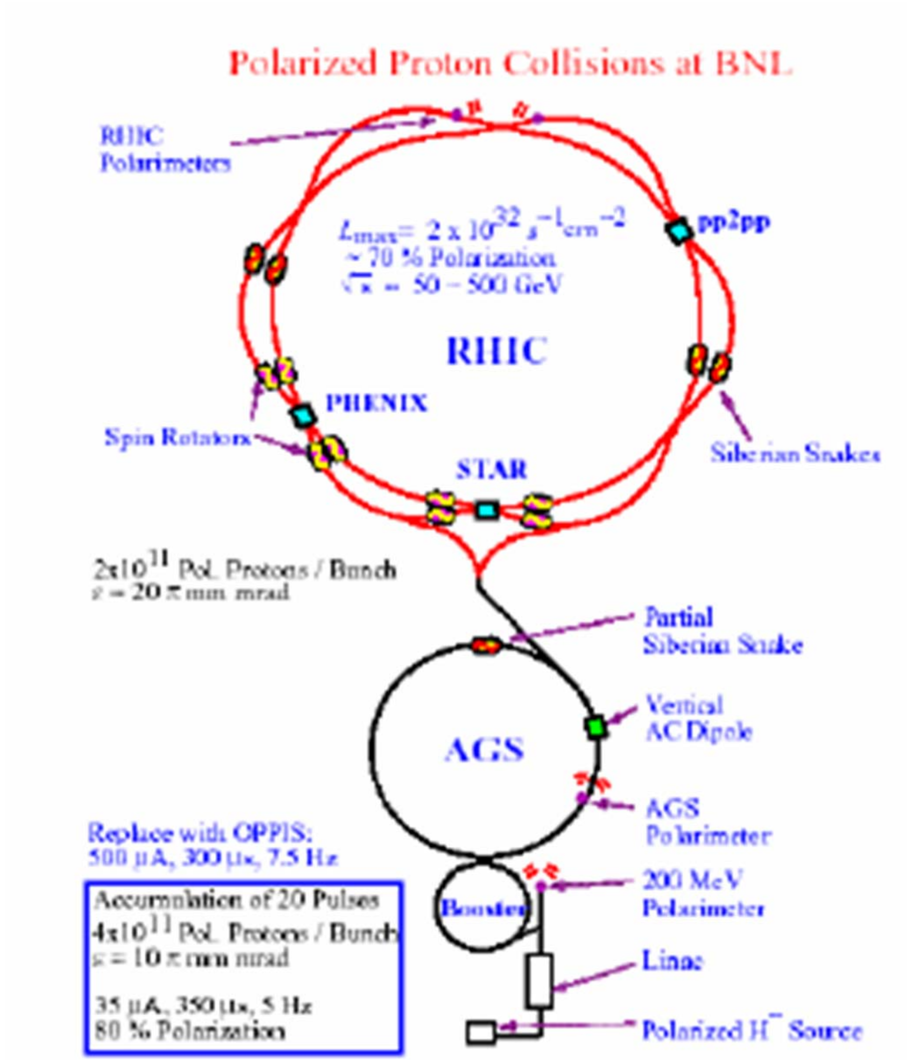


Figure 5.3: 1996 luminosity optics : linear intrinsic resonance strengths for the original lattice with vertical bends and without snakes (top), a flat model (BU00s and BVs switched off) (middle) and the original lattice with additional Flattening Snakes at the centres of the vertical bend sections around the East, North and South IPs (bottom).

The HERA Proton Chain





Polarized Proton Collisions at BNL
RHIC Polarimeters
pp2pp
RHIC
PHENIX
Spin Rotators
STAR
Siberian Snakes
2x10^11 Pol. Protons / Bunch
ε = 20 π mm mrad
Partial Siberian Snake
Vertical AC Dipole
AGS
AGS Polarimeter
Replace with OPPIS:
500 μA, 300 μs, 7.5 Hz
Accumulation of 20 Pulses
4x10^11 Pol. Protons / Bunch
ε = 10 π mm mrad
35 μA, 350 μs, 5 Hz
80 % Polarization
Booster
200 MeV Polarimeter
Linac
Polarized H⁻ Source

As one accelerates through a resonance, the damage depends on the time spent near the resonance:

If the passage is very fast, the spins are hardly perturbed
If the passage is slow, full spin flip (i.e., maintaining the value of the polarisation) can occur
At intermediate rates, polarisation can be lost.

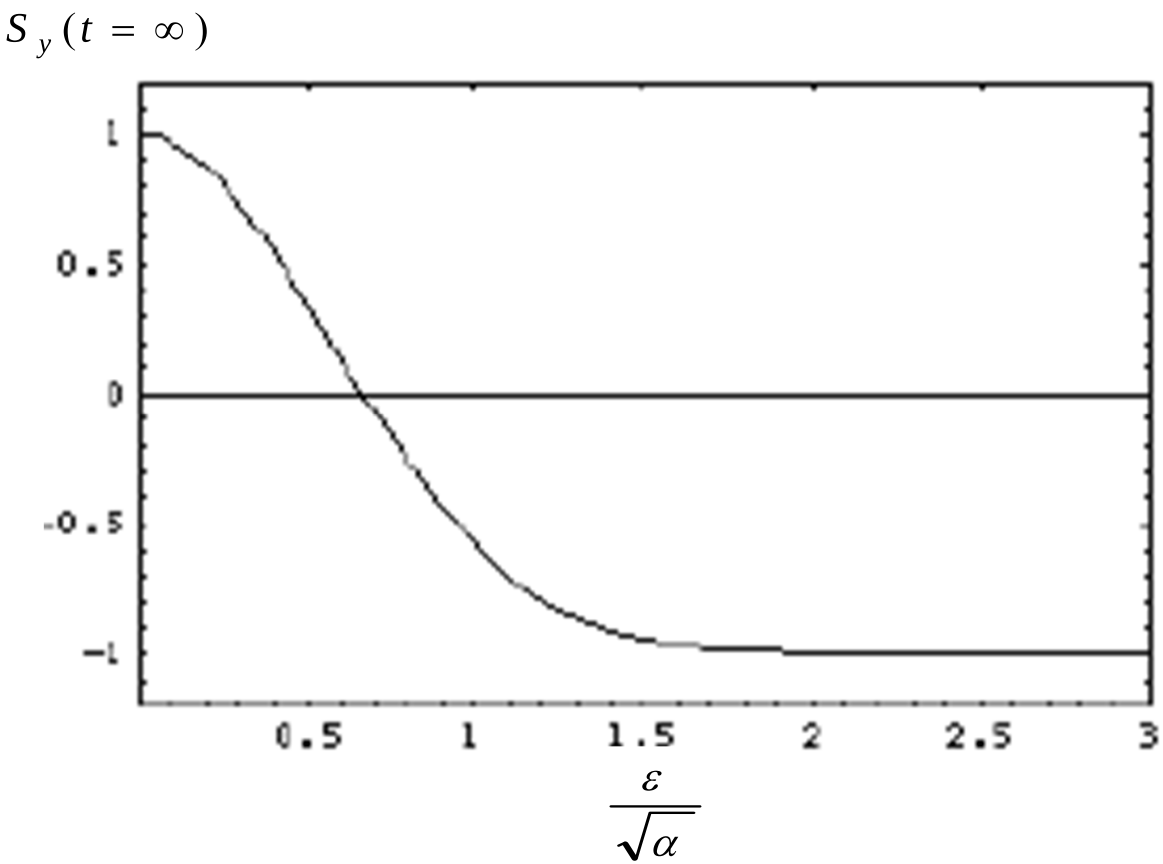
The Froissart–Stora formula for crossing resonances

$$\frac{P_{\text{final}}}{P_{\text{initial}}} = 2\, e^{-\frac{\pi|\epsilon|^2}{2\alpha}} - 1$$

- ϵ is the "resonance strength", a measure of the dominant spin perturbation at resonance (Fourier component),
- α expresses the rate of resonance crossing.

Very fast resonance crossing: Small $\frac{|\epsilon|^2}{2\alpha}$: polarisation preserved.

Very slow resonance crossing: Large $\frac{|\epsilon|^2}{2\alpha}$: adiabatic invariance ===> full spin flip without polarisation loss.



Survival of the vertical spin component from the Froissart-Stora formula

Some solutions

Make ν_0 independent of beam energy $\Longrightarrow$ Siberian snakes.

Impose massive known perturbations to overwhelm the uncontrollable effects of poorly known misalignments $\Longrightarrow$ partial snakes -- or try to get very good alignment (but it's never good enough.).

.

For Q_y resonances pulse the quadruples to make the tune jump so that the crossing rate is increased without increasing the acceleration rate.

For Q_y resonances blow up the vertical beam size, with a radial r.f. field to increase the average resonance strength – while trying not to lose the beam in the process..

Match the technique to the energy, the space in the lattice and the feasibility of the technology.

Siberian snakes: basic idea from Derbenev and Kondratenko 1976,1978

A.W. Chao, SLAC-PUB-9574
B. Montague Phys. Rep. 1984.

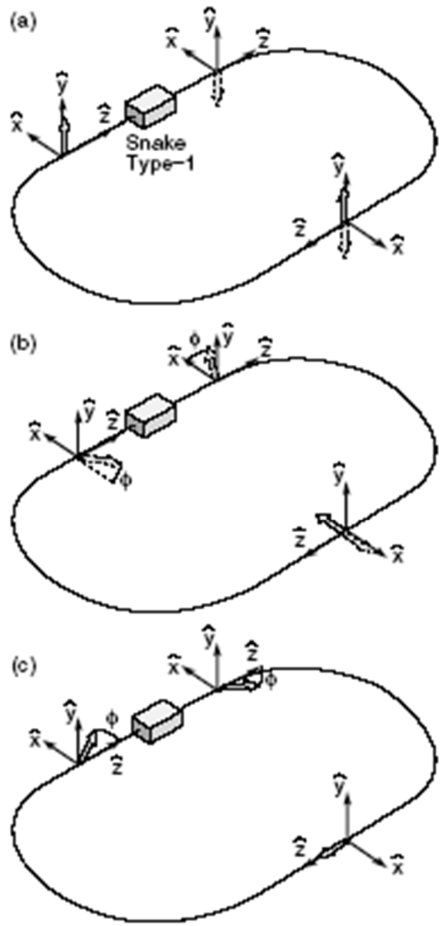


Figure 6.4: Spin motion in a storage ring with Siberian snake of type-1 [2]. (a), (b) are motion of spin perpendicular to the equilibrium direction, demonstrating the spin tune is 1/2. (c) shows the equilibrium spin direction. The angle $\phi = a\gamma/2$ lies in the x-z plane.

A compact system of magnets which rotates spin by 180 deg. around an axis in horizontal plane at all energies.

With 1 snake, $\hat{n}_0$ lies in the machine plane

at all energies and $\nu_0 = \frac{1}{2}$ at all energies.

At low energies: spin transparent solenoid systems. e.g. MIT-Bates, AmPs, IUCF.

A type-1 snake ensures longitudinal polarisation opposite the snake for an internal target.

A combination of a type-1 snake and a type-2 snake also gives $\nu_0 = \frac{1}{2}$ but with

$\hat{n}_0$ vertical so that the equilibrium polarisation direction is better defined.

