Higgs fields as nonunitary parallel transporters: a soluble model

C. Lehmann, G. Mack: *Gauge Theories with Nonunitary Parallel Transporters: A soluble Higgs model*

C. Lehmann, G. Mack, T. Prüstel: *Origin of quark masses and Kobayashi Maskawa matrix in a gauge theory with nonunitary parallel transporters*

Motivation

Dream:
Compute quark masses and Kobayashi Maskawa matrix of the standard model.

Basic hypothesis:
Higgs fields are nonunitary parallel transporters Φ along extra directions, e.g. across a defect (bilayered membrane).

Remember lattice gauge theory:
Lattice gauge fields $\mathcal{U}(x, y)$ sit on links of a lattice. They are also known as parallel transporters from x to y. In ordinary lattice gauge theories $\mathcal{U}(x, y)$ are unitary matrices.

Proposal: Abandon unitarity as a general requirement.

→ **geometric interpretation of Vierbein fields** as parts of gauge fields [Mack, Neugeborn, Prüstel]

- Vierbein of general relativity from nonunitary parallel transport in directions 0...3.
- Higgs from nonunitary parallel transport in further directions, e.g. across a defect (bilayered membrane).

→ **Higgs fields** in SM as nonunitary parallel transport between left- and right-handed fermions [Angermann, Mack, Olschewsky, Prüstel]

Flavor mass splitting
the essentials

1 Higgs fields are parallel transporters along links in extra directions.

Under gauge transformations

$$\Phi = \phi(x,y) \to g(x)\phi(x,y)g(y)^{-1},\ g(\cdot) \in \mathrm{G}, \Phi \in \mathrm{H}.$$

G Unitary gauge group (local symmetry)

H Holonomy group $\mathrm{H} \supset \mathrm{G}$
Matter fields form representation spaces of H, lattice gauge fields take values in H, typically noncompact.

2 The Higgs potential $V(\Phi)$ must be gauge invariant.
$\Rightarrow$ It has **G-biinvariance property**

$$V(u_1 \Phi u_2) = V(\Phi) \text{ for all } u_1, u_2 \in \mathrm{G}, \Phi \in \mathrm{H}.$$

3 For groups of interest

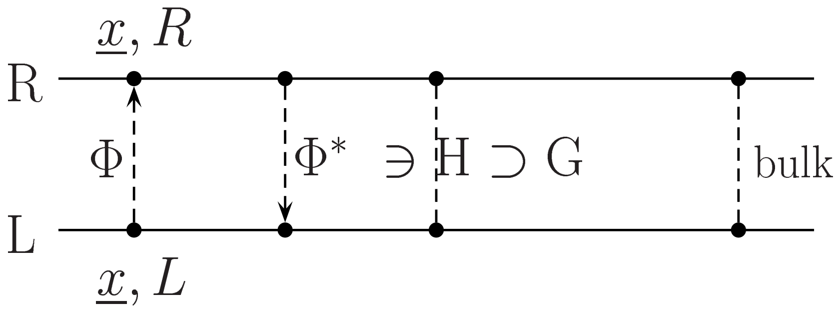
$$\Phi = u_1 d(\eta) u_2 \quad u_1, u_2 \in \mathrm{G},$$

$d(\eta)$ diagonal (make up the completely noncompact Cartan subgroup of H)

$$d(\eta) = \begin{pmatrix} e^{\eta_1} & & \\ & \ddots & \\ & & e^{\eta_r} \end{pmatrix},\ r = \text{rank of H},$$

$$\text{e.g. for } \mathrm{H} = \mathrm{SL}(2,\mathbb{C}):\ d(\eta) = \begin{pmatrix} e^{-\frac{\eta}{2}} & \\ & e^{\frac{\eta}{2}} \end{pmatrix}.$$

4 Bilayered membrane



Left- and right-handed fermions Ψ_L, Ψ_R on boundaries on different sides of a domain-wall (different points $(\underline{x}, R), (\underline{x}, L)$ in 5-dim. space time.)

