problem 1:

$$d\tau^2 = A\,dt^2 - \frac{1}{A}\,dr^2 - \underbrace{r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)}_{\text{unchanged}}$$

$$2r'dr' - 2t'dt' = T^2 \exp\left(\frac{r}{2GM}\right)\frac{r}{(2GM)^2}\,dr$$

$$(2dr't' - 2dt'r')\,\frac{(t'^2 - r'^2)}{(r'^2 + t'^2)^2}$$

$$= \frac{dt}{2MG}\,\frac{1}{\cosh^2}$$

$$dr = 2(r'dr' - t'dt')\,B$$

$$B = \frac{1}{T^2}\exp\left(-\frac{r}{2GM}\right)\frac{(2GM)^2}{r}$$

$$dt = 2(dr't' - dt'r')\,C$$

$$C = \frac{(t'^2 - r'^2)}{(r'^2 + t'^2)^2}\,(2MG)\,\cosh^2$$

$$\left(\cosh^2 = \frac{1}{1 - \tanh^2}\right) \qquad = \frac{2MG}{t'^2 - r'^2}$$

$$d\tau^2 = A\,dt^2 - \frac{1}{A}\,dr^2$$

$$= A\,C^2\left(dr't' - dt'r'\right)^2 \cdot 4$$

$$- \frac{1}{A}\,B^2\left(r'dr' - t'dt'\right)^2 \cdot 4$$

$$= dr'dt'\left[AC^2(-2\,t'r') + B^2/A\,(2\,r't')\right]$$

$$+ dr'^2\left(4AC^2\,t'^2 - \frac{4}{A}B^2 r'^2\right)$$

$$= 4dr'^2\,(-BC)\,(t'^2 - r'^2)$$

$$+ 4dt'^2\,(-BC)\,(r'^2 - t'^2)$$

$$A^2C^2 = B^2$$

$$\boxed{B = \pm A \cdot C}$$ ✓

$$\left(1 - \frac{2GM}{r}\right)\frac{2MG}{t'^2 - r'^2} = \pm\frac{1}{T^2}\underbrace{\exp\left(-r/_{2GM}\right)}_{T^2\left(\frac{r}{2GM} - 1\right)\frac{1}{r'^2 - t'^2}}\frac{(2GM)^2}{r}$$

$$-4BC\,(r'^2 - t'^2) = \frac{32\,(MG)^3}{r\,T^2}\exp\left(-\frac{r}{2MG}\right)$$

$$B = \frac{1}{T^2}\exp\left(-\frac{r}{2MG}\right)\frac{(2GM)^2}{r}$$

$$C = -\frac{2MG}{(r'^2 - t'^2)}$$

$r \to 0 \quad r'^2 - t'^2 \to -T^2$

$r \to 2MG \quad r'^2 - t'^2 = 0$

There is still a singularity in r->0
but no singularity for r->2MG

problem 2:

$$d\tau^2 = A(r)\,dt^2 - B(r)\,dr^2 - r^2 d\Omega$$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left\{ \frac{\partial g_{\mu\rho}}{\partial x^\nu} + \frac{\partial g_{\nu\rho}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right\}$$

$$\frac{\partial^2 x^\mu}{\partial \lambda^2} + \Gamma^\mu_{\kappa\nu} \frac{\partial x^\nu}{\partial \lambda} \frac{\partial x^\kappa}{\partial \lambda} = 0$$

$$g_{\mu\nu} = \begin{pmatrix} -A(r) & & & \\ & +B(r) & & \\ & & +r^2 & \\ & & & +\sin^2\theta\, r^2 \end{pmatrix}$$

$$\frac{\partial g_{rr}}{\partial r} \qquad \frac{\partial g_{tt}}{\partial r} \qquad \frac{\partial g_{\theta\theta}}{\partial r} \qquad \frac{\partial g_{\varphi\varphi}}{\partial r} \qquad \frac{\partial g_{\varphi\varphi}}{\partial \theta}$$

$$\Gamma^r_{rr} = \frac{B'}{2B}$$

$$\Gamma^r_{tt} = \frac{1}{2}\frac{A'}{B} \;; \qquad \Gamma^t_{rt} = \Gamma^t_{tr} = \frac{1}{2}\frac{A'}{A}$$

$$\Gamma^r_{\theta\theta} = -\frac{r}{B} \;; \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$$

$$\Gamma^r_{\varphi\varphi} = -\sin^2\frac{r}{B} \;; \qquad \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}$$

$$\Gamma^\theta_{\varphi\varphi} = -\cos\theta\sin\theta \qquad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \frac{\cos\theta}{\sin\theta}$$

geodesic equations can be found below.

