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Transformation: Two 2-component Dirac equations in one 4-component Dirac equation

To avoid too much matrix algebra, we show the equality of the two Dirac equations, page 19, with the four-dim denotation Dirac equation, also page 19, in this way:

$$|\sigma, p, 1> = [\frac{p_0}{m} - \frac{\vec{p}\vec{\sigma}}{m}]|\sigma, p, 2> \tag{1}$$

write as matrix for both solution of $\sigma = \pm\frac{1}{2}$:

$$\begin{pmatrix} |+\frac{1}{2}, p, 1> \\ |-\frac{1}{2}, p, 1> \end{pmatrix} = \frac{1}{m} \begin{pmatrix} (p_0 - p_3) & -(p_1 - ip_2) \\ -(p_1 + ip_2) & (p_0 + p_3) \end{pmatrix} \begin{pmatrix} |+\frac{1}{2}, p, 2> \\ |-\frac{1}{2}, p, 2> \end{pmatrix} \tag{2}$$

$\Rightarrow$

$$|+\frac{1}{2}, p, 1> = \frac{1}{m}(p_0 - p_3)|+\frac{1}{2}, p, 2> + \frac{1}{m}(-p_1 + ip_2)|-\frac{1}{2}, p, 2> \tag{3}$$

$$|-\frac{1}{2}, p, 1> = \frac{1}{m}(-p_1 - ip_2)|+\frac{1}{2}, p, 2> + \frac{1}{m}(p_0 + p_3)|-\frac{1}{2}, p, 2> \tag{4}$$

Analogous procedure with the second 2-component Dirac equation:

$$|\sigma, p, 2> = [\frac{p_0}{m} + \frac{\vec{p}\vec{\sigma}}{m}]|\sigma, p, 1> \tag{5}$$

Write it also explicitly as matrix:

$$\begin{pmatrix} |+\frac{1}{2}, p, 2> \\ |-\frac{1}{2}, p, 2> \end{pmatrix} = \frac{1}{m} \begin{pmatrix} (p_0 + p_3) & +(p_1 - ip_2) \\ (p_1 + ip_2) & (p_0 - p_3) \end{pmatrix} \begin{pmatrix} |+\frac{1}{2}, p, 1> \\ |-\frac{1}{2}, p, 1> \end{pmatrix} \tag{6}$$

$\Rightarrow$

$$|+\frac{1}{2}, p, 2> = \frac{1}{m}(p_0 + p_3)|+\frac{1}{2}, p, 1> + \frac{1}{m}(p_1 - ip_2)|-\frac{1}{2}, p, 1> \tag{7}$$

$$|-\frac{1}{2}, p, 2> = \frac{1}{m}(p_1 + ip_2)|+\frac{1}{2}, p, 1> + \frac{1}{m}(p_0 - p_3)|-\frac{1}{2}, p, 1> \tag{8}$$

Now, deriving the 4-dim denotation, page 19. First, the definition of the used γ matrices:

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & -\sigma_j \\ +\sigma_j & 0 \end{pmatrix}, \qquad j = 1, 2, 3, \tag{9}$$

where I denotes the 2×2 unit-matrix and σ^j the Pauli-matrices, page 8.

Keeping track of the order of the indices, eqn.(1) and (5) become:

$$(\gamma_\mu^\dagger p_\mu - m)_{\alpha'\alpha}|\sigma, p, \alpha' >= 0 \tag{10}$$

More explicitly:

$$\left\{ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^\dagger p_0 + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^\dagger p_1 + \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}^\dagger p_2 + \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}^\dagger p_3 - mI_{4\times 4} \right\} \begin{pmatrix} |p1> \\ |p2> \\ |p3> \\ |p4> \end{pmatrix} = 0 \tag{11}$$

$$\Rightarrow \begin{pmatrix} -m & 0 & p_0 + p_3 & p_1 - ip_2 \\ 0 & -m & p_1 + ip_2 & p_0 - p_3 \\ p_0 - p_3 & -p_1 + ip_2 & -m & 0 \\ -p_1 - ip_2 & p_0 + p_3 & 0 & -m \end{pmatrix} \begin{pmatrix} |+\frac{1}{2}, p, 2> \\ |-\frac{1}{2}, p, 2> \\ |+\frac{1}{2}, p, 1> \\ |-\frac{1}{2}, p, 1> \end{pmatrix} = 0 \tag{12}$$

This represents the following system of equations:

$$|+\frac{1}{2}, p, 2> \;=\; \frac{1}{m}(p_0 + p_3)|+\frac{1}{2}, p, 1> + \frac{1}{m}(p_1 - ip_2)|-\frac{1}{2}, p, 1> \tag{13}$$

$$|-\frac{1}{2}, p, 2> \;=\; \frac{1}{m}(p_1 + ip_2)|+\frac{1}{2}, p, 1> + \frac{1}{m}(p_0 - p_3)|-\frac{1}{2}, p, 1> \tag{14}$$

$$|+\frac{1}{2}, p, 1> \;=\; \frac{1}{m}(p_0 - p_3)|+\frac{1}{2}, p, 2> + \frac{1}{m}(-p_1 + ip_2)|-\frac{1}{2}, p, 2> \tag{15}$$

$$|-\frac{1}{2}, p, 1> \;=\; \frac{1}{m}(-p_1 - ip_2)|+\frac{1}{2}, p, 2> + \frac{1}{m}(p_0 + p_3)|-\frac{1}{2}, p, 2> \tag{16}$$

That's identical to Eqn.(7), (8), (3), (4)

Last step is only the definition of the wave functions:

$$u_\alpha(p) =< p\alpha|u > \tag{17}$$

Inserting this definition into Eq.(10) gives the Dirac equation in 4-dim denotation, page 19 (we have to do a hermitian conjugation, this removes the † in Eq.(10)):

$$(\gamma_\mu p_\mu - m)_{\alpha'\alpha} u_\alpha(p) = 0 \tag{18}$$