5 biinvariance $\Rightarrow$

$$V(u_1 d(\eta) u_2) = V(d(\eta)) = \mathcal{V}(\eta).$$

Let W_T be discrete group of $w \in \mathrm{G}$, so that

$wd(\eta)w^* = d(w\eta)$ diagonal, e.g. SL(2,$\mathbb{C}$): $d(w\eta) = \begin{pmatrix} e^{\frac{\eta}{2}} & \\ & e^{-\frac{\eta}{2}} \end{pmatrix}$.

then $\mathcal{V}$ has discrete symmetry

$$\mathcal{V}(w\eta) = \mathcal{V}(\eta) \text{ for } w \in W_T.$$

- W_T-symmetry is **spontaneously broken**:
  minima $\hat{\eta}$ on nontrivial orbit of W_T and $w\hat{\eta} \neq \hat{\eta}\ \forall w$
  flavor
- W_T-symmetry is **not spontaneously broken**:
  only one minimum $w\hat{\eta} = \hat{\eta}$, $\hat{\eta} = 0$
  color

6 case of **symmetry breaking**

$$\mathfrak{z}\bar{\Psi}_R\Phi\Psi_L = \mathfrak{z}\bar{\Psi}_R u_1 d(\hat{\eta}) u_2 \Psi_L, \;\; [\mathfrak{z}] = [\text{mass}]$$
$$= \mathfrak{z}\bar{\Psi}_R d(\hat{\eta})\Psi_L = \sum_\alpha \bar{\Psi}_{\alpha R} m_\alpha \Psi_{\alpha L}$$

with $m_\alpha = \mathfrak{z} e^{\hat{\eta}_\alpha}$ $\Rightarrow$ **different masses**

7 **RG-flow** should determine V. Start from ordinary gauge theory.

Study individual mechanisms in simplified models.

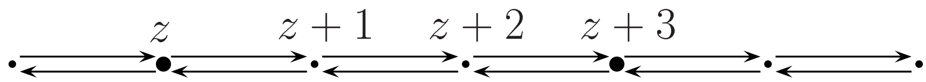
here:

- RG-flow of the Higgs potential in a 1-dimensional model, where only the extra direction is retained.
- Sketch the expected influence of the directions 0...3.
- Kobayashi Maskawa matrix is determined by the ground state of the full theory, including the boundaries R,L.

A simplified 1-dimensional model

retain only the extra dimension

1-dimensional Euclidean Lattice with fermions: Λ^0 with lattice spacing $a_0 = 1$, $z \in \mathbb{Z}$. Gauge group $\mathrm{G} = \mathrm{SU}(2)$ [or $\mathrm{U}(2)$].



Fermi-fields $\psi(z), \bar{\psi}(z)$ on the sites

- one spin component, two color components
- $\phi \in$ holonomy group $\mathrm{H} \subseteq \mathrm{GL}(2,\mathbb{C})$ act on them

Nonunitary parallel transporters = lattice gauge fields

$$\phi(z, z+1) = \phi(z+1, z)^* \in \mathrm{H}$$

ϕ transforms according to the fundamental representation of the gauge group $\mathrm{G} = \mathrm{SU}(2)$ and is integrated using the Haar measure on H.

$$\begin{aligned} \phi(z, z+1) &\to u(z)\phi(z, z+1)u(z+1)^{-1} \\ \psi(z) &\to u(z)\psi(z) \\ \bar{\psi}(z) &\to \bar{\psi}(z)u(z)^{-1} \end{aligned}$$

Initial action

$$S^0(\psi, \bar{\psi}, \phi) = \sum_z S^0(z, z+1) = \sum_z \left(S_F^0(z, z+1) - V(\phi(z, z+1)\right)$$

$$S_F^0(z, z+1) = \mathfrak{z}\bar{\psi}(z)\phi(z, z+1)\psi(z+1) + \bar{\mathfrak{z}}\bar{\psi}(z+1)\phi(z+1, z)\psi(z)$$

is required to be invariant under SU(2) gauge transformation ($\mathfrak{z}$ constant).