A.W. Chao, SLAC-PUB-9574
B. Montague Phys. Rep. 1984.

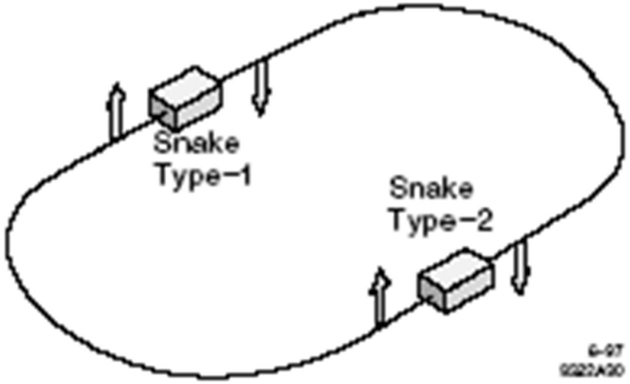


Figure 6.5: A storage ring with double Siberian snakes.

For 1 turn

$$R = R_s(\pi) R_y(\pi a \gamma) R_x(\pi) R_y(\pi a \gamma)$$

and $\nu_0 = \frac{1}{2}$ for all γ

A large variety of schemes (1,2,4,6,8…snakes .) can be devised.
See e.g., M. Vogt's thesis.
Conventional wisdom suggests that more is better than fewer but detailed study suggests otherwise. E.g., it was claimed that 26 snake would suffice for the SSC (20TeV) on the basis of hand waving and a strict avoidance of equations and 1st integrals. See G. Hoffstaetter's book..

RHIC has 2 snakes per ring.

As with rotators, exploit non-commutation of large spin rotations combined with commutation of small orbit deflections by interleaving rotations around different axes.

Ya.S. Derbenev, "Bending Snake", Preprint: UM HE 97-07.

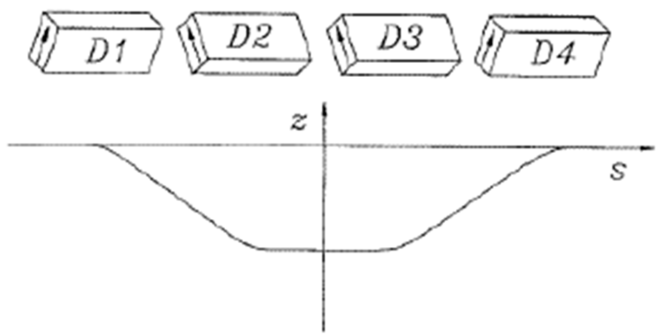


Figure 2: Bending snake with a radial spin rotation axis and a vertical orbit bump inside the snake.

V. Anferov: 8 dipole symmetric snake: May 99 DESY workshop.

2. Orbit excursion profile

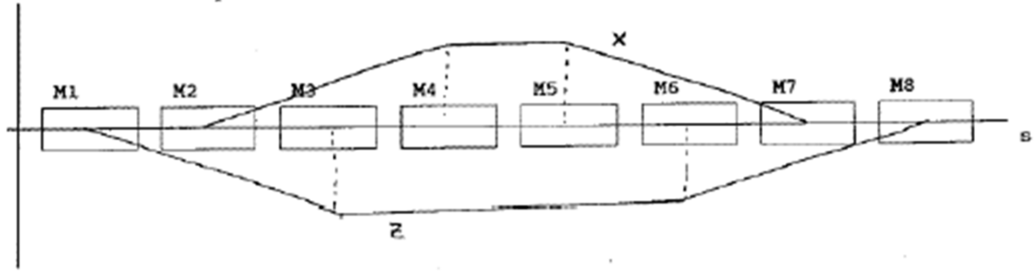


Figure 3: Radial axis, 90 degree spin rotation in each magnet, 20cm orbit excursion at 15.6 GeV.

Snakes with transverse fields
are well suited to high energy.

The fields can be kept constant
to ensure that the spin transformation
is independent of energy.

Need enough aperture for the large
orbit excursions at low energy.

Most compact form: the helical snakes of
V. Ptitsin and Yu. Shatunov.
4 modules, super cond. 4 Tesla, each 2.4 m.

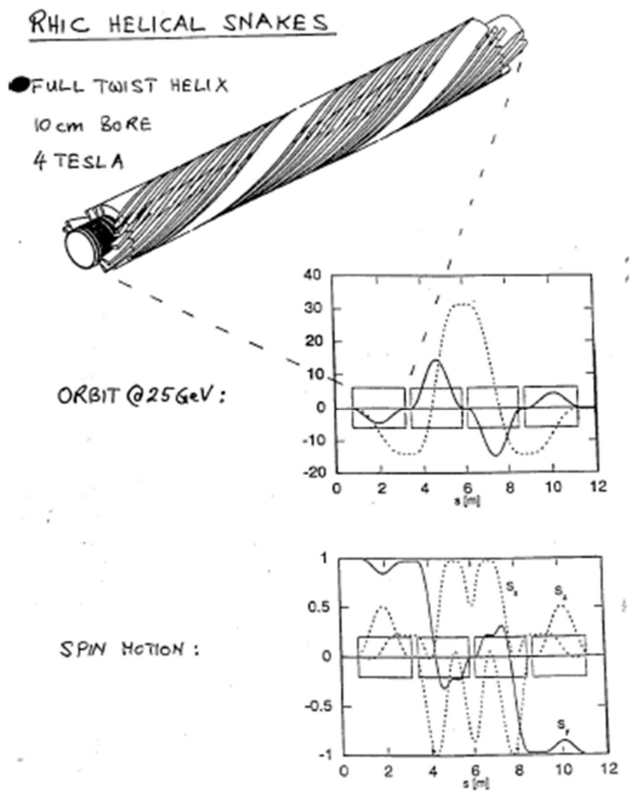
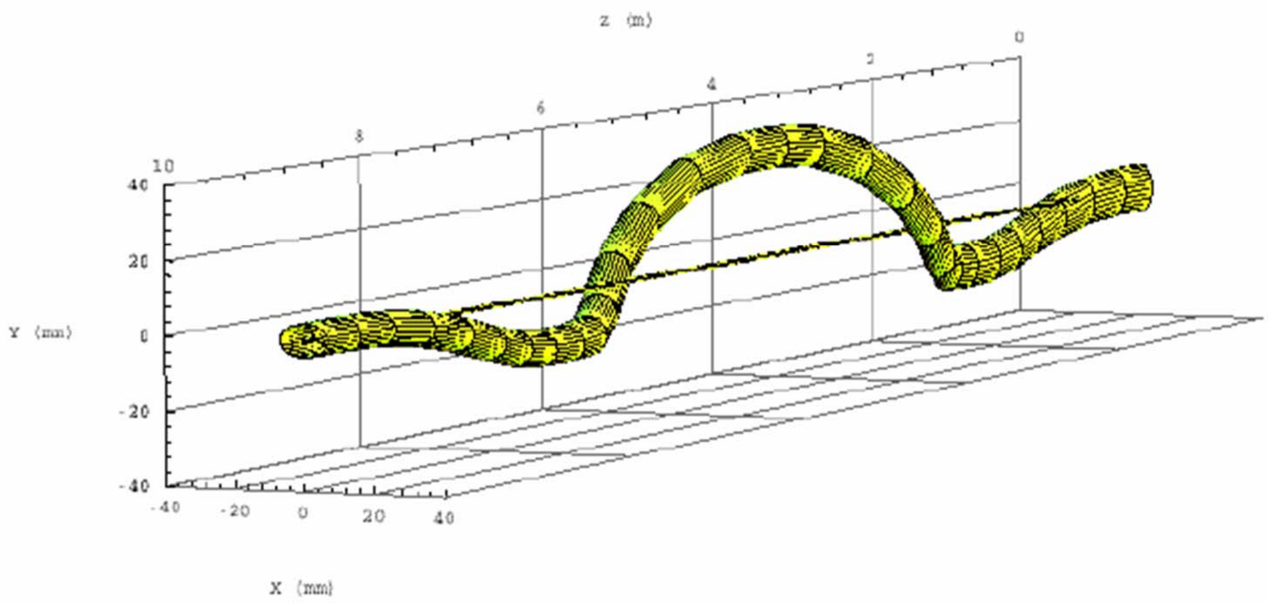


Figure 4: Rotators can be formed by choosing other combinations of the 4 modules
Sketch from T. Roser, May 99 DESY Workshop.

The snake–like orbit in a RHIC snake



Winding the coil on a RHIC helical snake module

With snakes, the crossing of 1st order resonances is avoided – but higher order resonances can be nearby. Usually many problems remain. Snake are not cure-all but are indispensable.

Snake usually help to reduce the spread of the ISF so that $|\langle \hat{n} \rangle|$ is high at the preferred running energy.

The best snake layouts must be chosen by analysis and by large amounts of spin-orbit tracking under realistic conditions.

Simulation of acceleration with more than a few particles needs enormous amounts of computer time. It is not the intelligent first step.
So the DESY philosophy was to study the ISF at the (fixed) chosen energy before embarking on simulations of acceleration --- the main initial reasons why we concentrated on it and invested so much effort in efficient ways to calculate it , to understand its properties, and why we continue to try to persuade other groups to confront it.

M. Vogt thesis.

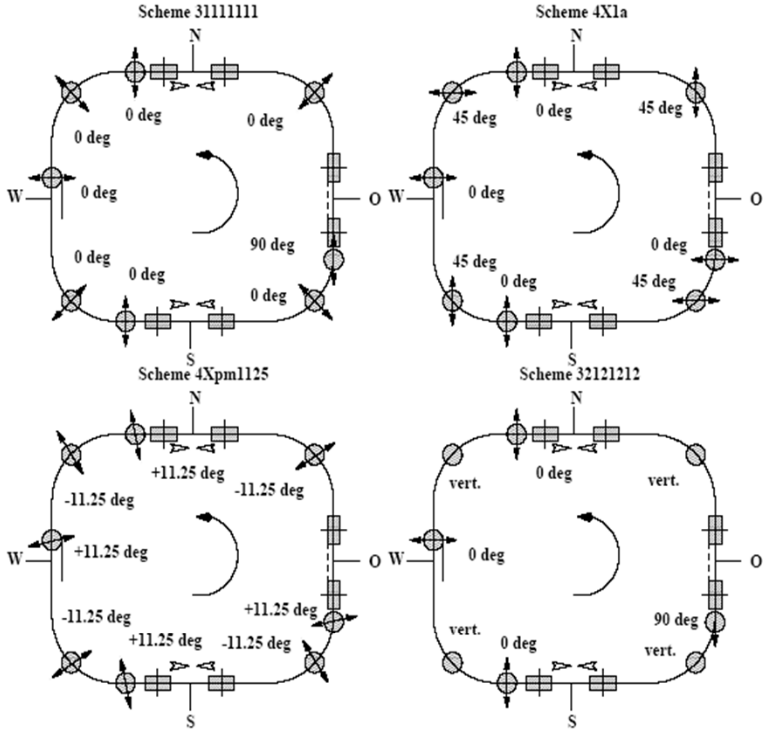


Figure 5.6: Standard 8–snake arrangements for HERA–p

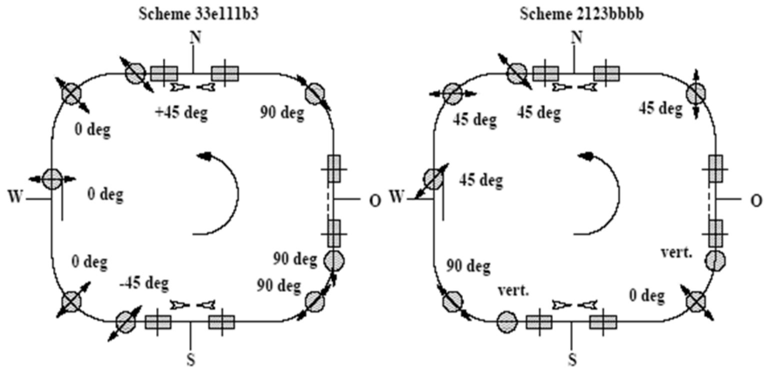


Figure 5.7: 8–snake arrangements for HERA–p found by filtering

At low energy in the pre-accelerators, the orbit excursion for full snakes is too large and solenoid snakes need impossibly high fields and introduce huge transverse coupling.