problem 3:

$$\mathcal{L} = \frac{1}{2}\left(-A\left(\frac{\partial t}{\partial\lambda}\right)^2 + B(r)\left(\frac{\partial r}{\partial\lambda}\right)^2 + r^2\left(\frac{\partial\theta}{\partial\lambda}\right)^2 + r^2\sin^2\theta\left(\frac{\partial\varphi}{\partial\lambda}\right)^2\right)$$

$$\partial_\lambda \frac{\delta\mathcal{L}}{\delta\left(\frac{\partial t}{\partial\lambda}\right)} - \frac{\partial\mathcal{L}}{\partial t} = 0$$

$$0 = \partial_\lambda\left(-A\frac{\partial t}{\partial\lambda}\right) = -A\frac{\partial^2 t}{(\partial\lambda)^2} - A'\frac{\partial t}{\partial\lambda}\frac{\partial r}{\partial\lambda}$$

$$\partial_\lambda \frac{\delta\mathcal{L}}{\partial\left(\frac{\partial r}{\partial\lambda}\right)} - \frac{\partial\mathcal{L}}{\partial r} = 0$$

$$0 = \partial_\lambda\left(B\frac{\partial r}{\partial\lambda}\right) - \frac{1}{2}\left[-A'\left(\frac{\partial t}{\partial\lambda}\right)^2 + B'\left(\frac{\partial r}{\partial\lambda}\right)^2 + 2r\left(\frac{\partial\theta}{\partial\lambda}\right)^2 + 2r\sin^2\theta\left(\frac{\partial\varphi}{\partial\lambda}\right)^2\right]$$

$$0 = B\frac{\partial^2 r}{(\partial\lambda)^2} + \frac{1}{2}B'\left(\frac{\partial r}{\partial\lambda}\right)^2 + \frac{1}{2}A'\left(\frac{\partial t}{\partial\lambda}\right)^2 - r\left(\frac{\partial\theta}{\partial\lambda}\right)^2 - r\sin^2\theta\left(\frac{\partial\varphi}{\partial\lambda}\right)^2$$

$$\partial_\lambda \frac{\delta\mathcal{L}}{\partial\left(\frac{\partial\theta}{\partial\lambda}\right)} - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

$$0 = \frac{\partial}{\partial\lambda}\left(r^2\frac{\partial\theta}{\partial\lambda}\right) - r^2\cos\theta\sin\theta\left(\frac{\partial\varphi}{\partial\lambda}\right)^2$$

$$= r^2\frac{\partial^2\theta}{(\partial\lambda)^2} + \frac{\partial\theta}{\partial\lambda}\frac{\partial r}{\partial\lambda}2r - r^2\cos\theta\sin\theta\left(\frac{\partial\varphi}{\partial\lambda}\right)^2$$

$$\partial_\lambda\left(\frac{\delta\mathcal{L}}{\frac{\partial\theta}{\partial\lambda}}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} : \quad \partial_\lambda\left(\sin^2\theta\, r^2\frac{\partial\varphi}{\partial\lambda}\right) = 0$$

$$0 = \sin^2\theta\, r^2\frac{\partial^2\varphi}{(\partial\lambda)^2} + 2\sin\theta\cos\theta\, r^2\frac{\partial\theta}{\partial\lambda}\frac{\partial\varphi}{\partial\lambda} + 2r\sin^2\theta\frac{\partial\varphi}{\partial\lambda}\frac{\partial r}{\partial\lambda}$$