Higgs potential $\quad V(\phi)$ is real function of $\phi \in \mathrm{H}$

$$V(u_1\phi u_2) = V(\phi) \quad \text{for} \quad u_1, u_2 \in \mathrm{SU}(2)$$

Boltzmann factor

$$B^0(\psi, \bar{\psi}, \phi) = \prod_z B_F^0(z, z+1) e^{-V^0(\phi(z,z+1))} = e^{S^0}$$

Real space renormalization group (RG) flow

After n RG steps, the model lives on a chain $\Lambda^n \subset \Lambda^0$ of lattice spacing $a_n = l^n a_0$ with fields $\psi^n, \bar{\psi}^n, \phi^n(z, z + a_n)$.

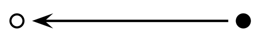
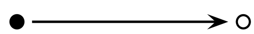
Decimation $\Rightarrow \psi^n(z) = \psi(z)$ for $z \in a_n\mathbb{Z}$, others integrated out.

Choose $l = 3$, since for $l = 2$ the effective Boltzmannian has no logarithm. (It has no term of 0th order in fermi fields.)

The effective actions remain strictly local.

$$S^n(\psi^n, \bar{\psi}^n, \phi) = \sum_{z \in \Lambda^n} S^n(z, z + a_n)$$

Notation of graphical representation of terms appearing:

$\bar{\psi}(z)\phi^n(z, z + a_n)\psi(z + a_n)$	$= L^n(z, z + a_n)$	
$\bar{\psi}(z + a_n)\phi^n(z + a_n, z)\psi(z)$	$= \bar{L}^n(z, z + a_n)$	
$\frac{1}{4}\bar{\psi}(z)\epsilon\bar{\psi}(z)\psi(z + a_n)\epsilon\psi(z + a_n)$	$= k^n(z, z + a_n)$	
$\frac{1}{4}\bar{\psi}\epsilon\bar{\psi}(z + a_n)\psi\epsilon\psi(z)$	$= \bar{k}^n(z, z + a_n)$	

n = number of RG-steps

In general case for arbitrary holonomy group H nonbilinear fermionic actions will be produced by RG flow and the integration of bosonic nonunitary parallel transporters.

Because of the Fierz-identities, the most general form of the effective Boltzmannian after n steps is

$$\begin{aligned} B^n(z, z+a_n) =& c_0^n + c_1^n L^n + \bar{c}_1^n \bar{L}^n + c_2^n k^n + \bar{c}_2^n \bar{k}^n \\ & + c_3^n L^n \bar{L}^n + c_4^n k^n \bar{L}^n + \bar{c}_4^n \bar{k}^n L^n + c_5^n k^n \bar{k}^n \end{aligned}$$

where $c_0^n = \exp\big(- V^n(\phi^n(z, z+a_n))\big)$.

The coefficients c_i^n are functions of $\phi^n(z, z+a_n)$

$$c_i^n = c_i^n(z, z+a_n) = c_i^n[\phi(z, z+a_n)].$$

The RG-flow will determine the dependence of c_i^n on the number n of RG-steps. The action is obtained by taking the logarithm and has the same form. The initial Boltzmannian is of the above form.

$$\Rightarrow S_F^0(z, z+1) = \mathfrak{z} L + \bar{\mathfrak{z}} \bar{L}$$

$$\begin{aligned} B_F^0(z, z+1) =& 1 + \mathfrak{z} L + \bar{\mathfrak{z}} \bar{L} - \mathfrak{z}^2 [\det \phi] k - \bar{\mathfrak{z}}^2 [\det \phi^*] \bar{k} \\ & + \mathfrak{z}\bar{\mathfrak{z}} L \bar{L} - \mathfrak{z}^2 \bar{\mathfrak{z}} [\det \phi] k \bar{L} - \bar{\mathfrak{z}}^2 \mathfrak{z} [\det \phi^*] \bar{k} L \\ & + \mathfrak{z}^2 \bar{\mathfrak{z}}^2 [\det \phi][\det \phi^*] \end{aligned}$$