Pre-accelerators are not designsed to ramp so slowly that one can guarentee spin flip with small, but deadly resonance strengths and one can't ramp quickly enough to avoid any significant perturbation.

$\Longrightarrow$ "partial snakes'' --- snake like magnet systems which rotate spins by a small fraction of 180 deg.

They still dominate the effect of C.O. distortions and preserve the value of the polarisation by enforcing multiple full flip.

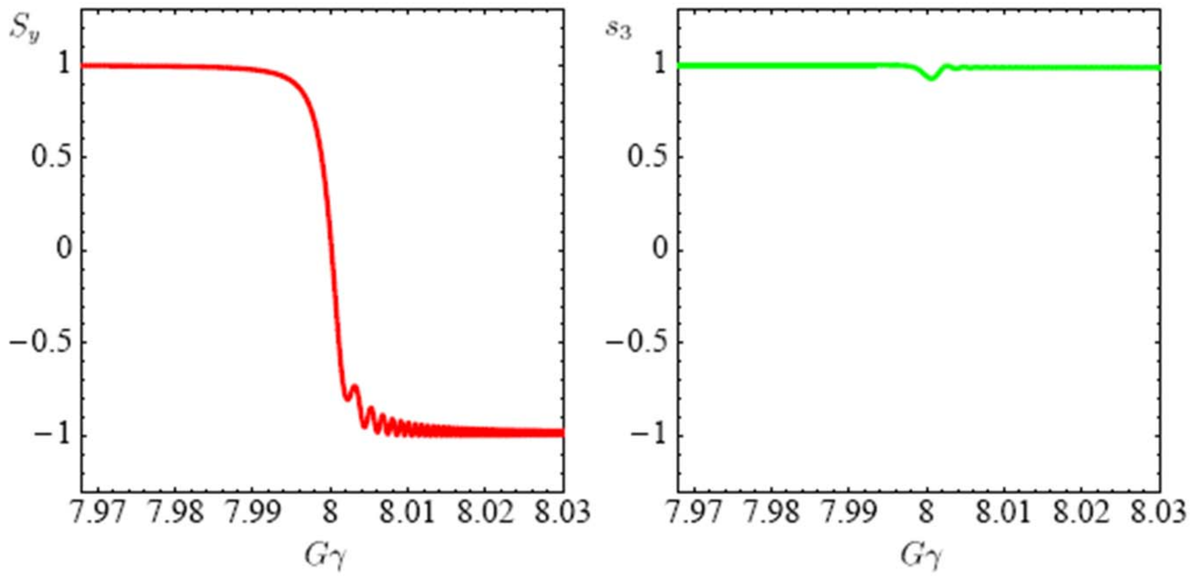
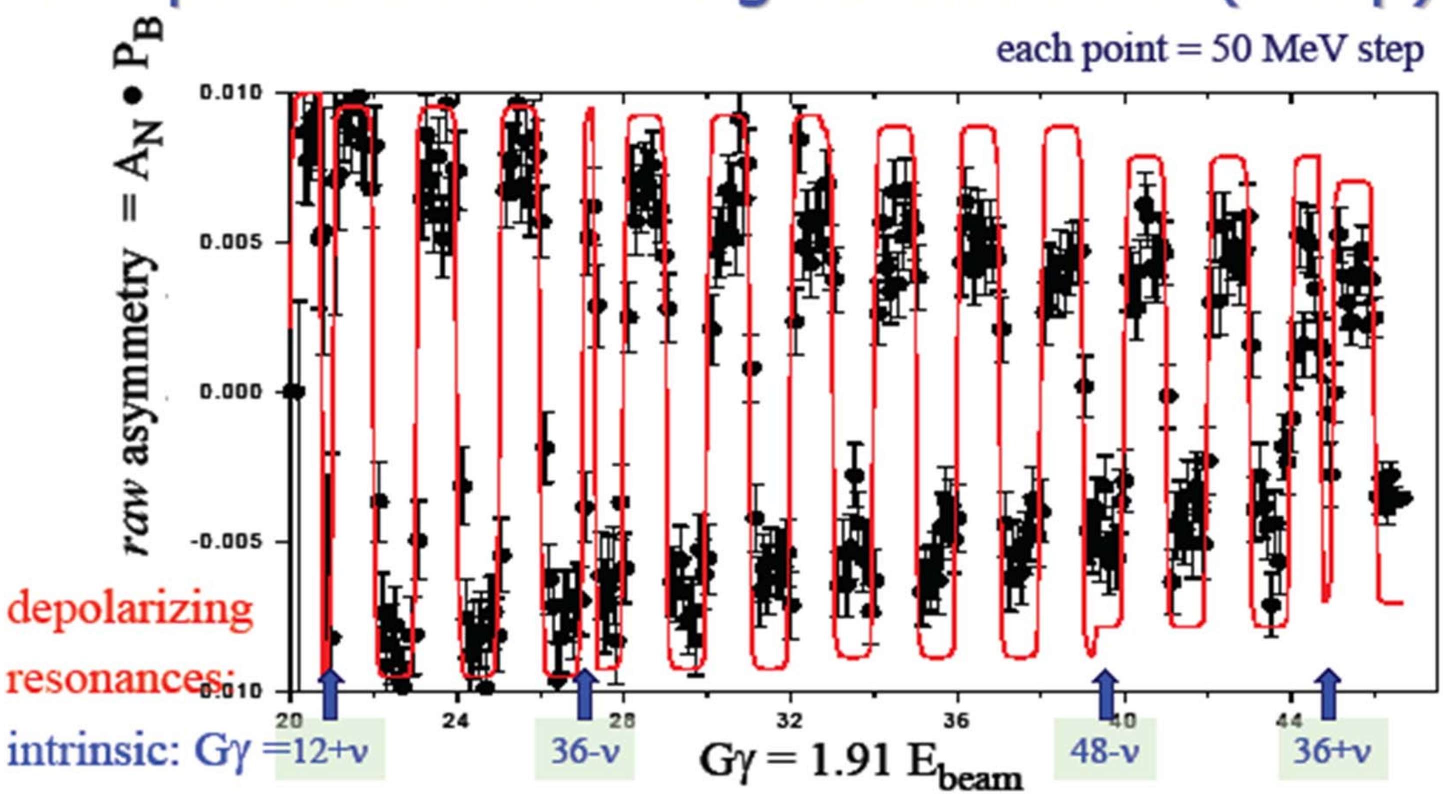


Figure 2.2: The change of $S_y = \vec{S} \cdot \vec{e}_y$ (left) and the change of $s_3 = \vec{S} \cdot \vec{n}_0$ (right) during the acceleration from $G\gamma = 7.97$ to $G\gamma = 8.03$ for particles on the closed orbit in DESY III in the presence of a 0.8° solenoid partial snake.

G. Hoffstaetter, thesis..

AGS polarization during acceleration (ramp)



imperfection: $G\gamma = n$

red line: simulation of polarization losses assuming constant A_N

BROOKHAVEN NATIONAL LABORATORY

DIS 2005

Alessandro Bravar

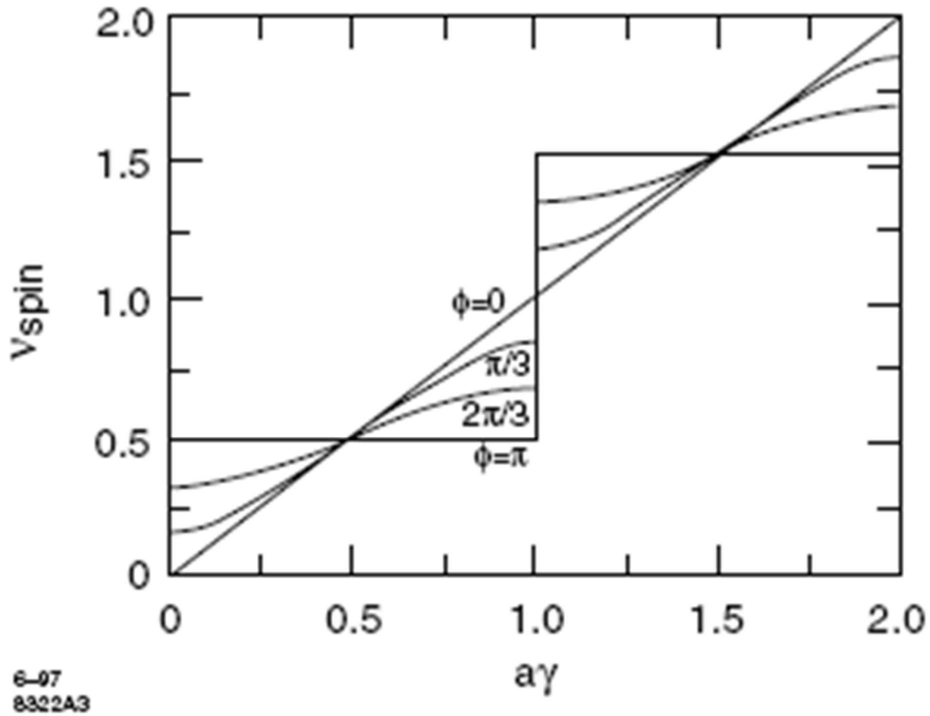


Figure 6.6: Spin tune ν_{spin} as a function of $a\gamma$ for partial Siberian snakes. The cases $\phi = 0, \pi/3, 2\pi/3$, and π correspond to no snake, 1/3 snake, 2/3 snake, and full snake, respectively.