Berezin integration of the Fermi fields $\psi, \bar{\psi}$: Four Grassmann variables $\psi_\alpha, \bar{\psi}_\alpha, \alpha = 1, 2$ are attached to each $z \in \Lambda^0$.

$$\int d^2\psi \psi_\alpha = 0, \quad \int d^2\psi \psi_\alpha \psi_\beta = \epsilon_{\alpha\beta}$$

Performing the integration

$$\int (k\bar{k})_2 \qquad = 1$$

$$\int L_1(\bar{L}k)_2 L_3 = L^1 \qquad = L^1$$

$$\int \bar{L}_1(L\bar{k})_2 \bar{L}_3 \qquad = \bar{L}^1$$

$$\int k_1 k_2 k_3 = k^1 \qquad = k^1$$

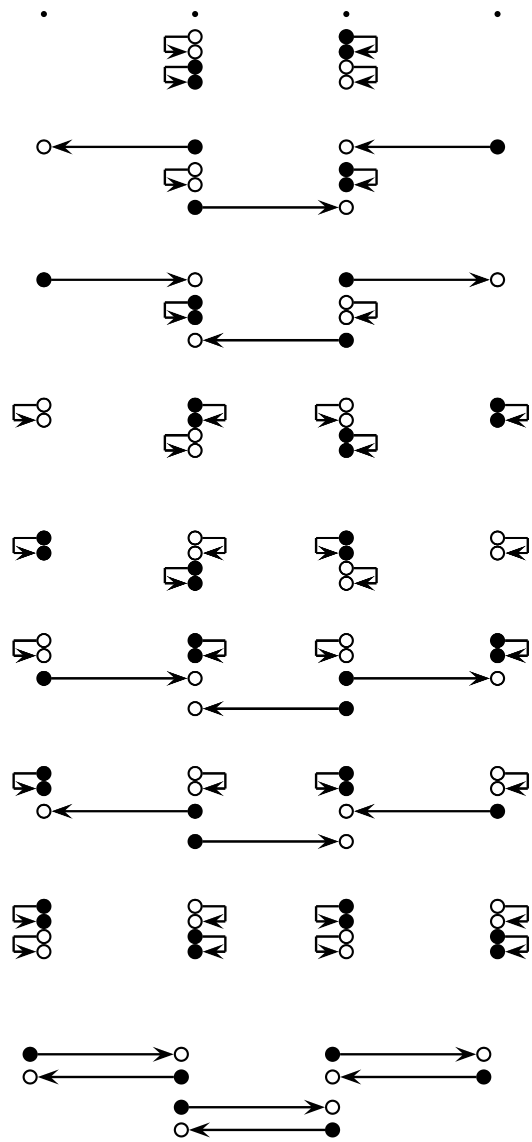
$$\int \bar{k}_1 \bar{k}_2 \bar{k}_3 \qquad = \bar{k}^1$$

$$\int (\bar{L}k)_1 L_2 (\bar{L}k)_3 \qquad = \bar{L}^1 k^1$$

$$\int (L\bar{k})_1 \bar{L}_2 (L\bar{k})_3 \qquad = L^1 \bar{k}^1$$

$$\int (k\bar{k})_1 (k\bar{k})_3 \qquad = k^1 \bar{k}^1$$

$$\int (L\bar{L})_1 (L\bar{L})_2 (L\bar{L})_3 = (L^1 \bar{L}^1)$$



Recursions relations for the coefficients in the semieffective Boltzmann factor

$$\hat{c}_i^{n+1}(z, z+3a_n) = \sum_{jkl} d^i_{jkl} c^n_j(z, z+a_n) c^n_k(z+a_n, z+2a_n) c^n_l(z+2a_n, z+3a_n)$$

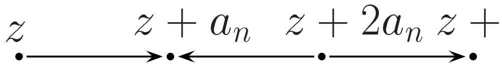
with one non vanishing term $d^i_{jkl} = 1$ in the sum $\sum_{jkl}$ for $i = 0, 1, \bar{1}, 2, \bar{2}, 3, 4, \bar{4}, 5$

e.g. $\quad \hat{c}_0^{n+1}(z, z+3a_n) = c^n_0(z, z+a_n) c^n_5(z+a_n, z+2a_n) c^n_0(z+2a_n, z+3a_n)$