A.W. Chao, SLAC-PUB-9574

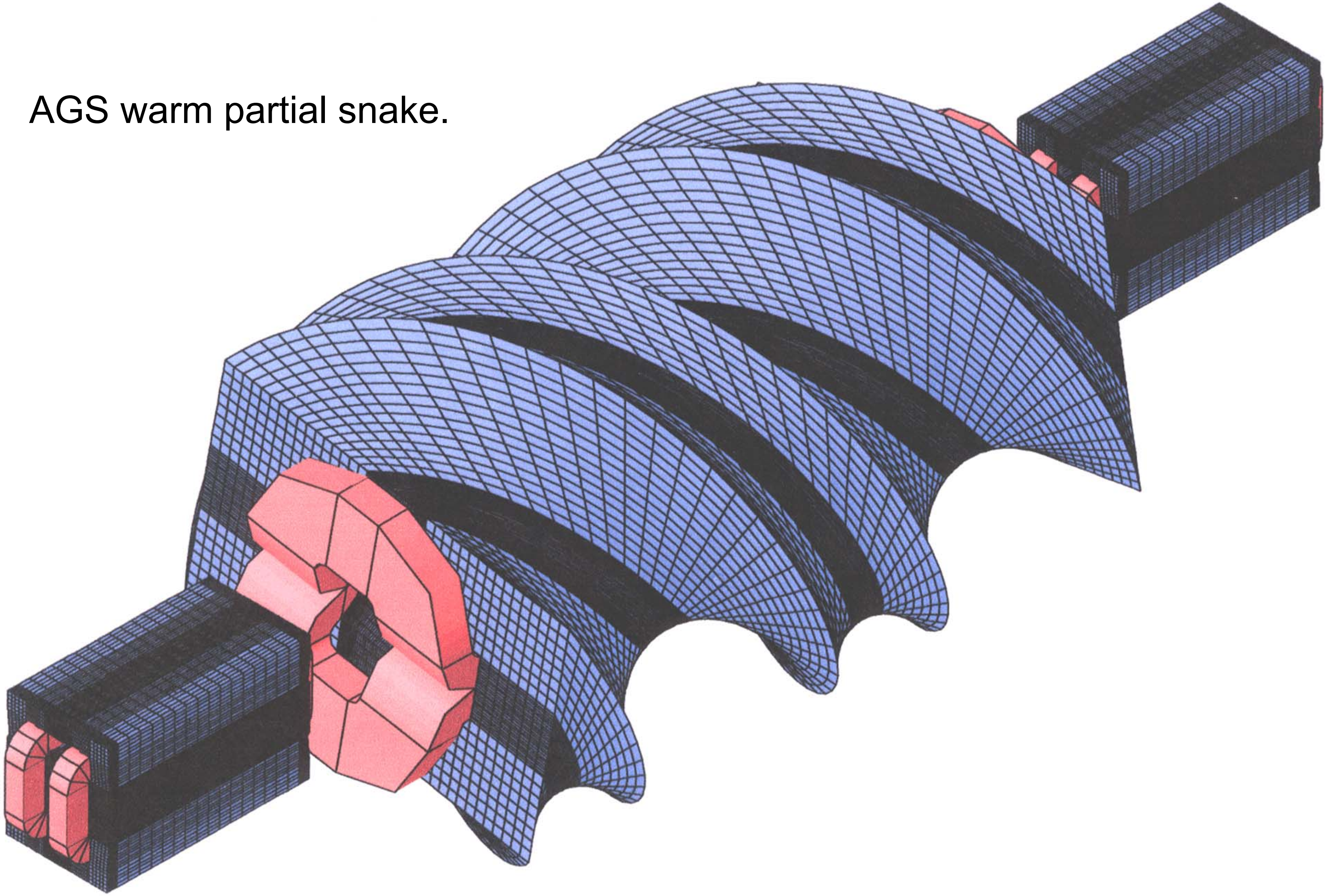
ν_0 vs. $a\gamma$ for various partial snake strengths.

Note the tune gap! – resonances are crossed, not hit!

If Q_y is close enough to an integer and the machine still runs, Q_y resonances can be avoided!

$\Longrightarrow$ 2 partial snakes now in the AGS enabling avoidance of . 1st order Q_y resonances too.

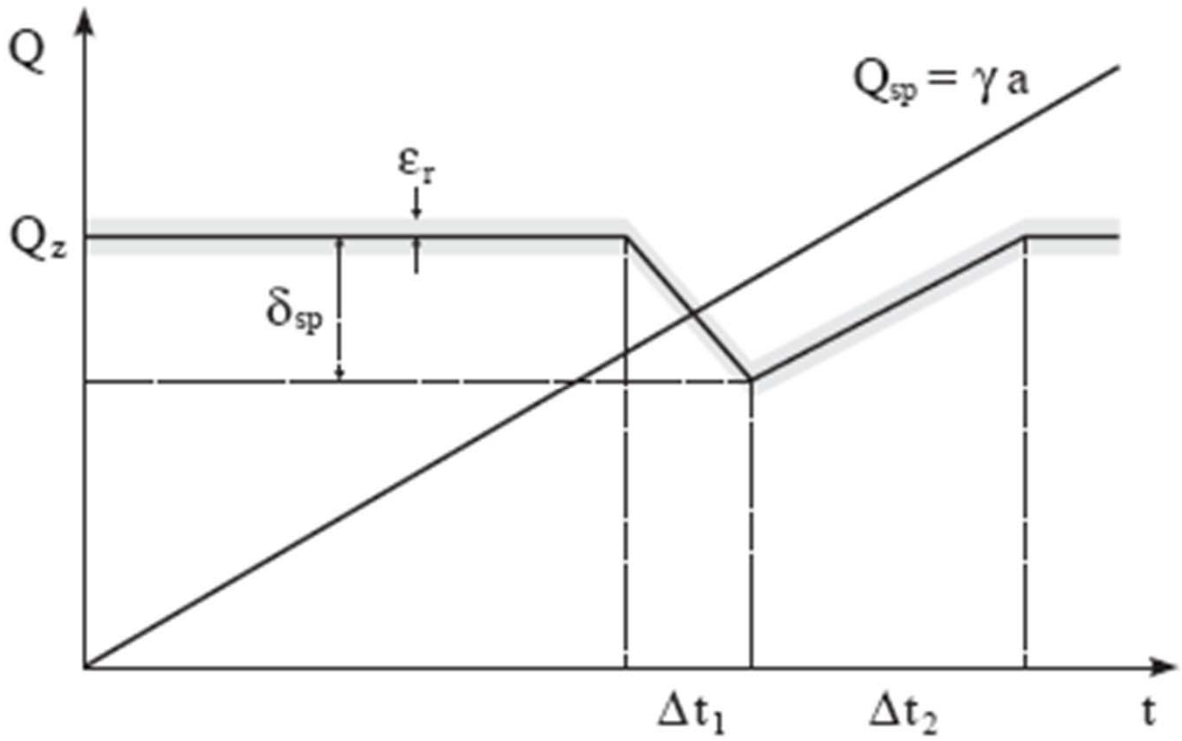
AGS warm partial snake.



And for real

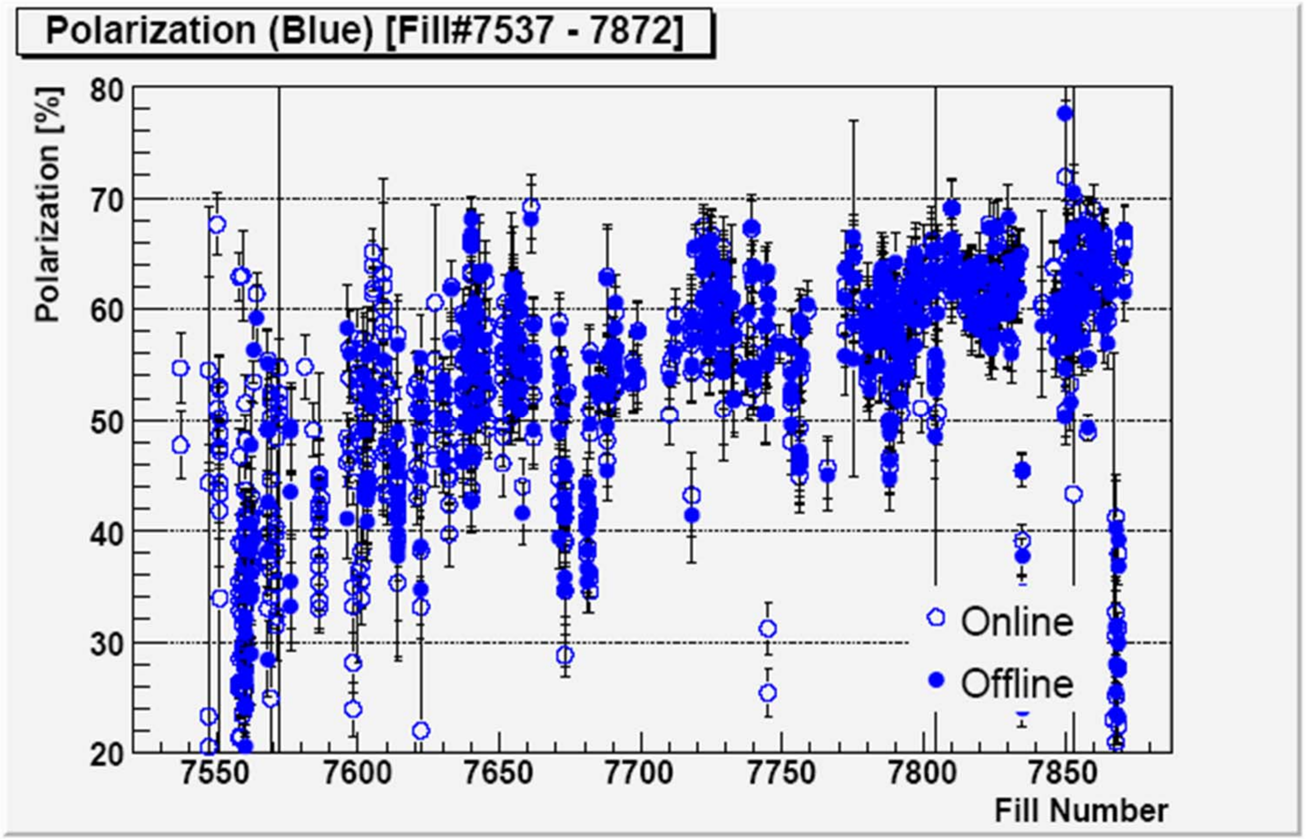
``Tune jumping'' to cross a Q_y resonance quickly. E.g., at ELSA

Not a comfortable way to have to proceed!



Christoph Steier, thesis, **BONN-IR-99-07**

Recent RHIC polarisation at 100 GeV..
Up to 100 GeV the limit is set by the polarisation supplied by the AGS
Down to 25% at 200 GeV..



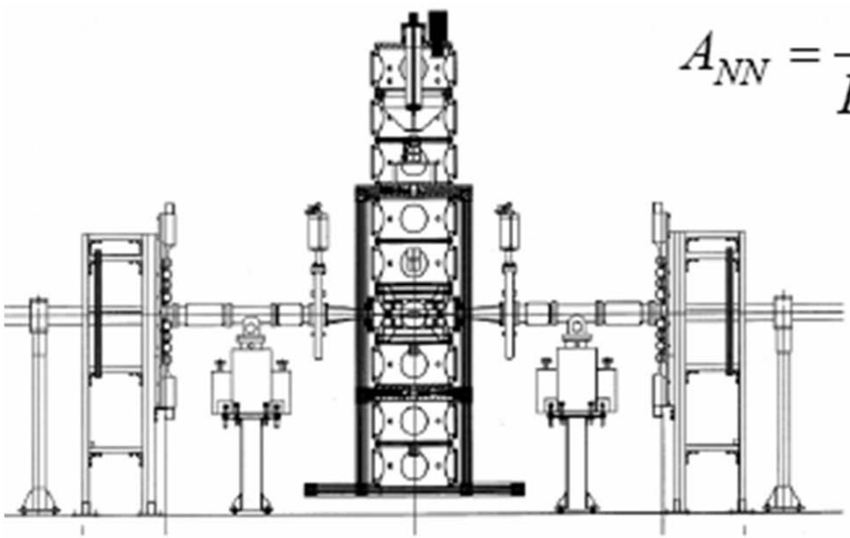
An enormous achievement!

$p\uparrow p \to pp$ and $p\uparrow p\uparrow \to pp$ with a Polarized Gas Jet Target

polarized gas JET target

$$A_N = \frac{1}{P_T}\frac{\left(N_L^{\Uparrow}+N_R^{\Downarrow}\right)-\left(N_R^{\Uparrow}+N_L^{\Downarrow}\right)}{\left(N_L^{\Uparrow}+N_R^{\Downarrow}\right)+\left(N_R^{\Uparrow}+N_L^{\Downarrow}\right)}$$

$$A_{NN} = \frac{1}{P_T P_B}\frac{N^{\uparrow\uparrow+\downarrow\downarrow}-N^{\uparrow\downarrow+\downarrow\uparrow}}{N^{\uparrow\uparrow+\downarrow\downarrow}+N^{\uparrow\downarrow+\downarrow\uparrow}}$$



RHIC polarized proton beams

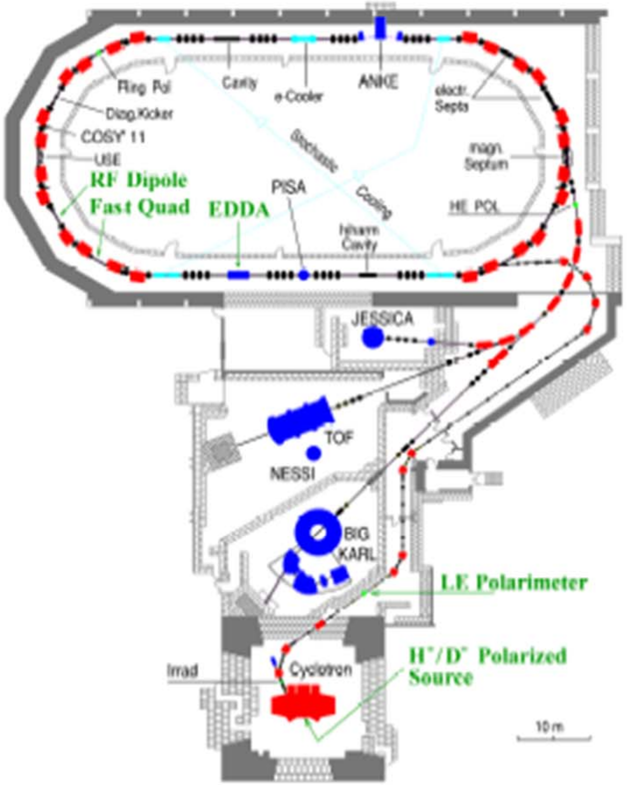
DIS 2005

Alessandro Bravar

BROOKHAVEN NATIONAL LABORATORY

COSY Accelerator Facility

Forschungszentrum Jülich
in der Helmholtz-Gemeinschaft



Ions: (pol. & unpol.) p and d

Momentum: 300 to 3650 MeV/c for p
540 to 3650 MeV/c for d

Circumference of the ring: 184 m

Electron Cooling at injection

Stochastic Cooling above 1.5 GeV/c

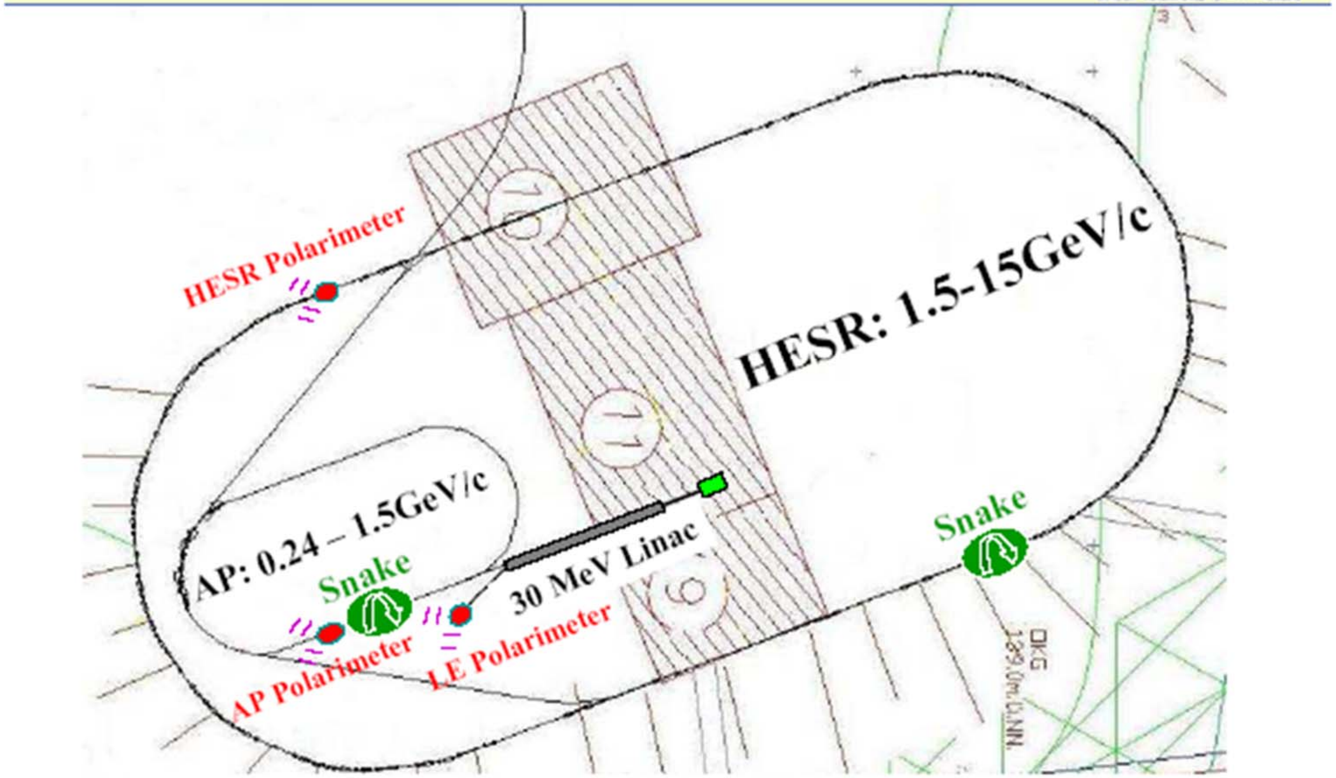
Targets:

- Internal: solid, cluster, atomic beam
- External: solid, liquid

09/14/2004 A. Lehrach, FZ Jülich, COSY Student Program 2004 13

HESR Accelerator Complex for Polarized Beams

Forschungszentrum Jülich



09/14/2004 A. Lehrach, FZ Jülich, COSY Student Program 2004 17

The amplitude dependent spin tune: the generalisation of ν_0 in the same way that $\hat{n}$ is the generalisation of $\hat{n}_0$ $\Rightarrow$ beautiful demonstration of the results of continual pushing for understanding and for finding computational methods.

$\Longrightarrow$ generalisation of F-S to beyond first order resonances.

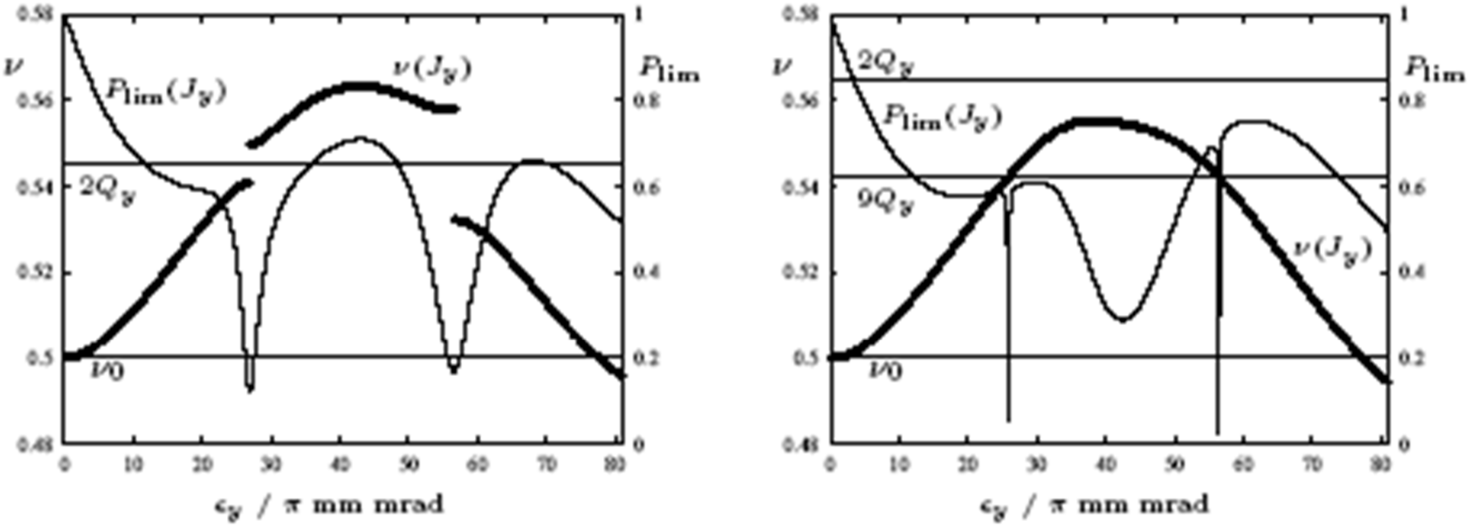


FIGURE 2. The amplitude dependent spin tune ν and P_{lim} on phase space ellipses with normalised vertical emittance ϵ_y as calculated with SPRINT for HERA–p at 805 GeV. Left: vertical tune $Q_y = 32.2725$, right: $Q_y = 32.2825$.

See the G.Hoffstaetter book and the M. Vogt theses for the other nice things emerging from the feasibility study for getting 920 GeV polarised protons in HERA, e.g., the stroboscopic averaging algorithm for constructing the ISF.