The integration of the bosons

In the semieffective Boltzmannian $\hat{B}^{n+1}$, the fermionic terms (but not the Higgs potential) depend on ϕ^n only through the following **blockspin**

$$\phi^{n+1}(z, z+3a_n) =$$
$$\phi^n(z, z+a_n)\epsilon\phi^n(z+2a_n, z+a_n)^T\epsilon^T\phi^n(z+2a_n, z+3a_n)$$



For the first step:
The semieffective Boltzmannian $\hat{B}^1$ still depends on the Higgs fields ϕ on the links of the original lattice. Integrate out the three lattice gauge fields $\phi_1 = \phi(z, z+a_n)$, $\phi_2 = \phi(z+a_n, z+2a_n)$ and $\phi_3 = \phi(z+2a_n, z+3a_n)$, holding the blockspin ϕ^1 fixed.

$$B^{n+1} = \int d\phi_1 d\phi_2 d\phi_3 \delta(\phi_1\epsilon\phi_2^{*T}\epsilon^T\phi_3\phi^1)\hat{B}_1^n\hat{B}_2^n\hat{B}_3^n$$

The RG-flow produces **recursions relations** for the coefficients c_j^n in the effective Boltzmannian after n RG-steps, e.g.

$$c_0^{n+1}[\Phi] = c_0^n[\Phi] * c_5^n[\Phi] * c_0^n[\Phi]$$
$$c_5^{n+1}[\Phi] = c_5^n[\Phi] * c_0^n[\Phi] * c_5^n[\Phi]$$

Their solution in terms of initial values c_j^0 is

$$\begin{aligned}
c_0^n[\Phi] &= c_0^0 * (c_5^0 * c_0^0)^{*N}[\Phi], \\
c_1^n[\Phi] &= c_1^0 * (\bar{c}_4^0 * c_1^0)^{*N}[\Phi]\ , \\
c_2^n[\Phi] &= c_2^0 * (\bar{c}_2^0 * c_2^0)^{*N}[\Phi]\ , \\
c_3^n[\Phi] &= c_3^0 * (c_3^0 * a_3^0)^{*N}[\Phi]\ , \\
c_4^n[\Phi] &= c_4^0 * (\bar{c}_1^0 * c_4^0)^{*N}[\Phi], \\
c_5^n[\Phi] &= c_5^0 * (c_0^0 * c_5^0)^{*N}[\Phi]\ ,
\end{aligned}$$

with $N = \frac{1}{2}(3^n - 1)$ and $(c)^{*N} = c * c * \ldots * c$ (N factors) for the N-fold convolution product.

Remember convolution product on a group H:

$$f_1 * f_2(a) = \int_H da' f_1(a') f_2(a a'^{-1}) = \int_H da_1 da_2 f_1(a_1) f_2(a_2) \delta(a_1 a_2 a^{-1})$$

Further analysis of the RG-flow

Apply **harmonic analysis** on H to convert convolution products of G-biinvariant functions into ordinary products.

SU(2)-biinvariant functions $f(\phi)$ on the group H

- fulfill $f(u_1 \phi u_2) = f(\phi) \forall u_1, u_2 \in \mathrm{SU}(2)$, $\phi \in \mathrm{H}$
  $\Rightarrow$ The coefficients $c_i^n(\phi)$ as well as the Higgs potential $V(\phi)$ and thus the Boltzmannian are SU(2)-biinvariant.
- $f(\phi)$ can be expanded in spherical functions $P^{\Xi}(\phi)$

  $$f[\phi] = \int d\Xi \tilde{f}(\Xi) P^{\Xi}(\phi)$$

  where Ξ are the unitary irreducible representations of H, $d\Xi$ is the restriction of the Plancherel measure on H to representations that contribute to the expansion of SU(2)-biinvariant functions. The inversion formula reads