Topic 10

“Kinetic polarisation” of electrons

``kinetic polarisation''

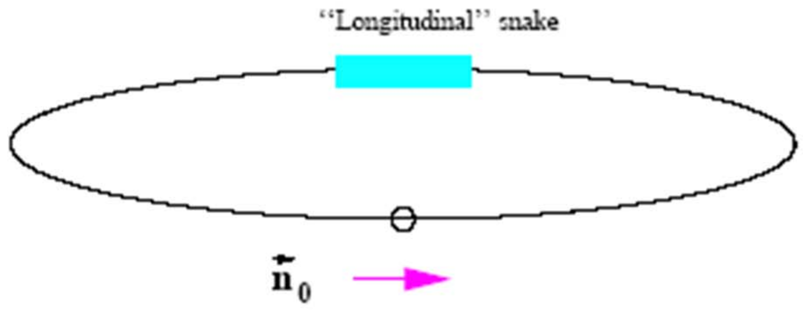
The Derbenev-Kondratenko formula

$$\Rightarrow P_{\mathrm{DK}}(s=\infty) = -\frac{8}{5\sqrt{3}} \frac{\oint ds \left\langle \frac{\hat{b}.\left(\hat{n} - \frac{\partial \hat{n}}{\partial \delta}\right)}{|\rho^3(s)|} \right\rangle_u}{\oint ds \left\langle \frac{1 - \frac{2}{9}(\hat{n}.\hat{\upsilon})^2 + \frac{11}{18}\left(\frac{\partial \hat{n}}{\partial \delta}\right)^2}{|\rho^3(s)|} \right\rangle_u}$$

The term $\hat{b}.\frac{\partial \hat{n}}{\partial \delta}$ is due to the dependence of the non-flip sync. rad. power on the orientation of the spin. This term is only present in a semi-classical calculation of the spin dynamics (D-K 1973, Mane 1987).

It is not seen in the simple "spin diffusion'' picture based on stochastic D.E.s

Snakes and wigglers for electron/positron polarization



	HERA	$\tau_{dep}=$ 260	millisecs at 27.5 GeV
$\vec{n}_0$ horizontal everywhere:	eRHIC	τ_{dep}	tens of seconds at 10 GeV
	MIT−Bates	τ_{dep}	hours at few hundred MeV

No Sokolov−Ternov ⟶ very exciting possibility to observe "kinetic polarization" at MIT−Bates ring.

since $\hat{b}.\frac{\partial \hat{n}}{\partial \delta} \neq 0$ in the dipoles.

An attempt to observe KP at AmPs , just before the machine was switched off in 1998, was not able to deliver conclusive results. If more time had been available, one could have perhaps observed it.

Appendix 1: coordinate systems for spin.

Internal Report
DESY M 92–04
Second Revision: September 1999

Notes on Spin Dynamics in Storage Rings

D.P. Barber, K. Heinemann and G. Ripken

This paper uses the symbol Q_{spin} instead of ν_0

$$\vec{e}_s \cdot \frac{d}{ds}\, \xi_{0s} + \vec{e}_x \cdot \frac{d}{ds}\, \xi_{0x} + \vec{e}_z \cdot \frac{d}{ds}\, \xi_{0z} \;=\; \vec{\Omega}^{(0)} \times \vec{\xi}_0 \tag{4.2b}$$

Appendix B: The Periodic Spin Frame $(\vec{n}_0, \vec{m}, \vec{l})$ along the Closed Orbit

In order to investigate spin motion along the closed orbit, we write eqn. (4.2b) [3] in the form:

$$\frac{d}{ds}\, \vec{\xi}_0(s) = \underline{\Omega}^{(0)}(s) \cdot \vec{\xi}_0(s) \tag{B.1}$$

where we have set

$$\vec{\xi}_0 \;=\; \begin{pmatrix} \xi_{0s} \\ \xi_{0x} \\ \xi_{0z} \end{pmatrix} \tag{B.2a}$$

and

$$\underline{\Omega}^{(0)}(s) = \begin{pmatrix} 0 & -\Omega_z^{(0)} & \Omega_x^{(0)} \\ \Omega_z^{(0)} & 0 & -\Omega_s^{(0)} \\ -\Omega_x^0 & \Omega_s^{(0)} & 0 \end{pmatrix} . \tag{B.2b}$$

The transfer matrix $\underline{M}_{(spin)}(s, s_0)$ for the spin motion defined by

$$\vec{\xi}_0(s) = \underline{M}_{(spin)}(s, s_0) \cdot \vec{\xi}_0(s_0)$$

satisfies the relationships:

$$\underline{M}^T_{(spin)}(s, s_0) \cdot \underline{M}_{(spin)}(s, s_0) \;=\; \underline{1} \;; \tag{B.3a}$$

$$\det\,[\underline{M}_{(spin)}(s, s_0)] = 1 \tag{B.3b}$$

[3]This equation can be solved by methods as described in Appendix C, using $\vec{\Omega}^{(0)}$ instead of $\vec{\tilde{\Omega}}$.

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since (using eqn. (B.1))

$$\frac{d}{ds}\,\underline{M}_{(spin)}(s,s_0) \;=\; \underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s,s_0)\;;$$

$$\underline{M}_{(spin)}(s_0,s_0) \;=\; \underline{1}$$

and therefore (with $[\underline{\Omega}^{(0)}]^T = -\underline{\Omega}^{(0)}$)

$$\begin{aligned}
\frac{d}{ds}\left[\underline{M}^T_{(spin)}(s,s_0)\cdot\underline{M}_{(spin)}(s,s_0)\right] &= \left[\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s,s_0)\right]^T\cdot\underline{M}_{(spin)}(s,s_0)\\
&\quad +\underline{M}^T_{(spin)}(s,s_0)\cdot\left[\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s,s_0)\right]\\
&= -\underline{M}_{(spin)}(s,s_0)^T\cdot\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s,s_0)\\
&\quad +\underline{M}^T_{(spin)}(s,s_0)\cdot\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s,s_0)\\
&= \underline{0}\;;
\end{aligned}$$

$$\det\,M_{(spin)}(s,s_0) \;=\; \det\,M_{(spin)}(s_0,s_0)=1\;,$$

i.e. $\underline{M}_{(spin)}(s,s_0)$ is an orthogonal matrix with determinant 1.

Let us now consider the eigenvalue problem for the revolution matrix $\underline{M}_{(spin)}(s_0+L,s_0)$ with the eigenvalues α_μ and eigenvectors $\vec{r}_\mu(s_0)$:

$$\underline{M}_{(spin)}(s_0+L,s_0)\;\vec{r}_\mu(s_0) = \alpha_\mu\cdot\vec{r}_\mu(s_0)\;; \qquad (\mu=1,\;2,\;3) \tag{B.4}$$

using the notation

$$\vec{r}_\mu \;\hat{=}\; \begin{pmatrix}\vec{r}_\mu\cdot\vec{e}_s\\ \vec{r}_\mu\cdot\vec{e}_x\\ \vec{r}_\mu\cdot\vec{e}_z\end{pmatrix}.$$

Because of (B.3a,b) we can write [35]:

$$\begin{aligned}
\alpha_1 &= 1\;;\\
\alpha_2 &= e^{i\cdot 2\pi\cdot Q_{spin}}\;;\\
\alpha_3 &= e^{-i\cdot 2\pi\cdot Q_{spin}}\;;
\end{aligned} \tag{B.5}$$

$$(Q_{spin} = \text{real number})$$

and

$$\begin{aligned} \vec{r}_1(s_0) &= \vec{n}_0(s_0) \; ; && \text{(B.6a)} \\ \vec{r}_2(s_0) &= \vec{m}_0(s_0) + i \cdot \vec{l}_0(s_0) \; ; && \text{(B.6b)} \\ \vec{r}_3(s_0) &= \vec{m}_0(s_0) - i \cdot \vec{l}_0(s_0) \; ; && \text{(B.6c)} \end{aligned}$$

$$(\vec{n}_0, \vec{m}_0, \vec{l}_0 = \text{real vectors}) \; .$$

If we require that

$$\begin{aligned} \vec{r}_1^{\,+} \cdot \vec{r}_1 &= 1 \; ; && \text{(B.7a)} \\ \vec{r}_2^{\,+} \cdot \vec{r}_2 &\equiv \vec{r}_3^{\,+} \cdot \vec{r}_3 = 2 \; ; && \text{(B.7b)} \end{aligned}$$

$$(\text{normalising conditions})$$

we find, using eqn. (B.3a) [35]:

$$|\vec{n}_0(s_0)| = |\vec{m}_0(s_0)| = |\vec{l}_0(s_0)| = 1 \; ; \qquad \text{(B.8a)}$$

$$\vec{n}_0(s_0) \perp \vec{m}_0(s_0) \perp \vec{l}_0(s_0) \; . \qquad \text{(B.8b)}$$

Thus the vectors $\vec{n}_0(s_0)$, $\vec{m}_0(s_0)$ and $\vec{l}_0(s_0)$ form an orthogonal system of unit vectors. Choosing the direction of $\vec{n}_0(s_0)$ such that

$$\vec{n}_0(s_0) = \vec{m}_0(s_0) \times \vec{l}_0(s_0) \qquad \text{(B.8c)}$$

these vectors form a righthanded coordinate system.

In this way we have found a coordinate frame for the position $s = s_0$.

An orthogonal system of unit vectors at an arbitrary position s can be defined by applying the transfer matrix $\underline{M}_{(spin)}(s, s_0)$ to the vectors $\vec{n}_0(s_0)$, $\vec{m}_0(s_0)$ and $\vec{l}_0(s_0)$:

$$\begin{aligned} \vec{n}_0(s) &= \underline{M}_{(spin)}(s, s_0)\, \vec{n}_0(s_0) \; ; && \text{(B.9a)} \\ \vec{m}_0(s) &= \underline{M}_{(spin)}(s, s_0)\, \vec{m}_0(s_0) \; ; && \text{(B.9b)} \\ \vec{l}_0(s) &= \underline{M}_{(spin)}(s, s_0)\, \vec{l}_0(s_0) \; . && \text{(B.9c)} \end{aligned}$$

Because of eqn. (B.3a,b) the orthogonality relations remain unchanged:

$$\begin{aligned} \vec{n}_0(s) &= \vec{m}_0(s) \times \vec{l}_0(s) && \text{(B.10a)} \\ \vec{m}_0(s) &\perp \vec{l}_0(s) \; ; && \text{(B.10b)} \\ |\vec{n}_0(s)| &= |\vec{m}_0(s)| = |\vec{l}_0(s)| = 1 \; . && \text{(B.10c)} \end{aligned}$$

The coordinate frame defined by $\vec{n}_0(s)$, $\vec{m}_0(s)$ and $\vec{l}_0(s)$ is not yet appropriate for a description of the spin motion, because it does not transform into itself after one revolution of the particles:

$$\begin{aligned} \vec{m}_0(s_0 + L) + i\, \vec{l}_0(s_0 + L) &= \underline{M}_{(spin)}(s_0 + L, s_0) \left[\vec{m}_0(s_0) + i\vec{l}_0(s_0)\right] \\ &= e^{i \cdot 2\pi \cdot Q_{spin}} \cdot \left[\vec{m}_0(s_0) + i\vec{l}_0(s_0)\right] \\ &\neq \vec{m}_0(s_0) + i\, \vec{l}_0(s_0) \end{aligned}$$

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$$(\text{if } Q_{spin} \neq \text{integer}).$$

i.e. although $\vec{n}_0(s)$ is periodic by eqns. (B.5), (B.6a), $\vec{m}_0(s)$ and $\vec{l}_0(s)$ are not periodic.

But by introducing a phase function $\psi(s)$ and using another orthogonal matrix $\underline{\hat{D}}(s)$:

$$\underline{\hat{D}}(s) = \begin{pmatrix} \cos[\psi_{spin}(s)] & \sin[\psi_{spin}(s)] \\ -\sin[\psi_{spin}(s)] & \cos[\psi_{spin}(s)] \end{pmatrix} \tag{B.11}$$

with

$$\underline{\hat{D}}^T(s) \cdot \underline{\hat{D}}(s) \;=\; \underline{1}\,; \tag{B.12a}$$

$$\det\,[\underline{\hat{D}}(s)] = 1 \tag{B.12b}$$

we can construct a periodic orthogonal system of unit vectors from $\vec{n}_0(s)$, $\vec{m}_0(s)$ and $\vec{l}_0(s)$. Namely, if we put [36]:

$$\begin{pmatrix} \vec{m}(s) \\ \vec{l}(s) \end{pmatrix} = \underline{\hat{D}}(s) \begin{pmatrix} \vec{m}_0(s) \\ \vec{l}_0(s) \end{pmatrix}$$

$$\begin{aligned} \Longrightarrow \qquad \vec{m}(s) + i\,\vec{l}(s) \;&=\; e^{-i\cdot\psi_{spin}(s)} \cdot \left[\vec{m}_0(s) + i\,\vec{l}_0(s)\right] \qquad \text{(B.13)} \\ &\neq\; \vec{m}_0(s_0) + i\,\vec{l}_0(s_0) \end{aligned}$$

we find, using eqns. (B.12a, b) :

$$\vec{n}_0(s) \;=\; \vec{m}(s) \times \vec{l}(s)\,; \tag{B.14a}$$

$$\vec{m}(s) \;\perp\; \vec{l}(s)\,; \tag{B.14b}$$

$$|\vec{n}_0(s)| \;=\; |\vec{m}(s)| \;=\; |\vec{l}(s)| \;=\; 1\,. \tag{B.14c}$$

Since

$$\begin{aligned} \vec{m}(s+L) + i\cdot\vec{l}(s+L) \;&=\; e^{-i\cdot\psi_{spin}(s+L)} \cdot \left[\vec{m}_0(s+L) + i\cdot\vec{l}_0(s+L)\right] \\ &=\; e^{-i\cdot\psi_{spin}(s+L)} \cdot e^{i\cdot 2\pi\,Q_{spin}} \cdot \left[\vec{m}_0(s) + i\cdot\vec{l}_0(s)\right] \\ &=\; e^{-i\cdot\psi_{spin}(s+L)} \cdot e^{i\cdot 2\pi\,Q_{spin}} \cdot e^{+i\cdot\psi(s)} \cdot \left\{\vec{m}(s) + i\cdot\vec{l}(s)\right\} \\ &=\; e^{-i\cdot[\psi_{spin}(s+L) - \psi_{spin}(s)]} \cdot e^{i\cdot 2\pi\,Q_{spin}} \cdot \left\{\vec{m}(s) + i\cdot\vec{l}(s)\right\} \end{aligned}$$

it follows, that the condition of periodicity for $\vec{n}_0$, $\vec{m}$ and $\vec{l}$:

$$\left(\vec{n}_0,\ \vec{m},\ \vec{l}\right)_{s=s_0+L} \;=\; \left(\vec{n}_0,\ \vec{m},\ \vec{l}\right)_{s=s_0} \tag{B.15}$$

can indeed be fulfilled if the phase function $\psi(s)$ satisfies the following relationship:

$$\psi_{spin}(s+L) - \psi_{spin}(s) = 2\pi \cdot Q_{spin} \; ; \tag{B.16a}$$

$$(Q_{spin} = \text{spin tune}).$$

For instance we can choose:

$$\psi_{spin}(s) = 2\pi \cdot Q_{spin} \cdot \frac{s}{L} \; . \tag{B.16b}$$

In this frame, spins on the closed orbit precess uniformly with respect to $\vec{m}$ and $\vec{l}$.

Note that the spin tune Q_{spin} can be separated into an arbitrary integer part κ and a fractional part $\tilde{Q}_{spin}$:

$$\begin{aligned} Q_{spin} &= \kappa + \tilde{Q}_{spin} \; ; \\ 0 &\le \tilde{Q}_{spin} < 1 \; . \end{aligned}$$

Taking the derivatives of $\vec{m}(s)$ and $\vec{l}(s)$ with respect to s, and taking into account eqns. (B.13), (B.9), and (4.2b) we get

$$\frac{d}{ds}\begin{pmatrix} \vec{m}\cdot\vec{e}_s \\ \vec{m}\cdot\vec{e}_x \\ \vec{m}\cdot\vec{e}_z \end{pmatrix} = \underline{\Omega}^{(0)}(s) \begin{pmatrix} \vec{m}\cdot\vec{e}_s \\ \vec{m}\cdot\vec{e}_x \\ \vec{m}\cdot\vec{e}_z \end{pmatrix} + \psi'(s)\cdot \begin{pmatrix} \vec{l}\cdot\vec{e}_s \\ \vec{l}\cdot\vec{e}_x \\ \vec{l}\cdot\vec{e}_z \end{pmatrix} \; ; \tag{B.17a}$$

$$\frac{d}{ds}\begin{pmatrix} \vec{l}\cdot\vec{e}_s \\ \vec{l}\cdot\vec{e}_x \\ \vec{l}\cdot\vec{e}_z \end{pmatrix} = \underline{\Omega}^{(0)}(s) \begin{pmatrix} \vec{l}\cdot\vec{e}_s \\ \vec{l}\cdot\vec{e}_x \\ \vec{l}\cdot\vec{e}_z \end{pmatrix} - \psi'(s)\cdot \begin{pmatrix} \vec{m}\cdot\vec{e}_s \\ \vec{m}\cdot\vec{e}_x \\ \vec{m}\cdot\vec{e}_z \end{pmatrix} \tag{B.17b}$$

and $\vec{n}_0(s)$ satisfies (see (B.9a))

$$\frac{d}{ds}\begin{pmatrix} \vec{n}_0\cdot\vec{e}_s \\ \vec{n}_0\cdot\vec{e}_x \\ \vec{n}_0\cdot\vec{e}_z \end{pmatrix} = \underline{\Omega}^{(0)}(s) \begin{pmatrix} \vec{n}_0\cdot\vec{e}_s \\ \vec{n}_0\cdot\vec{e}_x \\ \vec{n}_0\cdot\vec{e}_z \end{pmatrix} \; . \tag{B.17c}$$

Finally, the vectors

$$\vec{r}_1(s) = \vec{n}_0(s) \equiv \underline{M}_{(spin)}(s,s_0)\, \vec{r}_1(s_0) \; ; \tag{B.18a}$$

$$\vec{r}_2(s) = \vec{m}_0(s) + i\cdot\vec{l}_0(s) \equiv \underline{M}_{(spin)}(s,s_0)\, \vec{r}_2(s_0) \; ; \tag{B.18b}$$

$$\vec{r}_3(s) = \vec{m}_0(s) - i\cdot\vec{l}_0(s) \equiv \underline{M}_{(spin)}(s,s_0)\, \vec{r}_3(s_0) \tag{B.18c}$$

are eigenvectors of the revolution matrix $\underline{M}_{(spin)}$ with the same eigenvalues as in (B.5):

$$\underline{M}_{(spin)}(s+L,s)\, \vec{r}_\mu(s) = \alpha_\mu \cdot \vec{r}_\mu(s) \; . \tag{B.19}$$

Thus, the eigenvalues α_μ and the quantity Q_{spin} defined by eqn. (B.5) are independent of the chosen initial position s_0.

Finally, we remark that $(\vec{n}_0, \vec{m}_0, \vec{l}_0)$ are all T–BMT solutions whereas $(\vec{m}, \vec{l})$ in general are not T–BMT solutions.

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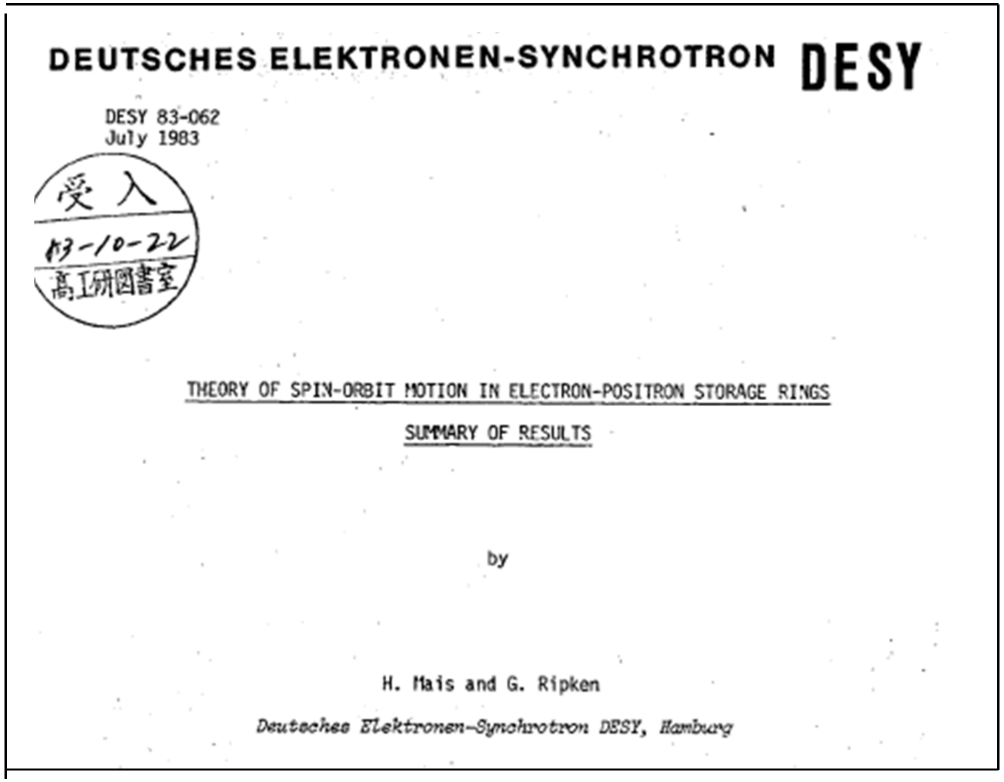
Appendix 2:
Symplecticity, orthogonality of eigenvectors etc.



DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 83-062
July 1983

THEORY OF SPIN-ORBIT MOTION IN ELECTRON-POSITRON STORAGE RINGS

SUMMARY OF RESULTS

by

H. Mais and G. Ripken

Deutsches Elektronen-Synchrotron DESY, Hamburg

Note some differences in notation wrt these lectures..