  $$\tilde{f}(\Xi) = \int_H d\phi f(\phi) P^{\Xi}(\phi^{-1}).$$

IR-behaviour: Asymptotic behaviour after many RG-steps

- **convolution product** $e^{-V^n[\phi]} = c_0^0 * (c_5^0 * c_0^0)^{*N}[\phi]$
- **central limit theorem**
  gives asymptotic behaviour of such convolution products, when $c_j^0[\phi] d\phi$ are normalizable measures

  $$c_0^0 = e^{-V^0} \qquad\qquad c_5^0 = |\mathfrak{z}|^4 |\det \phi|^2 e^{-V^0}$$

Classical result:

Lemma (Fourier Transform of normalized measures $\mathbb{R}$)
Let f be an integrable and square integrable positive definite function on the real line with $\int f(x)dx = 1$. Then the modulus of its Fourier transform $\tilde{f}(k) = \int e^{ikx} f(x)dx$ has a unique absolute maximum at $k = 0$ with $\tilde{f}(0) = 1$.

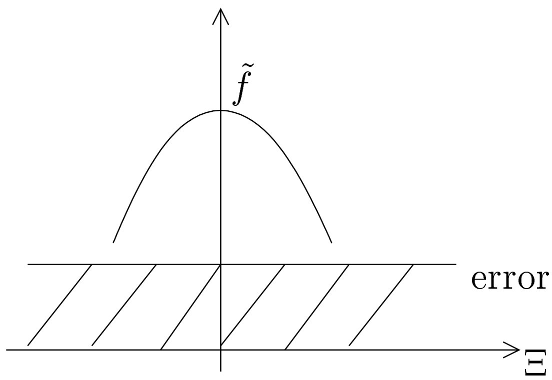
Generalization to H:

Under appropriate conditions ("good functions f" on H) the Fourier coefficient $\tilde{f}(\Xi)$ for normalized measures $f d\Phi$ has an isolated absolute maximum at a special representation Ξ_0, hence $[\tilde{f}(\Xi)]^N$ is nearly Gaussian.

Central limit theorem:

$\Rightarrow$ therefore $\underbrace{f * \ldots * f}_{N \text{ factors}}(\Xi)$ can be approximated by the inverse

Fourier transform of a Gaussian. $\Rightarrow \tilde{f}(\Xi) = \exp(-\frac{b}{2}\Xi^2) + \ldots$



For H = SL(2,$\mathbb{C}$), $\Xi = (l, \rho)$ with $l = 0$, ρ real. Read Ξ as ρ.

Results for the asymptotic behaviour of the Higgs potential

H $=$ **SL(2,$\mathbb{C}$)** $\ni a = u_1 d(\eta) u_2, u_1, u_2 \in \mathrm{SU}(2), d(\eta) = \begin{pmatrix} e^{-\frac{\eta}{2}} & 0 \\ 0 & e^{\frac{\eta}{2}} \end{pmatrix}$

$$e^{-V^n(d(\eta))} \sim \frac{1}{(2\pi B)^{\frac{3}{2}}} \frac{\eta}{4\sinh\eta} e^{-\frac{1}{8B}\eta^2} \quad B = \mathcal{O}(N = 3^n)$$

For $n \to \infty$ the Gaussian factor $\to 1$.

H $= \mathbb{R}_+$ **SU(2)** $\ni \phi = ru, r \in \mathbb{R}_+, u \in \mathrm{SU}(2)$

$$e^{-V^n} \sim \frac{\mathcal{N}}{(2\pi B)^{\frac{1}{2}}} \exp[-\frac{1}{2B}\ln^2(re^A)] \quad A, B = \mathcal{O}(N = \frac{1}{2}(3^n + 1))$$

Transform A to 0 by modifying the blockspin

$\phi^{n+1}(z, z + 3a_n) = Z^{\frac{1}{2}} \phi^n(z, z + a_n)\epsilon\phi^n(z + 2a_n, z + a_n)^T \epsilon\phi^n(z + 2a_n, z + 3a_n).$