8. The unperturbed problem

In order to investigate the spin-orbit motion it is reasonable to neglect in a first approximation the small terms $\delta\hat{\underline{A}}$ and $\delta\vec{\tilde{c}}$ and to consider only the "unperturbed problem"

$$\frac{d}{ds}\vec{u} = \hat{\underline{A}}\,\underline{u} \tag{8.1}$$

with the orbital part

$$\frac{d}{ds}\vec{\tilde{y}} = \underline{A}\,\vec{\tilde{y}} \tag{8.2a}$$

and the spin part

$$\frac{d}{ds}\vec{\zeta} = \underline{G}_o\,\vec{\tilde{y}} + \underline{D}_o\,\vec{\zeta}\ . \tag{8.2b}$$

The radiative perturbations described by $\delta\hat{\underline{A}}$ and $\delta\vec{\tilde{c}}$ will then be treated in a second step with perturbation theory.

8.1 The unperturbed orbital motion

8.1.1 Symplectic structure of the transfer matrices

The unperturbed orbital motion is described by (8.2a). The solution of this equation is given by

$$\vec{\tilde{y}}(s) = \underline{M}(s,\ s_o)\ \vec{\tilde{y}}(s_o) \tag{8.3}$$

with $\underline{M}(s,\ s_o)$ being the transfer matrix belonging to (8.2a). The elements of $\underline{M}(s,\ s_o)$ have been calculated already in chapter 5.1.2: $\underline{M}(s,\ s_o)$ is a submatrix of the enlarged transfer matrix $\hat{\underline{M}}(s,\ s_o)$ such that

$$M_{ik} = \hat{M}_{ik} \qquad (i,\ k = 1,\ 2.....6)\ . \tag{8.4}$$

It is important for our further considerations that the orbital equations can be written in canonical form

$$\tilde{x}' = \frac{\partial \mathcal{H}}{\partial \tilde{p}_x} \quad ; \quad \tilde{p}_x' = -\frac{\partial \mathcal{H}}{\partial \tilde{x}}$$

$$\tilde{z}' = \frac{\partial \mathcal{H}}{\partial \tilde{p}_z} \quad ; \quad \tilde{p}_z' = -\frac{\partial \mathcal{H}}{\partial \tilde{z}}$$

$$\sigma' = \frac{\partial \mathcal{H}}{\partial \tilde{p}_\sigma} \quad ; \quad p_\sigma' = -\frac{\partial \mathcal{H}}{\partial \tilde{\sigma}}$$

with the Hamiltonian ($\tilde{p}_\sigma \equiv \tilde{\eta}$)

$$\mathcal{H} = \frac{1}{2}\{G_1 \tilde{x}^2 + G_2 \tilde{z}^2 - 2N\tilde{x}\tilde{z} + (\tilde{p}_x + H\tilde{z})^2 + (\tilde{p}_z - H\tilde{x})^2\}$$
$$- \frac{1}{2}\tilde{\sigma}^2 \frac{e\hat{V}}{E_o} k \cdot \frac{2\pi}{L} \cos\phi \sum_\nu \delta(s - s_\nu) - (K_x\tilde{x} + K_z\tilde{z})\tilde{p}_\sigma .$$

The canonical structure of the equations of motion then implies that the transfer matrices $\underline{M}(s, s_o)$ must be symplectic which means that the following relations are valid [1]

$$\underline{M}^T(s, s_o) \cdot \underline{S} \cdot \underline{M}(s, s_o) = \underline{S} \tag{8.5a}$$

with

$$\underline{S} = \begin{pmatrix} \underline{S}_2 & \underline{0} & \underline{0} \\ \underline{0} & \underline{S}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_2 \end{pmatrix} \quad ; \quad \underline{S}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{8.5b}$$

8.1.2 The eigenvalue spectrum of the revolution matrix $\underline{M}(s_o + L, s_o)$

The characteristic features of the synchro-betatron oscillations show up in the symplectic structure of the transfer matrices, in particular in the revolution matrix $\underline{M}(s_o + L, s_o)$.

The following statements are valid for the eigenvalue problem of $\underline{M}(s_o + L, s_o)$

$$\underline{M}(s_o + L, s_o) \cdot \vec{v}_\mu(s_o) = \lambda_\mu \vec{v}_\mu(s_o)$$

1) The eigenvectors can be divided into three groups

$$(\vec{v}_k(s_o), \ \vec{v}_{-k}(s_o)) \quad ; \quad k = I, II, III$$

with the properties

$$\underline{M}\,\vec{v}_k = \lambda_k\,\vec{v}_k\ ;\ \ \underline{M}\,\vec{v}_{-k} = \lambda_{-k}\,\vec{v}_{-k}\ ;\ \ \lambda_k\,\lambda_{-k} = 1 \qquad (8.6a)$$

$$\vec{v}_{-k}^{\,T}(s_o)\cdot\underline{S}\cdot\vec{v}_k(s_o) = -\vec{v}_k^{\,T}(s_o)\cdot\underline{S}\cdot\vec{v}_{-k}(s_o) \neq 0 \qquad (8.6b)$$

$$\vec{v}_\mu^{\,T}(s_o)\cdot\underline{S}\cdot\vec{v}_\nu(s_o) = 0 \quad \text{otherwise}$$

$$(k = I,\ II,\ III)\,.$$

In the following we shall put

$$\lambda_k = e^{-i2\pi Q_k}$$
$$\lambda_{-k} = e^{-i2\pi Q_{-k}} \qquad (8.7a)$$
$$(k = I,\ II,\ III)$$

Using (8.6a) we get

$$Q_{-k} = -Q_k \qquad (8.7b)$$

where the quantity Q_k can be a real or complex number.

2) Eqs. (8.6a) or (8.7) imply that the eigenvalues of the matrix $\underline{M}(s_o + L,\ s_o)$ always appear in reciprocal pairs

$$(\lambda_k,\ \lambda_{-k} = 1/\lambda_k) \qquad (k = I,\ II,\ III)$$

If λ is an eigenvalue, λ^* is also an eigenvalue because $\underline{M}(s_o + L,\ s_o)$ is a real matrix.
With these statements we find the following possibilities for the eigenvalue spectrum of the revolution matrix $\underline{M}(s_o + L,\ s_o)$ [1]:

a) All of the six eigenvalues are complex and lie on the unit circle in the complex plane

$$|\lambda_k| = |\lambda_{-k}| = 1 \qquad (8.8)$$
$$(k = I,\ II,\ III)$$

then $\vec{v}_\mu(s)$ is an eigenvector of the revolution matrix $\underline{M}(s + L, s)$ belonging to the same eigenvalue λ_μ [1]:

$$\underline{M}(s + L, s)\, \vec{v}_\mu(s) = \lambda_\mu\, \vec{v}_\mu(s) \tag{8.11}$$

Thus, the eigenvalue itself is independent of s:

$$\lambda_\mu(s) = \lambda_\mu(s_o)\,. \tag{8.12}$$

4) Defining

$$\begin{aligned} \vec{v}_\mu(s) &= \hat{\vec{v}}_\mu(s)\, e^{-i2\pi Q_\mu \cdot \frac{s}{L}} \\ \hat{\vec{v}}_\mu(s) &= \vec{v}_\mu\, e^{+i2\pi Q_\mu \cdot \frac{s}{L}} \end{aligned} \tag{8.13a}$$

we find

$$\hat{\vec{v}}_\mu(s + L) = \hat{\vec{v}}_\mu(s) \tag{8.13b}$$

which can be verified easily by putting (8.13a) into (8.11).

Eq. (8.13) is called the Floquet-theorem. It states that the vectors $\vec{v}_\mu(s)$, which are special solutions of the equations of motion (8.2a) can be written as the product of a periodic function $\hat{\vec{v}}_\mu(s)$ and a (generally aperiodic) harmonic function

$$e^{-i2\pi Q_\mu \cdot \frac{s}{L}}\,.$$

5) The general solution of the equation of motion (8.2a) is a linear combination of the special solutions (8.13a). Therefore it can be written in the form

$$\vec{y}(s) = \sum_{\substack{k=I,II\\III}} \{ A_k\, \hat{\vec{v}}_k(s)\, e^{-i2\pi Q_k \cdot \frac{s}{L}} + A_{-k}\, \hat{\vec{v}}_{-k}(s)\, e^{-i2\pi Q_{-k} \cdot \frac{s}{L}} \}\,. \tag{8.14}$$

This equation implies that the amplitudes of the synchro-betatron oscillations remain bounded (stable motion) only if the quantities Q_k are real numbers, which also means, that the eigenvalues must lie on the unit circle, as mentioned already:

$$|\lambda_k| = |\lambda_{-k}| = 1 \qquad (k = I, II, III) \tag{8.15}$$

(criterion of orbital stability).

If at least one of the exponents Q_k is complex, Q_k, or Q_{-k} has a positive imaginary part. In this case the components of $\vec{y}(s)$ grow exponentially and the particle motion becomes unstable.

6) For the following discussions we shall always assume that the criterion of stability (8.15) is satisfied.
Then, it follows from (8.9c)

$$\vec{v}_{-k} = \vec{v}_k^{\,*} \qquad (k = I, II, III)$$

and eq. (8.6b) reduces to ($\vec{v}^{\,+} = (\vec{v}^{\,*})^T$)

$$\vec{v}_k^{\,+}(s_o) \cdot \underline{S}\, \vec{v}_k(s_o) = - \vec{v}_{-k}^{\,+}(s_o) \cdot \underline{S}\, \vec{v}_{-k}(s_o) \neq 0 \tag{8.16a}$$

$$\vec{v}_\mu^{\,+}(s_o) \cdot \underline{S}\, \vec{v}_\nu(s_o) = 0 \quad \text{otherwise.} \tag{8.16b}$$

The terms

$$\vec{v}_\mu^{\,+}(s_o) \cdot \underline{S}\, \vec{v}_\mu(s_o)$$

appearing in (8.16a) are purely imaginary:

$$[\vec{v}_\mu^{\,+}(s_o) \cdot \underline{S}\, \vec{v}_\mu(s_o)]^+ = \vec{v}_\mu^{\,+}(s_o) \cdot \underline{S}^+ \vec{v}_\mu(s_o) = - [\vec{v}_\mu^{\,+}(s_o) \cdot \underline{S}\, \vec{v}_\mu(s_o)]$$

$$(\text{since } \underline{S}^+ = - \underline{S})$$

so that the following normalizing conditions can be used for the vectors $\vec{v}_k(s_o)$ and $\vec{v}_{-k}(s_o)$ $(k = I, II, III)$

$$\vec{v}_k^{\,+}(s_o) \cdot \underline{S}\, \vec{v}_k(s_o) = - \vec{v}_{-k}^{\,+}(s_o) \cdot \underline{S}\, \vec{v}_{-k}(s_o) = i \tag{8.17}$$

$$(k = I, II, III) .$$

The validity of the symplectic condition (8.5a) then implies that the eigenvectors $\vec{v}_k(s)$ and $\vec{v}_{-k}(s)$ $(k = I, II, III)$ at the position s satisfy the same conditions (8.16) and (8.17)

$$\left.\begin{aligned} &\vec{v}_k^{\,+}(s)\, \underline{S}\, \vec{v}_k(s) = - \vec{v}_{-k}^{\,+}(s)\, \underline{S}\, \vec{v}_{-k}(s) = i ; \\ &\vec{v}_\nu^{\,+}(s)\, \underline{S}\, \vec{v}_\mu(s) = 0 \quad \text{otherwise .} \end{aligned}\right. \tag{8.18}$$