H $=$ **GL(2,$\mathbb{C}$)** $\ni \phi = \zeta a, a \in \mathrm{SL}(2,\mathbb{C}), \zeta \in \mathbb{C}\backslash\{0\}$

"combines" $\mathbb{R}_+$ SU(2) and SL(2,$\mathbb{C}$)

The asymptotic behaviour is independent of the phase of ζ:

$$\exp[-V(\zeta a)] = \exp[-V(\zeta d(\eta))] \overset{n\to\infty}{=} \exp[-V(|\zeta| d(\eta))]$$

$$e^{-V^n(\zeta d(\eta))} \sim \frac{\mathcal{N}}{(2\pi B')^{\frac{3}{2}}(2\pi B)^{\frac{1}{2}}} \exp[-\frac{1}{8B'}\eta^2 - \frac{1}{2B}\ln^2(|\zeta| e^A)]$$

UV-Fixpoint $\xrightarrow{\textbf{RG-flow}}$ IR-Fixpoint

There is an **UV-fixpoint** describing a SU(2) or U(2) gauge theory with fermions and unitary parallel transporters $\phi(z, z+1) \in$ SU(2) or U(2).

For G = SU(2) and H = $\mathbb{R}_+$ SU(2) the Higgs potential becomes flat in the limit of large n, it tends to a definite function for H = SL(2,$\mathbb{C}$) with G = SU(2) and H = GL(2,$\mathbb{C}$) with G = U(2).
In all cases there is a **flow away from this fixpoint towards an IR-fixpoint**.

Beyond one dimension

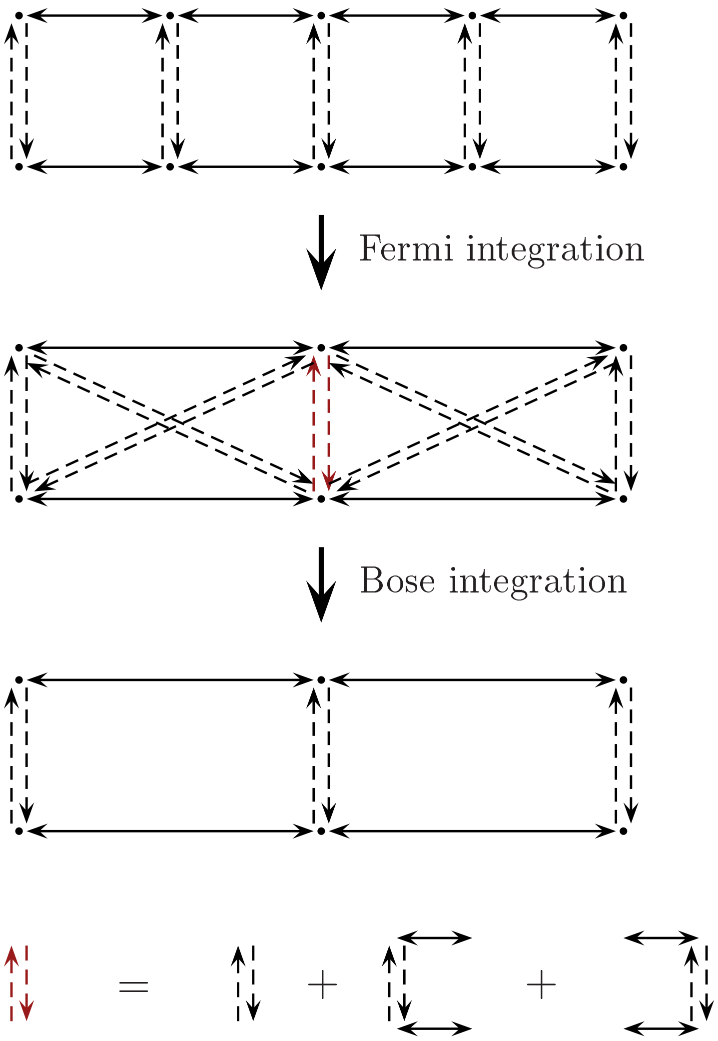
How can the usual 4 dimensions be involved?

e.g. $\mathrm{G} = \mathrm{SU}(2)$, $\mathrm{H} = \mathbb{R}_+\mathrm{SU}(2)$, $\phi = ru$, $u \in \mathrm{SU}(2)$, $r \equiv$ “**length**”

$\underbrace{\textbf{Gauge theory}}_{\textbf{fixpoint}} \implies$ **Higgs model with Higgs potential**

Fluctuations of r are needed to start the flow in one dimension but not in more dimensions.

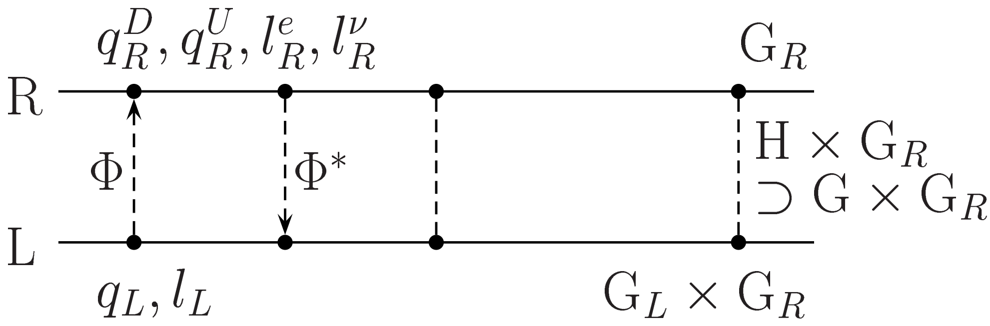
Coarsening: Summation leads to length $\neq 1$.



Will we be forced into a Higgs model?

It may be possible **or not** to reduce to ordinary gauge theory by integrating out the length r at some scale. It depends on whether the resulting effective action stays local after the r-integration.

LMP-model:



G_R: right-handed gauge group $\mathrm{G}_R = \mathrm{U}(1)$
with two 1-dimensional representation spaces

$\mathrm{G}_L{\times}\mathrm{G}_R$: left-handed gauge group $\mathrm{G}_L \times \mathrm{G}_R = \mathrm{SU}(2) \times \mathrm{U}(1)$,
with 2-dimensional representation space

$\mathrm{G}{\times}\mathrm{G}_R$: unitary gauge group of the bulk $\mathrm{G} \times \mathrm{G}_R$, $\mathrm{G} = \mathrm{SO}(3) \times \mathrm{SU}(3)$

$\mathrm{H}{\times}\mathrm{G}_R$: bulk holonomy group $\mathrm{H} = \mathbb{R}_* \, \mathrm{H}^D \times \mathbb{R}_* \, \mathrm{H}^U$
$\mathrm{H}^D = \mathrm{SL}(3,\mathbb{R})$, $\mathrm{H}^D = \mathrm{SL}(3,\mathbb{C})$ or vice versa

Salient feature:
G_L and G can both act on q_L but their action does not commute.

$\Rightarrow$ $\bar{q}_L \gamma^\mu U_{L\mu} q_L$-terms on L-boundary break the bulks G-symmetry.

$q_L = (d_L, u_L; s_L, c_L; b_L, t_L)$

$$\Phi = \begin{pmatrix} \phi^D & 0 \\ 0 & \phi^U \end{pmatrix} = diag(\rho^D \phi^D u_R^D, \rho^U \phi^U u_U^D)$$

$$\phi_D = \mathcal{A}_L^D d(\eta^D) \mathcal{A}_R^{D*} \in \mathrm{H}^D, \;\; \phi_U = \mathcal{A}_L^U d(\eta^U) \mathcal{A}_R^{U*} \in \mathrm{H}^U$$

The **Kobayashi Maskawa matrix** $C = \mathcal{A}_L^D \mathcal{A}_L^{U*}$ could be transformed away if it were not for the G-symmetry breaking by L-boundary.

C is determined by ground state of full theories (bulk and boundary).